# A characterization of the Grassmann embedding of $\mathsf{H}(q)$, with $q$ even

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**Abstract**

In this note, we characterize the Grassmann embedding of $\mathsf{H}(q)$, $q$ even, as the unique full embedding of $\mathsf{H}(q)$ in $\mathsf{PG}(12,q)$ for which each ideal line of $\mathsf{H}(q)$ is contained in a plane. In particular, we show that no such embedding exists for $\mathsf{H}(q)$, with $q$ odd. As a corollary, we can classify all full polarized embeddings of $\mathsf{H}(q)$ in $\mathsf{PG}(12,q)$ with the property that the lines through any point are contained in a solid; they necessarily are Grassmann embeddings of $\mathsf{H}(q)$, with $q$ even.

## 1 Introduction

In the theory of finite geometries, full projective embeddings play a central role. Indeed, it attaches a field to a geometry, it provides a representation of the geometry which can be used to investigate certain properties, and in many cases, it tells something about the nature of the collineations of that geometry. The basic question is always to try to classify all full projective embeddings for a certain class of geometries. For the class of finite generalized quadrangles, this has been done by Buekenhout & Lefèvre [1]. A similar result for finite generalized hexagons has not yet been established, and it seems that we are still very far from it. In particular, we even do not know yet an absolute and general upper bound on the dimension of the projective space. The additional assumption of being flat (which means that the lines of the hexagon through a point are contained in a plane of the projective space) allows to generalize some arguments of quadrangles to hexagons, and in fact shows that the dimension of the projective space is at most 7, see [5, 6]. But a complete classification of all flat full projective embeddings of finite generalized hexagons has not yet been obtained. Another condition that makes a partial classification possible is that of being polarized—which means that the set of points not opposite a given point in the hexagon does not span the whole projective space. The combination—flat and polarized—has been completely settled, even in the infinite case; see [5] and [4].

Another project is to investigate embeddings of the classical examples, which in the case of finite generalized hexagons, are all known examples. The central question here is to classify all full projective embeddings of the split Cayley hexagons, and to start with to determine the possible dimensions. In particular the question of an upper bound for the dimension shows up again. In this approach, one would already be satisfied with a classification of embeddings in the case of maximal projective dimension (since projections usually behave much less nice and regular, and are much harder to characterize or to recover from general assumptions). Surprisingly, such an upper bound, namely 13, has been determined for the finite *dual* split Cayley hexagons, and all full projective embeddings in that top dimensional space have been determined, see [7]. A similar result for the finite split Cayley hexagons seems out of reach. The standard conjecture here is that the known embeddings are the ones providing the upper bounds on the dimension. This would mean that the upper bound is 6 for all split Cayley hexagons over fields of characteristic unequal to 3, except for the smallest one, of order 2, which admits an embedding in a 13-dimensional space. In characteristic 3, the upper bound is 13, as these hexagons are self-dual.

In this paper, we investigate an embedding of the split Cayley hexagons in characteristic 2 in a 12-dimensional projective space, which we call the Grassmann embedding because it is the image on the Grassmannian of all planes of $\mathsf{PG}(5,q)$ of the set of planes containing $q+1$ lines of the ordinary 5-dimensional embedding of the hexagon. This embedding is not so well known, and it does not appear explicitly in the literature, although implicitly, it is around in the theory of hyperplanes of dual polar spaces, see [2]; and it also has been discovered during the writing of this paper independently by Coolsaet (personal communication, unpublished). While we cannot prove yet that 12 is the upper bound for the dimension of any full projective embedding of the split Cayley hexagons in characteristic 2, and neither we can classify the full projective embeddings of these hexagons in a 12-dimensional space, we are able to provide a natural characterization of these embeddings, which should ultimately be useful for a possible complete classification. In particular we show that this embedding is characterized by the dimension and the properties of being polarized and *solid*—all lines of the hexagon through a point are contained in a solid. This follows from a slightly stronger result, in which we only assume that *ideal lines* are contained in planes, see below for more details. Not insisting on this stronger result, we can state our main result as follows:

**Main Result, first version.** *If the split Cayley hexagon* $\mathsf{H}(q)$*,* $q>2$*, is embedded in* $\mathsf{PG}(d,q)$*, with* $d\geq 12$*, so that the embedding is polarized and solid, then it is the*

*Grassmann embedding.*

The case $q = 2$ is a true exception since there exists a (unique) polarized and solid embedding of $\mathsf{H}(2)$ in $\mathsf{PG}(13, 2)$. However, all embeddings of $\mathsf{H}(2)$ are classified; for an account on this, see [9].

