On collineations and dualities of finite generalized polygons

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Abstract

In this paper we generalize a result of Benson to all finite generalized polygons. In particular, given a collineation θ of a finite generalized polygon $\mathcal{S}$, we obtain a relation between the parameters of $\mathcal{S}$ and, for various natural numbers i, the number of points x which are mapped to a point at distance i from x by θ. As a special case we consider generalized $2n$-gons of order $(1, t)$ and determine, in the generic case, the exact number of absolute points of a given duality of the underlying generalized n-gon of order t.

1   Introduction

Given a finite generalized quadrangle of order (s, t), and a collineation θ, there is a connection between the parameters s, t, the number f_0 of fixed points and the number f_1 of points mapped under θ to collinear points, given by Benson's theorem [1]:

$$(1 + t)f_0 + f_1 \equiv 1 + st \mod s + t.$$

The natural question arising here is whether there exists a similar formula for a *duality* of a generalized quadrangle $\mathcal{S}$ of order s. Of course, a duality cannot fix points or lines,

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but elements can be mapped to elements at distance 1 or 3. In the former case, we say that these elements are *absolute*. It will turn out that, in the generic case, i.e., in the case where $2s$ is not a square, there are always exactly $1 + s$ absolute points, for every duality. This yields an alternative proof for the fact that polarities in $\mathcal{S}$ can only exist if $2s$ is a square (as a geometric argument immediately shows that there must be $1 + s^2$ absolute points).

More generally, one can ask for similar restrictions on collineations and dualities for an arbitrary finite generalized polygon. This is exactly what we are going to do in the present paper. Roughly speaking, we first generalize Benson's formula to finite generalized hexagons, octagons and dodecagons with parameters s, t, and then show how these can be used to obtain new results on dualities in projective planes, generalized quadrangles and hexagons. The extension of Benson's formula has a straightforward part, but also a less trivial observation which precisely allows us to draw some rather strong conclusions in the case of dualities, thereby producing new results even for finite projective planes.

The main application of our results lies in the classification of finite generalized polygons whose collineation or duality group satisfies some given transitivity property like flag-transitivity, or point- or line-sharply-transitivity. For an explicit application, see [10].

2   Notation and Main Results

Throughout, we shall deal with finite generalized polygons of order (s, t). A *generalized n-gon* $\mathcal{S} = (\mathcal{P}, \mathcal{L} \text{ I})$ *of order* (s, t) is a point-line geometry with point set $\mathcal{P}$, line set $\mathcal{L}$ and symmetric incidence relation I, such that every line is incident with $1 + s$ points, every point with $1 + t$ lines, the diameter of the incidence graph (i.e. the graph with vertex set $\mathcal{P} \cup \mathcal{L}$ and adjacency given by I) is n and its girth is $2n$. Here, $n \geq 2$, and for $n = 2$, the incidence graph is a complete bipartite graph and hence $\mathcal{S}$ is a trivial geometry in which every point is incident with every line. So in the sequel we will always assume $n \geq 3$.

Note that in a generalized n-gon with $n \geq 3$ each line is determined by the set of points it is incident with. So, we can view a generalized n-gon $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \text{I})$ as the set of points $\mathcal{P}$ endowed with a set $\mathcal{B}$ of subsets of $\mathcal{P}$, where $\mathcal{B} = \{B \subseteq \mathcal{P} : (\exists L \in \mathcal{L})(x \in B \Leftrightarrow x \text{I} B)\}$. In this case, there is no need for an incidence relation and we denote $\mathcal{S} = (\mathcal{P}, \mathcal{B})$. Related to this point of view is the *point graph* of $\mathcal{S}$, which is the graph with vertex set $\mathcal{P}$ and adjacency given by *collinearity* (two points x, y are collinear, in symbols $x \sim y$, if they are distinct and incident with a common line).

If $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \text{I})$ is a generalized polygon of order (s, t), then $\mathcal{S}^{\text{dual}} = (\mathcal{L}, \mathcal{P}, \text{I})$ is also a

generalized polygon, now of order (t, s), called the *dual* of $\mathcal{S}$. This gives rise the *principle of duality*, which states that every theorem in the theory of generalized polygons can be dualized by interchanging the roles of $\mathcal{P}$ and $\mathcal{L}$, and of s and t. If $s = t$, then we say that $\mathcal{S}$ has order s.

Generalized 3-gons of order (s, t) only exist for $s = t$ and are exactly projective planes of order $s = t$. Generalized 4-gons are usually called generalized quadrangles and these structures have been investigated in detail; see [9]. Generalized 6-gons, 8-gons and 12-gons are also called generalized hexagons, octagons and dodecagons, respectively. We refer to [12] and [15] for extensive surveys on (finite and infinite) generalized polygons. We content ourselves here with mentioning that generalized polygons were introduiced by Jacques Tits [13] in 1959, and that they are the spherical buildings of rank 2, and as such they provide the natural geometries for Chevalley groups of rank 2, and, more generally, algebraic, classical and mixed groups, and twisted versions, all of relative rank 2.

The main result on finite generalized polygons is due to Feit & Higman [6] and states that for a finite generalized $n - gon$, $n \geq 3$, of order (s, t), we always have either $n = 3$ (and then $s = t$), or $n = 4$ (and then $s + t$ divides $st(1 + st)$), or $n = 6$ (and then st is a perfect square), or $n = 8$ (and then $2st$ is a perfect square), or $n = 12$ (and then $1 \in \{s, t\}$). Each generalized n-gon $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ of order (s, s) gives rise to a unique generalized $2n$-gon $2\mathcal{S} = (\mathcal{P} \cup \mathcal{L}, \mathcal{E})$ of order $(1, s)$, called the *double of* $\mathcal{S}$, where $\mathcal{E}$ is the set of *flags* of $\mathcal{S}$ (and a flag is an incident point-line pair). This construction has an inverse which is unique up to duality (see [14], cp. [15], Theorem 1.6.2).

A *collineation* θ of a generalized polygon $\mathcal{S} = (\mathcal{P}, \mathcal{B})$ is a permutation of $\mathcal{P}$ that induces a permutation of $\mathcal{B}$. If $\mathcal{S}' = (\mathcal{P}', \mathcal{B}')$ is a second generalized polygon, then an *isomorphism* $\sigma : \mathcal{S} \to \mathcal{S}'$ is a bijection from $\mathcal{P}$ to $\mathcal{P}'$ that induces a bijection from $\mathcal{B}$ to $\mathcal{B}'$. A *duality* of $\mathcal{S}$ is an isomorphism from $\mathcal{S}$ to $\mathcal{S}^{\mathrm{dual}}$. In this case we call $\mathcal{S}$ *self-dual*. Note that for a self-dual generalized polygon of order (s, t) necessarily holds that $s = t$.

If θ is a duality of $\mathcal{S}$, then an *absolute* element for θ is a point or a line incident with its image.

Now let θ be a collineation of a finite generalized n-gon $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ with parameters s, t. Let f_i, $0 \leq i \leq \frac{n}{2}$, be the number of points x of $\mathcal{S}$ that are mapped under θ onto a point ate distance i from x, measured in the point graph (or, equivalently, distance $2i$ in the incidence graph). Also, the point graph has one eigenvalue (namely, $s(1 + t)$) with multiplicity 1, it has always an eigenvalue equal to $-1 - t$ and it has $\frac{n}{2} - 1$ eigenvalues different from $-1 - t$ with multiplicity > 1. We denote the latter with ξ_j, $1 \leq j \leq \frac{n}{2} - 1$.

Benson's theorem [1] says that, if $n = 4$, then $(1 + t)f_0 + f_1 \equiv 1 + st \mod s + t$. Equivalently, there exists an integer k_1 such that $1 + st + k_1(s + t) = (1 + t)f_0 + f_1$. Our

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Main Result generalizes this equivalent formulation as follows:

Main Result *With the above notation, for $n \in \{4, 6, 8, 12\}$, there exist integers k_j, $1 \leq j \leq \frac{n}{2} - 1$, and, for each $k \in \{1, 2, \ldots, \frac{n}{2} - 1\}$, explicitly defined polynomial expressions $P_{k,i}(s,t)$, $0 \leq i \leq k - 1$, in s and t, such that*

$$(1+s)^k (1+t)^k + \sum_{j=1}^{n/2-1} k_j (\xi_j + 1 + t)^k = f_k + \sum_{i=0}^{k-1} P_{k,i}(s,t) f_i.$$

We explicitly determine the expressions $P_{k,i}$ in this theorem below, for each n separately: for $n = 4$, see Theorem 4.1 (this is Benson's theorem); for $n = 6$, see Theorems 5.1 and 5.4; for $n = 8$, see Theorems 6.1, 6.4 and 6.5; for $n = 12$, see Theorems 7.1, 7.2, 7.3, 7.4 and 7.5.

Although the expressions are, especially for the cases $n = 8, 12$, rather involved and cumbersome, we are able to draw some interesting conclusions. In particular we obtain strong restrictions on the number of absolute elements of a duality. Generically, we obtain the following result.

Main Corollary *Let θ be a duality of a finite generalized n-gon with parameter s. Then there is at least one absolute point for θ. Suppose now that s is not a square if $n = 3$, that $2s$ is not a square if $n = 4$, and that none of s and $3s$ are squares if $n = 6$. Then there are exactly $1 + s$ absolute points for θ.*

Much more precise information is contained in the corollaries below. The consequences of our Main Result seem endless, and we have included only a few of them. They are related to ovoids, subpolygons, involutions and dualities.

The paper is organized as follows. In Section 3, we prove some general statements about eigenvalues and multiplicities, and recall a useful result concerning the adjacency matrix of a generalized polygon. In Section 4 we repeat Benson's theorem and write down some consequences, as a warming-up for the more general cases treated in Section 5 (generalized hexagons), Section 6 (generalized octagons) and Section 7 (generalized dodecagons).

