Veronesian Embeddings of Hermitian Unitals

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Abstract

In this paper, we determine the Veronesian embeddings of Hermitian unitals, i.e., the representations of Hermitian unitals as points of a 7-dimensional projective space where the blocks are plane ovals. As an application, we derive that the following objects coincide: (1) the generic hyperplane sections of Hermitian Veronesians in 8-dimensional projective space, (2) the Grassmannians of the classical spreads of non-degenerate quadrics of Witt index 2 in 5-dimensional projective space, (3) the sets of absolute points of trialities of Witt index 1. As a consequence, we prove that the set of absolute points of a triality without fixed lines, but with absolute points, determines the triality quadric and the triality itself uniquely.

1 Introduction

Veroneseans of finite projective spaces play an important role in finite geometry. In [13], they are characterized as representations of $\mathsf{PG}(d,q)$ in $\mathsf{PG}(n,q)$, with $n = \frac{1}{2}n(n+3)$, such that the points of $\mathsf{PG}(d,q)$ correspond with (not all) points of $\mathsf{PG}(n,q)$, and the lines of $\mathsf{PG}(d,q)$ correspond with plane ovals in $\mathsf{PG}(n,q)$. This characterization, however, also holds in the infinite case. In general, for a given field $\mathbb{K}$ and for any point-line geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L})$, one can consider injective mappings $\alpha : \mathcal{P} \to \mathsf{PG}_0(n,\mathbb{K})$, where $\mathsf{PG}_0(n,\mathbb{K})$ denotes the 0-dimensional subspaces (hence the points) of $\mathsf{PG}(n,\mathbb{K})$, such that α maps de point sets of every line (a line is an element of $\mathcal{L}$) onto an oval in a certain plane of $\mathsf{PG}(n,\mathbb{K})$, and such that the image of $\mathcal{P}$ generates $\mathsf{PG}(n,\mathbb{K})$. (The lack of "classical" ovals in projective planes over noncommutative skew fields leads to considering only commutative fields, and not noncommutative ones.) We call such an embedding of $\mathcal{S}$ a *Veronesean embedding*. Motivated by the existence of some classes of examples of Veronesean embeddings of Hermitian unitals, we take a closer look at this case in the present paper. The starting observation is that most examples we know are embedded in

7-dimensional projective space. In fact, all these examples are, for the same values of q, isomorphic. This will be proved by showing that for a given Hermtitian unital, say related to the quadratic Galois extension $\mathbb{L}$ of $\mathbb{K}$, any Veronesean embedding of it in $\mathsf{PG}(7, \mathbb{K})$ is projectively unique.

It turns out that Veronesean embeddings of Hermitian unitals are intimately connected to triality. Not only do trialities of Type II over fields of characteristic different from 3 having no nontrivial cubic roots of unity (for characteristic different from 2, this is equivalent with -3 being a nonzero nonsquare) give direct rise to such objects, also the trialities of Type II over other fields of characteristic different from 3 produce such objects, and trialities of Type I_{id} can also sometimes be used to construct them. In fact, our approach will allow us to reconstruct the triality quadric and the triality itself from a given set of absolute points of a triality of Type II over a field of characteristic different from 3 having no nontrivial cubic roots of unity. We will also discuss other beautiful, mainly geometric, properties of Veronesean embeddings of Hermitian unitals in $\mathsf{PG}(7, \mathbb{K})$. Our results hold in both the finite and infinite case.

The paper is structured as follows. In Section 2, we collect all notions that we need to successfully prove our main results, and to discuss the main properties of Veronesean embeddings of Hermitian unitals. Notice that we use a wide spectrum of notions, so this section is rather large. We also state our Main Result in that section.

In Section 3, we prove our Main result. This section is the main body of our paper, and we have divided it into subsections for clarity's sake. It contains seemingly unrelated results, that also might be of independent interest somewhere else, and we have stated these results as lemmas (see Lemma 1 and Lemma 3).

In Section 4, we show how our Main Result proves that the objects mentioned in the abstract are isomorphic.

Section 5, finally, is devoted to more properties of Veronesean embeddings of Hermitian unitals, and, in particular, we will prove in that section our above claim about the uniqueness of the triality of Type II over a suitable field, given its set of absolute points.

2 Preliminaries

Throughout, let $\mathbb{K}$ be a field, and let $\mathbb{L}$ be a quadratic Galois extension of $\mathbb{K}$. We denote the nontrivial element (which is an involution) of the corresponding Galois group by $x \mapsto \overline{x}$. We occasionally call this map *(complex) conjugation.*

2.1 Hermitian unitals

A *Hermitian curve over* $\mathbb{L}/\mathbb{K}$ in the projective plane $\mathsf{PG}(2,\mathbb{L})$ is any set of points of $\mathsf{PG}(2,\mathbb{L})$ that is projectively equivalent to the set of points satisfying, after introducing coordinates (X_0, X_1, X_2), the equation

$$X_0\overline{X}_1 + \overline{X}_0X_1 = X_2\overline{X}_2.$$

This set of points, say $\mathcal{P}$, can be given the structure of a linear space by collecting all nontrivial (i.e., containing at least two points) line intersections in a set $\mathcal{B}$, and defining the point-block geometry $\mathcal{C} = (\mathcal{P}, \mathcal{B})$ in the obvious way (incidence is given by containment; the elements of $\mathcal{B}$ are usually called *blocks*, because in the finite case this structure is a 2-design). This point-block geometry will be called the *Hermitian unital over* $\mathbb{L}/\mathbb{K}$, and will be denoted by $\mathsf{U}(\mathbb{L}/\mathbb{K})$.

We will denote the set of points of a projective space $\mathsf{PG}(d,\mathbb{F})$, for $\mathbb{F}$ any field, by $\mathsf{PG}_0(d,\mathbb{F})$.

Let L be a line of $\mathsf{PG}(2,\mathbb{L})$. Recall that, with respect to a given coordinatization of $\mathsf{PG}(2,\mathbb{L})$, the cross ratio of four points a, b, c, d on L, with $a \neq b \neq c \neq a$, is equal to

$$(a, b; c, d) = \frac{r_c - r_a}{r_c - r_b} : \frac{r_d - r_a}{r_d - r_b},$$

where $r_a, \ldots, r_d$ are nonhomogeneous coordinates on L with respect to an arbitrary base. The *Baer subline* containing the three distinct points a, b, c on L is the set of points x on L with $x = a$ or $(a, b; c, x) \in \mathbb{K}$. Dually, one defines a *Baer subpencil* of the *pencil* of lines through some fixed point p.

Now let p be an arbitrary point of $\mathsf{PG}(2,\mathbb{L})$. Let $\mathcal{L}$ be the set of lines through p. The Baer subpencils of the pencil in p define the block set of a geometry $\mathcal{M} = (\mathcal{L}, \mathcal{E})$, which is a *Möbius plane*. The elements of $\mathcal{E}$ are called *circles*. The geometry $\mathcal{M}$ has the following property: for every element $L \in \mathcal{L}$, the structure $\mathcal{M}_L = (\mathcal{L} \setminus \{L\}, \mathcal{E}_L)$ obtained from $\mathcal{M}$ by deleting the element L and all blocks not through L (and removing L from the blocks that do contain L), is an affine plane (over $\mathbb{L}$). The structure $\mathcal{M}_L^{\mathrm{Aff}} = (\mathcal{L} \setminus \{L\}, \mathcal{E})$, where one removes L from every circle that contains L, will be called a *pointed Möbius plane*. A circle through L with L removed will be called a *pointed circle*.

If $p \in \mathcal{P}$, then there is a unique line L in $\mathcal{L}$ meeting $\mathcal{C}$ trivially (i.e., in $\{p\}$); it is often called the *tangent line* in p to $\mathcal{C}$. The corresponding pointed Möbius plane will be referred to as the the pointed Möbius plane *at* p.

Abstractly, $\mathcal{M}_L^{\mathrm{Aff}}$ can be defined as follows. Its point set is the point set of the affine plane $\mathsf{AG}(2,\mathbb{K})$. Extend $\mathsf{AG}(2,\mathbb{K})$ to $\mathsf{AG}(2,\mathbb{L})$ and choose two *imaginary* points p, p' at infinity that are mutually (complex) conjugate (i.e., they have complex conjugate slopes that belong to $\mathbb{L} \setminus \mathbb{K}$). Then the pointed circles are the affine lines of $\mathsf{AG}(2,\mathbb{K})$ and the ordinary circles are the nondegenerate conics in $\mathsf{AG}(2,\mathbb{K})$ whose extensions to $\mathsf{AG}(2,\mathbb{L})$ contain p, p' at infinity of $\mathsf{AG}(2,\mathbb{L})$.

2.2 Hermitian Veronesean

Now choose an arbitrary element $\eta \in \mathbb{L}\backslash\mathbb{K}$, and consider the following map $\beta : \mathsf{PG}_0(2,\mathbb{L}) \to \mathsf{PG}_0(8,\mathbb{K})$:

$$\begin{aligned}\beta:\quad (x_0,x_1,x_2) \mapsto \quad & (x_0\overline{x}_0, x_1\overline{x}_1, x_2\overline{x}_2; x_0\overline{x}_1 + \overline{x}_0x_1, x_1\overline{x}_2 + \overline{x}_1x_2, x_2\overline{x}_0 + \overline{x}_2x_0;\\ & \eta x_0\overline{x}_1 + \overline{\eta x}_0x_1, \eta x_1\overline{x}_2 + \overline{\eta x}_1x_2, \eta x_2\overline{x}_0 + \overline{\eta x}_2x_0).\end{aligned}$$

The direct image of β is called the *Hermitian Veronesean of* $\mathsf{PG}(2,\mathbb{L})$ *with respect to* $\mathbb{L}$, and we denote it by $\mathcal{H}$. It is, up to isomorphism, independent of η. Also, in any one or two of the last three coordinates, one may substitute η by $\overline{\eta}$ to still obtain a projective equivalent set.

Here are some properties of the Hermitian Veroneseans $\mathcal{H}$.

(HV1) The inverse image under β of any hyperplane section of $\mathcal{H}$ is the null set of a Hermitian polynomial, i.e., a polynomial of the form

$$\sum_{i=0}^{2} a_iX_i\overline{X}_i + \sum_{0\le i<j\le 2} a_{ij}X_i\overline{X}_j + \overline{a}_{ij}\overline{X}_iX_j,$$

with $a_k \in \mathbb{K}$, $k \in \{0,1,2\}$ and $a_{ij} \in \mathbb{L}$, $0 \le i < j \le 2$.

(HV2) The direct image of a line L of $\mathsf{PG}(2,\mathbb{L})$ is an elliptic quadric (Witt index 1) in some 3-dimensional subspace of $\mathsf{PG}(8,\mathbb{K})$.

(HV3) Let p be a point of $\mathsf{PG}(2,\mathbb{L})$. Then the space generated by the tangent lines at p^β of the elliptic quadrics on $\mathcal{H}$ containing p^β is a 4-dimensional space P, and each point of $P \setminus \{p^\beta\}$ lies on exactly one tangent line. Moreover, no line through p^β is tangent to more than one such elliptic quadric.

