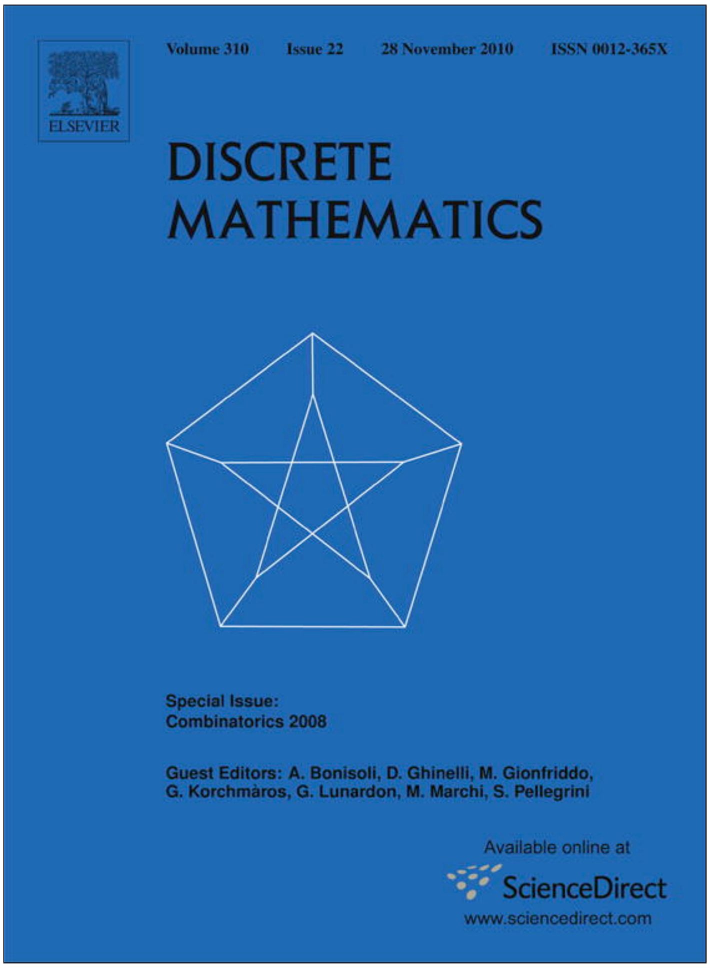
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Affine twin $\mathbb{R}$-buildings

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ABSTRACT

In this paper, we define twinnings for affine $\mathbb{R}$-buildings. We thus extend the theory of simplicial twin buildings of affine type to the non-simplicial case. We show how classical results can be extended to the non-discrete case, and, as an application, we prove that the buildings at infinity of a Moufang twin $\mathbb{R}$-building have the induced structure of a Moufang building. The latter is not true for ordinary "Moufang" $\mathbb{R}$-buildings.

1. Introduction

Spherical buildings were introduced by Jacques Tits in the 1960s as a geometric tool for investigating groups of Lie type, Chevalley groups, groups of mixed type and classical groups. The main result here is the classification of all spherical buildings of rank at least 3 (by Tits [10]), and of all so-called Moufang spherical buildings of rank 2 (by Tits and Weiss [13]). This theory was extended by Bruhat and Tits [5], on the one hand, to affine buildings when Iwahori and Matsumoto [6,7] considered the above groups over p-adic fields, thus constructing affine BN-pairs in algebraic groups over local fields. Here the main result is the classification of all affine buildings of rank at least 4 (by Tits [11]). Also, Tits considered in [11] not only pure simplicial affine buildings, but "non-discrete" analogues of these, called "systems of apartments" in [11], and nowadays simply referred to as "non-discrete affine buildings", or briefly (affine) $\mathbb{R}$-buildings. These relate to groups of Lie type over fields with a non-discrete real valuation.