3 Some general observations

We will use the following notation. Suppose that $\mathcal{S}$ is a finite generalized polygon of order (s,t). Let v be the number of points of $\mathcal{S}$ and b the number of lines. Put $\mathcal{P} = \{x_i : 1 \leq i \leq v\}$ and $\mathcal{L} = \{L_j : 1 \leq j \leq b\}$. Let D be an incidence matrix of $\mathcal{S}$, i.e., the rows of D are

labelled by the points of $\mathcal{S}$, the columns by the lines of $\mathcal{S}$ and the (x, L)-entry of D (where $x \in \mathcal{P}$ and $L \in \mathcal{L}$) is equal to 1 if $x\mathtt{I}L$; otherwise it is 0. Then $M := DD^T = A + (t+1)I$, where A is an adjacency matrix of the point graph of $\mathcal{S}$. Let θ be an automorphism of $\mathcal{S}$ of order n and let $Q = (q_{ij})$ be the $v \times v$ matrix with $q_{ij} = 1$ if $x_i^\theta = x_j$ and $q_{ij} = 0$ otherwise (in fact Q is the permutation matrix belonging to θ with respect to the action on $\mathcal{P}$). Similarly, let $R = (r_{ij})$ be the permutation matrix belonging to θ with respect to the action of θ on $\mathcal{L}$ (so $r_{ij} = 1$ if $L_i^\theta = L_j$ and $r_{ij} = 0$ otherwise). Then $DR = QD$. Because Q and R are permutation matrices, it follows that $Q^T = Q^{-1}$ and $R^T = R^{-1}$, so we have $QM = QDD^T = DRD^T = DRR^T D^T (Q^{-1})^T = DD^T Q = MQ$. Hence $QM = MQ$. Because n is the order of θ and $QM = MQ$, we have $(QM)^n = Q^n M^n = M^n$. It follows that the eigenvalues of QM are of the form $\xi\lambda$ with λ an eigenvalue of M and ξ an nth root of unity. Note that the eigenvalue $(1+s)(1+t)$ of M is also an eigenvalue of QM with multiplicity 1.

Lemma 3.1 *Suppose that ξ and ξ' are both primitive d^{th} roots of unity, with d a divisor of n, and let λ be an eigenvalue of M. If at least one of $\xi\lambda$ and $\xi'\lambda$ is an eigenvalue of QM, than they both are and they have the same multiplicity.*

Proof. The coefficients of the characteristic polynomial of QM are integers. The minimal polynomials (over $\mathbb{Q}$) of $\xi\lambda$ and $\xi'\lambda$ coincide, hence $\xi\lambda$ and $\xi'\lambda$ have the same multiplicity. $\square$

Lemma 3.2 *Let ξ be an nth root of unity and λ an eigenvalue of M. Then the multiplicity of $\xi\lambda$ as an eigenvalue of QM is equal to the multiplicity of $\xi\lambda^j$ as an eigenvalue of QM^j, with $j = 2, 3, \cdots$.*

Proof. We first claim that there exists a basis of eigenvectors with eigenvalues $\xi\lambda$ for QM (with ξ ranging over the nth roots of unity) in the eigenspace V which corresponds with the eigenvalue λ for M. First note that, if $\vec{v} \in V$, then

$$
\begin{aligned}
MQ\vec{v} \ &= QM\vec{v} \\
&= Q\lambda\vec{v} \\
&= \lambda Q\vec{v},
\end{aligned}
$$

and hence $Q\vec{v} \in V$. So Q preserves V and induces an isometry in V, V viewed as an Euclidian space with metric $d(\vec{v}, \vec{w}) = \parallel \vec{w} - \vec{v} \parallel = \overline{(\vec{w} - \vec{v})} \cdot (\vec{w} - \vec{v})$ with $\cdot$ the ordinary dot product. Hence there exists an orthonormal basis B of V of eigenvectors of Q. Since

$Q^n = 1$, each eigenvalue ξ is an nth root of unity. If $\vec{v} \in B$ is an eigenvector for Q with eigenvalue ξ, then $MQ\vec{v} = \xi\lambda\vec{v}$. This shows our claim.

Now we will prove that the multiplicity of $\xi\lambda$ for QM is equal to the multiplicity of $\xi\lambda^j$ for QM^j, which is the dimension of the eigenspace which corresponds with this eigenvalue. Let $\vec{v}$ be an eigenvector of QM with eigenvalue $\xi\lambda$, then we have

$$
\begin{aligned}
QM^j\vec{v} &= M^{j-1}(QM\vec{v}) \\
&= M^{j-1}\xi\lambda\vec{v} \\
&= \xi\lambda M^{j-1}\vec{v} \\
&= \xi\lambda^j\vec{v}.
\end{aligned}
$$

Hence $\vec{v}$ is an eigenvector of QM^j with eigenvalue $\xi\lambda^j$. So the eigenvectors of QM with eigenvalue $\xi\lambda$ form a subset of the eigenvectors of QM^j with eigenvalue $\xi\lambda^j$. Hence, since QM, and similarly also QM^j, is diagonalizable,

$$
v = \sum_{m=1}^{n} \text{dimension of eigenspace corresponding with eigenvalue } e^{\frac{m2\pi i}{n}}\lambda^j \text{ of } QM^j
$$

$$
\geq \sum_{m=1}^{n} \text{dimension of eigenspace corresponding with eigenvalue } e^{\frac{m2\pi i}{n}}\lambda \text{ of } QM = v.
$$

Hence every two corresponding terms in these sums must be equal.

$\square$

In the sequel, we will also consider *thin* generalized polygons, by which we mean generalized $2n$-gons $\mathcal{S}$ of order $(1, t)$. As already noted, in this case $\mathcal{S}$ is the double $2\mathcal{S}'$ of a — up to duality uniquely — defined generalized n-gon $\mathcal{S}'$ of order t. Every collineation of $\mathcal{S}$ induces either a unique collineation of $\mathcal{S}'$ or a unique duality of $\mathcal{S}'$. In the first case a point x of Γ is mapped onto a point at even distance from x; in the second case a point x of Γ is mapped onto a point at odd distance from x. In the sequel we will call $\mathcal{S}'$ the *underlying generalized n-gon*.

The following lemma is well known, e.g. see [2].

Lemma 3.3 *Let A be the adjacency-matrix of a generalized $2n$-gon of order (s, t). Since for two points x and y, the (x, y)-entry only depends on the distance $2i$ between x and y, the same is true for A^k with $k \geq 1$. Hence we can denote this entry by $a_i^{(k)}$. Let p_j^i be the number of points at distance $2j$ from x and collinear to y, with x and y as above. Then $a_i^{(k+1)} = \sum_{j=0}^{n} p_j^i a_j^k$, with $i \in \{0, 1, \cdots, n\}$.*

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For a generalized n-gon p_j^i is equal to:

$$
\begin{cases}
s(t+1) & \text{if } j = 1, i = 0 \\
1 & \text{if } 0 \leq j = i - 1 \leq n - 2 \\
s - 1 & \text{if } 1 \leq j = i \leq n - 1 \\
st & \text{if } 2 \leq j = i + 1 \leq n \\
t + 1 & \text{if } j = n - 1, i = n \\
(s-1)(1+t) & \text{if } j = i = n \\
0 & \text{otherwise.}
\end{cases}
$$

We end with a straightforward observation.

Lemma 3.4 *A duality θ of a generalized polygon has as many absolute points as absolute lines.*

Proof. This follows from the fact that, if $x \mathtt{I} x^\theta$, then $x^\theta \mathtt{I} (x^\theta)^\theta$, which implies that, if x is an absolute point, then x^θ is an absolute line. It is now easy to see that θ induces a bijection from the set of absolute points tot the set of absolute lines. $\square$

4   Collineations of generalized quadrangles

At first we will have a look at a generalized quadrangle $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ of order (s,t). Let D, M, A, Q and θ be defined as before. If $M = A + (t+1)I$, then M has eigenvalues $\tau_0 = (1+s)(1+t)$, $\tau_1 = 0$ and $\tau_2 = s + t$, with respective multiplicities $m_0 = 1$, $m_1 = s^2(1 + st)/(s + t)$ and $m_2 = st(1 + s)(1 + t)/(s + t)$ (cf. Table 6.4 in [2]). Now we have the following theorem:

Theorem 4.1 (Benson [1]) *If f_0 is the number of points fixed by the automorphism θ and if f_1 is the number of points x for which $x^\theta \neq x \sim x^\theta$, then*

$$
\text{tr}(QM) = (1+t)f_0 + f_1 \quad and \quad (1+t)f_0 + f_1 \equiv 1 + st \,(\text{mod } s + t).
$$

Proof. For the proof of this theorem, we refer to [1]. $\square$

Remark 4.2 We can also write the conclusion of Theorem 4.1 as follows:

$$\mathrm{tr}(QM) = (1+t)f_0 + f_1 = k(s+t) + (1+s)(1+t)$$

Corollary 4.3 *Let $\mathcal{S}$ be a generalized quadrangle of order (s,t) and let θ be an automorphism of $\mathcal{S}$. Suppose that s and t are not relatively prime, then there exists at least one fixpoint or at least one point which is mapped to a point collinear to itself.*

Proof. Suppose that there are no fixpoints and no points which are mapped to a collinear point. Then $f_0 = f_1 = 0$. Because of the previous theorem $1 + st + k(s+t)$ has to be 0 for some k, or $st + k(s+t) = -1$. But because s and t are not relatively prime, there exists an integer $m > 1$ which divides both s and t. Hence m divides $st + k(s+t)$. But m does not divide -1 and we have a contradiction. $\square$