# 2 Preliminaries

## 2.1 Generalized polygons

Generalized polygons were introduced by Jacques Tits in [11]. The claims below can be found in the monograph [12].

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be a point-line geometry, where we view the elements of $\mathcal{L}$ as sets of points. The *incidence graph* $\Gamma(\mathcal{S})$ is the graph with vertex set $\mathcal{P} \cup \mathcal{L}$ and $p \in \mathcal{P}$ is adjacent to $L \in \mathcal{L}$ if $p \in L$. Then we call $\mathcal{S}$ a *generalized $n$-gon*, or *generalized polygon*, if the diameter of $\Gamma(\mathcal{S})$ is equal to $n$, and the girth of $\Gamma(\mathcal{S})$ is equal to $2n$; the *girth* being the length of the smallest cycle in $\Gamma$. In fact, in the present paper, we will only need generalized 6-gons, or *generalized hexagons*. Another, more common example, is any projective plane, which is a generalized 3-gon.

It follows directly from the definition that elements of a generalized $n$-gon can have mutual distance at most $n$ in the incidence graph. Elements at distance exactly $n$ will be called *opposite*. If the valency of every vertex of $\Gamma(\mathcal{S})$ is at least 3, then we call $\mathcal{S}$ *thick*; otherwise just *non-thick*. In a thick generalized polygon, the valency of a line is some constant $s+1$, and the valency of a point is some (possibly different) constant $t + 1$, We then say that the order of the polygon is $(s, t)$. If $s = t$, we say that the order is $s$.

## 2.2 Linear and Veronesean embeddings

A plane oval $\mathcal{O}$ of $\mathsf{PG}(d, \mathbb{K})$ is a set of points of $\mathsf{PG}(d, \mathbb{K})$ contained in a plane $\pi$, such that, for any point $x \in \mathcal{O}$, there is a unique line of $\pi$ through $x$ intersecting $\mathcal{O}$ in just $x$, and all other lines through $x$ intersect $\mathcal{O}$ in exactly two points (including $x$!). Examples are conics.

Let again $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be a point-line geometry. A *linear projective full embedding*, or *linear embedding* for short, of $\mathcal{S}$ in the projective space $\mathsf{PG}(d, \mathbb{K})$ over a field $\mathbb{K}$ is an injective mapping $\alpha : \mathcal{P} \to \mathsf{PG}(d, \mathbb{K})$ such that the image of any line of $\mathcal{S}$ is a full line of $\mathsf{PG}(d,\mathbb{K})$, and such that the image of $\mathcal{P}$ under $\alpha$ generates $\mathsf{PG}(d,\mathbb{K})$. A *Veronesean projective embedding*, or *Veronesean embedding* for short, of $\mathcal{S}$ in the projective space $\mathsf{PG}(d,\mathbb{K})$ over a field $\mathbb{K}$ is an injective mapping $\alpha : \mathcal{P} \to \mathsf{PG}(d,\mathbb{K})$ such that the image of any line is a plane oval, and such that the image of $\mathcal{P}$ under $\alpha$ generates $\mathsf{PG}(d,\mathbb{K})$.

We define equivalent projective embeddings in the usual way, i.e., two projective embeddings $\alpha$ and $\alpha'$, be they Veronesean or linear, in $\mathsf{PG}(d,\mathbb{K})$ and $\mathsf{PG}(d,\mathbb{K})'$, respectively, are *equivalent* if there is a projectivity $\sigma : \mathsf{PG}(d,\mathbb{K}) \to \mathsf{PG}(d,\mathbb{K})'$ mapping $p^\alpha$ to $p^{\alpha'}$, for all points $p \in \mathcal{P}$.

For example, the ordinary conic Veronesean of $\mathsf{PG}(2,\mathbb{K})$ yields a Veronesean embedding of $\mathsf{PG}(2,\mathbb{K})$ in $\mathsf{PG}(5,\mathbb{K})$. Conversely, every Veronesean embedding of $\mathsf{PG}(2,\mathbb{K})$ into $\mathsf{PG}(5,\mathbb{K})$, $q > 2$, is equivalent to the conic Veronesean of $\mathsf{PG}(2,\mathbb{K})$, see [8].

Some more terminology concerning linear embeddings of generalized hexagons: if in an embedding, the set of points not opposite any given point does not generate the whole projective space, then we call the embedding *polarized*; if the points collinear to any given point are contained in a plane or solid, then we call the embedding *flat* or *solid*, respectively.