On the other hand, Ronan and Tits extended the notion of a spherical building to the one of a twin building, thus creating a natural geometric framework for the Kac–Moody groups (unpublished, but see [12]). However, when a Kac–Moody group is of affine type, then the interplay of the spherical building defined by the group, the affine building defined by the valuation, the spherical building defined by the group over the residue field, and the twinning defined by the Kac–Moody group has proved very fruitful in investigating these groups. As an example we mention various finiteness properties of arithmetic groups, due to ground-breaking work of Abels [1] and Abramenko [2]; see also [3] for an introduction and an overview.

In this paper, we join the two above branches of extensions of spherical building theory by defining (affine) twin $\mathbb{R}$-buildings. These objects must be the primary tools for investigating groups of Lie type over most non-discrete function fields. Our intention is to lay the foundation for such a study by establishing the basic theory. It is by no means either a slight rewriting of the discrete case, nor a simple adaptation of the ordinary affine $\mathbb{R}$-building theory. The arguments are very different and do indeed provide alternative proofs for the discrete case. However, note that the buildings are restricted to the affine type; hence it is not surprising that the arguments strongly depend on the affine structure of the apartments. As a consequence, our results and proofs, read in the discrete case, give a revision of the basics of the theory of twin buildings of affine type, rewritten in such a way that the affine structure is prominently used.

Our main result is a generalisation of a result of the second author and Van Steen [14]. Namely, we show that any Moufang affine twin $\mathbb{R}$-building induces the structure of a Moufang building at infinity. This joins the new world of Moufang twin $\mathbb{R}$-buildings with the older world of Moufang spherical buildings. At the same time, it feeds the conjecture that affine twin

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$\mathbb{R}$-buildings should be classifiable. Note that the main result is not true for ordinary Moufang $\mathbb{R}$-buildings. It is precisely the twin structure that allows the conclusion. Also, work of Ronan [9] on the discrete case already showed that the buildings at infinity of an affine twin building are rather restricted.

As already mentioned, the theory of simplicial twin affine buildings is included in this extended theory. However, one must be careful since certain notions known in simplicial building theory are different when formulated in an $\mathbb{R}$-building environment. For instance, the convex closure of two points in a discrete affine building is, though defined in completely the same way as the intersection of all roots containing the two points, different when we view the building as a simplicial complex than when we view the building as an $\mathbb{R}$-building (see below for the explanation; basically there are more roots when the building is conceived as an $\mathbb{R}$-building).

Finally, we remark that, although this theory is highly non-discrete, our methods use the combinatorics of the Coxeter groups involved, and in particular the discrete spherical buildings that arise as residues. As a consequence, we can regard this work as an approach for a non-discrete structure by discrete methods.

2. Definitions

2.1. $\mathbb{R}$-buildings

Let $(\overline{W}, S)$ be a finite irreducible Coxeter system. So $\overline{W}$ is presented by the set S of involutions subject to the relations which specify the order of the products of every pair of involutions. This group has a natural action on a real vector space V of dimension $|S|$ (see for instance [3, Section 2.5]). Let $\mathbf{A}$ be the affine space associated with V. Let T be the group of translations of $\mathbf{A}$. We define the affine reflection group W to be the group generated by T and $\overline{W}$.

Let $\mathcal{H}_0$ be the set of hyperplanes of V corresponding to the axes of the reflections in S and their conjugates. Let $\mathcal{H}$ be the orbit of all elements of $\mathcal{H}_0$ under W. The elements of $\mathcal{H}$ are called *walls* and the (closed) half-spaces that they bound are called *half-apartments* or *roots*. A *vector sector* is the intersection of all roots that (1) are bounded by elements of $\mathcal{H}_0$, and (2) contain a given point x that does not belong to any element of $\mathcal{H}_0$. The bounding walls of these roots will be referred to as the *side-walls* of the vector sector. A vector sector can also be defined as the closure of a connected component of $V \setminus (\cup\mathcal{H}_0)$. Any translate under T of a vector sector is a *sector*, with corresponding translated *side-walls*. A *sector-facet* is an infinite intersection of a given sector with a finite number of its side-walls. This number can be zero, in which case the sector-facet is the sector itself; if this number is 1, then we call the sector-facet a *sector-panel*. The intersection of a sector with all its side-walls is a point which is called the *base point* of the sector, and of every sector-facet defined from it.