Corollary 4.4 *Suppose that $\mathcal{S}$ is a generalized quadrangle of order s and θ is a nontrivial automorphism of $\mathcal{S}$. If s is even, then θ cannot fix any ovoid pointwise.*

Proof. Suppose that $\mathcal{O}$ is an ovoid which is fixed pointwise by θ. Then (with the notation of the previous theorem) $f_0 = 1 + s^2 = |\mathcal{O}|$. Note that by 2.4.1 in [9] θ cannot fix anything else. Suppose that there exists a point x which is mapped to a collinear point by θ. Take a line through x different from xx^θ. This line contains a point y of the ovoid. Now the line xy is mapped to the line $x^\theta y$, hence we have a triangle which is a contradiction. So $f_1 = 0$. Because of Benson's theorem it follows that $(1+s)(1+s^2) \equiv 1 + s^2 \,(\mathrm{mod}\, 2s)$ or $s(1+s^2) \equiv 0 \,(\mathrm{mod}\, 2s)$. Hence $1 + s^2$ is even and so s has to be odd. $\square$

Let $\mathsf{Q}(4,q)$ be a nondegenerate (parabolic) quadric in the projective space $\mathsf{PG}(4,q)$, and let H be a hyperplane of $\mathsf{PG}(4,q)$ meeting $\mathsf{Q}(4,q)$ in a nondegenerate elliptic quadric $\mathsf{Q}^-(3,q)$. If q is odd, then H has a pole x (the intersection of all hyperplanes tangent to $\mathsf{Q}(4,q)$ in points of $\mathsf{Q}^-(3,q)$), and the unique involutive perspectivity of $\mathsf{PG}(4,q)$ with center x and axis H induces a nontrivial collineation in the generalized quadrangle $\mathsf{Q}(4,q)$ fixing the ovoid $\mathsf{Q}^-(3,q)$ pointwise. Hence Corollary 4.4 is not valid for s odd.

Corollary 4.5 *Suppose that $\mathcal{S}$ is a generalized quadrangle of order s and θ is a nontrivial automorphism of $\mathcal{S}$. If s is even, then θ cannot fix any thin subquadrangle of order $(1, s)$ pointwise.*

Proof. Suppose that $\mathcal{S}'$ is a thin subquadrangle of $\mathcal{S}$ of order $(1, s)$ which is fixed pointwise by θ. Then we claim $f_0 = 2(s+1) = |\mathcal{S}'|$. Indeed, if θ fixes a point not belonging to $\mathcal{S}'$, then θ would fix a subquadrangle $\mathcal{S}''$ of order (s'', s) with $\mathcal{S}' \subset \mathcal{S}'' \subseteq \mathcal{S}$. By [9] it follows

that $s \geq s''t$ so $s \geq s''s$. Hence $s'' = 1$ a contradiction. Now suppose that x is a point of $\mathcal{S}'$. There are $s + 1$ lines trough x in $\mathcal{S}'$ and on each of these lines there is precisely one other point which belongs to $\mathcal{S}'$. Take such a point y. On the line xy of $\mathcal{S}$ there are $s - 1$ other points which are mapped to each other (not fixed). So we already have $(s-1)(s+1)^2$ points which are mapped to a collinear point by θ. Suppose that there is a point z of $\mathcal{S}$ which does not lie on a line of $\mathcal{S}'$ and which is mapped to a point collinear to itself. Then by [9] the line zz^θ intersects a line of $\mathcal{S}'$ which is fixed under θ. But then the line zz^θ also has to be fixed and we obtain a contradiction. Hence $f_1 = (s-1)(s+1)^2$. Now Theorem 4.1 implies that there exists an integer k with

$$k(s+t) + (1+s)(1+t) = 2(1+s)^2 + (s-1)(s+1)^2.$$

Hence $k = \frac{(s+1)^2}{2}$ and so s is odd. $\qquad \square$

5 Collineations of hexagons and dualities of projective planes

Now we will generalize Benson's theorem for hexagons.

Suppose that $\mathcal{S}$ is a generalized hexagon. Let θ be an automorphism of $\mathcal{S}$ and let f_0 be the number of fixpoints, f_1 the number of points which are mapped on a collinear point ($\mathrm{d}(x^\theta, x) = 1$ in the point graph of $\mathcal{S}$) and f_2 the number of points which are mapped on a point at distance 2 from itself (in the point graph). The matrix M is again equal to $A + (t+1)I$, with A an adjacency matrix of the point graph of $\mathcal{S}$. And Q is the matrix with $q_{ij} = 1$ if $x_i^\theta = x_j$ and $q_{ij} = 0$ otherwise.

We start from the eigenvalues of A which are $-1-t$, $s(t+1)$, $-1+s+\sqrt{st}$ and $-1+s-\sqrt{st}$, with respective multiplicities m_0, $m_1 = 1$, m_2 and m_3. Because $M = A + (t+1)I$ the eigenvalues of M are as follows (cf. table 6.4 in [2]):

eigenvalues of M	multiplicity
0	$m_0 = \dfrac{s^3(1+st+s^2t^2)}{s^2+st+t^2}$
$(s+1)(t+1)$	$m_1 = 1$
$s+t+\sqrt{st}$	$m_2 = \dfrac{(1+t)st(1+s)(1+st+s^2t^2)}{2(s(t-1)^2+t(s-1)^2+3st+(s-1)(t-1)\sqrt{st})}$
$s+t-\sqrt{st}$	$m_3 = \dfrac{(1+t)st(1+s)(1+st+s^2t^2)}{2(s(t-1)^2+t(s-1)^2+3st-(s-1)(t-1)\sqrt{st})}$

We now have the following result.

Theorem 5.1 *Let $\mathcal{S}$ be a generalized hexagon of order (s,t) and let θ be an automorphism of $\mathcal{S}$. If f_i, $i = 0, 1$, is defined as above, then for some integers k_1 and k_2 holds*

$$k_1(s+t+\sqrt{st}) + k_2(s+t-\sqrt{st}) + (1+s)(1+t) = (1+t)f_0 + f_1.$$

Proof. Suppose that θ has order n, so that $(QM)^n = Q^n M^n = M^n$. It follows that the eigenvalues of QM are the eigenvalues of M multiplied by the appropriate roots of unity. Let J be the $v \times v$ matrix with all entries equal to 1. Since $MJ = (1+s)(1+t)J$, we have $(QM)J = (1+s)(1+t)J$, so $(1+s)(1+t)$ is an eigenvalue of QM. Because $m_1 = 1$, it follows that this eigenvalue of QM has multiplicity 1. Further it is clear that 0 is an eigenvalue of QM with multiplicity m_0. For each divisor d of n, let ξ_d denote a primitive d^{th} root of unity, and put $U_d = \sum \xi_d^i$, where the summation is over those integers $i \in \{1, 2, \ldots, d-1\}$ that are relatively prime to d. Then U_d is an integer by [7]. For each divisor d of n, the primitive d^{th} roots of unity all contribute the same number of times to the eigenvalues φ of QM with $|\varphi| = s+t+\sqrt{st}$ and also the primitive d^{th} roots of unity all contribute the same number of times to the eigenvalues φ' of QM with $|\varphi'| = s+t-\sqrt{st}$, because of Lemma 3.1. Let a_d denote the multiplicity of $\xi_d(s+t+\sqrt{st})$ and let b_d denote the multiplicity of $\xi_d(s+t-\sqrt{st})$ as eigenvalues of QM, with $d|n$ and ξ_d a primitive d^{th} root of unity. Then we have:

$$\operatorname{tr}(QM) = \sum_{d|n} a_d(s+t+\sqrt{st})U_d + \sum_{d|n} b_d(s+t-\sqrt{st})U_d + (1+s)(1+t),$$

or

$$\mathrm{tr}(QM) = k_1(s + t + \sqrt{st}) + k_2(s + t - \sqrt{st}) + (1 + s)(1 + t),$$

with k_1 and k_2 integers.

Since the entry on the ith row and i^{th} column of QM is the number of lines incident with x_i and x_i^θ, we have $\mathrm{tr}(QM) = (1 + t)f_0 + f_1$. Hence

$$k_1(s + t + \sqrt{st}) + k_2(s + t - \sqrt{st}) + (1 + s)(1 + t) = (1 + t)f_0 + f_1,$$

with k_1 and k_2 integers. $\qquad\square$

The following corollary is the analogue of Corollary 4.3

Corollary 5.2 *Let $\mathcal{S}$ be a generalized hexagon of order (s, t) and let θ be an automorphism of $\mathcal{S}$. If s and t are not relatively prime, then there exists at least one fixpoint or at least one point which is mapped to a point collinear to itself.*

Proof. Suppose that there are no fixpoints and no points which are mapped to a collinear point. Then $f_0 = f_1 = 0$. Because of the previous theorem, $k_1(s+t+\sqrt{st})+k_2(s+t-\sqrt{st})+ (1+s)(1+t)$ has to be equal to 0. Hence $k_1(s+t+\sqrt{st})+k_2(s+t-\sqrt{st})+s+t+st = -1$. But because s and t are not relatively prime, there exists an integer $m > 1$ which divides s and t. Hence m divides $k_1(s + t + \sqrt{st}) + k_2(s + t - \sqrt{st}) + s + t + st$, but m does not divide -1 and we have a contradiction. $\qquad\square$

Corollary 5.3 *Let $\mathcal{S}$ be a generalized hexagon of order (s, t) and let θ be an involution of $\mathcal{S}$. If s and t are not relatively prime, then there exists at least one fixpoint or at least one fixline.*

Proof. This follows immediately from the previous corollary because if there is a point x which is mapped to a point collinear to x by the involution θ, then the line xx^θ is a fixline. $\qquad\square$

Now we have a look at the formula in Theorem 5.1 in the special case of a thin hexagon $(s = 1)$. Then we have:

$$k_1(1 + t + \sqrt{t}) + k_2(1 + t - \sqrt{t}) + 2(1 + t) = (1 + t)f_0 + f_1.$$

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If t is not a square, then it follows that $k_1 = k_2$, hence we obtain $k_1 2(t+1) + 2(t+1) = (t+1)f_0 + f_1$ and so $t+1$ has to divide f_1. But we will improve this below (see Corollary 5.6). Note that either f_0 or f_1 is 0, according to wether the corresponding collineation θ of $\mathcal{S}$ induces a duality or a collineation in the underlying projective plane.