## 2.3 Split Cayley hexagons

In the present paper, we will be concerned with the finite split Cayley hexagon $\mathsf{H}(q)$ of order $q$, for each prime power $q$. It can be constructed as follows—and this construction is due to Jacques Tits [11]. Choose coordinates in the projective space $\mathsf{PG}(6,q)$ in such a way that the parabolic quadric $\mathsf{Q}(6,q)$ has equation $X_0X_4 + X_1X_5 + X_2X_6 = X_3^2$. Then the points of $\mathsf{H}(q)$ are all points of $\mathsf{Q}(6,q)$. The lines of $\mathsf{H}(q)$ are the lines on $\mathsf{Q}(6,q)$ whose Grassmannian coordinates $(p_{01}, p_{02}, \ldots, p_{56})$ satisfy the six relations $p_{12} = p_{34}, p_{56} = p_{03}, p_{45} = p_{23}, p_{01} = p_{36}, p_{02} = -p_{35}$ and $p_{46} = -p_{13}$. This defines a linear embedding of $\mathsf{H}(q)$ into $\mathsf{PG}(6,q)$, which we call the *standard embedding of* $\mathsf{H}(q)$ *in* $\mathsf{PG}(6,q)$. It is linear, flat and polarized.

The lines of $\mathsf{PG}(6,q)$ entirely contained in $\mathsf{Q}(6,q)$, but which are not lines of $\mathsf{H}(q)$, are called *ideal lines* of $\mathsf{H}(q)$. Likewise, the planes of $\mathsf{PG}(6,q)$ entirely contained in $\mathsf{Q}(6,q)$ but not containing any line of $\mathsf{H}(q)$, are called *ideal planes* of $\mathsf{H}(q)$.

If $q$ is even, then the projection of the standard embedding of $\mathsf{H}(q)$ from the point $(0,0,0,1,0,0,0)$ of $\mathsf{PG}(6,q)$ onto some hyperplane is faithful and yields a flat and polarized linear embedding of $\mathsf{H}(q)$ in $\mathsf{PG}(5,q)$, which we call the *standard embedding of* $\mathsf{H}(q)$ *in* $\mathsf{PG}(5,q)$. To each point $x$ of $\mathsf{PG}(5,q)$ there corresponds a plane $\pi_x$ of $\mathsf{PG}(5,q)$

obtained by taking the span of the lines of $\mathsf{H}(q)$ through $x$. Moreover, if $x$ runs through the set of points of a line $L$ of $\mathsf{H}(q)$, then $\pi_x$ runs through all planes of a certain solid $\xi_L$. Consequently, the composition $\theta$ of the map $x \mapsto \pi_x$ with the Grassmann embedding of all planes of $\mathsf{PG}(5,q)$ takes a point $x$ of $\mathsf{H}(q)$ to a point $x'$ of $\mathsf{PG}(19,q)$, and the set of points on a line $L$ is taken to the set of points on a line $L'$ of $\mathsf{PG}(19,q)$. Using the explicit form of the lines, it takes an elementary but tedious calculation to see that the image under $\theta$ of the point set of $\mathsf{PG}(5,q)$ spans a 12-dimensional subspace $\mathsf{PG}(12,q)$ of $\mathsf{PG}(19,q)$. Hence we obtain a linear embedding of $\mathsf{H}(q)$ in $\mathsf{PG}(12,q)$. We call this embedding the *Grassmann embedding of* $\mathsf{H}(q)$. We are not aware of an explicit mentioning in the literature of this embedding, although it is implicit in the work of B. De Bruyn [2], and it has independently been discovered by K. Coolsaet during the writing of this paper.

One interesting property of the Grassmann embedding is that every ideal line of $\mathsf{H}(q)$ is mapped onto a plane conic in $\mathsf{PG}(12,q)$. In particular, ideal lines in $\mathsf{PG}(12,q)$ are contained in planes.

We can now state our Main Result in some more detail.

**Main Result, second version.** *The Grassmann embedding of* $\mathsf{H}(q)$ *in* $\mathsf{PG}(12,q)$*,* $q>2$ *and* $q$ *even, is the unique embedding of* $\mathsf{H}(q)$ *in* $\mathsf{PG}(d,q)$*,* $d \geq 12$*, with the property that every ideal line is contained in a plane. For* $q$ *odd, there is no embedding of* $\mathsf{H}(q)$ *with these properties.*

As a corollary one immediately has that this embedding is also the unique solid and polarized one in $\mathsf{PG}(d,q)$, $d \geq 12$, of $\mathsf{H}(q)$, $q>2$. Hence the first version follows from the second one. Conversely, if every ideal line is contained in a plane, then the embedding is obviously solid, but in order to see that it is also polarized, we need a large part of our proof below. So the converse is not immediate.