An *affine* $\mathbb{R}$-*building* (also called an *affine apartment system*, or for short an $\mathbb{R}$-building) of type (W, S) (introduced by Tits in [11]) is an object $(\Lambda, \mathcal{F})$ consisting of a set Λ together with a collection $\mathcal{F}$ of injections of $\mathbf{A}$ into Λ obeying the five conditions below. The image of $\mathbf{A}$ under an $f \in \mathcal{F}$ will be called an *apartment*, and the image of a wall, sector, half-apartment, $\ldots$, of $\mathbf{A}$ under a certain $f \in \mathcal{F}$ will be called a *wall*, *sector*, *half-apartment*, $\ldots$, of Λ.

(A1) If $w \in W$ and $f \in \mathcal{F}$, then $f \circ w \in \mathcal{F}$.
(A2) If $f, f' \in \mathcal{F}$, then $X := f^{-1}(f'(\mathbf{A}))$ is closed and convex in $\mathbf{A}$, and $f|_X = f' \circ w|_X$ for some $w \in W$.
(A3) Any two points of Λ lie in a common apartment.

The last two axioms allow us to define a function $\mathsf{d} : \Lambda \times \Lambda \to \mathbb{R}^+$ such that for any $a, b \in \mathbf{A}$ and $f \in \mathcal{F}$, $\mathsf{d}(f(a), f(b))$ is equal to the Euclidean distance between a and b in $\mathbf{A}$.

(A4) Any two sectors contain subsectors lying in a common apartment.
(A5′) Given $f \in \mathcal{F}$ and a point $\alpha \in \Lambda$, there is a retraction $\rho : \Lambda \to f(\mathbf{A})$ such that the preimage of α is $\{\alpha\}$ and which does not increase d.

We call $|S|$, which is also equal to $\dim \mathbf{A}$, the *dimension* of $(\Lambda, \mathcal{F})$. We will usually denote $(\Lambda, \mathcal{F})$ briefly by Λ, with a slight abuse of notation.

One can associate spherical buildings of type $(\overline{W}, S)$ with these $\mathbb{R}$-buildings in two ways. The first way to do so is to construct the building at infinity. Two sector-facets of Λ will be called *asymptotic* if the Hausdorff distance between them is finite. One can show that d is a metric (i.e. it satisfies the triangle inequality; see for instance [8]), and so this is an equivalence relation. The equivalence classes (named *facets at infinity*) form a spherical building Λ_∞ of type $(\overline{W}, S)$ called the *building at infinity* of $(\Lambda, \mathcal{F})$. The chambers of Λ_∞ are the equivalence classes of asymptotic sectors. An apartment Σ of Λ corresponds to an apartment Σ_∞ of Λ_∞ in a bijective relation. The *direction* of a sector-facet will be the facet at infinity that it belongs to.

A second way to construct a spherical building is to look at the "local" structure instead of the asymptotic one. Let α be a point of Λ, and F, F' two sector-facets with base point α. Then these two facets will *locally coincide* if their intersection is a neighbourhood of α in both F and F'. This relation forms an equivalence relation defining *germs of facets* as equivalence classes (notation $[F]_\alpha$). These germs of facets form a (possibly weak) spherical building $[\Lambda]_\alpha$ of type $(\overline{W}, S)$, called the *residue* at α. We will use the notion *germ* by itself exclusively for germs of sectors.

There exist various equivalent definitions of $\mathbb{R}$-buildings which can be found in [8] by Parreau. One of these definitions demands that (A1), (A2), (A4) and the following stronger version of (A3) hold:

(A3′) Any two germs of sectors lie in a common apartment.

If for a germ of facet there is a corresponding sector-facet and a neighbourhood of the base point of that sector-facet completely lying in a certain set, then we say that this germ of facet is in that set. The notion of an *interior point* is the one in a topological sense.