(HV4) Every Baer subline in $\mathsf{PG}(2,\mathbb{L})$ maps under β onto a conic in some plane of $\mathsf{PG}(8,\mathbb{K})$.

These properties can be verified directly using elementary calculations with coordinates; most of them are also contained in [6] and [14].

2.3 Generalized polygons

Generalized polygons were introduced by Jacques Tits in [15]. The claims below can be found in the monograph [16]. The finite case for generalized quadrangles is extensively studied in [9].

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be a point-line geometry, where we view the elements of $\mathcal{L}$ as sets of points. The *incidence graph* $\Gamma(\mathcal{S})$ is the graph with vertex set $\mathcal{P} \cup \mathcal{L}$ and $p \in \mathcal{P}$ is adjacent to $L \in \mathcal{L}$ if $x \in L$. Then we call $\mathcal{S}$ a *generalized n-gon*, or *generalized polygon*, if the diameter of $\Gamma(\mathcal{S})$ is equal to n, and the girth of $\Gamma(\mathcal{S})$ is equal to $2n$ (the *girth* is the length of the smallest cycle in Γ). In fact, in the present paper, we will only need generalized 4-gons, also called *generalized quadrangles*, and generalized 6-gons, or *generalized hexagons*. Another, more common example, is any projective plane, which is a generalized 3-gon.

It follows directly from the definition that elements of a generalized n-gon can have mutual distance at most n in the incidence graph. Such elements will be called *opposite*. If the valency of every vertex of $\Gamma(\mathcal{S})$ is at least 3, then we call $\mathcal{S}$ *thick*; otherwise just *non-thick*.

The prototype of examples of generalized quadrangles are the symplectic quadrangles, which we will need in "dual form". Let $\mathsf{Q}(4, \mathbb{K})$ be a nondegenerate quadric of Witt index 2 in $\mathsf{PG}(4, \mathbb{K})$, e.g., with equation $X_0X_1 + X_2X_3 = X_4^2$. Then the points and lines of $\mathsf{PG}(4, \mathbb{K})$ entirely contained in $\mathsf{Q}(4, \mathbb{K})$ form a generalized quadrangle, which we also denote by $\mathsf{Q}(4, \mathbb{K})$. Let H be a hyperplane of $\mathsf{PG}(4, \mathbb{K})$, and suppose that H contains two opposite lines of $\mathsf{Q}(4, \mathbb{K})$. Then the intersection of $\mathsf{Q}(4, \mathbb{K})$ with H is a hyperbolic quadric $\mathsf{Q}^+(3, \mathbb{K})$ (Witt index 2), and $\mathsf{Q}^+(3, \mathbb{K})$ is a generalized quadrangle in its own right. But it is also a *subquadrangle*. It is called a non-thick *full* subquadrangle since every point of $\mathsf{Q}(4, \mathbb{K})$ on every line of $\mathsf{Q}^+(3, \mathbb{K})$ is a point of $\mathsf{Q}^+(3, \mathbb{K})$. It is easily seen that every pair of opposite lines of $\mathsf{Q}(4, \mathbb{K})$ is contained in a unique non-thick full subquadrangle, necessarily isomorphic to $\mathsf{Q}^+(3, \mathbb{K})$. For short, we will sometimes call such a non-thick full subquadrangle a *grid*.

Another class of examples arises from quadrics of Witt index 2 in $\mathsf{PG}(5, \mathbb{K})$. Let $x^2 + x + n$ be an irreducible polynomial over $\mathbb{K}$ defining $\mathbb{L}$, i.e., this polynomial is reducible over $\mathbb{L}$. Then the equation $X_0X_1 + X_2X_3 = X_4^2 + X_4X_5 + nX_5^2$ defines a quadric $\mathsf{Q}^-(5, \mathbb{K})$, which defines on its turn a generalized quadrangle, as above. In fact, the quadrangle $\mathsf{Q}(4, \mathbb{K})$ is a thick full subquadrangle of $\mathsf{Q}^-(5, \mathbb{K})$.

Finally, let $\mathsf{H}(3, \mathbb{L}/\mathbb{K})$ be the null set in $\mathsf{PG}(3, \mathbb{L})$ of the Hermitian polynomial $X_0\overline{X}_1 + \overline{X}_0X_1 + X_2\overline{X}_3 + \overline{X}_2X_3$. Then, again as before, $\mathsf{H}(3, \mathbb{L}/\mathbb{K})$ defines a generalized quadrangle. But the incidence graphs of $\mathsf{Q}^-(5, \mathbb{K})$ and of $\mathsf{H}(3, \mathbb{L}/\mathbb{K})$ are isomorphic graphs (and every isomorphism maps points to lines and vice versa); we say that these generalized quadrangles are *dual to each other*. Geometrically, such an isomorphism can be obtained by the Klein correspondence.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be a generalized quadrangle. A set $\mathfrak{O}$ of points is called an *ovoid*, if every line of $\mathcal{S}$ is incident with exactly one point of $\mathfrak{O}$. Dually, a set $\mathfrak{S}$ of lines is called a *spread* if every point of $\mathcal{S}$ is incident with exactly one element of $\mathfrak{S}$. For example, a plane intersection of $\mathsf{H}(3, \mathbb{L}/\mathbb{K})$ not containing any line of $\mathsf{H}(3, \mathbb{L}/\mathbb{K})$ intersects $\mathsf{H}(3, \mathbb{L}/\mathbb{K})$ in the points of an ovoid. This ovoid is called a *Hermitian ovoid* of $\mathsf{H}(3, \mathbb{L}/\mathbb{K})$.

Next, we turn to hexagons. Since we will need the notion of triality anyway, it is convenient to introduce the relevant class of examples with triality.

2.4 Triality

Consider, in $\mathsf{PG}(7, \mathbb{K})$, the *triality* quadric $\mathsf{Q}(7, \mathbb{K})$, which we denote by Q for short, of Witt index 4 with equation $X_0X_4 + X_1X_5 + X_2X_6 + X_3X_7 = 0$. We will call subspaces of projective dimension 3 *solids*, for short. It is well known that this quadric contains 2 families of generators (*generators* are solids entirely contained in Q), such that two generators of the same family meet in an odd dimensional projective subspace, and two belonging to different families meet in an even dimensional projective subspace. Now define the *triality graph* $T\Gamma$ as follows. The vertices are the points (gathered in Γ_0), lines (gathered in the set Γ_1) and 3-spaces (gathered in the sets Γ_3 and Γ_3', corresponding with the two families above) of $\mathsf{PG}(7, \mathbb{K})$ entirely contained in Q; adjacency is defined as follows: a line is adjacent to every point it contains, and to every solid it is contained in (notice that every edge contains a vertex in Γ_1 and that we thus do not obtain the incidence graph of the corresponding building; the given adjacencies are enough since they define collinearity of points and this, on its turn, completely determines the structure of the quadric). Then this graph admits graph automorphisms that preserve the partition $\{\Gamma_0, \Gamma_1, \Gamma_3, \Gamma_3'\}$, but induce any permutation of the set $\{\Gamma_0, \Gamma_3, \Gamma_3'\}$. More in particular, there exist graph automorphisms τ of order 3 inducing the cyclic permutation $\Gamma_0 \to \Gamma_3 \to \Gamma_3' \to \Gamma_0$. These are called *trialities*. Now, call a point $x \in \Gamma_0$ *absolute* as soon as the (graph) distance in $T\Gamma$ from x to x^τ is 2. Similarly as for polarities in projective spaces, we can define the *Witt index* of a triality as the 'dimension' of its 'absolute geometry': if there are no absolute points, then the Witt index is 0; if there are absolute points, but no fixed lines, then the Witt index is 1; if there are fixed lines—and hence also absolute points since it is easily seen that every point of a fixed line is absolute—then the Witt index is equal to 2. With this terminology, all trialities of Witt index at least 1 are classified in [15]. With the notation of [15], trialities of Type I have Witt index 2 and produce generalized hexagons, i.e., the geometry of absolute points and fixed lines, with natural incidence, is a generalized hexagon. More precisely, the set of absolute points of a triality of Type I_{id} is precisely the set of points of Q lying in some fixed hyperplane H isomorphic to $\mathsf{PG}(6,\mathbb{K})$, and, identifying H with $\mathsf{PG}(6,\mathbb{K})$, this intersection is a quadric $\mathsf{Q}(6,\mathbb{K})$ of Witt index 3 in $\mathsf{PG}(6,\mathbb{K})$. The corresponding generalized hexagon is thick and is denoted by $\mathsf{H}(\mathbb{K})$, called the *split Cayley hexagon (over* $\mathbb{K}$*)*. The lines of $\mathsf{H}(\mathbb{K})$ are some lines of $\mathsf{Q}(6,\mathbb{K})$, certainly not all. But the precise description is not important for our purposes.

A subset $\mathfrak{S}$ of the line set of a generalized hexagon is called a *spread* if every line not in $\mathfrak{S}$ meets exactly one element of $\mathfrak{S}$, and if all elements of $\mathfrak{S}$ are mutually opposite. In Subsection 4.2, we will encounter an example.

Trialities τ of Type II come in three flavours, depending on the field $\mathbb{K}$. If $\mathbb{K}$ has characteristic 3, then τ has Witt index 2. Also, every absolute point is on a fixed line, and every fixed line is concurrent with a certain given fixed line. This case is not important for us. If $\mathrm{char}\mathbb{K} \neq 3$ and $\mathbb{K}$ contains nontrivial cubic roots of unity, then τ again has Witt index 2. The absolute points and absolute lines form a non-thick generalized hexagon, which can be constructed abstractly as follows: the points are the incident point-line pairs of $\mathsf{PG}(2,\mathbb{K})$; the lines are the points and lines of $\mathsf{PG}(2,\mathbb{K})$; incidence is natural. We will see this more in detail in Subsection 4.1. Finally, when $\mathrm{char}\mathbb{K} \neq 3$ and $\mathbb{K}$ does not contain nontrivial cubic roots of unity, then the Witt index is equal to 1, and the set of absolute points admits the unitary group $\mathsf{PGU}_3(\mathbb{L}'/\mathbb{K})$ (with obvious notation), where $\mathbb{L}'$ is the quadratic Galois extension corresponding to the polynomial X^2-X+1, and where $\mathsf{PGU}_3(\mathbb{L}'/\mathbb{K})$ acts on the set of absolute points in the same way as it acts on a Hermitian unital over $\mathbb{L}'/\mathbb{K}$. We remark that the set of absolute points is in this case an *ovoid* of the triality quadric, i.e., a set of points with the property that every generator of $\mathsf{Q}(7,\mathbb{K})$ contains exactly one absolute point. Applying triality, we obtain a *spread*, i.e., a set of generators that partitions the point set of $\mathsf{Q}(7,\mathbb{K})$ (and both the ovoid and spread admit a 2-transitive group). Intersecting with suitable hyperplanes gives spreads of the quadric $\mathsf{Q}(6,\mathbb{K})$, which yield, on their turn, distance-3 ovoids of the associated near hexagon. But we will not use these well known properties in the present paper. We refer to [10] for more about ovoids and spreads of (finite) polar spaces.