The case $q=2$ in the above is again a true exception, see [9].

Concerning the infinite case: the Grassmann embedding exists for the split Cayley hexagon over every field of characteristic 2, perfect or not, but our methods of characterization only work in the finite case.

## 2.4 Some more properties of $\mathsf{H}(q)$

We need some additional properties of the split Cayley hexagon $\mathsf{H}(q)$.

First, ideal planes come in pairs. Indeed, more precisely, given an ideal plane $\mathfrak{P}$, then every point of $\mathfrak{P}$ is at distance 4 in the incidence graph from every other point of $\mathfrak{P}$.

By considering the set $\mathfrak{Q}$ of points of $\mathsf{H}(q)$ which are at distance 2 from at least two points of $\mathfrak{P}$, we obtain a second ideal plane. Starting with $\mathfrak{Q}$ and carrying out the same construction yields $\mathfrak{P}$ again. Hence we call the ideal planes $\mathfrak{P}$ and $\mathfrak{Q}$ *twins*.

If $\mathfrak{P}$ and $\mathfrak{Q}$ are twin ideal planes, then $\mathfrak{P} \cup \mathfrak{Q}$, together with all lines of $\mathsf{H}(q)$ that contain at least (and hence exactly) two points of $\mathfrak{P} \cup \mathfrak{Q}$, form a non-thick generalized hexagon of order $(1, q)$, which we call a *non-thick ideal subhexagon*. The word *ideal* comes from the fact that all lines of $\mathsf{H}(q)$ through a point of that subhexagon are lines of that subhexagon.

Let $x, y$ be two opposite points in $\mathsf{H}(q)$. Then the set $\mathcal{R}$ of lines at distance 3 from both $x$ and $y$ is called a *regulus*, and it has the property that, whenever a point of $\mathsf{H}(q)$ is at distance 3 from at least two of its members, then this point is at distance 3 from all of its members. The points on the lines of such a regulus $\mathcal{R}$ and at distance 2 from a point that is at distance 3 from all members of $\mathcal{R}$, form an ideal line. In $\mathsf{PG}(6, q)$ and in $\mathsf{PG}(5, q)$, $q$ even, a regulus is the set of generators of one class of a hyperbolic quadric in some solid; the ideal lines on it constitute the other class of generators.

For a point $x$ of $\mathsf{H}(q)$, we denote by $x^{\perp}$ the set of points of $\mathsf{H}(q)$ collinear with $x$. We note that the union of all lines of $\mathsf{H}(q)$ through $x$ in $\mathsf{PG}(6, q)$ and in $\mathsf{PG}(5, q)$, $q$ even, is a plane of $\mathsf{Q}(6, q)$ and $\mathsf{PG}(5, q)$, respectively. Consequently, two ideal lines contained in $x^{\perp}$ either coincide or meet in a unique point.

Finally, a line of $\mathsf{PG}(5, q)$, $q$ even, not belonging to $\mathsf{H}(q)$, and which is not an ideal line, will be called an *imaginary line*.

# 3 Proof of the Main Result

## 3.1 Projective planes

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be isomorphic to $\mathsf{PG}(2, q)$ with $\mathcal{P} \subseteq \mathsf{PG}(d, q)$, $\langle \mathcal{P} \rangle = \mathsf{PG}(d, q)$, $d \geq 5$, and such that every member $L$ of $\mathcal{L}$ is a subset of points of a plane in $\mathsf{PG}(d, q)$, which we denote by $\pi_L$ if it is unique; if it is not unique, then $\pi_L$ is the intersection of all such planes, and so $\pi_L$ is the line of $\mathsf{PG}(d, q)$ containing all points of $L$. We call such a representation of $\mathcal{S}$ in $\mathsf{PG}(d, q)$ a *generalized Veronesean embedding*.

We denote the line of $\mathsf{PG}(d, q)$ spanned by two points $a, b \in \mathcal{P}$ by $\langle a, b \rangle$, whereas the line of $\mathcal{S}$ through $a, b$ is denoted by $ab$. More generally, we use the symbol $\langle A \rangle$ to denote the subspace of $\mathsf{PG}(12, q)$ generated by the elements of $A$.

We will assume that $q > 2$.