The method exploited in the proof of Theorem 5.1 is completely similar to the original approach of Benson. However, in order to be able to say more, an additional idea is needed. Motivated by Lemma 3.2, our idea is now to apply Benson's approach to the matrix M^2 (which, in case of generalized quadrangles, does not give anything new). Without Lemma 3.2, this would not give too much, but in combination with that lemma, we will obtain new and quite interesting results, even for the case of projective planes!.

Theorem 5.4 *Let $\mathcal{S}$ be a generalized hexagon of order (s,t) and let θ be an automorphism of $\mathcal{S}$. If f_i, $i = 0, 1, 2$, is as before, then for the integers k_1 and k_2 of Theorem 5.1 holds*

$$k_1(s+t+\sqrt{st})^2 + k_2(s+t-\sqrt{st})^2 + ((1+s)(1+t))^2 = (1+s+t)(1+t)f_0 + (1+s+2t)f_1 + f_2.$$

Proof. Suppose that M, A and Q are defined as before. Suppose that θ has order n, so that $(QM^2)^n = Q^n M^{2n} = M^{2n}$. It follows that the eigenvalues of QM^2 are the eigenvalues of M^2 multiplied by the appropriate roots of unity. Since $M^2 J = ((1+s)(1+t))^2 J$, we have $(QM^2)J = ((1+s)(1+t))^2 J$, so $((1+s)(1+t))^2$ is an eigenvalue of QM^2. By Lemma 3.2 $m_1 = 1$ and it follows that this eigenvalue of QM^2 has multiplicity 1. Further it is clear that 0 is an eigenvalue of QM^2 with multiplicity m_0. For each divisor d of n, let ξ_d again denote a primitive d^{th} root of unity, and put $U_d = \sum \xi_d^i$, where the summation is over those integers $i \in \{1, 2, \ldots, d-1\}$ that are relatively prime to d. Then U_d is an integer [7]. For each divisor d of n, the primitive d^{th} roots of unity all contribute the same number of times to the eigenvalues φ of QM^2 with $|\varphi| = (s+t+\sqrt{st})^2$ and also the primitive d^{th} roots of unity all contribute the same number of times to the eigenvalues φ' of QM with $|\varphi'| = (s+t-\sqrt{st})^2$, because of Lemma 3.1. Let a_d denote the multiplicity of $\xi_d(s+t+\sqrt{st})^2$ and let b_d denote the multiplicity of $\xi_d(s+t-\sqrt{st})^2$ as eigenvalues of QM^2, with $d|n$ and ξ_d a primitive d^{th} root of unity. Then we have:

$$\text{tr}(QM^2) = \sum_{d|n} a_d(s+t+\sqrt{st})^2 U_d + \sum_{d|n} b_d(s+t-\sqrt{st})^2 U_d + ((1+s)(1+t))^2,$$

or

$$\text{tr}(QM^2) = k_1(s+t+\sqrt{st})^2 + k_2(s+t-\sqrt{st})^2 + ((1+s)(1+t))^2,$$

with k_1 and k_2 integers. On the other hand we have

$$
\begin{aligned}
& M = A + (1+t)I \\
\Rightarrow \quad & \mathrm{tr}(M) = \mathrm{tr}(A) + (1+t)\mathrm{tr}(I) \\
\Rightarrow \quad & \mathrm{tr}(QM) = \mathrm{tr}(QA) + (1+t)\mathrm{tr}(Q) \\
\Rightarrow \quad & (1+t)f_0 + f_1 = \mathrm{tr}(QA) + (1+t)f_0 \\
\Rightarrow \quad & \mathrm{tr}(QA) = f_1.
\end{aligned}
$$

The matrix $A^2 = (a_{ij})$ is the matrix with $s(1+t)$ along the main diagonal and on the other entries we have $a_{ij} = s-1$ if $x_i \sim x_j$, $a_{ij} = 1$ if $\mathrm{d}(x_i, x_j) = 4$ and $a_{ij} = 0$ otherwise. Hence $\mathrm{tr}(QA^2) = s(1+t)f_0 + (s-1)f_1 + f_2$. It follows that

$$
\begin{aligned}
& \mathrm{tr}(QM^2) \\
= \quad & \mathrm{tr}(Q(A + (1+t)I)^2) \\
= \quad & \mathrm{tr}(QA^2) + 2(1+t)\mathrm{tr}(QA) + (1+t)^2\mathrm{tr}(Q) \\
= \quad & s(1+t)f_0 + (s-1)f_1 + f_2 + 2(1+t)f_1 + (1+t)^2 f_0 \\
= \quad & (1+s+t)(1+t)f_0 + (1+s+2t)f_1 + f_2.
\end{aligned}
$$

Finally, as already notices before, the integers k_1 and k_2 are the same integers as in Theorem 5.1 by Lemma 3.2.

This completes the proof of the theorem. $\qquad \square$

Corollary 5.5 *Suppose that we have a thin hexagon of order $(1, t)$. Consider a duality in the underlying projective plane. If t is not a square, then $f_1 = 2(1+t)$. If t is a square, then $f_1 \equiv 2 \mod 2\sqrt{t}$. In particular there is at least one absolute line and one absolute point.*

Proof. Since we have a duality in the underlying projective plane, we know that $f_0 = 0$ and $f_2 = 0$. Because of Theorems 5.1 and 5.4, we have the following equations:

$$
\begin{cases}
k_1(1+t+\sqrt{t}) + k_2(1+t-\sqrt{t}) + 2(1+t) = f_1, \\
k_1(1+t+\sqrt{t})^2 + k_2(1+t-\sqrt{t})^2 + (2(1+t))^2 = (2+2t)f_1,
\end{cases}
$$

hence

$$
\begin{cases}
k_1 = \frac{f_1 - 2(1+t)}{2\sqrt{t}}, \\
k_2 = -\frac{f_1 - 2(1+t)}{2\sqrt{t}}.
\end{cases}
$$

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So $\frac{f_1 - 2(1+t)}{2\sqrt{t}}$ has to be an integer. In the case that t is not a square, this only holds if $f_1 - 2(1+t) = 0$. Hence $f_1 = 2(1+t)$ if t is not a square. If t is a square, then $f_1 - 2$ has to be a multiple of $2\sqrt{t}$. Hence $f_1 \equiv 2 \mod 2\sqrt{t}$. $\qquad\square$

In view of Lemma 3.4, this immediately implies:

Corollary 5.6 *Suppose that θ is a duality of a projective plane of order t, then it has $1 + t$ absolute points and $1 + t$ absolute lines if t is not a perfect square, and it has $1 \mod \sqrt{t}$ absolute points and just as many absolute lines if t is a perfect square.*

If t is a square, then the lower bound of this corollary can be obtained. Indeed, let b be an element of the finite (Galois) field $\mathbb{F}_t$ (of t elements) not belonging to the subfield $\mathbb{F}_{\sqrt{t}}$. With the usual representation of the Desarguesian projective plane $\mathsf{PG}(2,t)$ of order t by means of triples of elements of $\mathbb{F}_t$ (with round brackets to denote points and square brackets for lines), the map

$$\theta : \mathsf{PG}(2,t) \rightarrow \mathsf{PG}(2,t) : \begin{cases} (x,y,z) & \mapsto & [x^{\sqrt{t}}, z^{\sqrt{t}}, y^{\sqrt{t}} - bz^{\sqrt{t}}], \\ [u,v,w] & \mapsto & (u^{\sqrt{t}}, w^{\sqrt{t}} + bv^{\sqrt{t}}, v^{\sqrt{t}}) \end{cases}$$

is a duality. A point (x,y,z) is absolute if and only if (x,y,z) is incident with $[x^{\sqrt{t}}, z^{\sqrt{t}}, y^{\sqrt{t}} - bz^{\sqrt{t}}]$. If $z \neq 0$, then this obviously implies that

$$b = \frac{xx^{\sqrt{t}} + (yz^{\sqrt{t}} + y^{\sqrt{t}}z)}{zz^{\sqrt{t}}}$$

is fixed by the field automorphism $a \mapsto a^{\sqrt{t}}$, and hence belongs to $\mathbb{F}_{\sqrt{t}}$, a contradiction. Hence, if (x,y,z) is absolute, then $z = 0$ and so $xx^{\sqrt{t}} = 0$. We obtain a unique absolute point $(0,1,0)$. Likewise, $[0,0,1]$ is the unique absolute line.

For the next corollary we need the following lemma.

Note that for a generalized hexagon of order (s,t) with $s \neq t$ no ovoids exist because of [8].