The collection $\mathcal{F}$ of injections is not determined by the metric space (Λ, d). However the union of all the possible collections producing the same metric space is again a viable collection of injections, giving us a maximal set of apartments. We will always consider that we are in this case. It has the advantage that a subset of Λ isometric to **A** will be an apartment (see [8]).

The conditions (A2) and (A3) imply the existence of unique geodesics between any two points; these will be line segments in apartments. There is also another form of convexity: the *Weyl convex hull* of two points in an apartment is the intersection of all half-apartments of this apartment containing both points (this is independent of the choice of apartment; again see [8]). For a general set X the Weyl convex hull will be the minimal subset Y of Λ containing X such that for each $p, q \in Y$, the Weyl convex hull of p and q is contained in Y.

The metric realisation of an affine building will be an $\mathbb{R}$-building; this case is called the discrete case.

2.2. A Weyl distance between germs

In the discrete case each two chambers have a Coxeter group valued distance function (the Weyl distance). Here we will define a W-distance function on the germs, which we will also call the Weyl distance. Fix a sector Ψ of **A**. Now consider two germs C and D in $(\Lambda, \mathcal{F})$; then according to (A3′) there exists an $f \in \mathcal{F}$ and a unique $w \in W$ such that the germ of $f(\Psi)$ is C and the germ of $(f \circ w)(\Psi)$ is D. We set the *Weyl distance* $\delta(C, D)$ between C and D to be w. Note that this is independent of the choice of Ψ or f.

One can also define a partial order relation on W: for $v, w \in W$ we say that $v \leq w$ if the Weyl convex hull of any two neighbourhoods in Ψ and $w^{-1}(\Psi)$ of the respective base points of these sectors contains a neighbourhood of the base point of $v^{-1}(\Psi)$ in $v^{-1}(\Psi)$.

When restricting this partial order relation to $\overline{W}$, if $w, v \in \overline{W}$, then $v \leq w$ if and only if there is a $v' \in \overline{W}$ such that $w = v'v$ and $l(w) = l(v') + l(v)$ with l the word length of an element in $\overline{W}$ defined by its set of generators S. For more information see [3].

2.3. Twinnings

Let $(\Lambda_+, \mathcal{F}_+)$ and $(\Lambda_-, \mathcal{F}_-)$ be two $\mathbb{R}$-buildings of type (W, S), and Weyl distances δ_+ and δ_- on their respective sets of germs $\mathcal{C}_+$ and $\mathcal{C}_-$. A *codistance* function $\delta^* : (\mathcal{C}_+ \times \mathcal{C}_-) \cup (\mathcal{C}_- \times \mathcal{C}_+) \to W$ forms a *twinning* if for each $\epsilon \in \{+, -\}$, any $C \in \mathcal{C}_\epsilon$ and any $D \in \mathcal{C}_{-\epsilon}$, where $w = \delta^*(C, D)$:

(Tw1) $\delta^*(C, D) = \delta^*(D, C)^{-1}$,
(Tw2) if $C' \in \mathcal{C}_\epsilon$ satisfies $\delta_\epsilon(C', C) = v$ with $vw \leq w$, then $\delta^*(C', D) = vw$,
(Tw3) for any $v \in W$, there exists a germ $C' \in \mathcal{C}_\epsilon$ with $\delta_\epsilon(C', C) = v$ and $\delta^*(C', D) = vw$.

The collection $(\Lambda_+, \mathcal{F}_+, \Lambda_-, \mathcal{F}_-, \delta^*)$ will be called a *twin* $\mathbb{R}$*-building of type* (W, S). The buildings $(\Lambda_+, \mathcal{F}_+)$ and $(\Lambda_-, \mathcal{F}_-)$ are the *components* of the twin building. If two germs C, D have the neutral element e of W as codistance, then we say that C and D are *opposite*, or $C \operatorname{op} D$.