**Lemma 3.1** *If $L, M$ are two distinct lines of $\mathcal{S}$, meeting in the point $z \in \mathcal{P}$, and $x \in \mathcal{P}$ is a point off $L \cup M$ not contained in $\langle \pi_L, \pi_M \rangle$, then every point $y \in \mathcal{P}$ off $xz$ is contained in the space $W := \langle \pi_L, \pi_M, x \rangle$.*

**Proof** The line $xy$ meets $L \cup M$ in two distinct points $u, v$, with $u \in L$ and $v \in M$. So the space $\pi_{xy}$ has a line in common with $\langle \pi_L, \pi_M \rangle$ and contains $x$. Hence $\pi_{xy} = \langle x, u, v \rangle \subseteq W$. $\square$

**Lemma 3.2** *If $d \geq 5$ and if a proper subspace $H$ of $\mathsf{PG}(d, q)$ contains all points of $\mathcal{S}$ off a certain line $L \in \mathcal{L}$. Then one of the following holds:*

(A) *$d = 5$, the points of $\mathcal{S}$ off $L$ form an affine plane $\mathsf{AG}(2, q)$ in $\mathsf{PG}(5, q)$, and $L$ defines a plane $\pi_L$ skew to the projective completion of $\mathsf{AG}(2, q)$;*

(B) *all points of $\mathcal{S}$ except exactly one are contained in $H$.*

**Proof** Suppose some proper subspace $H$ contains all points of $\mathcal{S}$ off a line $L \in \mathcal{L}$. We may assume that $H$ is spanned by the points of $\mathcal{S}$ off $L$. Then there is at least one point $x \in L$ not contained in $H$. For each line $M \in \mathcal{L}$ containing $x$, with $M \neq L$, the space $\pi_M$ is a plane and intersects $H$ in a line. It follows that all points of $M$ except $x$ are contained in a line $L_M$ of $H$. If there is a second point $x' \in L$ not contained in $H$, then we similarly obtain for each line $M' \in \mathcal{L}$ through $x'$, with $M' \neq L$, a line $L'_{M'}$ in $H$ containing all points of $M'$ except for $x'$. Each of the lines $L_M$ meets each of the lines $L'_{M'}$.

First assume that no third point of $L$ lies outside $H$. As $q \geq 3$, this implies that at least two points of $L$ are contained in $H$, and so $\pi_L$ is a plane meeting $H$ in a line. This implies that $H$ is a hyperplane. From the above, we see that, if at least two points of $L$ are not contained in $H$, then $H$ is at most 3-dimensional, a contradiction. Hence in such a case exactly one point $x$ of $L$ is not contained in $H$.

Next, assume that at least three points $x, x', x''$ of $L$ lie outside $H$. In $H$, there arise three sets of $q$ lines, and two lines of different sets always intersect. It follows that all points of $\mathcal{S}$ off $L$ lie in a plane, which by definition coincides with $H$. Hence, deleting $L$ and its points gives rise to an affine plane $\mathsf{AG}(2, q)$ in $H$. As $\mathcal{S}$ generates a $d$-dimensional space, with $d \geq 5$, $\pi_L$ is a plane which is disjoint from the plane $H$, and $d = 5$. $\square$

These two lemmas have now the following important consequence.

**Corollary 3.3** *We have $d \leq 5$. Also, if $d = 5$, then for every line $L \in \mathcal{L}$, the space $\pi_L$ is a plane. Moreover, if $d = 5$ and we do not have Case (A) above, then for every two distinct lines $L, M \in \mathcal{L}$, the planes $\pi_L$ and $\pi_M$ meet in a unique point.*

**Proof** Suppose $d > 5$. Take two arbitrary lines $L, M \in \mathcal{L}$. Then $W := \langle \pi_L, \pi_M \rangle$ is at most 4-dimensional. Let $d'$ be the dimension of $W$. Hence there are two points $x, y \in \mathcal{P}$ with $\langle W, x, y \rangle$ a $(d'+2)$-dimensional space. From Lemma 3.1 it follows that $xy$ contains the intersection point $z$ of $L$ and $M$, and that all points of $\mathcal{S}$ off $xy$ are contained in both $\langle W, x \rangle$ and $\langle W, y \rangle$, hence in $W$. This now contradicts Lemma 3.2.

If $\pi_L$ were a line, then $d' \leq 3$ and the argument of the previous paragraph can be copied and leads to a contradiction with $d = 5$. Hence $\pi_L$ is a plane, and so is $\pi_M$. If these planes generated a subspace of dimension at most 3, then again the same argument leads to a contradiction.