Lemma 5.7 *Suppose that $\mathcal{S}$ is a generalized hexagon of order s and θ is a collineation of $\mathcal{S}$. If $\mathcal{O}$ is an ovoid of $\mathcal{S}$ which is fixed pointwise by θ and θ additionally fixes some point $x \notin \mathcal{O}$, then θ is the identity.*

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Proof. There exists a unique point of the ovoid which is collinear to x, we call this point y. Every point x' collinear to x which is not incident with the line xy is collinear to a unique point $y' \neq y$ of the ovoid. Since both x and y' are fixed, also x' is fixed. Now take any point $x'' \notin \{x, y\}$ that is incident with the line xy. Take a point a collinear to x'', not incident with xy. This point is collinear to a unique point b of $\mathcal{O}$. Since b is fixed by θ and the line xy is fixed by θ, also the point x'' has to be fixed. So every point collinear to x is fixed. Since $\mathcal{O}$ contains a point at distance 6 from x we can apply Theorem 4.4.2(v) in [15] to obtain that θ is the identity. $\qquad\square$

It is well known that the dual $\mathsf{H}(q)^{\text{dual}}$ of the split Cayley hexagon $\mathsf{H}(q)$ of order q admits an ovoid stabilized by the group $\mathsf{SU}_3(q)$, and every element of the center of $\mathsf{SU}_3(q)$ fixes the ovoid pointwise (see [3]). If q is divisible by 3, however, this center is trivial and the ovoid does not admit a nontrivial collineation fixing it pointwise. This is a special case of the following more general phenomenon.

Corollary 5.8 *Suppose that $\mathcal{S}$ is a generalized hexagon of order s and θ is an automorphism of $\mathcal{S}$. If s is a multiple of 3, then θ cannot fix any ovoid pointwise.*

Proof. Suppose that $\mathcal{O}$ is an ovoid of $\mathcal{S}$ which is fixed pointwise by θ. Then $f = 1 + s^3 = |\mathcal{O}|$. Because of the previous lemma there are no other fixpoints. Suppose that there exists a point x which is mapped to a collinear point by θ. Suppose that there is a point y of $\mathcal{O}$ which is collinear to x and not collinear to x^θ. Then the line xy will be mapped to the line $x^\theta y$, hence we have a triangle, so this is not possible. Now there has to be a point y' at distance 4 from x which belongs to the ovoid. By a similar reasoning we become a pentagon. Hence we have a contradiction and g has to be equal to 0. Because θ fixes an ovoid every point is mapped to a point at distance at most 4 from it. So h is the number of points not on the ovoid, hence $h = (1 + s^3)(s + s^2)$. Because of Theorem 5.1 and Theorem 5.4 we have the following equations

$$\begin{cases} k_1 3s + k_2 s + (1+s)^2 = (1+s)(1+s^3), \\ k_1 9s^2 + k_2 s^2 + (1+s)^4 = (1+2s)(1+s)(1+s^3) + (1+s^3)(s+s^2). \end{cases}$$

Hence

$$\begin{cases} k_1 = \frac{(1+s)(s^2-1)}{3}, \\ k_2 = 0. \end{cases}$$

Because k_1 has to be an integer, it follows that s cannot be a multiple of 3. $\qquad\square$

The split Cayley hexagon $\mathsf{H}(q)$ admits a subhexagon of order $(1, q)$, stabilized by the group $\mathsf{SL}_3(q)$ (which has index two in the full stabilizer, see e.g. [4]). The pointwise stabilizer

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of that subhexagon is the center of $\mathsf{SL}_3(q)$, which is again trivial if 3 divides q. More generally, we can now show:

Corollary 5.9 *Suppose that $\mathcal{S}$ is a generalized hexagon of order s and θ is a nontrivial automorphism of $\mathcal{S}$. If s is a multiple of 3, then θ cannot fix any thin subhexagon of order $(1, s)$ pointwise.*

Proof. Suppose that $\mathcal{S}'$ is a thin subhexagon of $\mathcal{S}$ of order $(1, s)$ which is fixed pointwise by θ. Then we claim $f_0 = 2(1 + s + s^2) = |\mathcal{S}'|$. Indeed, if θ fixes a point not belonging to $\mathcal{S}'$, then θ would fix a subhexagon $\mathcal{S}''$ of order (s'', s) with $\mathcal{S}' \subset \mathcal{S}'' \subseteq \mathcal{S}$. By [11] it follows that $s \geq s''^2 t$ so $s \geq s''^2 s$. Hence $s'' = 1$, a contradiction. Now suppose that x is a point of $\mathcal{S}'$. There are $s + 1$ lines through x in $\mathcal{S}'$ and on each of these lines there is precisely one other point which belongs to $\mathcal{S}'$. Take such a point y. On the line xy of $\mathcal{S}$ there are $s - 1$ other points which are mapped to each other (not fixed). So we already have $(1 + s + s^2)(s + 1)(s - 1)$ points which are mapped to a collinear point by θ. Suppose that there is a point z of $\mathcal{S}$ which does not lie on a line of $\mathcal{S}'$ and which is mapped to a point collinear to itself. Then by [11] the line zz^θ intersects a line of $\mathcal{S}'$ which is fixed under θ. But then the line zz^θ also has to be fixed and we obtain a contradiction. Hence $f_1 = (1 + s + s^2)(s + 1)(s - 1)$. Now suppose that there is a point u which is mapped to a point at distance 4 from itself. There exists a point u' which is collinear to u and also to u^θ. We have two possibilities, either the point u' is incident with a line L of $\mathcal{S}'$ or it is not. In the first case the point u' should be fixed, because it is the unique point from L collinear to both u and u^θ. But it cannot be a point of $\mathcal{S}'$ so we obtain a contradiction. In the second case, again by [11] the line uu' intersects a line of $\mathcal{S}'$, so the line uu' is mapped to a line trough u^θ which intersects the same line of $\mathcal{S}'$. Hence we obtain a quadrangle and we have a contradiction. Consequently $f_2 = 0$. Because of Theorem 5.1 and Theorem 5.4 we have the following equations

$$\begin{cases} k_1 3s + k_2 s + (1 + s)^2 = (1 + s)2(1 + s + s^2) + (1 + s + s^2)(1 + s)(s - 1), \\ k_1 9s^2 + k_2 s^2 + (1 + s)^4 = (1 + 2s)(1 + s)2(1 + s + s^2) + \\ \quad (1 + 3s)(1 + s + s^2)(1 + s)(s - 1). \end{cases}$$

Hence

$$\begin{cases} k_1 = \frac{(1+s)(1+s+s^2)}{3}, \\ k_2 = s + s^2. \end{cases}$$

We see that k_2 is an integer, but k_1 also has to be an integer. It follows that s cannot be a multiple of 3.

$\square$

6 Collineations of octagons and dualities of quadrangles

Suppose that $\mathcal{S}$ is a generalized octagon. Let θ be an automorphism of $\mathcal{S}$ and let f_i be the number of points for which $d(x, x^\theta) = i$ in the point graph. The matrices M, A and Q are defined analogously as before. The eigenvalues of M are as follows (cf. table 6.4 in [2]):

eigenvalues of M	multiplicity
0	$m_0 = \dfrac{s^4(1+st)(1+s^2t^2)}{(s+t)(s^2+t^2)}$
$(s+1)(t+1)$	$m_1 = 1$
$s+t+\sqrt{2st}$	$m_2 = \dfrac{(1+t)st(1+s)(1+st)(1+s^2t^2)}{4(s(t-1)^2+t(s-1)^2+2st+(s-1)(t-1)\sqrt{2st})}$
$s+t-\sqrt{2st}$	$m_3 = \dfrac{(1+t)st(1+s)(1+st)(1+s^2t^2)}{4(s(t-1)^2+t(s-1)^2+2st-(s-1)(t-1)\sqrt{2st})}$
$s+t$	$m_4 = \dfrac{(1+t)st(1+s)(1+st)(1+s^2t^2)}{2(s(t-1)^2+t(s-1)^2+4st)}$

Theorem 6.1 *Let $\mathcal{S}$ be a generalized octagon of order (s,t) and let θ be an automorphism of $\mathcal{S}$. If f_0 and f_1 are as before, then for some integers k_1, k_2 and k_3 holds*

$$k_1(s+t+\sqrt{2st}) + k_2(s+t-\sqrt{2st}) + k_3(s+t) + (1+s)(1+t) = (1+t)f_0 + f_1.$$

Proof. This proof is totally analogous to the proof of Theorem 5.1. $\qquad\square$

Corollary 6.2 *Let $\mathcal{S}$ be a generalized octagon of order (s,t) and let θ be an automorphism of $\mathcal{S}$. If s and t are not relatively prime, then there exists at least one fixpoint or at least one point which is mapped to a point collinear to itself.*

Proof. This proof is totally analogous to the proof of Corollary 5.2. $\qquad\square$

Note that in a thick generalized octagon s and t cannot be odd at the same time since $\sqrt{2st}$ is an integer and hence either the number of points or the number of lines is odd, or both are. So we have:

Corollary 6.3 *Let $\mathcal{S}$ be a thick generalized octagon of order (s,t) and let θ be an involution of $\mathcal{S}$. Then there exists at least one fixpoint or at least one fixline.*

Note that the previous corollary in the case that s and t are not relatively prime, also follows from Corollary 6.2 in the same way as Corollary 5.3 follows from Corollary 5.2 for hexagons.

Now suppose that we have a thin octagon, then we obtain:

$$k_1(1 + t + \sqrt{2t}) + k_2(1 + t - \sqrt{2t}) + k_3(1 + t) + 2(1 + t) = (1 + t)f_0 + f_1.$$

If $2t$ is not a square, then it follows that $k_1 = k_2$, hence:

$$k_1 2(1 + t) + k_3(1 + t) + 2(1 + t) = (1 + t)f_0 + f_1.$$

Note that this implies that in a generalized quadrangle of order t, with $2t$ not a square, the number of absolute elements of a duality is divisible by $1 + t$. But we will do better in Corollary 6.7.