3. Properties of the Weyl distance in $\mathbb{R}$-buildings

In this section, we list some properties of the Weyl distance in $\mathbb{R}$-buildings for future reference. Let $(\Lambda, \mathcal{F})$ be an $\mathbb{R}$-building, $\mathcal{C}$ the set of germs of it, and δ the corresponding Weyl distance. The proofs of the following lemmas are direct or can be derived from [8] easily.

Lemma 1. *If $v, w \in W$ and $vw \geq w$, then also $vw \geq v^{-1}$.* □

Lemma 2.
- *If $C, D \in \mathcal{C}$, then $\delta(C, D)^{-1} = \delta(D, C)$.*
- *If C, D, E are germs in the same apartment, then $\delta(C, E) = \delta(C, D)\delta(D, E)$.* □

Lemma 3. *If $C, D, E \in \mathcal{C}$, such that $\delta(C, D) = v$, $\delta(D, E) = w$ and $vw \geq w$, then $\delta(C, E) = vw$.* □

Lemma 4. *If $C, D \in \mathcal{C}$, such that $\delta(C, D) = v$ and $w \in W$ such that $vw \leq w$, then there is a unique germ E such that $\delta(D, E) = w$ and $\delta(C, E) = vw$.* □

Lemma 5. *If $C, D \in \mathcal{C}$, then $\delta(C, D) \in \overline{W}$ if and only if C and D have the same base point.* □

Lemma 6. *Let $C, D \in \mathcal{C}$ be germs in an apartment Σ with Weyl distance $s \in S$; then there is a unique root containing D but not C, formed by the base points of all the germs E with $\delta(C, E) > \delta(D, E)$.* □

4. Properties of twin $\mathbb{R}$-buildings

We study some properties of twin $\mathbb{R}$-buildings in analogy with the discrete case. In particular we show the existence of twin apartments, and study the local behaviour in opposite points and coconvexity. Let $(\Lambda_+, \mathcal{F}_+, \Lambda_-, \mathcal{F}_-, \delta^*)$ be a twin $\mathbb{R}$-building of type (W, S) and $\mathcal{C}_+, \mathcal{C}_-$ the sets of germs of its components. The symbol ϵ designates -1 or 1.

4.1. Direct consequences of the definition

Lemma 7. *If $C \in \mathcal{C}_\epsilon$, $D \in \mathcal{C}_{-\epsilon}$ and $\delta^*(C, D) = w$, then if $C' \in \mathcal{C}_\epsilon$ and $\delta_\epsilon(C, C') = w$, then C' op D.*

Proof. It is trivial that $e = w^{-1}w \le w$, so using (Tw2) and Lemma 2, it follows that $\delta^*(C', D) = e$ and C' op D. □

Corollary 8. *If $C \in \mathcal{C}_\epsilon$ and $\Sigma_{-\epsilon}$ is an apartment of $(\Lambda_{-\epsilon}, \mathcal{F}_{-\epsilon})$, then $\Sigma_{-\epsilon}$ contains at least one germ opposite C.*

Proof. Choose a germ D in $\Sigma_{-\epsilon}$, and let $w = \delta^*(D, C)$. There exists a germ D' in $\Sigma_{-\epsilon}$ such that $\delta_{-\epsilon}(D, D') = w$. The above lemma now implies that D' op C. □

4.2. Local behaviour in opposite points

We say that two points $\alpha_+ \in \Lambda_+$ and $\alpha_- \in \Lambda_-$ are *opposite* if they are the base points of opposite germs.

With these two opposite points there are associated two (possibly non-thick) spherical buildings: the residues $[\Lambda_+]_{\alpha_+}$ and $[\Lambda_-]_{\alpha_-}$. As chambers of these buildings are germs of the $\mathbb{R}$-buildings, they have a codistance δ^* associated. Denote the chamber sets of these two spherical buildings as $\mathcal{C}_{\alpha_+}$ and $\mathcal{C}_{\alpha_-}$.