2.5 Grassmannians

For our applications, we will need the concept of Grassmannians and Grassmann coordinates. We refer to Chapter 24 of [7] for more about this. Here, we content ourselves with the basic definitions, and with the special case of line-Grassmannians.

Let L be a line in $\mathsf{PG}(d,\mathbb{K})$, with $d>1$. After coordinatization, we can pick two arbitrary different points x and y on L, with coordinates $(x_0,x_1,\ldots,x_d)$ and $(y_0,y_1,\ldots,y_d)$, respectively. This defines a unique point p_L in $\mathsf{PG}(d,\mathbb{K})$, with $d=\frac{1}{2}(n^2+n-2)$, with coordinates $(p_{ij})_{0\leq i<j\leq d}$, where $p_{ij}=x_iy_j-x_jy_i$. This point is independent of the choices of x and y. The mapping $L\mapsto(p_{ij})_{0\leq i<j\leq d}$ is called the *(line) Grassmann map*. The coordinates p_{ij}, $0\leq i<j\leq d$, are called the *(line) Grassmann coordinates*, and the collection of all points of $\mathsf{PG}(n,\mathbb{K})$ thus obtained is called the *(line) Grasmannian of* $\mathsf{PG}(d,\mathbb{K})$. One can prove that the points of the line Grassmannian of $\mathsf{PG}(d,\mathbb{K})$ really generate the whole space $\mathsf{PG}(n,\mathbb{K})$. Two easy direct properties are: (1) the set of lines of $\mathsf{PG}(d,\mathbb{K})$ through a fixed point and contained in a fixed plane is mapped under the Grassmann map onto the set of points of a line of $\mathsf{PG}(n,\mathbb{K})$; (2) the set of lines of $\mathsf{PG}(d,\mathbb{K})$ belonging to one family of generators of a hyperbolic quadric in some solid of $\mathsf{PG}(d,\mathbb{K})$ is mapped onto the set of points of a conic in some plane of $\mathsf{PG}(n,\mathbb{K})$.

2.6 Projective embeddings and our Main Result

Let $\mathcal{S}=(\mathcal{P},\mathcal{L})$ be a point-line geometry, as before, where lines are completely determined by their points, and let $\alpha:\mathcal{P}\rightarrow\mathsf{PG}_0(d,\mathbb{K})$ be a mapping such that the image of $\mathcal{P}$ generates $\mathsf{PG}(d,\mathbb{K})$. If for any line $L\in\mathcal{L}$, either the image under α is a point of $\mathsf{PG}(d,\mathbb{K})$, or α is injective on the set of points on L and maps this point set into a line L^α, then we call α a *linear projective stacking* of $\mathcal{S}$. If moreover, α is injective, then we call it a *linear projective embedding* of $\mathcal{S}$. If moreover, every point of $\mathsf{PG}(d,\mathbb{K})$ on a line L^α has an inverse image under α on L, then we call the linear projective embedding *full*.

Examples of full linear projective embeddings are given by the generalized quadrangles $\mathsf{Q}^+(3,\mathbb{K})$, $\mathsf{Q}(4,\mathbb{K})$ and $\mathsf{Q}^-(5,\mathbb{K})$ (α is the identity map), and the generalized hexagon $\mathsf{H}(\mathbb{K})$.

A plane oval $\mathcal{O}$ of $\mathsf{PG}(d,\mathbb{K})$ is a set of points of $\mathsf{PG}(d,\mathbb{K})$ contained in a plane π, such that, for any point $x\in\mathcal{O}$, there is a unique line of π through x intersecting $\mathcal{O}$ in just x, and all other lines through x intersect $\mathcal{O}$ in exactly two points (including x!). Examples are conics.

Let again $\mathcal{S}=(\mathcal{P},\mathcal{L})$ be a point-line geometry. A *Veronesean (projective) embedding* of $\mathcal{S}$ in the projective space $\mathsf{PG}(d,\mathbb{K})$ over a field $\mathbb{K}$ is an injective mapping $\alpha:\mathcal{P}\rightarrow\mathsf{PG}_0(d,\mathbb{K})$ such that the image of the points of any line is a plane oval, and such that the image of $\mathcal{P}$ under α generates $\mathsf{PG}(d,\mathbb{K})$.

We define equivalent projective embeddings in the usual way, i.e., two projective embeddings α and α', be it Veronesean or linear, in $\mathsf{PG}(d,\mathbb{K})$ and $\mathsf{PG}(d,\mathbb{K})'$, respectively, are *equivalent* if there is a projectivity $\sigma : \mathsf{PG}(d,\mathbb{K}) \to \mathsf{PG}(d,\mathbb{K})'$ mapping p^α to $p^{\alpha'}$, for all points $p \in \mathcal{P}$.

For example, the ordinary (quadric) Veronesean of $\mathsf{PG}(2,\mathbb{K})$ yields a Veronesean embedding of $\mathsf{PG}(2,\mathbb{K})$ in $\mathsf{PG}(5,\mathbb{K})$. Conversely, every Veronesean embedding of $\mathsf{PG}(2,\mathbb{K})$ into $\mathsf{PG}(5,\mathbb{K})$ is equivalent to the (quadric) Veronesean of $\mathsf{PG}(2,\mathbb{K})$, see [13].

Now consider, with respect to a quadratic extension $\mathbb{L}$ of $\mathbb{K}$, the Hermitian Veronesean $\mathcal{H}$ of $\mathsf{PG}(2,\mathbb{L})$. Then $\mathcal{H} \subseteq \mathsf{PG}(8,\mathbb{K})$. Let H be a hyperplane of $\mathsf{PG}(8,\mathbb{K})$ related to a non-degenerate Hermitian curve $\mathcal{C}$ in $\mathsf{PG}(2,\mathbb{L})$. Then $\mathcal{H} \cap H$ is a Veronesean embedding of $\mathcal{C}$, viewed as a Hermitian unital, in $\mathsf{PG}(7,\mathbb{K})$. If we call this the *standard Veronesean embedding* of $\mathcal{C}$, then our main result reads as follows.

Main Result. *Let $\mathcal{C} = (\mathcal{P},\mathcal{B})$ be a Hermitian unital over $\mathbb{L}/\mathbb{K}$. Then every Veronesean embedding of $\mathcal{C}$ in $\mathsf{PG}(d,\mathbb{F})$, with $\mathbb{F}$ any field, and with $d \geq 7$ is equivalent to the standard Veronesean embedding of $\mathcal{C}$.*

3 Proof of the Main Result

3.1 A lemma on $\mathsf{Q}(4,\mathbb{K})$

An important and handy tool for the sequel will be the study of linear projective stackings of so-called affine quadrangles of hyperbolic type related to $\mathsf{Q}(4,\mathbb{K})$. These are defined as follows. Consider a full non-thick subquadrangle $\mathsf{Q}^+(3,\mathbb{K})$ on $\mathsf{Q}(4,\mathbb{K})$, then the subgeometry induced by $\mathsf{Q}(4,\mathbb{K})$ on the complement of $\mathsf{Q}^+(3,\mathbb{K})$ is called an *affine quadrangle of hyperbolic type*, but we will simply and briefly call it an *affine quadrangle.* We will use the notation $\mathsf{AQ}(4,\mathbb{K}) := \mathsf{AG}(4,\mathbb{K}) \cap \mathsf{Q}(4,\mathbb{K})$ for it and we note that it determines $\mathsf{Q}(4,\mathbb{K})$ uniquely (see also [8]).

It is to be expected that linear projective embeddings of $\mathsf{AQ}(4,\mathbb{K})$ behave like restrictions of linear projective embeddings of $\mathsf{Q}(4,\mathbb{K})$, and in fact, this is not so difficult to prove. Also, it is to be expected that linear projective stackings of $\mathsf{Q}(4,\mathbb{K})$ for large enough projective dimensions are just embeddings, and this is not so difficult to prove either. But it is a bit more involved to see what happens when we consider linear projective stackings of $\mathsf{AQ}(4,\mathbb{K})$. Nevertheless, the result is what we expect.

Lemma 1 *Let α be a linear projective stacking of $\mathsf{AQ}(4,\mathbb{K})$ in some projective space $\mathsf{PG}(d,\mathbb{F})$, with $|\mathbb{K}| > 2$ and $d \geq 4$. Then $d = 4$ and α is injective. If for each line K*

of $\mathsf{AQ}(4,\mathbb{K})$*, there is a unique point of the line* K^α *not belonging to the image under* α *of a point of* $\mathsf{AQ}(4,\mathbb{K})$ *incident with* K*, then* α *can be uniquely extended to a full linear projective embedding of* $\mathsf{Q}(4,\mathbb{K})$ *in* $\mathsf{PG}(4,\mathbb{F})$*.*

Proof First we claim that α is injective when restricted to the point set incident with an arbitrary line. Indeed, suppose some line L is mapped onto a point L^α of $\mathsf{PG}(d,\mathbb{F})$. Let L' be any line of $\mathsf{AQ}(4,\mathbb{K})$ disjoint from L, opposite L in $\mathsf{Q}(4,\mathbb{K})$ and such that L and L' meet the same line of $\mathsf{Q}^+(3,\mathbb{K})$. It is easy to see that all points of the non-thick full subquadrangle generated by L and L' are mapped onto points in $\langle L^\alpha, L'^\alpha\rangle$. Consider a line L'' disjoint from L' but not opposite L' in $\mathsf{Q}(4,\mathbb{K})$. Playing the same game with L'' and any line in $\{L,L'\}^{\perp\perp}\cap\mathsf{AQ}(4,\mathbb{K})$, and doing it again with the thus obtained lines yields that, if $|\mathbb{K}|>2$, the image under α of $\mathsf{AQ}(4,\mathbb{K})$ is contained in $\langle L^\alpha, L'^\alpha, L''^\alpha\rangle$, which is only at least 4-dimensional if at most one of $L^\alpha, L'^\alpha, L''^\alpha$ is a point. Since L^α is a assumed to be a point, we have that both , L'^α and L''^α are lines and $d=4$. Now we note that the lines of $\mathsf{AQ}(4,\mathbb{K})$ meeting L' outside $\mathsf{Q}^+(3,\mathbb{K})$, structured by the grids through L' form a *bi-affine plane*, i.e., a projective plane with all points on a line removed, and with all lines through a point on the first line removed. Projecting $\mathsf{AQ}(4,\mathbb{K})^\alpha$ from L'^α onto a $(d-2)$-dimensional skew subspace U (hence a plane, since $d=2$), we now see that one of these lines G (corresponding to the grid containing L and L') is projected onto a point. Let M be any line concurrent with L', and such that M^α is a line. Then for any line M' concurrent with L and such that M'^α is a line, such that M and M' are collinear in the above mentioned bi-affine plane, and such that the joining line meets G, the projection of M'^α is contained in the span of the projection of G^α and M^α. By connectivity, we now see that the projection is contained in a projective line, and hence $d=3$, a contradiction.

The claim is proved.