Theorem 6.4 *Let $\mathcal{S}$ be a generalized octagon of order (s,t) and let θ be an automorphism of $\mathcal{S}$. If f_0, f_1 and f_3 are as before, then for the integers k_1, k_2 and k_3 obtained in Theorem 6.1* holds

$$k_1(s + t + \sqrt{2st})^2 + k_2(s + t - \sqrt{2st})^2 + k_3(s + t)^2 + ((1 + s)(1 + t))^2 = $$
$$(1 + s + t)(1 + t)f_0 + (1 + s + 2t)f_1 + f_2.$$

Proof. This proof is totally analogous to the proof of Theorem 5.4. □

For a thin octagon we obtain:

$$k_1(1+t+\sqrt{2t})^2+k_2(1+t-\sqrt{2t})^2+k_3(1+t)^2+(2(1+t))^2 = (2+t)(1+t)f_0+(2+2t)f_1+f_2.$$

If $2t$ is not a square, then it follows that $k_1 = k_2$ and we obtain:

$$k_1 2(1 + 4t + t^2) + k_3(1 + t)^2 + (2(1 + t))^2 = (2 + t)(1 + t)f_0 + (2 + 2t)f_1 + f_2.$$

Theorem 6.5 *Let $\mathcal{S}$ be a generalized octagon of order (s,t) and let θ be a nontrivial automorphism of $\mathcal{S}$. If f_0, f_1, f_2 and f_3 are defined as above, then for the integers k_1, k_2 and k_3 of* Theorem 6.1 *holds*

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$$k_1(s+t+\sqrt{2st})^3 + k_2(s+t-\sqrt{2st})^3 + k_3(s+t)^3 + ((1+s)(1+t))^3 =$$
$$(s(s-1)(1+t) + 3s(1+t)^2 + (1+t)^3)f_0$$
$$+(s(1+t) + (s-1)^2 + st + 3(1+t)(s-1) + 3(1+t)^2)f_1$$
$$+(2(s-1) + 3(1+t))f_2 + f_3.$$

Proof. Suppose that M, A and Q are defined as before. In the same way as in the proof of Theorems 5.1 and 5.4 we can prove that $\text{tr}(QM^3) = k_1(s+t+\sqrt{2st})^3 + k_2(s+t-\sqrt{2st})^3 + k_3(s+t)^3 + ((1+s)(1+t))^3$, with k_1, k_2 and k_3 integers. On the other hand, because of Lemma 3.3 and the values for p_j^i given after that lemma, we can calculate that $A^3 = (a_{ij})$ is the matrix with $s(s-1)(1+t)$ along the main diagonal and on the other entries we have $a_{ij} = s(1+t) + (s-1)^2 + st$ if $x_i \sim x_j$, $a_{ij} = 2(s-1)$ if $\text{d}(x_i, x_j) = 4$, $a_{ij} = 1$ if $\text{d}(x_i, x_j) = 6$ and $a_{ij} = 0$ otherwise. Hence $\text{tr}(QA^3) = s(s-1)(1+t)f_0 + (s(1+t) + (s-1)^2 + st)f_1 + 2(s-1)f_2 + f_3$. Because of the proof of Theorem 5.4 we know that $\text{tr}(QA^2) = s(1+t)f_0 + (s-1)f_1 + f_2$, $\text{tr}(QA) = f_1$ and $\text{tr}(Q) = f_0$. Hence

$$
\begin{aligned}
&\text{tr}(QM^3)\\
=\ &\text{tr}(Q(A+(1+t)I)^3)\\
=\ &\text{tr}(QA^3) + 3(1+t)\text{tr}(QA^2) + 3(1+t)^2\text{tr}(QA) + (1+t)^3\text{tr}(Q)\\
=\ &s(s-1)(1+t)f_0 + (s(1+t) + (s-1)^2 + st)f_1 + 2(s-1)f_2 + f_3\\
&+3(1+t)(s(1+t)f_0 + (s-1)f_1 + f_2) + 3(1+t)^2f_1 + (1+t)^3f_0\\
=\ &(s(s-1)(1+t) + 3s(1+t)^2 + (1+t)^3)f_0\\
&+(s(1+t) + (s-1)^2 + st + 3(1+t)(s-1) + 3(1+t)^2)f_1\\
&+(2(s-1) + 3(1+t))f_2 + f_3.
\end{aligned}
$$

Using Lemma 3.2 as before, the proof of the theorem is complete. $\square$

For a thin octagon with $s = 1$ we obtain:

$$k_1(1+t+\sqrt{2t})^3 + k_2(1+t-\sqrt{2t})^3 + k_3(1+t)^3 + (2(1+t))^3 =$$
$$(3(1+t)^2 + (1+t)^3)f_0 + (1 + 2t + 3(1+t)^2)f_1 + 3(1+t)f_2 + f_3.$$

If $2t$ is not a square, then it follows that $k_1 = k_2$ and we obtain:

$$k_1 2(1 + 9t + 9t^2 + t^3) + k_3(1+t)^3 + (2(1+t))^3 =$$
$$(3(1+t)^2 + (1+t)^3)f_0 + (1 + 2t + 3(1+t)^2)f_1 + 3(1+t)f_2 + f_3.$$

Corollary 6.6 *Suppose that we have a thin octagon of order $(1, t)$. Consider a duality in the underlying generalized quadrangle. If $2t$ is not a square, then $f_1 = 2(1 + t)$. If $2t$ is a square, then $f_1 \equiv 2 \mod 2\sqrt{2t}$. In particular there is at least one absolute point and one absolute line.*

Proof. Since we have a duality in the underlying generalized quadrangle, we know that $f_0 = 0$ and $f_2 = 0$. Because of Lemma 3.2, Theorems 6.1, 6.4 and 6.5, we have the following equations:

$$
\begin{cases}
k_1(1 + t + \sqrt{2t}) + k_2(1 + t - \sqrt{2t}) + k_3(1 + t) + 2(1 + t) = f_1, \\
k_1(1 + t + \sqrt{2t})^2 + k_2(1 + t - \sqrt{2t})^2 + k_3(1 + t)^2 + (2(1 + t))^2 = (2 + 2t)f_1, \\
k_1(1 + t + \sqrt{2t})^3 + k_2(1 + t - \sqrt{2t})^3 + k_3(1 + t)^3 + (2(1 + t))^3 = \\
\qquad (1 + 2t + 3(1 + t)^2)f_1 + f_3.
\end{cases}
$$

Because f_0 and f_2 are 0, we know that $f_1 + f_3 = 2(1 + t)(1 + t^2)$. Hence:

$$
\begin{cases}
k_1 = \frac{\sqrt{2}(-2t + f_1 - 2)}{4\sqrt{t}}, \\
k_2 = -\frac{\sqrt{2}(-2t + f_1 - 2)}{4\sqrt{t}}, \\
k_3 = 0, \\
f_3 = 2(t^3 + t^2 + t + 1) - f_1.
\end{cases}
$$

So $\frac{\sqrt{2}(-2t + f_1 - 2)}{4\sqrt{t}}$ has to be an integer. In the case that $2t$ is not a square, this only holds if $-2t + f_1 - 2 = 0$. Hence $f_1 = 2(1 + t)$ if $2t$ is not a square. If t is a square, then $f_1 - 2$ has to be a multiple of $2\sqrt{2t}$. Hence $f_1 \equiv 2 \mod 2\sqrt{2t}$. $\qquad\square$

From Lemma 3.4 now immediately follows :

Corollary 6.7 *Suppose that θ is a duality of a generalize quadrangle of order t. If $2t$ is not a square, then it has $1 + t$ absolute points and $1 + t$ absolute lines, and there are $(1 + t)t^2$ points which are mapped to a line at distance 3 and $(1 + t)t^2$ lines which are mapped to a point at distance 3. If $2t$ is a perfect square, then it has $1 \mod \sqrt{2t}$ absolute points and equally many absolute lines.*

If $2t$ is a square, then one can again construct examples of dualities in a generalized quadrangle of order t, namely in the symplectic quadrangle $\mathsf{W}(t)$, admitting the lower bound 1 of absolute points given in the previous corollary. Indeed, consider a polarity ρ and compose it with a nontrivial central root elation τ whose center is an absolute point

x. The resulting duality $\theta = \rho\tau$ has x as unique absolute point. Indeed, suppose by way of contradiction that the point $y \neq x$ is absolute for θ. Since τ is involutive, this implies that $y^\rho \mathtt{I} y^\tau$. Since τ is central, the unique point z on y^ρ collinear with x is also collinear with y. Applying ρ to the chain $y\mathtt{I}yz\mathtt{I}z\mathtt{I}y^\rho$, we deduce that $z^\rho = yz$, hence z is an absolute point for ρ, contradicting the fact that z is collinear with x.

7 Collineations of dodecagons and dualities of hexagons

Suppose that $\mathcal{S}$ is a generalized dodecagon. Let θ be an automorphism of $\mathcal{S}$ and let f_i be the number of points for which $\mathrm{d}(x, x^\theta) = i$ in the point graph, with $i \in \{0, 1, 2, 3, 4, 5\}$. The matrices M, A and Q are defined analogously as before. The eigenvalues of M are as follows (cf. [5]):

eigenvalues of M	multiplicity
0	m_0
$(s+1)(t+1)$	$m_1 = 1$
$s+t+\sqrt{st}$	m_2
$s+t-\sqrt{st}$	m_3
$s+t+\sqrt{3st}$	m_4
$s+t-\sqrt{3st}$	m_5
$s+t$	m_6

Thick finite generalized dodecagons do not exist, but nevertheless we formulate the following results with general s and t. In real life, either s or t is equal to 1, but the formulae do not seem to be equivalent. Afterwards, we apply our results to the case $s = 1$, implying results for dualities of generalized hexagons.