Lemma 9. *The codistance of two germs $C \in \mathcal{C}_+$ and $D \in \mathcal{C}_-$ lies in $\overline{W}$ if and only if their base points are opposite points.*

Proof. First let $C \in \mathcal{C}_+$ and $D \in \mathcal{C}_-$ with $\delta^*(C, D) = w \in \overline{W}$. Choose a germ $C' \in \mathcal{C}_+$ such that $\delta_+(C, C') = w$. The germs C and C' have the same base point because of Lemma 5. According to Lemma 7, C' and D are opposite, so the base points of C and D are opposite points.

Conversely, let α_+ and α_- be two opposite points. Suppose that two germs $E \in \mathcal{C}_{\alpha_\epsilon}$ and $F \in \mathcal{C}_{\alpha_{-\epsilon}}$ have as codistance $\delta^*(E, F) = w \in \overline{W}$. Choose an $s \in S$, and $E' \in \mathcal{C}_{\alpha_\epsilon}$ such that $\delta_\epsilon(E, E') = s$ with $s \in S$.

- If $l(sw) < l(w)$ (or equivalently $sw < w$), then due to (Tw2) $\delta^*(E', F) = sw \in \overline{W}$.
- If $l(sw) > l(w)$ (or equivalently $sw > w$), then due to (Tw3) there is an E'' such that $\delta_\epsilon(E, E'') = s$ and $\delta^*(E'', F) = sw \in \overline{W}$. If $E' \neq E''$, then $\delta_\epsilon(E', E'') = s$ and (Tw2) gives us that $\delta^*(E', F) = s(sw) = w \in \overline{W}$.

So we can conclude that $\delta^*(E', F) \in \overline{W}$. Repeating the above argument starting from the opposite germs that two opposite points need to have yields the desired result. □

This lemma allows us to consider $[\Lambda_+]_{\alpha_+}$ and $[\Lambda_-]_{\alpha_-}$ and the $\overline{W}$-valued codistance between their chambers as a spherical twin building.

Corollary 10. 1. *If α_+ and α_- are opposite points then the residues $[\Lambda_+]_{\alpha_+}$ and $[\Lambda_-]_{\alpha_-}$ are isomorphic.*
2. *If $C \in \mathcal{C}_\epsilon$ and $D \in \mathcal{C}_{-\epsilon}$ are opposite germs and $w \in \overline{W}$, then there is a unique germ $C' \in \mathcal{C}_\epsilon$ such that $\delta_\epsilon(C', C) = \delta^*(C', D)$.*

Proof. This follows directly from known properties of (discrete) twin buildings; see for example [3, Section 5.8]. □

4.3. Twin apartments

A *twin apartment* of a twin $\mathbb{R}$-building is a pair $\Sigma = (\Sigma_+, \Sigma_-)$ such that Σ_+ is an apartment of $(\Lambda_+, \mathcal{F}_+)$, and Σ_- an apartment of $(\Lambda_-, \mathcal{F}_-)$, such that each germ of Σ_ϵ has at most one (and so exactly one by Corollary 8) opposite germ in $\Sigma_{-\epsilon}$.

Let op_Σ be the function that maps each germ of Σ to its unique opposite.

Lemma 11. *Let $\Sigma = (\Sigma_+, \Sigma_-)$ be a twin apartment and C, D germs of Σ_ϵ; then $\delta_\epsilon(C, D) = \delta^*(C, \mathrm{op}_\Sigma D)$.*

Proof. Let $w = \delta_\epsilon(C, D)$, and $v = \delta^*(C, \mathrm{op}_\Sigma D)$. There is a unique germ D' in Σ_ϵ such that $\delta_\epsilon(C, D') = v$. This germ is opposite $\mathrm{op}_\Sigma D$ because of Lemma 7. According to the definition of the twin apartment, D is equal to D' and $v = w$. □

Corollary 12. *Let $\Sigma = (\Sigma_+, \Sigma_-)$ be a twin apartment and $C, D \in \mathcal{C}_\epsilon$ germs in Σ_ϵ; then $\delta_\epsilon(C, D) = \delta_{-\epsilon}(\mathrm{op}_\Sigma C, \mathrm{op}_\Sigma D)$.*

This is obtained by applying the above lemma twice. □

This also implies that the induced map of op_Σ on the base points of the germs is an isometry between both apartments.