A similar argument as the last one in the previous paragraph shows that α is injective on the set of points. Indeed, if two points of $\mathsf{AQ}(4,\mathbb{K})$ are mapped onto the same point by α, then we can find a line L contained in a grid which is mapped by α into a plane of $\mathsf{PG}(d,\mathbb{K})$ (and that grid contains two points not on L mapped to the same point by α). Projection from L^α and arguing as above leads to $d<4$.

But the very same argument now shows that $d=4$. Indeed, projection from a line L^α yields a sub-bi-affine plane of some plane in $\mathsf{PG}(d,\mathbb{K})$, hence $d\leq 2+1+1=4$.

Note that the argument above also shows that, if two lines of $\mathsf{AQ}(4,\mathbb{K})$ are opposite in $\mathsf{Q}(4,\mathbb{K})$, then their images under α do not meet in $\mathsf{PG}(4,\mathbb{F})$.

Now assume that for each line K of $\mathsf{AQ}(4,\mathbb{K})$, there is a unique point, which we denote by x_K, of the line K^α not belonging to the image under α of a point of $\mathsf{AQ}(4,\mathbb{K})$ incident with K. If K and K' do not meet in $\mathsf{AQ}(4,\mathbb{K})$, but are not opposite in $\mathsf{Q}(4,\mathbb{K})$, then we first claim that $x_K = x_{K'}$. Indeed, we again project onto a suitable plane π from K the bi-affine plane corresponding to K. It is easy to see that our assumption above implies that there are a unique point z and a unique line R in π such that the projection of that bi-affine plane is the bi-affine plane obtained from π by deleting z and R and all elements incident with these. It follows directly that the projection of K'^α is one of the deleted points, and so K^α and K'^α are concurrent. If K^α and K'^α met in a point t different from x_K and $x_{K'}$, then there would be some line M of $\mathsf{AQ}(4,\mathbb{K})$ with $t \in M^\alpha$ and $K \neq M \neq K'$ and M meeting one of K, K' in $\mathsf{AQ}(4,\mathbb{K})$, say it meets K. But then M is opposite K' and so M^α, K'^α cannot be contained in a common plane, a contradiction. The claim follows.

So the points of $\mathsf{Q}(4,\mathbb{K})$ not in $\mathsf{AQ}(4,\mathbb{K})$ have a uniquely defined image under α. Suppose R is a line of $\mathsf{Q}(4,\mathbb{K})$ disjoint from $\mathsf{AQ}(4,\mathbb{K})$. Then R^α is contained in a unique line of the regulus of $\mathsf{PG}(4,\mathbb{K})$ defined by any grid of $\mathsf{Q}(4,\mathbb{K})$ containing R and at least one line of $\mathsf{AQ}(4,\mathbb{K})$. It follows easily that R^α is the complete set of points of a line of $\mathsf{PG}(4,\mathbb{F})$.

The proof of the lemma is complete. $\square$

3.2 Bounding the dimension

Let $\mathcal{C} = (\mathcal{P}, \mathcal{B})$ be a Hermitian unital over the field $\mathbb{K}$ corresponding to the field extension $\mathbb{L}$. Let α be a Veronesean embedding of $\mathcal{C}$ in $\mathsf{PG}(d,\mathbb{F})$, with $\mathbb{F}$ some field, and with $d \geq 7$. Our first aim is to prove that $d = 7$.

Let us first recall the following representation of $\mathcal{C}$.

Let $\mathsf{AG}(4,\mathbb{K})$ be a 4-dimensional affine space over $\mathbb{K}$, and let $\mathsf{PG}(4,\mathbb{K})$ be the corresponding projective space with hyperplane $\mathsf{PG}(3,\mathbb{K})$ at infinity. Let L be any line of $\mathsf{PG}(2,\mathbb{L})$ meeting $\mathcal{C}$ in at least two points, and let $\mathsf{AG}(2,\mathbb{L})$ be the associated affine plane. Then $\mathsf{AG}(2,\mathbb{L})$ has a natural representation in $\mathsf{AG}(4,\mathbb{K})$ using the fact that $\mathbb{L}$ is two-dimensional over $\mathbb{K}$. The lines of $\mathsf{AG}(2,\mathbb{L})$ are in bijective correspondence with the planes of $\mathsf{AG}(4,\mathbb{K})$ whose line at infinity belongs to a certain fixed line spread Σ of $\mathsf{PG}(3,\mathbb{K})$. The elements of Σ are in bijective correspondence with the points of L. In this setting, the equation of $\mathcal{C}$ reduces to a quadratic equation in the coordinates in $\mathsf{AG}(4,\mathbb{K})$. It is easy to calculate that this implies that there is a unique non-degenerate quadric $\mathsf{Q}(4,\mathbb{K})$ of Witt index 2 in $\mathsf{PG}(4,\mathbb{K})$ such that

(BBB1) The points of $\mathcal{C}$ off L are precisely the points of $\mathsf{Q}(4,\mathbb{K})$ in $\mathsf{AG}(4,\mathbb{K})$;

(BBB2) The intersection of $\mathsf{Q}(4,\mathbb{K})$ with $\mathsf{PG}(3,\mathbb{K})$ is a non-degenerate hyperbolic quadric (Witt index 2) $\mathsf{Q}^+(3,\mathbb{K})$ where one class of generators entirely belongs to Σ;

(BBB3) The blocks of $\mathcal{C}$ intersecting L in a point x of $\mathcal{C}$ are precisely the lines of $\mathsf{Q}(4,\mathbb{K})$ not in $\mathsf{Q}^+(3,\mathbb{K})$ intersecting the element of Σ corresponding to x.

The letters "BBB" stand for Bruck & Bose [2] and Buekenhout [3]. The representation of $\mathsf{AG}(2,\mathbb{K})$ above is attributed to Bruck & Bose (although André [1] discovered it ten years earlier, too), while Buekenhout discovered the correspondence of the Hermitian unitals with the non-degenerate quadrics in this representation.

So it is now clear that the points off L, together with the blocks of $\mathcal{C}$ meeting L non-trivially form an affine quadrangle which we denote by AQ^L and which is isomorphic to $\mathsf{AQ}(4,\mathbb{K})$.

Now fix a point x on L, with $x \in \mathcal{P}$. The lines through x in $\mathsf{PG}(2,\mathbb{L})$ correspond bijectively to the points of an elliptic quadric $\mathsf{Q}^-(3,\mathbb{K})$ in a 3-dimensional projective space $\mathsf{PG}'(3,\mathbb{K})$ with equation $X_0X_1 = X_2^2 - tX_2X_3 + nX_3^2$, where $\mathbb{L} = \mathbb{K}(z)$, with z a root of $Z^2 - tZ + n = 0$ (which is irreducible over $\mathbb{K}$). The Baer subpencils over $\mathbb{K}$ of the pencil at x in $\mathsf{PG}(2,\mathbb{L})$ correspond bijectively with the non-trivial plane intersections of $\mathsf{Q}^-(3,\mathbb{K})$. Let M be the tangent line of $\mathcal{C}$ at x in $\mathsf{PG}(2,\mathbb{L})$ and let M correspond to the point m of $\mathsf{Q}^-(3,\mathbb{K})$. Then the following assertions hold (they can be easily verified using appropriate coordinates).

(BBB4) If a Baer subpencil $\mathcal{B}$ of the pencil at x does not contain the line M, but does contain the line L, then the corresponding set of lines of $\mathsf{Q}(4,\mathbb{K})$ together with a unique line K of $\mathsf{Q}^+(3,\mathbb{K})$ not belonging to Σ forms one set of generators of a full non-thick subquadrangle Γ of $\mathsf{Q}(4,\mathbb{K})$. Also, Γ contains a unique element of Σ, and all other lines of Γ correspond to blocks of $\mathcal{C}$ meeting all members of $\mathcal{B}$ in a point of $\mathcal{C}$. All these blocks are disjoint and cover all points of $\mathcal{C}$ that lie on elements of $\mathcal{B}$, except for x.

(BBB5) If a Baer subpencil $\mathcal{B}$ of the pencil at x contains both L and M, then the corresponding set of lines of $\mathsf{Q}(4,\mathbb{K})$ together with two intersecting lines of $\mathsf{Q}^-(3,\mathbb{K})$ form a line pencil of $\mathsf{Q}(4,\mathbb{K})$. Hence no block of $\mathcal{C}$ meets at least 3 blocks that correspond to members of $\mathcal{B}$.

With slight abuse of language, we will call a set of blocks of $\mathcal{C}$ on x corresponding with the lines of a Baer subpencil in $\mathsf{PG}(2,\mathbb{L})$ not containing M, a *projective Baer subpencil of* *blocks*. If a Baer subpencil in $\mathsf{PG}(2,\mathbb{L})$ does contain M, then the corresponding set of $|\mathbb{K}|$ blocks of $\mathcal{C}$ is referred to as an *affine Baer subpencil of blocks*.

We are now ready to prove that $d = 7$.

Lemma 2 *If α is a Veronesean embedding of $\mathcal{C}$ in $\mathsf{PG}(d,\mathbb{F})$, with $d \geq 7$ and with $\mathcal{C}$ defined over $\mathbb{K}$, with $|\mathbb{K}| > 2$, then $d = 7$.*

Proof Let L be any block of $\mathcal{C}$. We project $\mathcal{C}^\alpha \setminus L^\alpha$ from the plane $\langle L^\alpha \rangle$ onto a suitable $(d-3)$-dimensional subspace $\mathsf{PG}(d-3,\mathbb{F})$ of $\mathsf{PG}(d,\mathbb{F})$. Clearly, the obtained set is a linear projective stacking of $\mathsf{AQ}(4,\mathbb{K})$ and satisfies the condition that, if a line M of $\mathsf{AQ}(4,\mathbb{K})$ is not represented by a point of $\mathsf{PG}(d-3,\mathbb{F})$, then there is a unique point on the corresponding line of $\mathsf{PG}(d-3,\mathbb{F})$ which is not the image of a point of $\mathsf{AQ}(4,\mathbb{K})$. Lemma 1 now tells us that $d - 3 \leq 4$, implying $d = 7$ $\square$

The previous lemma fails for $|\mathbb{K}| = 2$ as in this case every three points of any projective space over $\mathsf{GF}(2)$ constitute a conic in the plane they generate, and so any generating set of points of $\mathsf{PG}(8,2)$ provides a Veronesean embedding of the Hermitian unital over $\mathsf{GF}(4)/\mathsf{GF}(2)$ (this unital is, by the way, isomorphic to the affine plane $\mathsf{AG}(2,3)$).

The previous proof combined with Lemma 1 and the classification of full linear projective embeddings of generalized quadrangles in [4] uniquely determines the projection of $\mathcal{C}^\alpha$ from the plane determined by any of the blocks of $\mathcal{C}$. In particular, we see that all ovals on $\mathcal{C}^\alpha$ must be conics.

3.3 The projection from a point

Next, we want to prove uniqueness of the projection of $\mathcal{C}^\alpha$ from a point of $\mathcal{C}^\alpha$. Therefor, we first determine the dimension of the space ξ_x generated by the tangent lines to the various conics through a point x^α, $x \in \mathcal{P}$, at the point x^α.