Theorem 7.1 *Let $\mathcal{S}$ be a generalized dodecagon of order (s, t) and let θ be an automorphism of $\mathcal{S}$. If f_0 is the number of points fixed by θ and f_1 is the number of points x for which $x \sim x^\theta$, then for some integers k_1, k_2, k_3, k_4 and k_5 holds*

$$k_1(s+t+\sqrt{st})+k_2(s+t-\sqrt{st})+k_3(s+t+\sqrt{3st})+k_4(s+t-\sqrt{3st})+k_5(s+t)+(1+s)(1+t) =$$
$$(1+t)f_0 + f_1.$$

Proof. This proof is totally analogous to the proof of Theorem 5.1. □

Theorem 7.2 *Let $\mathcal{S}$ be a generalized dodecagon of order (s,t) and let θ be an automorphism of $\mathcal{S}$. If f_i, $i = 0, 1, 2$, is defined as above, then for the integers k_1, k_2, k_3, k_4 and k_5 obtained in* Theorem 7.1 *holds*

$$k_1(s+t+\sqrt{st})^2 + k_2(s+t-\sqrt{st})^2 + k_3(s+t+\sqrt{3st})^2 + k_4(s+t-\sqrt{3st})^2 + k_5(s+t)^2 +$$
$$((1+s)(1+t))^2 = (1+s+t)(1+t)f_0 + (1+s+2t)f_1 + f_2.$$

Proof. This proof is totally analogous to the proof of Theorem 5.4. □

Theorem 7.3 *Let $\mathcal{S}$ be a generalized dodecagon of order (s,t) and let θ be an automorphism of $\mathcal{S}$. If f_i, $i = 0, 1, 2, 3$, is as before, then for the integers k_1, k_2, k_3, k_4 and k_5 of* Theorem 7.1 *holds*

$$k_1(s+t+\sqrt{st})^3 + k_2(s+t-\sqrt{st})^3 + k_3(s+t+\sqrt{3st})^3 + k_4(s+t-\sqrt{3st})^3 + k_5(s+t)^3 +$$
$$((1+s)(1+t))^3 =$$
$$(s(s-1)(1+t) + 3s(1+t)^2 + (1+t)^3)f_0$$
$$+(s(1+t) + (s-1)^2 + st + 3(1+t)(s-1) + 3(1+t)^2)f_1$$
$$+(2(s-1) + 3(1+t))f_2 + f_3.$$

Proof. This proof is totally analogous to the proof of Theorem 6.5. □

Theorem 7.4 *Let $\mathcal{S}$ be a generalized dodecagon of order (s,t) and let θ be an automorphism of $\mathcal{S}$. If f_i, $i = 0, 1, 2, 3, 4$, is as before, then for the integers k_1, k_2, k_3, k_4 and k_5 of* Theorem 7.1 *holds*

$$k_1(s+t+\sqrt{st})^4 + k_2(s+t-\sqrt{st})^4 + k_3(s+t+\sqrt{3st})^4 + k_4(s+t-\sqrt{3st})^4 + k_5(s+t)^4 +$$
$$((1+s)(1+t))^4 =$$
$$((s(1+t) + (s-1)^2 + st)(1+t)s + 4s(s-1)(1+t)^2 + 6s(1+t)^3 + (1+t)^4)f_0$$
$$+(s(s-1)(1+t) + (s-1)(s(1+t) + (s-1)^2 + st) + 2(s-1)st + 4(1+t)(s(1+t)$$
$$+(s-1)^2 + st) + 6(1+t)^2(s-1) + 4(1+t)^3)f_1$$
$$+(s(1+t) + 3(s-1)^2 + 2st + 4(1+t)(s-1) + 6(1+t)^2)f_2$$
$$+(3(s-1) + 4(1+t))f_3 + f_4.$$

Proof. Suppose that M, A and Q are defined as before. In the same way as in the proof of Theorems 5.1, 5.4 and 6.5 we can prove that $\text{tr}(QM^4) = k_1(s + t + \sqrt{st})^4 + k_2(s + t - \sqrt{st})^4 + k_3(s + t + \sqrt{3st})^4 + k_4(s + t - \sqrt{3st})^4 + k_5(s + t)^4 + ((1 + s)(1 + t))^4$, with k_1, k_2, k_3, k_4 and k_5 integers. On the other hand, because of Lemma 3.3 and the values for p_j^i given after that lemma, we can calculate that $A^4 = (a_{ij})$ is the matrix with $(s(1 + t) + (s - 1)^2 + st)(1 + t)s$ on the main diagonal and on the other entries we have

$$
\begin{aligned}
a_{ij} &= s(s - 1)(1 + t) + (s - 1)(s(1 + t) + (s - 1)^2 + st) + 2(s - 1)st && \text{if } x_i \sim x_j \\
a_{ij} &= s(1 + t) + 3(s - 1)^2 + 2st && \text{if } d(x_i, x_j) = 4 \\
a_{ij} &= 3(s - 1) && \text{if } d(x_i, x_j) = 6 \\
1 && && \text{if } d(x_i, x_j) = 8 \\
0 && && \text{otherwise.}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\text{tr}(QA^4) = {}& ((s(1 + t) + (s - 1)^2 + st)(1 + t)s)f_0 \\
& + (s(s - 1)(1 + t) + (s - 1)(s(1 + t) + (s - 1)^2 + st) + 2(s - 1)st)f_1 \\
& + (s(1 + t) + 3(s - 1)^2 + 2st)f_2 + 3(s - 1)f_3 + f_4
\end{aligned}
$$

The rest of the proof is totally analogous to the proof of Theorem 6.5. $\qquad\square$

Theorem 7.5 *Let $\mathcal{S}$ be a generalized dodecagon of order (s, t) and let θ be an automorphism of $\mathcal{S}$. If f_i, $i = 0, 1, 2, 3, 4, 5$, is as before, then for the integers k_1, k_2, k_3, k_4 and k_5 of* Theorem 7.1 *holds*

$$
\begin{aligned}
&k_1(s{+}t{+}\sqrt{st})^5 + k_2(s{+}t{-}\sqrt{st})^5 + k_3(s{+}t{+}\sqrt{3st})^5 + k_4(s{+}t{-}\sqrt{3st})^5 + k_5(s{+}t)^5 + ((1{+} \\
&s)(1{+}t))^5 = (((((t{+}1)^2 + s(t{+}1))(t{+}1) + s(t{+}1)^2 + s(t{+}1)(s{+}t))(t{+}1) + ((t{+}1)^2 + \\
&s(t{+}1))s(t{+}1) + (s(t{+}1)^2 + s(t{+}1)(s{+}t))(s{+}t) + s^2(t{+}1)t)(t{+}1) + (((t{+}1)^2 + s(t{+} \\
&1))(t{+}1) + s(t{+}1)^2 + s(t{+}1)(s{+}t))s(t{+}1) + (((t{+}1)^2 + s(t{+}1))s(t{+}1) + (s(t{+}1)^2 + \\
&s(t{+}1)(s{+}t))(s{+}t) + s^2(t{+}1)t)(s{+}t) + (s(t{+}1)^2 + s(t{+}1)(s{+}t))st + s^2(t{+}1)t(s{+}t))f_0 \\
&+ ((((2t{+}1{+}s)(t{+}1) + s(t{+}1) + (s{+}t)^2 + st)(t{+}1) + (2t{+}1{+}s)s(t{+}1) + (s(t{+}1) + \\
&\quad (s{+}t)^2 + st)(s{+}t) + 2(s{+}t)st)(t{+}1) + ((2t{+}1{+}s)(t{+}1) + s(t{+}1) + (s{+}t)^2 + \\
&st)s(t{+}1) + ((2t{+}1{+}s)s(t{+}1) + (s(t{+}1) + (s{+}t)^2 + st)(s{+}t) + 2(s{+}t)st)(s{+}t) + \\
&\quad\quad\quad\quad (s(t{+}1) + (s{+}t)^2 + st)st + 2(s{+}t)^2st + s^2t^2)f1 \\
&+ (((3t{+}1{+}2s)(t{+}1) + s(t{+}1) + (2s{+}2t)(s{+}t) + 2st + (s{+}t)^2)(t{+}1) + (3t{+}1{+}2s)s(t{+}1) + \\
&(s(t{+}1) + (2s{+}2t)(s{+}t) + 2st + (s{+}t)^2)(s{+}t) + (2s{+}2t)st + (2st + (s{+}t)^2)(s{+}t) + 2(s{+}t)st)f_2 \\
&+ ((4t{+}1{+}3s)(t{+}1) + s(t{+}1) + (3s{+}3t)(s{+}t) + 3st + (2s{+}2t)(s{+}t) + (s{+}t)^2)f_3 \\
&\quad\quad\quad\quad\quad\quad\quad + (5t{+}1{+}4s)f_4 + f_5.
\end{aligned}
$$