Lemma 13. *If $C \in \mathcal{C}_\epsilon$, $D \in \mathcal{C}_{-\epsilon}$, $\delta^*(C, D) = w$ and $v \in W$ such that $vw \geq w$, then there is a unique germ $C' \in \mathcal{C}_\epsilon$ such that $\delta_\epsilon(C', C) = v$, $\delta^*(C', D) = vw$.*

Proof. Due to (Tw3) we only have to prove uniqueness here, not existence. A first observation is that one can suppose that w is the identity element e of W. This is true, because if one chooses $E \in \mathcal{C}_\epsilon$ such that $\delta_\epsilon(C, E) = w$, then Lemma 7 implies that $\delta^*(E, D) = e$, and then for a germ C' with the properties mentioned in the statement of this lemma, $\delta_\epsilon(C', E)$ will be vw according to Lemma 3. So we can suppose $w = e$ without loss of generality.

Such a germ C' exists by (Tw3), and we suppose that there is different germ C'' also satisfying these conditions.

A second assumption that one can make without loss of generality is that the geodesic between the base points of C and C' is disjoint from the boundary of the Weyl convex hull of these base points, except for the base points themselves. The reason for this is that if $v'v \geq v$ for a certain $v' \in W$, then a germ F' and F'' will exist by (Tw3) such that $\delta_\epsilon(F', C') = \delta_\epsilon(F'', C'') = v'$ and $\delta^*(F', D) = \delta^*(F'', D) = v'v$. Because of Lemma 3, we have that $\delta_\epsilon(F', C) = \delta_\epsilon(F'', C) = v'v$. Notice that $F' \neq F''$ because of Lemma 4. Using a suitable v' reduces the problem to the desired situation. So from now on we assume this extra condition.

Choose an apartment Σ' containing both C and C' (possible by (A3$'$)) and Σ'', an apartment containing C and C''. Let Ξ' be the Weyl convex hull of the base points of C and C' and Ξ'' the Weyl convex hull of the base points of C and C''.

Consider the geodesic from the base point of C to the base point of C', and another geodesic to the base point of C''. These geodesics start at the same point, but split at a certain point (possibly an endpoint); denote this point by α.

Suppose that α is not an endpoint of these geodesics. Consider the germ E'' of the sector in Σ'' with base point α containing C''. Using (A3$'$) one obtains an apartment Σ containing C' and this germ. Let $f, f', f'' \in \mathcal{F}$ be three injections with images Σ, Σ' and Σ'' respectively, such that $f''((f')^{-1}(C)) = C$, $f((f')^{-1}(C')) = C'$ and $f''((f')^{-1}(C')) = C''$ (this is possible because of (A1) and (A2)). Let Ξ be the image of $(f')^{-1}(\Xi')$ under f. Note that α is an interior point of Ξ, so E'' is contained in both Ξ and Ξ''. Using $f' \circ (f'')^{-1}$ on Ξ'' we can map the germ E'' to a germ E' in Ξ', and using the map $f' \circ f^{-1}$ one obtains another germ E in Ξ'. Condition (Tw2) applied to both C' and C'' now implies that E, E' and E'' all have the same codistance to D. As E and E' both lie in Ξ they have to be identical, but E' lies in the convex hull of α and the base point of C' and hence in the intersection of Σ and Σ', implying that E' equals E'', which is a contradiction.

Now suppose that α is an endpoint. If α is the base point of both C' and C'', then we can, like in the reasoning after the second extra assumption, find two other germs such that we are back in the previous case.