Let L be a block of $\mathcal{C}$ and x a point on L. The blocks through x distinct from L are the lines of AQ_L that meet a certain but fixed line at infinity. If $B \in \mathcal{B}$ denotes such block, then the tangent line at x^α to B^α is projected from the plane $\langle L^\alpha \rangle$ onto the unique point of the projection of $\mathcal{B} \setminus \{x\}$ that is not a projection of a point of $\mathcal{B} \setminus B$. All such points lie on a unique line, by Lemma 1. Hence ξ_x is contained in the inverse image of that line under the projection ρ from $\langle L^\alpha \rangle$, which is a subspace of dimension 4. It also follows that ξ_x does not contain the points of $\mathcal{C}^\alpha$ not on L^α, and hence, by varying L though x, ξ_x cannot contain any element of $(\mathcal{P} \setminus \{x\})^\alpha$. So ξ_x is 3-dimensional.

Now consider a point y of $\mathcal{C}$ not on L. The tangent space ξ_y is projected by ρ onto the tangent space of AQ_L at the projection of y^α, which is indeed 3-dimensional. But there, by considering the blocks through y meeting L, we clearly see that the tangent lines belonging to a fixed projective Baer subpencil form a quadratic cone with vertex y^α.

Now consider an arbitrary affine Baer subpencil in x containing L. Since there can be no block of $\mathcal{C}$ meeting L not in x and meeting another member of that pencil, we see that in AQ_L, all corresponding lines go through the same point at infinity, that is to say, they all share the same point in projection from ρ. Hence, all the tangent lines at x^α to the images under α of the members of this affine Baer subpencil are contained in a solid. Varying L, we conclude similarly as above that all these lines are contained in a plane.

Now let π_x be the plane obtained by projecting the solid ξ_x from x^α. Then each block through x is represented by a point of π_x and this representation is injective. Moreover, each projective Baer subpencil corresponds to (all points of) a conic in π_x and each affine Baer subpencil to (some points on) a line of π_x. Now note that the affine Baer subpencils in x form an affine plane. Let B be such a pencil, and let B_x be the corresponding line in π_x. Let C be an arbitrary projective Baer subpencil in x and let C_x be the corresponding conic in π_x. Consider a point $z \in C_x$, with $z \notin B_x$. Let u vary over C_x, then the line zu of π determines a unique affine Baer subpencil in x, and exactly one line zu does not meet B_x in a point of π_x that represents a block through x. It follows easily that the points of B_x that represent blocks through x of $\mathcal{C}$ form an affine line of π, and so the affine plane determined by the affine Baer subpencils, which is an affine plane over $\mathbb{K}$, is isomorphic to an affine plane in π, hence an affine plane over $\mathbb{F}$. This also proves that $\mathbb{K}$ and $\mathbb{F}$ are isomorphic.

It also follows that the pointed Möbius plane at x is represented in π_x with points all points of an affine plane, with pointed circles all lines of this affine plane, and with circles some conics in that affine plane. We now observe that this representation is unique.

Lemma 3 *If the pointed Möbius plane at x is represented in $\mathsf{AG}(2,\mathbb{K})$ such that its point set is the point set of $\mathsf{AG}(2,\mathbb{K})$ and its pointed circles are the affine lines of $\mathsf{AG}(2,\mathbb{K})$, then its circles are all (elliptic) conics in $\mathsf{AG}(2,\mathbb{K})$ that share two conjugate imaginary points at infinity in the extension $\mathsf{AG}(2,\mathbb{L})$. Hence this representation is projectively equivalent with the one given at the end of* Subsection 2.1.

Proof This follows from the fact that every collineation of $\mathsf{AG}(2,\mathbb{K})$ preserves the family of elliptic conics, due to the Fundamental Theorem of Projective Geometry. $\square$

Let us now project $\mathcal{C}^\alpha$ from the point x^α, and denote the projection map by σ. Then ξ_x is projected onto the plane π_x, and by the above, the structure in π_x of the pointed Möbius plane at x is projectively unique. Let M be a block of $\mathcal{C}$ not through x but meeting L. The plane $\langle M^\alpha \rangle$ does not meet the solid ξ_x, as no plane of $\mathsf{PG}(4, \mathbb{K})$ intersecting $\mathsf{AQ}(4, \mathbb{K})$ in a (full) conic contains a point at infinity of $\mathsf{AQ}(4, \mathbb{K})$. Hence the projection of M^α under σ is a conic C_M in some plane π_M skew to π_x. This is projectively unique. For each point z on C_M, there is a corresponding point z' of $\mathcal{C}$, and there is a corresponding block xz' of $\mathcal{C}$, hence there is a corresponding conic through x^α, and so a corresponding tangent line in ξ_x, which determines a unique point z_x in π_x. The line zz_x is the projection of $(xz')^\alpha$. Hence the projection under σ of the image under α of all blocks through x meeting M is projectively uniquely determined. Moreover, let M' be another block meeting all blocks through x that meet M. Then $M'^{\alpha\sigma}$ is a conic meeting all lines zz_x above. There is exactly one conic through each point of each line zz_x (indeed, let u be the point, and let $z^* z_x^*, z^{**} z_x^{**}$ be two such lines, then the unique such conic through u meets the line $z^{***} z_x^{***}$ in the intersection point $xz_x^{***} \cap \langle u, z^*, z^{**}, z_x^*, z_x^{**} \rangle$). Hence also this structure is projectively unique. Notice that everything up to now is defined in a 5-dimensional space W, and notice that the planes defined by disjoint blocks are also disjoint. Since we may regard such a pair as arbitrary, we have, in general, that disjoint blocks define disjoint planes.

For a line Z through x, there is a unique corresponding point x_Z of π_x which is the projection of the tangent line at x at the conic Z^α.

Now let K be a block through x not meeting M. Then $M^{\alpha\sigma}$ is not contained in W (this can be seen using the projection from $\langle L^\alpha \rangle$ and Lemma 1). Now there exists a block Y not through x meeting K and meeting two distinct blocks B_1, B_2 of the projective Baerpencil $\mathfrak{B}_M$ in x determined by M, say in the points u_K, x_1, x_2, respectively. There is also a second block Y' not through x meeting K, B_1, B_2 in, say, $u_{K'}, x_1', x_2'$, respectively. Since M is not contained in W, all choices for $x_K^{\alpha\sigma}$ and $x_{K'}^{\alpha\sigma}$ on $M^{\alpha\sigma}$ are projectively equivalent. But once these choices are made, we claim that the rest is uniquely determined. Indeed, let $\mathfrak{B}_Y$ be the projective Baer subpencil in x determined by Y. Then, for each element $Z \in \mathfrak{B}_Y$, the line $Z^{\alpha\sigma}$ is the unique line through x_Z meeting both planes $\langle u_K^{\alpha\sigma}, x_1^{\alpha\sigma}, x_2^{\alpha\sigma} \rangle$ and $\langle u_{K'}^{\alpha\sigma}, {x_1'}^{\alpha\sigma}, {x_2'}^{\alpha\sigma} \rangle$ (the fact that this defines a unique point is due to the fact that the two planes are skew).

From the previous paragraph follows that, whenever three blocks B_1, B_2, B_3 through x are contained in a projective Baer subpencil, then $B_1^{\alpha\sigma} \cup B_2^{\alpha\sigma} \cup B_3^{\alpha\sigma}$ uniquely determines the image under $\alpha\sigma$ of all points on all blocks of that subpencil. But now our claim follows since in any Möbius plane with circle size at least 4, given two circles C_1, C_2 meeting in two distinct points, every point x is contained in a circle meeting $C_1 \cup C_2$ in at least 3 distinct points (indeed, choose points $x_i \in C_i$, $i = 1, 2$, with $x_i \notin C_1 \cap C_2$; then there is a unique circle C through x, x_1, x_2. If C intersects both C_1 and C_2 in respective unique points, then replace x_2 by another point in $C_2 \setminus C_1$—this is possible since circles are large enough).

3.4 End of the proof of the Main Result

Now, since the image under $\alpha\sigma$ of $\mathfrak{B}$ spans a 5-dimensional space, the image under α of $\mathfrak{B}_Y$ spans a 6-dimensional space. So, if we replace $\mathfrak{B}$ by a projective Baer subpencil not containing x, we see that the image under α of any projective Baer subpencil is projectively uniquely determined. In particular, $\mathfrak{B}_M^\alpha$ is projectively uniquely determined. One checks that $\mathfrak{B}_M^\alpha$ is embedded as follows in a hyperplane $\mathcal{H}_M$. The space ξ_x is contained in $\mathcal{H}_M$; the plane $\langle C^\alpha \rangle$, for every block C not through x and meeting all elements of $\mathfrak{B}_M$, is skew to ξ_x, and the conics B^α, for $B \in \mathfrak{B}_M$, meet every such plane (which we will call *horizontal* for $\mathfrak{B}_M$) in unique points and the tangent lines to these conics (which we call *vertical* for $\mathfrak{B}_M$) at x form a quadratic cone in ξ_x.

Let us make an intermediate but important remark. It is easy to see that the horizontal planes define a perspectivity between the vertical conics, fixing the point x^α. As an immediate consequence, the set of horizontal planes is determined by three vertical conics and two horizontal planes.

All tangent lines at x in ξ_x can be considered as given and their structure is projectively unique (follows from Lemma 3). Moreover, with the above notation, K^α can be chosen (outside $\mathcal{H}_M$) in a projectively unique way. It follows from our remark in the previous paragraph that all horizontal planes for $\mathfrak{B}_Y$ are determined. Since we also know the planes corresponding to all vertical conics for $\mathfrak{B}_Y$ (as we know their image under σ), and a horizontal plane meets the plane of any vertical conic in a unique point (belonging to $\mathcal{C}^\alpha$), all points of $\mathfrak{B}_Y^\alpha$ are uniquely determined. But now, similarly as for $\alpha\sigma$, the embedding α is completely and projectively uniquely determined.

The proof of our Main Result is complete.

4 Some consequences

4.1 Segre varieties

Here, we describe Veronesean embeddings of Hermitian unitals arising from Segre varieties.

We keep using our notation $\mathbb{L}, \mathbb{K}$, $\overline{x}$, etc.

The following initial paragraph holds for every field, but we will only apply it to the fields $\mathbb{L}$ and $\mathbb{K}$. Therefor, we can use the notation $\mathbb{L}$, for the time being.