The corollary is completely proved. $\square$

**Remark 3.4** All the above holds for arbitrary finite projective planes. An interesting question is whether one can classify all generalized Veronesean embeddings of all projective planes of order $q$ in $\mathsf{PG}(5, q)$. This has been solved and will appear in a different paper, along with other, related, results. Notice that not every generalized Veronesean embedding is a conic Veronesean embedding, and there are two classes of counterexamples. For more information, we refer to [10]

## 3.2 The split Cayley hexagon $\mathsf{H}(q)$

From now on we assume that $\mathsf{H}(q)$, $q > 2$, is fully embedded in $\mathsf{PG}(d, q)$, $d \geq 12$, and that the points of every ideal line are contained in a not necessarily unique plane. Then Corollary 3.3 implies that the points of a given ideal plane $\mathfrak{P}_1$ span a subspace $S_1$ of dimension $d_1 \leq 5$. The same argument shows that the points of its twin, say $\mathfrak{P}_2$, span a subspace $S_2$ of dimension $d_2 \leq 5$. Let $\mathfrak{H}$ be the non-thick ideal subhexagon with point set $\mathfrak{P}_1 \cup \mathfrak{P}_2$. Then it is well known and, in fact, easy to see, that for $q > 2$ the set $\mathfrak{P}$ of points of $\mathsf{H}(q)$ incident with some line of $\mathfrak{H}$ is a geometric hyperplane of $\mathsf{H}(q)$, whose complement induces a connected subgeometry of $\mathsf{H}(q)$. Hence, it follows that $12 \geq d_1 + d_2 + 1 + 1 \geq d \geq 12$. Consequently $d_1 = d_2 = 5$, $d = 12$, and the points of every ideal line span a unique plane of $\mathsf{PG}(12, q)$; use Corollary 3.3. It also follows that $S_1 \cap S_2 = \emptyset$. In particular, the points of two ideal lines, one of $\mathfrak{P}_1$ and one of $\mathfrak{P}_2$, span a subspace of dimension 5.

Let $L_1, M_1$ be two ideal lines of $\mathfrak{P}_1$, and let $x_1$ be a point in $\mathfrak{P}_1$ not on $L_1 \cup M_1$. Let $X_2$ be the ideal line in $\mathfrak{P}_2$ all of whose points are collinear to $x_1$ (in $\mathsf{H}(q)$). Then each point of $L_1$ and $M_1$ is collinear in $\mathsf{H}(q)$ with a unique point of $X_2$ and the connecting lines between $L_1$ and $X_2$ form a line-regulus $\mathcal{R}_L$; similarly for $M_1$ and $X_2$, we obtain the line-regulus $\mathcal{R}_M$. By a previous observation, the points on each of these line-reguli span a 5-dimensional space, say $S_L$ and $S_M$, respectively. Let $K$ be a third ideal line all of whose points are contained in $\mathcal{R}_L$. Let $\pi_K$ be the plane generated by the points of $K$. Projecting $X_2$ in $S_L$ from $\pi_K$ onto the plane generated by the points of $L_1$, we see that $\mathcal{R}_L$ induces the restriction of an isomorphism $\varphi_L$ from the plane generated by the points of $X_2$ onto the plane generated by the points of $L_1$, and so $\varphi_L$ maps $X_2$ onto $L_1$. Similarly, we obtain an isomorphism $\varphi_M$ mapping $X_2$ onto $M_1$. We now easily see that the product $\varphi_L^{-1}\varphi_M$ maps $L_1$ isomorphically onto $M_1$ and the image of a point $p_1$ on $L_1$ is the point $q_1$ of $M_1$ with the property that $x_1, p_1, q_1$ are collinear in the ideal plane $\mathfrak{P}_1$, i.e., $\varphi_L^{-1}\varphi_M$ corresponds to a perspectivity from $L_1$ to $M_1$ with center $x_1$. By considering appropriate compositions of such maps, we see that $L_1$ is invariant under a group isomorphic to the group of projectivities of $L_1$, which is $\mathsf{PGL}_2(q)$, where, moreover, this (permutation) group is a subgroup of the projective group $\mathsf{PGL}_3(q)$ of the plane $\pi_{L_1}$. By the 3-transitivity of this group, it easily follows that no three points of $L_1$ are contained in a line of $\mathsf{PG}(12, q)$. Hence the points of $L_1$ form a plane oval. Using [8], this now implies that each ideal plane is isomorphic to a conic Veronesean and that all ideal lines are conics.