Proof. Suppose that M, A and Q are defined as before. In the same way as in the proof of Theorems 5.1, 5.4 and 6.5 we can prove that $\text{tr}(QM^5) = k_1(s+t+\sqrt{st})^5 + k_2(s+t-\sqrt{st})^5 + k_3(s+t+\sqrt{3st})^5 + k_4(s+t-\sqrt{3st})^5 + k_5(s+t)^5 + ((1+s)(1+t))^5$, with k_1, k_2, k_3, k_4 and k_5 the integers of Theorem 7.1 (by Lemma 3.2). On the other hand, because of Lemma 3.3 and the values for p_j^i given after that lemma, we can calculate that $A^4 = (a_{ij})$ is the matrix with $s(t+1)(s(s-1)(1+t) + (s-1)(s(1+t) + (s-1)^2 + st) + 2(s-1)st)$ on the main diagonal and on the other entries we have

$$
\begin{aligned}
a_{ij} &= (s(1+t) + (s-1)^2 + st)(1+t)s + (s-1)(s(s-1)(1+t)\\
&\quad +(s-1)(s(1+t) + (s-1)^2 + st) + 2(s-1)st)\\
&\quad +st(s(1+t) + 3(s-1)^2 + 2st) & \text{if } x_i \sim x_j\\
a_{ij} &= s(s-1)(1+t) + (s-1)(s(1+t) + (s-1)^2 + st) + 2(s-1)st\\
&\quad +(s-1)(s(1+t) + 3(s-1)^2 + 2st) + 3st(s-1) & \text{if } d(x_i, x_j) = 4\\
a_{ij} &= s(1+t) + 3(s-1)^2 + 2st + 3(s-1)^2 + st & \text{if } d(x_i, x_j) = 6\\
a_{ij} &= 4(s-1) & \text{if } d(x_i, x_j) = 8\\
1 & & \text{if } d(x_i, x_j) = 10\\
0 & & \text{otherwise.}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\text{tr}(QA^5) &= (s(t+1)(s(s-1)(1+t) + (s-1)(s(1+t) + (s-1)^2 + st) + 2(s-1)st))f_0\\
&\quad +((s(1+t) + (s-1)^2 + st)(1+t)s + (s-1)(s(s-1)(1+t) + (s-1)(s(1+t)\\
&\quad +(s-1)^2 + st) + 2(s-1)st) + st(s(1+t) + 3(s-1)^2 + 2st))f_1\\
&\quad +(s(s-1)(1+t) + (s-1)(s(1+t) + (s-1)^2 + st) + 2(s-1)st\\
&\quad +(s-1)(s(1+t) + 3(s-1)^2 + 2st) + 3st(s-1))f_2\\
&\quad +(s(1+t) + 3(s-1)^2 + 2st + 3(s-1)^2 + st)f_3 + 4(s-1)f_4 + f_5.
\end{aligned}
$$

We have

$$
\begin{aligned}
&\text{tr}(QM^5)\\
=\ &\text{tr}(Q(A + (1+t)I)^5)\\
=\ &\text{tr}(QA^5) + 5(1+t)\text{tr}(QA^4) + 10(1+t)^2\text{tr}(QA^3) + 10(1+t)^3\text{tr}(QA^2)\\
&+5(1+t)^4\text{tr}(QA) + (1+t)^5\text{tr}(Q).
\end{aligned}
$$

If we substitute the formula for $\text{tr}(QA^5)$ which we obtained above and the formulas which we obtained in Theorems 5.1, 5.4, 6.5 and 7.4 for $\text{tr}(QA^4)$, $\text{tr}(QA^3)$, $\text{tr}(QA^2)$, $\text{tr}(QA)$ and $\text{tr}(Q)$, then we obtain the assertion. $\qquad\square$

Corollary 7.6 *Suppose that we have a thin dodecagon of order $(1, t)$. Consider a duality in the underlying generalized hexagon. If $3t$ and t are no squares, then $f_1 = 2(1 + t)$, $f_3 = 2(t^2 + t^3)$ and $f_5 = 2(t^4 + t^5)$. If $3t$ is a square and t is not a square, then $f_1 equiv 2$ mod $2\sqrt{3t}$ and $f_3 \equiv 0 \mod 2\sqrt{3t}$. If t is a square and $3t$ is not a square, then $f_1 \equiv 2$ mod $2\sqrt{t}$ and $f_3 \equiv 0 \mod 2\sqrt{t}$. In particular there is always at least one absolute point and one absolute line.*

Proof. Since we have a duality in the underlying generalized hexagon, we know that $f_0 = 0$, $f_2 = 0$ and $f_4 = 0$. Because of Theorems 7.1, 7.2, 7.3, 7.4 and 7.5, we have the following equations:

$$
\begin{cases}
k_1(1+t+\sqrt{t}) + k_2(1+t-\sqrt{t}) + k_3(1+t+\sqrt{3t}) + k_4(1+t-\sqrt{3t}) + k_5(1+t) \\
\quad +2(1+t) = f_1, \\
k_1(1+t+\sqrt{t})^2 + k_2(1+t-\sqrt{t})^2 + k_3(1+t+\sqrt{3t})^2 + k_4(1+t-\sqrt{3t})^2 + k_5(1+t)^2 \\
\quad +(2(1+t))^2 = (2+2t)f_1, \\
k_1(1+t+\sqrt{t})^3 + k_2(1+t-\sqrt{t})^3 + k_3(1+t+\sqrt{3t})^3 + k_4(1+t-\sqrt{3t})^3 + k_5(1+t)^3 \\
\quad +(2(1+t))^3 = (1+2t+3(1+t)^2)f_1 + f_3, \\
k_1(1+t+\sqrt{t})^4 + k_2(1+t-\sqrt{t})^4 + k_3(1+t+\sqrt{3t})^4 + k_4(1+t-\sqrt{3t})^4 + k_5(1+t)^4 \\
\quad +(2(1+t))^4 = (4(1+t)(1+2t) + 4(1+t)^3)f_1 + 4(1+t)f_3, \\
k_1(1+t+\sqrt{t})^5 + k_2(1+t-\sqrt{t})^5 + k_3(1+t+\sqrt{3t})^5 + k_4(1+t-\sqrt{3t})^5 + k_5(t+1)^5 \\
\quad +(2(t+1))^5 = (5t^4 + 40t^3 + 85t^2 + 64t + 16)f_1 + (10t^2 + 24t + 11)f_3 + f_5.
\end{cases}
$$

Because f_0, f_2 and f_4 are 0, we know that $f_1 + f_3 + f_5 = \frac{2(t^6-1)}{t-1}$. Hence:

$$
\begin{cases}
k_1 = -\frac{-2t^3 - f_1 t + f_3 + f_1 - 2}{4\sqrt{t^3}}, \\
k_2 = \frac{-2t^3 - f_1 t + f_3 + f_1 - 2}{4\sqrt{t^3}}, \\
k_3 = \frac{(-2t^3 - 4t^2 - 4t - 2 + f_1 t + f_3 + f_1)\sqrt{3}}{12\sqrt{t^3}}, \\
k_4 = -\frac{(-2t^3 - 4t^2 - 4t - 2 + f_1 t + f_3 + f_1)\sqrt{3}}{12\sqrt{t^3}}, \\
k_5 = 0, \\
f_5 = 2(t^5 + t^4 + t^3 + t^2 + t + 1) - f_1 - f_3.
\end{cases}
$$

So $\frac{-2t^3 - f_1 t + f_3 + f_1 - 2}{4\sqrt{t^3}}$ and $\frac{(-2t^3 - 4t^2 + f_1 t - 2 - 4t + f_3 + f_1)\sqrt{3}}{12\sqrt{t^3}}$ have to be integers. In the case that $3t$ and t are no squares, this only holds if $-2t^3 - f_1 t + f_3 + f_1 - 2 = 0$ and $-2t^3 - 4t^2 - 4t - 2 + f_1 t + f_3 + f_1 = 0$. Hence $f_1 = 2(1 + t)$ and $f_3 = 2(t^2 + t^3)$ if $3t$ and t are no squares. If $3t$ is a square and t is no square, then $-2t^3 - f_1 t + f_3 + f_1 - 2 = 0$ and $-2t^3 - 4t^2 + f_1 t - 2 - 4t + f_3 + f_1$ has to be a multiple of $4t\sqrt{3t}$. Combining these,

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we see that $f_1 - 2$ has to be a multiple of $2\sqrt{3t}$, which means that $f_1 \equiv 2 \mod 2\sqrt{3t}$. Substituting this in the former equality yields $f_3 \equiv 0 \mod 2\sqrt{3t}$. On the other hand, if t is a square and $3t$ is no square, then $-2t^3 - 4t^2 + f_1 t - 2 - 4t + f_3 + f_1 = 0$ and $-2t^3 - f_1 t + f_3 + f_1 - 2$ has to be a multiple of $4t\sqrt{t}$. Hence $f_1 - 2$ has to be a multiple of $2\sqrt{t}$, which means that $f_1 \equiv 2 \mod 2\sqrt{t}$. Similarly as above, $f_3 \equiv 0 \mod 2\sqrt{t}$. □

This immediately implies, in view of Lemma 3.4:

Corollary 7.7 *Suppose that θ is a duality of a generalize hexagon of order t. If none of $3t$ and t are perfect squares, then θ has $1 + t$ absolute points and $1 + t$ absolute lines, there are $t^2 + t^3$ points which are mapped to a line at distance 3 and $t^2 + t^3$ lines which are mapped to a point at distance 3, and there are $t^4 + t^5$ points which are mapped to a line at distance 5 and $t^4 + t^5$ lines which are mapped to a point at distance 5. If t is a perfect square, then there are $1 \mod \sqrt{t}$ absolute points and equally many absolute lines; the number of point mapped onto a line at distance 3 in the incidence graph is divisible by $\sqrt{t}$. If $3t$ is a perfect square, then there are $1 \mod \sqrt{3t}$ absolute points and equally many absolute lines; the number of point mapped onto a line at distance 3 in the incidence graph is divisible by $\sqrt{3t}$.*

We currently do not know of any finite self-dual generalized hexagon of order t, with neither t nor $3t$ a perfect square. If $3t$ is a square, then similarly as for symplectic quadrangles of order s, with $2s$ a square, one can easily construct dualities of the split Cayley hexagon $\mathsf{H}(t)$ with exactly one absolute point as the composition of a polarity with a nontrivial central collineation with center one of the absolute points of the polarity. If t is a square, then our conjecture is that the number of absolute points of a duality is also exactly $1 + t$.

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