Let $\mathsf{H}(\mathbb{L},1)$ denote the non-thick generalized hexagon with points set $\mathcal{F}$ the set of incident point-line pairs of $\mathsf{PG}(2,\mathbb{L})$, and line set $\mathcal{P}\cup\mathcal{L}$ the union of the point and line sets ($\mathcal{P}$ and $\mathcal{L}$, respectively) of $\mathsf{PG}(2,\mathbb{L})$. We embed $\mathsf{H}(\mathbb{L},1)$ in $\mathsf{PG}(d,\mathbb{L})$, with $d\in\{7,8\}$, as follows. Let θ be any field endomorphism of $\mathbb{L}$. Define the map $\gamma:\mathcal{F}\to\mathsf{PG}_0(8,\mathbb{L})$ as follows. For a given incident point line pair $\{(x_0,x_1,x_2),[a_0,a_1,a_2]\}$, with $a_0x_0+a_1x_1+a_2x_2=0$, we define the image under γ as the point with coordinates

$$(a_0x_0^\theta, a_0x_1^\theta, a_0x_2^\theta; a_1x_0^\theta, a_1x_1^\theta, a_1x_2^\theta; a_2x_0^\theta, a_2x_1^\theta, a_2x_2^\theta).$$

If $\theta\neq\mathrm{id}$, then the set of images generates $\mathsf{PG}(8,\mathbb{L})$; if $\theta=\mathrm{id}$, then all images are (clearly) contained in the hyperplane (with obvious notation for the variables) with equation $X_{0,0}+X_{1,1}+X_{2,2}=0$. In the latter case we obtain part of the *Segre variety corresponding with* $\mathsf{PG}(2,\mathbb{L})$.

Now let $\mathcal{C}$ be the Hermitian unital in $\mathsf{PG}(2,\mathbb{L})$ whose corresponding curve has equation $X_0\overline{X}_1+\overline{X}_0X_1=X_2\overline{X}_2$. We gather the incident point-line pairs consisting of a point of $\mathcal{C}$ and the tangent line to $\mathcal{C}$ at that point in the set $\mathcal{C}^{\dashv}$; they have generic coordinates $\{(x_0,x_1,x_2),[\overline{x}_1,\overline{x}_0,-\overline{x}_2]\}$. In order to obtain a Veronesean embedding of $\mathcal{C}$ after applying γ to $\mathcal{C}^{\dashv}$, two necessary conditions must be satisfied. The first one is, that the points of a block of $\mathcal{C}$ are transferred under $\dashv\gamma$ into a set of planar points of $\mathsf{PG}(8,\mathbb{L})$. The second one is that we can find a subspace $\mathsf{PG}(8,\mathbb{K})$ containing all the images.

Let us start with the first one. We consider the block B induced by the line with equation $X_2=0$. Then we obtain as image under γ the following set of points:

$$\{(x_0^\theta\overline{x}_1, x_1^\theta\overline{x}_1, 0; x_0^\theta\overline{x}_0, x_1^\theta\overline{x}_0, 0, ; 0,0,0)\mid x_0\overline{x}_1+\overline{x}_0x_1=0, x_0,x_1\in\mathbb{L}, (x_0,x_1)\neq(0,0)\}.$$

Putting $x_0=0$ and $x_1=0$, respectively, we see that the points $(0,1,0;0,\ldots,0)$ and $(0,0,0;1,0,\ldots,0)$ are contained in that image. Hence the first and fifth coordinate must satisfy a linear equation. Putting $x_0=1$, we deduce that $\overline{x}_1=\ell x_1^\theta$, for some $\ell\in\mathbb{L}$, and for all $x_1\in\mathbb{L}$ such that $x_1+\overline{x}_1=0$. Since this set of x_1's is nonempty, we may choose such a fixed x_0; then, for any $k\in\mathbb{K}$, also kx_0 satisfies the given condition. This implies $k=k^\theta$, for all $k\in\mathbb{K}$, which implies that θ is either the identity, or coincides with complex conjugation.

Suppose that θ coincides with complex conjugation. Labelling the coordinates as ($X_{0,0}$, $X_{0,1}$, …, $X_{2,1}$, $X_{2,2}$), we easily see that the image under γ of $\mathcal{C}^{\dashv}$ is contained in the 5-space with equations $X_{0,0} = X_{1,1}$, $X_{0,2} + X_{2,1} = 0$ and $X_{1,2} + X_{2,1} = 0$. In fact, we here obtain the image of $\mathcal{C}$ under the ordinary quadric Veronesean of $\mathsf{PG}(2,\mathbb{L})$. Since the dimension is here $5 < 7$, we do not consider this case further.

Hence θ is the identity. The image of the block B now lies on the conic with equations $X_{0,0}^2 = X_{0,1}X_{1,0}$, $X_{0,0} + X_{1,1} = X_{0,2} = X_{1,2} = X_{2,0} = X_{2,1} = X_{2,2} = 0$. We now check the second condition. Let $\eta \in \mathbb{L} \setminus \mathbb{K}$ be arbitrary. We perform the following coordinate change:

$$\begin{cases} X'_{0,1} = X_{0,1}, \\ X'_{0,0} = X_{0,0} + X_{1,1}, \\ X'_{1,1} = \eta X_{0,0} + \overline{\eta} X_{1,1}, \\ X'_{1,0} = X_{1,0}, \\ X'_{0,2} = X_{0,2} - X_{2,1}, \\ X'_{2,1} = \eta X_{0,2} - \overline{\eta} X_{2,1}, \\ X'_{2,2} = X_{2,2}, \\ X'_{1,2} = X_{1,2} - X_{2,0}, \\ X'_{2,0} = \eta X_{1,2} - \overline{\eta} X_{2,0}. \end{cases}$$

We now see that $\mathcal{C}^{\dashv\gamma}$ is contained in a subspace $\mathsf{PG}(7,\mathbb{K})$ of $\mathsf{PG}(8,\mathbb{L})$ with equation $X'_{0,0} + X'_{2,2} = 0$. Hence we obtain a Veronesean embedding of $\mathcal{C}$.

We have shown:

Corollary 4 *If $\mathcal{C}$ is a Hermitian curve over $\mathbb{L}/\mathbb{K}$ in $\mathsf{PG}(2,\mathbb{L})$, then the image of $\mathcal{C}$ on the corresponding Segre variety is a standard Veronesean embedding of $\mathcal{C}$.*

4.2 Grassmannians of Hermitian spreads

Consider the generalized quadrangle $\mathsf{H}(3,\mathbb{L}/\mathbb{K})$ in $\mathsf{PG}(3,\mathbb{L})$. Let π be a plane of $\mathsf{PG}(3,\mathbb{L})$ intersecting $\mathsf{H}(3,\mathbb{L}/\mathbb{K})$ in an ovoid $\mathfrak{O}$, which is a non-degenerate Hermitian curve. Consider the Klein correspondence mentioned in Subsection 2.3, which maps the lines of $\mathsf{H}(3,\mathbb{L}/\mathbb{K})$ onto the points of a the generalized quadrangle $\mathsf{Q}^-(5,\mathbb{K})$ in $\mathsf{PG}(5,\mathbb{K})$, and the points of $\mathsf{H}(3,\mathbb{L}/\mathbb{K})$ are mapped onto the lines of $\mathsf{Q}^-(5,\mathbb{K})$. The points of $\mathfrak{O}$ are mapped onto the lines of a spread $\mathfrak{S}$ of $\mathsf{Q}^-(5,\mathbb{K})$, called a *Hermitian* spread. Embed $\mathsf{Q}^-(5,\mathbb{K})$ into a quadric $\mathsf{Q}(6,\mathbb{K})$ of Witt index 3. Then there exists a naturally embedded split Cayley hexagon $\mathsf{H}(\mathbb{K})$ on $\mathsf{Q}(6,\mathbb{K})$ such that $\mathfrak{S}$ is precisely the set of lines of $\mathsf{Q}^-(5,\mathbb{K})$ that also belong to $\mathsf{H}(\mathbb{K})$. It is also well known, see [12], that the line Grassmannian takes the lines of $\mathsf{H}(\mathbb{K})$ to a generating set of points of some 13-dimensional projective subspace $\mathsf{PG}(13,\mathbb{K})$

of the space $\mathsf{PG}(20,\mathbb{K})$ generated by the full line Grassmannian. Intersecting $\mathsf{PG}(13,\mathbb{K})$ with the full line Grassmannian of $\mathsf{PG}(5,\mathbb{K})$ — which has dimension 14 — we see that the line Grassmannian takes $\mathfrak{S}$ to a set of points $\mathcal{C}$ in a 7-dimensional space $\mathsf{PG}(7,\mathbb{K})$. Since, viewed as a unital, the blocks are reguli, it follows easily that the blocks are transformed by the line Grassmannian to conics. Also, using an easy explicit calculation, it is easily seen that $\mathcal{C}$ really generates $\mathsf{PG}(7,\mathbb{K})$, and by our Main Result, we obtain the classical Veronesean embedding.

Hence we have shown:

Corollary 5 *The line Grassmannian of a Hermitian spread of the quadric $\mathsf{Q}^-(5,\mathbb{K})$ of Witt index 2 of $\mathsf{PG}(5,\mathbb{K})$ is projectively isomorphic to the standard Veronesean embedding of the corresponding Hermitian unital.*

This can also be seen in a different way. Indeed, In the above description, we extend $\mathsf{PG}(6,\mathbb{K})$ to $\mathsf{PG}(6,\mathbb{L})$. Then $\mathsf{Q}(6,\mathbb{K})$ is extended to $\mathsf{Q}(6,\mathbb{L})$, and $\mathsf{H}(\mathbb{K})$ to $\mathsf{H}(\mathbb{L})$. The space $\mathsf{PG}(5,\mathbb{L})$, as the extension of $\mathsf{PG}(5,\mathbb{K})$ now intersects $\mathsf{Q}(6,\mathbb{L})$ in a hyperbolic quadric $\mathsf{Q}^+(5,\mathbb{L})$. The lines of $\mathsf{H}(\mathbb{L})$ that belong to $\mathsf{Q}^+(5,\mathbb{L})$ define a subhexagon $\mathcal{H}$ the line Grassmannian of which defines part of a Segre variety corresponding with $\mathsf{PG}(2,\mathbb{L})$, see [12] (the arguments of this construction in the latter easily generalize to the infinite case). In fact, the lines of $\mathcal{H}$ correspond with the flags of $\mathsf{PG}(2,\mathbb{L})$, and the elements of $\mathfrak{S}$ correspond with the fixed flags under a Hermitian polarity, see [17] (the arguments in the latter also easily extend to the infinite case). The corollary now follows from Corollary 4.

4.3 Trialities

We show our last corollary.

Corollary 6 *Let τ be a triality of Witt index 1, i.e., it has type II (notation as in [15]) and is defined over a field $\mathbb{K}$ with characteristic different from 3 and such that $\mathbb{K}$ does not contain nontrivial cubic roots of unity. Let $\mathbb{L}$ be the quadratic extension of $\mathbb{K}$ defined by the nontrivial cubic roots of unity. Then the set of absolute points of τ has the structure of a Hermitian unital $\mathcal{C}$ over $\mathbb{L}/\mathbb{K}$, where blocks correspond to plane conics. This set of absolute points is again isomorphic to the standard Veronesean embedding of $\mathcal{C}$.*

Proof Over $\mathbb{L}$, the triality τ extends to a triality τ', still of type II, but now with Witt index 2, and the fixed lines form part of the Segre variety corresponding with the projective plane $\mathsf{PG}(2,\mathbb{L})$, in which the set of absolute points of τ corresponds precisely to the set of flags of $\mathsf{PG}(2,\mathbb{L})$ fixed under a Hermitian polarity, see [15, Théorème 9.2.1]. The assertion now follows from Corollary 4. $\square$

5 Some more properties of Veronesean embeddings of Hermitian unitals

In this section, we collect some more properties and connections with other geometric objects. Some proofs require tedious calculations, and we will only be sketchy about these; other arguments are purely synthetic (once the ground work with coordinates is finished) and we write these down in detail.