It also follows that $\mathfrak{P}_1 \cup \mathfrak{P}_2$ is projectively unique. Moreover, the lines of $\mathsf{H}(q)$ joining a point of $\mathfrak{P}_1$ to a point of $\mathfrak{P}_2$ induce an anti-isomorphism from $\mathfrak{P}_1$ to $\mathfrak{P}_2$, both endowed with a projective plane structure. It follows that the set $\mathfrak{P}$ of points of $\mathsf{H}(q)$ contained in a line of $\mathsf{H}(q)$ meeting $\mathfrak{P}_1 \cup \mathfrak{P}_2$ in two points, is unique in $\mathsf{PG}(12, q)$, up to a semilinear automorphism.

We can now prove that $q$ must be even.

**Lemma 3.5** *We have that $q$ is even.*

**Proof** Consider a point $x$ of $\mathsf{H}(q)$, and an arbitrary ideal line $L$ in $x^\perp$. Since we may think of $x$ belonging to $\mathfrak{P}_1$ and $L$ to $\mathfrak{P}_2$, we see that $\langle x, L\rangle$ is a solid $S$ in $\mathsf{PG}(12, q)$, and $x^\perp$ is a quadratic cone $\mathcal{Q}$. Consider now two distinct ideal lines $L_1, L_2$ in $x^\perp$ such that no point of $x^\perp$ is contained in $L_1 \cap L_2 \cap L$. We have $|L_1 \cap L_2| = |L \cap L_1| = |L \cap L_2| = 1$ (see Subsection 2.4). So the conics $L_1$ and $L_2$ share the tangent line at their common point $x_{12}$; similarly for $L$ and $L_1$ with common point $x_1$, and for $L$ and $L_2$ with common point $x_2$. It easily follows that these tangent lines share a common point $\widetilde{x}$. Clearly, the points $x_{12}, x_1, x_2$ lie on three different generators of $\mathcal{Q}$. Projecting $\mathcal{Q}$, $\widetilde{x}$ and the three previous tangent lines from the vertex $x$ of $\mathcal{Q}$, we now obtain three concurrent tangent lines to a plane conic. Consequently $q$ is even. $\square$

Let $x$ be a point of $\mathsf{H}(q)$ and let $\xi_x$ be the solid spanned by the lines of $\mathsf{H}(q)$ through $x$; the first part of this section implies that $x^\perp$ is the set of all points of $\mathsf{H}(q)$ in $\xi_x$. From the foregoing proof it follows that there is a unique point $\widetilde{x}$ contained in every plane spanned by an ideal line contained in $\xi_x$. This point is the nucleus of every conic corresponding to an ideal line in $\xi_x$.

Let us denote the point set of $\mathsf{H}(q)$ by $\mathcal{P}$.

**Lemma 3.6** *The set of points $\widetilde{\mathcal{P}} = \{\widetilde{x} \mid x \in \mathcal{P}\}$ is the set of points of a 5-dimensional subspace $\widetilde{W}$ of $\mathsf{PG}(12, q)$. Also, to the lines of $\mathsf{H}(q)$ the lines correspond of a standard embedding of $\mathsf{H}(q)$ in $\widetilde{W}$.*

**Proof** Let $I$ be an imaginary line of $\mathsf{H}(q)$. The set $\widetilde{I} = \{\widetilde{x} \mid x \in I\}$ is the set of nuclei of conics contained in a line set $\mathfrak{L}$ of lines of $\mathsf{H}(q)$ joining points of one conic $C_1$ to another conic $C_2$ in two planes spanning a 5-dimensional subspace; observe that the elements of $\mathcal{L}$ are maximal spaces of a Segre variety $\mathcal{S}_{1;2}$ of a line and a plane, see §25.5 of [3]. Hence $\widetilde{I}$ is the point set of a line of $\mathsf{PG}(12, q)$. Similarly, if $C$ is an ideal line, then the point set $\widetilde{C} = \{\widetilde{x} \mid x \in C\}$ is a line in the nucleus plane of the conic Veronesean corresponding to the twin ideal plane of any ideal plane containing $C$. If the distance of the points $x$ and $y$ of $\mathcal{P}$ measured in the incidence graph of $\mathsf{H}(q)$ is either 4 or 6, then the foregoing implies that $\widetilde{x} \neq \widetilde{y}$; if $x$ and $y$ are collinear, but distinct, then, by considering ideal lines through $x$ and $y$ in twin ideal planes, it follows that $\widetilde{x} \neq \widetilde{y}$. Hence the mapping $x \mapsto \widetilde{x}$ is injective.