5.1 Some subspaces related to points and blocks

Let again $\mathcal{C}=(\mathcal{P},\mathcal{B})$ be a Hermitian unital over the field $\mathbb{K}$, and α a Veronesean embedding of $\mathcal{C}$ in $\mathsf{PG}(7,\mathbb{K})$. For convenience, we shall conceive $\mathcal{C}$ as a Hermitian curve in the projective plane $\mathsf{PG}(2,\mathbb{L})$. Let $x\in\mathcal{P}$ be arbitrary. In the beginning of Subsection 3.3, we showed that the set of tangents at x^α to the various conics through x^α is contained in a solid ξ_x. Moreover, we showed that the lines through x^α of this solid which are not tangent lines to any conic form a plane, which we will denote by π_x. In fact, it is easy to see that, if we conceive $\mathcal{C}^\alpha$ as a hyperplane section in $\mathsf{PG}(8,\mathbb{K})$ of the Hermitian Veronesean $\mathcal{H}$ of $\mathsf{PG}(2,\mathbb{L})$ (with corresponding Hermitian Veronesean map ν), then π_x is the tangent plane at x^α to the unique elliptic quadric on $\mathcal{H}$ through x that intersects the hyperplane spanned by $\mathcal{C}$ in just x (and this elliptic quadric corresponds with the tangent line at x to $\mathcal{C}$).

From our description of the projection from x^α of $\mathcal{C}^\alpha$ in Subsection 3.3, we also deduce that the projection of $\mathcal{C}^\alpha$ from the entire subspace ξ_x is an elliptic quadric with one point a removed, in some 3-dimensional projective space. The inverse image under this projection of the tangent plane at a of that elliptic quadric is a hyperplane which we will denote by χ_x and call the *tangent hyperplane at x^α*. In the finite case, the above tangent plane to the ellitpic quadric is the unique plane in the corresponding 3-dimensional space space that does not meet the elliptic quadric minus the point. It follows that, still in the finite case, χ_x is the unique hyperplane in $\mathsf{PG}(7,\mathbb{K})$ containing ξ_x and intersecting $\mathcal{C}^\alpha$ in just x. This implies that, in the finite case, this hyperplane is unique without the assumption of containing ξ_x, since any hyperplane meeting $\mathcal{C}^\alpha$ in just x must obviously intersect the planes spanned by the conics through x^α in the tangent lines at x^α, and hence contain ξ_x.

Recall that, for any block $L \in \mathcal{B}$, the projection from $\langle L^\alpha \rangle$ of $\mathcal{C}^\alpha \setminus L^\alpha$ is a natural embedding of $\mathsf{AQ}(4, \mathbb{K})$. Hence, there is a unique plane in that projection not containing any projected point of $\mathcal{C}^\alpha \setminus L^\alpha$. The inverse image under the projection of this point is thus the unique hyperplane of $\mathsf{PG}(7, \mathbb{K})$, denoted χ_L, meeting $\mathcal{C}^\alpha$ in L^α. We call it the *conic hyperplane at L*. It can equivalently be obtained as follows. For each line M of $\mathsf{PG}(2, \mathbb{L})$, we can consider the reducible Hermitian curve that consists of M only (if M has equation $V = 0$, the this curve has equation $V \cdot \overline{V} = 0$, where $\overline{a}$ denotes the image of $a \in \mathbb{L}$ under the unique nontrivial element of the Galois group of $\mathbb{L}/\mathbb{K}$). The corresponding hyperplane χ_M of $\mathsf{PG}(8, \mathbb{K})$ meets $\mathcal{H}$ in M^ν, and is called the *quadric hyperplane at* M^ν. If L is the intersection of M with $\mathcal{C}$, then one easily sees that $\chi_L = \chi_M \cap \mathsf{PG}(7, \mathbb{K})$. Also, if $x \in \mathcal{C}$, and M is the line in $\mathsf{PG}(2, \mathbb{L})$ tangent to $\mathcal{C}$ at x, then $\chi_x = \chi_M \cap \mathsf{PG}(7, \mathbb{K})$.

We first prove a lemma.

Lemma 7 *The quadric hyperplanes of a Hermitian Veronesean of a projective plane constitute the dual of the Hermitian Veronesean of the dual projective plane.*

Proof Let, with abbreviated but obvious notation, the Hermitian Veronesean map be given by

$$\beta : (x_i) \mapsto (x_i\overline{x}_i; x_j\overline{x}_{j+1} + \overline{x}_j x_{j+1}; \eta x_j \overline{x}_{j+1} + \overline{\eta x}_j x_{j+1}), \quad i, j = 1, 2, 3$$

where $\eta \in \mathbb{L} \setminus \mathbb{K}$, arbitrary but fixed, and $j + 1$ must be read cyclically. One calculates that the dual coordinates of the quadric hyperplane related to the line with equation $\sum a_i x_i = 0$ equal

$$[a_i\overline{a}_i; \frac{\overline{\eta} a_j \overline{a}_{j+1} - \eta \overline{a}_j a_{j+1}}{\overline{\eta} - \eta}; \frac{\overline{a}_j a_{j+1} - a_j \overline{a}_{j+1}}{\overline{\eta} - \eta}].$$

An easy coordinate change now concludes the proof. □

From this lemma, it follows that the set of quadric hyperplanes of $\mathcal{H}$ at quadrics which intersect $\mathsf{PG}(7, \mathbb{K})$ in just one point, is the dual of the standard Veronesean embedding of $\mathcal{C}$ in the space it generates. In particular, all these hyperplanes intersect in a fixed point N of $\mathsf{PG}(8, \mathbb{K})$. It is our intention to show that N belongs to $\mathsf{PG}(7, \mathbb{K})$ if and only if $\mathrm{char}\mathbb{K} = 3$. To this aim, we must use coordinates again. While using these, we will show along the way that the map $x^\alpha \mapsto \chi_x$, $x \in \mathcal{C}$, is part of a polarity if and only if $\mathrm{char}\mathbb{K} \neq 3$. In fact, we will do this first. We will also deduce some other consequences, related to triality later.

5.2 Points vs. tangent hyperplanes

Proposition 1 *The map $x^\alpha \mapsto \chi_x$, $x \in \mathcal{C}$, is part of a polarity ρ in $\mathsf{PG}(7,\mathbb{K})$ if and only if* $\mathrm{char}\mathbb{K} \neq 3$. *More in particular,*

(*i*) *If* $\mathrm{char}\mathbb{K} \notin \{2,3\}$, *then ρ is an orthogonal polarity of Witt index* 3 *or* 4. *The Witt index is* 4 *precisely and only when $\mathbb{K}$ has no nontrivial cubic roots of unity and $\mathbb{L}$ is the quadratic Galois extension of $\mathbb{K}$ with respect to the nontrivial cubic roots of unity (in other words, $\mathbb{L}$ does contain nontrivial cubic roots of unity).*

(*ii*) *If* $\mathrm{char}\mathbb{K} = 2$, *then ρ is a symplectic polarity. However, there is a unique quadric Q in $\mathsf{PG}(7,\mathbb{K})$ of Witt index* 3 *or* 4, *containing all points of $\mathcal{C}^\alpha$ and such that the tangent hyperplane at x^α, $x \in \mathcal{P}$, is precisely χ_x. The Witt index is* 4 *precisely and only when $\mathbb{K}$ has no nontrivial cubic roots of unity and $\mathbb{L}$ is the quadratic Galois extension of $\mathbb{K}$ with respect to the nontrivial cubic roots of unity (in other words, $\mathbb{L}$ does contain nontrivial cubic roots of unity).*

(*iii*) *If* $\mathrm{char}\mathbb{K} = 3$, *then there is a unique point N in $\mathsf{PG}(7,\mathbb{K})$ such that the map $Nx^\alpha \mapsto \chi_x$ is part of an orthogonal polarity of Witt index 3 in the projection $\mathsf{PG}(6,\mathbb{K})$ of $\mathsf{PG}(7,\mathbb{K})$ from N.*

Proof This is a 'computational' proof, and we will only mention the important steps (all calculations, however, can be easily done by hand).

First of all, in order to cover all characteristics at the same time, we choose as irreducible quadratic polynomial defining $\mathbb{L}$ the polynomial $X^2 + X + n = 0$, where $n \in \mathbb{K}$ (this is always possible). We will write the zeros over $\mathbb{L}$ of this polynomial as η and $\overline{\eta}$, and we use the same η to define the Hermitian Veronesean, see above. The Hermitian curve $\mathcal{C}$ is defined by the equation $X_0\overline{X}_1 + \overline{X}_0X_1 = X_2\overline{X}_2$. A generic point of this curve has coordinates

$$(a + \eta(b\overline{b} - 2a), 1, b), a \in \mathbb{K}, b \in \mathbb{L}.$$

Applying β, we obtain the generic point of $\mathcal{C}^\alpha$ (for our own convenience and ease of computations, we have replaced the coordinate corresponding to $\eta X_0\overline{X}_1 + \overline{\eta}\overline{X}_0X_1$ with $\overline{\eta}X_0\overline{X}_1 + \eta\overline{X}_0X_1$) :

$$\begin{gathered}(a^2 + a(b\overline{b} - 2a) + n(b\overline{b} - 2a)^2, 1, b\overline{b};\\ b\overline{b}, b + \overline{b}, a(b + \overline{b}) + (b\overline{b} - 2a)(\overline{etab} + \eta\overline{b});\\ a + 2n(b\overline{b} - 2a), \overline{etab} + \eta\overline{b}, a(\eta b + \overline{eta}\overline{b}) + nj(b + \overline{b})(b\overline{b} - 2a)),\end{gathered}$$

and $\mathsf{PG}(7,\mathbb{K})$ clearly has equation $Y_2 = Y_3$, denoting coordinates in $\mathsf{PG}(8,\mathbb{K})$ by $Y_0, Y_1, \ldots, Y_8$.

The tangent line in $\mathsf{PG}(2,\mathbb{K})$ to $\mathcal{C}$ at this generic point has equation

$$(X_0 + (a + \overline{\eta}(b\overline{b} - 2a))X_1 - \overline{b}X_2)(\overline{X}_0 + (a + \eta(b\overline{b} - 2a))\overline{X}_1 - b\overline{X}_2) = 0.$$

If we denote $b = b_r + b_i\eta$, then the dual coordinates of the corresponding quadric hyperplane in $\mathsf{PG}(8,\mathbb{K})$ are:

$$\begin{gathered}[1, a^2 + a(b\overline{b} - 2a) + n(b\overline{b} - 2a)^2, b\overline{b}; \\ b\overline{b} - a, -b_r a - (b\overline{b} - 2a)(b_r + nb_i), -b_r - b_i; \\ -b\overline{b} + 2a, b_r(b\overline{b} - 2a) - ab_i, b_i].\end{gathered}$$

We now see that this generic hyperplane contains the fixed point $(0, 0, -1; 2, 0, 0; 1, 0, 0)$. This point is contained in $\mathsf{PG}(7,\mathbb{K})$ if, and only if, $-1 = 2$, so if, and only if, $\mathrm{char}\mathbb{K} = 3$.