Now let $x_1, x_2, x_3$ be three collinear points of $\mathsf{H}(q)$, and let $p_1, p_2$ be two further points on the line $x_1x_2$ of $\mathsf{H}(q)$. In $p_1^\perp$, we can choose three concurrent ideal lines, say through the common point $x$, containing respectively $x_1, x_2, x_3$, and one ideal line $L$ not containing $x$ to see that $\widetilde{x}_1, \widetilde{x}_2$ and $\widetilde{x}_3$ lie in a common plane $\pi_1$, which contains the "tildes" of at least $q^2 + 1$ points of $p_1^\perp$, namely, all points on ideal lines containing $x$ and meeting $L$. Analogously, using $p_2$, the points $\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3$ lie in a common plane $\pi_2$ containing the "tildes" of at least $q^2 + 1$ points of $p_2^\perp$. It follows that $\pi_1 \neq \pi_2$ and that $\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3$ are contained in the intersection line of $\pi_1$ and $\pi_2$. We easily deduce from this that, for any line $M$ of $\mathsf{H}(q)$, the set $\{\widetilde{x} \mid x \in M\}$ forms a line of $\mathsf{PG}(12, q)$. Hence $\widetilde{\mathcal{P}}$ is the set of points of a 5-dimensional subspace $\widetilde{W}$ of $\mathsf{PG}(12, q)$. To the set of all lines of $\mathsf{H}(q)$ there corresponds the set of all lines of a flatly embedded generalized hexagon, isomorphic to $\mathsf{H}(q)$, in $\widetilde{W}$. The assertion now follows from the Main Result (*ii*) and (*iii*) of [6]. □

Since for any two points $x, y$ of $\mathsf{H}(q)$, $x \neq y$, with $\mathrm{d}(x, y) \neq 4$, we can choose ideal lines in $x^\perp$ and $y^\perp$ in a such a way that they are contained in a common pair of twin ideal planes, we see that $\widetilde{x}$ is never contained in $\xi_y$; for $x, y$ in $\mathsf{H}(q)$, with $\mathrm{d}(x, y) = 4$, consider an ideal plane containing $x$ and $y$ to see that $\widetilde{x}$ is not contained in $\xi_y$. It follows that the projection of $\mathcal{P}$ from $\widetilde{W}$ is a flat embedding (say, $\alpha$) of $\mathsf{H}(q)$ in some $\mathsf{PG}(6,q)$. Hence this embedding is uniquely determined. It immediately follows that our original embedding is polarized. We now show that our original embedding is also unique. Indeed, first we remark that $\widetilde{W}$ is uniquely determined by $\mathfrak{P}$, as it is the span of the nucleus planes of the twin conic Veroneseans in $\mathfrak{P}$. Now we choose a point $x$ of $\mathsf{H}(q)$ outside $\mathfrak{P}$. Then the configuration $\mathfrak{P} \cup \{x\}$ is again projectively unique. Let $y$ be any point of $\mathsf{H}(q)$, not belonging to $\mathfrak{P}\cup\{x\}$. By the connectivity of the complement of $\mathfrak{P}$ in $\mathsf{H}(q)$, we may assume that $y$ is collinear with $x$. Let $L$ be the line incident with $x$ and $y$. Let $z$ be the unique point on $L$ belonging to $\mathfrak{P}$. The point $z$ does not belong to $\mathfrak{P}_1 \cup \mathfrak{P}_2$, since all lines of $\mathsf{H}(q)$ through points of this set are contained in $\mathfrak{P}$. Hence there are unique points $z_i \in \mathfrak{P}_i$, $i = 1, 2$, such that $z, z_1, z_2$ are collinear. We may suppose that $\alpha(y)$ is known; indeed, it is any point of $\alpha(\mathsf{H}(q)) \setminus \alpha(x)$ collinear in $\alpha(\mathsf{H}(q)$ with $\alpha(x)$. Then $\alpha(z), \alpha(z_1), \alpha(z_2)$ are uniquely defined. The point $z$ is uniquely determined by the intersection of the line $z_1z_2$ in $\mathsf{PG}(12,q)$ with the hyperplane spanned by the points of $\alpha(\mathsf{H}(q))$ not opposite $\alpha(z)$ and $\widetilde{W}$. Hence also $L$ is determined, and so is $y$, as the intersection of $L$ with the hyperplane spanned by the points of $\alpha(\mathsf{H}(q))$ not opposite $\alpha(y)$ and $\widetilde{W}$.

The proof of our Main Result is now complete.

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