Let us first assume that $\mathrm{char}\mathbb{K} \neq 3$.

One checks that, intersecting the above hyperplane with $\mathsf{PG}(7,\mathbb{K})$, we have to delete the third and fourth entries, but keep the sum of those as third entry. In the coordinates of the points, we can simply delete the fourth entries. This way, we obtain a coordinate system in $\mathsf{PG}(7,\mathbb{K})$ consisting of 8 coordinates, so that we can calculate with matrices of the right dimension to recognize polarities.

It is an easy exercise to calculate the coordinates $[A_0, A_1, \ldots, A_7]$ of the generic tangent hyperplane (which is the intersection of the generic quadric hyperplane above with $\mathsf{PG}(7,\mathbb{K})$) in function of the coordinates $(Z_0, Z_1, \ldots, Z_7)$ of the corresponding generic point of $\mathcal{C}^\alpha$. We obtain, in matrix form, and multiplied with $1 - 4n$, which cannot be zero anyway:

$$\begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \end{bmatrix} = \begin{bmatrix} 0 & 1-4n & 0 & 0 & 0 & 0 & 0 & \\ 1-4n & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2-6n & 0 & 0-1 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 2n-1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2n-1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 & -2 & 0 \end{bmatrix} \cdot \begin{bmatrix} Z_0 \\ Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \\ Z_5 \\ Z_6 \\ Z_7 \end{bmatrix}.$$

One easily verifies that this is a symplectic polarity if $\mathrm{char}\mathbb{K} = 2$, and an orthogonal polarity otherwise (assuming $\mathrm{char}\mathbb{K} \neq 3$).

The corresponding quadric is easy to calculate, in case that the characteristic is not 2 or 3. We obtain:

$$(1-4n)Z_0Z_1 + (2n-1)Z_3Z_4 + Z_3Z_7 + Z_4Z_6 - 2Z_6Z_7 = (3n-1)Z_2^2 + Z_2Z_5 - Z_5^2.$$

By a simple inspection, one sees that also in characteristics 2 and 3, all points of $\mathcal{C}^\alpha$ are contained in the quadric with the above equation. We denote this quadric by Q. In characteristic 3, it is immediate that Q is degenerate, since $(3n-1)Z^2 + Z_2Z_5 - Z_5^2 = -(Z_2+Z_5)^2$. Moreover, the point N is the unique vertex of the quadric, and the quadric projects from N onto a non-degenerate quadric of Witt index 3 (proved similarly to the general case below). This concludes (iii).

Now suppose $\mathrm{char}\mathbb{K} = 2$. First we claim that Q is unique. Indeed, if we view the corresponding symplectic space as an orthogonal quadric in some 8-dimensional projective space, then it is easy to see that Q must arise from a hyperplane section. This section is completely determined by a set of generating points. Since the points x^α, for x varying over $\mathcal{C}$, generate $\mathsf{PG}(7,\mathbb{K})$, the claim follows.

Now we write the equation of Q as

$$Z_0Z_1 + Z_3(Z_7+Z_4) + Z_4Z_6 = Z_5^2 + Z_2Z_5 + (n+1)Z_2^2,$$

which clearly implies that the Witt index is either 3 or 4. Now, the Witt index equals 4 if, and only if, the equation $X^2+X+n+1=0$ has a solution in $\mathbb{K}$. But we know that $X^2+X+n=0$ does not admit any solution in $\mathbb{K}$. Now, since over any field (with characteristic 2 of course) reducibility of two of the three polynomial X^2+X+1, X^2+X+n, $X^2+X+n+1$ implies reducibility of the third (add two respective zeros together to obtain a zero of the third polynomial), we see that $X^2+X+n+1$ is reducible over $\mathbb{K}$ if and only if X^2+X+1 is irreducible over K but reducible over $\mathbb{L}$. Noting that nontrivial cubic roots of unity satisfy $X^2+X+1=0$, this completes the proof of (ii).

Now suppose $\mathrm{char}\mathbb{K} \notin \{2,3\}$. Substituting $2Z_7$ by Z_4+Z_7' and $2Z_6$ by Z_3-Z_6', the equation of Q becomes (after multiplying with 2):

$$2(1-4n)Z_0Z_1 + (4n-1)Z_3Z_4 + Z_6'Z_7' = (6n-2)Z_2^2 + 2Z_2Z_5 - 2Z_5^2.$$

It is now clear that the Witt index is either 3 or 4. It equals 4 precisely when $-3(1-4n)$ is not a square in $\mathbb{K}$. But since X^2-X+n is irreducible over $\mathbb{K}$, the element $1-4n$ is not a square in $\mathbb{K}$. Note also that -3 is a square in $\mathbb{K}$ if, and only if, $X_2 + X + 1$ is reducible over $\mathbb{K}$, i.e., if, and only if, $\mathbb{K}$ contains nontrivial cubic roots of unity. These remarks are enough to conclude, as above, that the Witt index is 4 if, and only if, $\mathbb{K}$ has no nontrivial cubic roots of unity, but $\mathbb{L}$ does.

This completes the proof of the proposition. $\square$

So we know now what the collection of tangent hyperplanes looks like in de dual space (it is again isomorphic to a standard Veronesean embedding of $\mathcal{C}$, if $\mathrm{char}\mathbb{K} \neq 3$). But we also know what the collection of tangent hyperplanes and conic hyperplanes forms. Moreover, in characteristic 2, there is a nice duality, which has no counterpart in other characteristics, as far as we can see.

Proposition 2 *The collection of all tangent hyperplanes and all conic hyperplanes of* $\mathsf{PG}(7, \mathbb{K})$ *(with respect to $\mathcal{C}^\alpha$) is anti-isomorphic to the projection from N of $\mathcal{H}$ onto a hyperplane not containing N. Moreover, if* $\mathrm{char}\mathbb{K} = 2$*, then this set is also isomorphic to $\mathcal{C}^\alpha$ union the collection of the nuclei of all conics on $\mathcal{C}^\alpha$.*

Proof The first part follows from noting that in the description of the Hermitian Veronesean of the dual of $\mathsf{PG}(2, \mathbb{K})$, the role of the hyperplane $\mathsf{PG}(7, \mathbb{K})$ is played by the point N.

Now let $\mathrm{char}\mathbb{K} = 2$. By transitivity of the unitary group on the blocks of $\mathcal{C}$ and thus on the conics of $\mathcal{C}^\alpha$, it suffices to show for one conic $C \subseteq \mathcal{P}^\alpha$ that its nucleus K, the point N and the image under β of the point P of $\mathsf{PG}(2, \mathbb{K})$ obtained by applying the unitary polarity corresponding to $\mathcal{C}$ on the line L_C defining C, are collinear.

If we take for $\mathcal{C}$ the Hermitian curve as above, and for L_C the line $X_2 = 0$, then one easily calculates that $P = (0, 0, 1)$, $P^\beta = (0, 0, 1; 0, 0, 0; 0, 0, 0)$, $K = (0, 0, 0; 0, 0, 0; 1, 0, 0)$. Hence the line $P^\beta K$ passes through the point $N = (0, 0, 1; 0, 0, 0; 1, 0, 0)$. $\square$

5.3 Triality

We now return to triality and show that and how any Veronesean embedding of a Hermitian unital over $\mathbb{K}$ in $\mathsf{PG}(7, \mathbb{K})$ uniquely and geometrically determines a triality, if $\mathbb{K}$ has no nontrivial cubic roots of unity, but $\mathbb{L}$ does. Of course, we already know that, conversely, under these assumptions, every such embedding (by uniqueness!) comes from a triality. But here we show that the triality is unique. In fact, it is geometrically beautiful to see that we can, starting with the rather sparse embedding of $\mathcal{C}$ in $\mathsf{PG}(7, \mathbb{K})$, reconstruct the triality quadric and the triality itself. Here is a sketch.

Proposition 3 *Every triality of Witt index* 1 *is determined by its set of absolute points.*

Proof We have already reconstructed the triality quadric Q of type D_4, under the given assumptions. Let τ be any triality with as set of absolute points the set $\mathcal{C}^\alpha$. Now we show that the plane π_x, $x \in \mathcal{C}^\alpha$, is precisely the intersection π'_x of $x^{\alpha\rho}$ and $x^{\alpha\rho^2}$. We know that π'_x is invariant under the stabilizer in $\mathsf{U}_3(\mathbb{L}/\mathbb{K})$ of x^α. If $\pi'_x \cap \xi_x = \{x^\alpha\}$, then the projection from ξ_x of π'_x would be an invariant line in the quotient space. But this quotient space is equipped with an elliptic quadric with one point removed, and the above stabilizer induces a group which does not fix any line in that quotient space (leaving the details for the reader). If π_x meets ξ_x in a line, then we use the fact (which we again do not prove, but the reader might do this for himself) that no line of ξ_x is invariant (this is easy for lines tangent to conics; for the other lines one uses the fact that they are tangent lines to an elliptic quadric, which corresponds with the tangent line at x to $\mathcal{C}$; invariance of a line would mean that on this line a Baer subline is invariant, and this is a contradiction). Hence π'_x must be contained in ξ_x, and with the same arguments as before, we must then have $\pi'_x = \pi_x$. If we now number the two classes of generators of Q with 1 and 2, then we can assume that $x^{\alpha\rho^i}$ is the unique generator of type i through π_x. If z is a point of Q not contained in a π_x, for any $x \in \mathcal{C}$, then z is contained in $x^{\alpha\rho}$ and in $y^{\alpha\rho^2}$, for unique $x, y \in \mathcal{C}$ (because the image under the triality of $\mathcal{C}$ is a so-called *spread* of Q). Moreover, our assumption implies that $x^{\alpha\rho} \cap y^{\alpha\rho^2} = \{z\}$. It then follows that z^ρ is the unique generator of type 1 through y and meeting $x^{\alpha\rho^2}$ in a plane. Notice that we now also know the images of generators that at the same time do not contain a point x^α and share a plane with $x^{\alpha\rho}$ or $x^{\alpha\rho^2}$. Finally, if z is contained in the plane π_x, then we can take two generators of which we already know the image and which just meet in z and apply the same method as before.

This completes to proof of the proposition. $\square$

Remark 8 If $\mathrm{char}\mathbb{K} = 3$, then the planes π_x project from N onto so-called *hexagon planes* of a generalized hexagon naturally embedded on the projected quadric. Moreover, the projection from N of $\mathcal{C}^\alpha$ is a natural Hermitian ovoid of that hexagon. This shall be pursued in a sequel paper. Note that, in this case, the point N behaves partly as a kind of a "nucleus", whence the notation for it.

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