If α is the base point of C, then we can use Corollary 10 and the two germs of the sectors with base point α and containing C' and C'' to derive a contradiction. □

Proposition 14. *Two opposite germs lie in exactly one common twin apartment.*

Proof. Let $C \in \mathcal{C}_+$ and $D \in \mathcal{C}_-$ be opposite germs. For every $w \in W$ let C_w be the unique germ in $\mathcal{C}_+$ such that $\delta_+(C_w, C) = \delta^*(C_w, D) = w$. We now claim that the collection of germs $\{C_w | w \in W\}$ is exactly the set of germs in a certain apartment Σ_+ of $(\Lambda_+, \mathcal{F}_+)$.

Consider v and w in W such that $vw \geq w$. The unique germ E such that $\delta_+(E, C_w) = v$ and $\delta^*(E, D) = vw$ (uniqueness follows from the above lemma and $vw \geq w$) satisfies $\delta_+(E, C) = vw$ due to Lemma 3. Because of the uniqueness of construction of C_{vw}, E equals this germ. So we have the following property:

$$vw \geq w \Rightarrow \delta_+(C_{vw}, C_w) = v.$$

This implies that C_w lies in the Weyl convex hull of C and C_{vw}. An argument similar to the one used by A. Parreau in [8, Prop. 2.17] now shows that the germs C_w with w in a certain sector are exactly the germs in a certain sector with the same base point as C. Because $C_v = C_w$ implies $v = w$, all these sectors only have their borders in common with each other and hence form an apartment.

Similarly, one can define germs D_w ($w \in W$) and show that they are the germs of an apartment Σ_- of $(\Lambda_+, \mathcal{F}_+)$.

The next step is to show that $\Sigma := (\Sigma_+, \Sigma_-)$ is indeed a twin apartment. The property that we need to prove is satisfied trivially for the opposite germs C and D. Choose a $w \in W$; then Lemma 7 implies $C_w \operatorname{op} D_w$. Set $C' = C_w$ and $D' = D_w$ and construct germs C'_v, D'_v for $v \in W$ and apartments Σ'_+ and Σ'_- similarly to above. If $vw > w$, then (Tw2) yields $\delta^*(C_{vw}, D_w) = v$, implying $C'_v = C_{vw}$ in this case. So Σ_+ and Σ'_+ share asymptotic sectors. Trivially $C'_{w^{-1}} = C$ holds, so we can re-apply the same argument with the roles of C and C' switched, which leads to Σ_+ and Σ'_+ sharing two pairs of asymptotic sectors with an opposite direction at infinity, and hence equality. And analogously for $\Sigma_- = \Sigma'_-$. So the property that we need to show for twin apartments is also satisfied for $C' = C_w$ and $D' = D_w$, and hence we obtain that Σ is a twin apartment.

Uniqueness follows from the two lemmas above. □

Corollary 15. *Any two germs C and D lie in a twin apartment.*

Proof. If C and D lie in a different component of the twin building ($C \in \mathcal{C}_\epsilon$, $D \in \mathcal{C}_{-\epsilon}$) and $\delta^*(C, D) = w$, let C' be a germ in $\mathcal{C}_\epsilon$ such that $\delta_\epsilon(C, C') = w$. Applying the construction of the above lemma on C' and D (which are opposite due to Lemma 7) produces a twin apartment containing both C and D.

If C and D lie in the same component of the twin building ($C, D \in \mathcal{C}_\epsilon$), then one can easily construct a germ $E \in \mathcal{C}_{-\epsilon}$ with codistance w from C using (Tw3). Then E and D are opposite and the unique twin apartment that contains both will also contain C. □

4.4. Coconvexity

In ($\mathbb{R}$-)buildings the following property is well known: if two points lie in a certain apartment, then their Weyl convex hull also does. For twin apartments a similar property is true.

Lemma 16. *Let $C, C' \in \mathcal{C}_\epsilon$ and $D \in \mathcal{C}_{-\epsilon}$ with $w = \delta^*(C, D)$, $v = \delta_\epsilon(C', C)$ such that $vw \geq w$ and $\delta^*(C', D) = vw$; then every twin apartment containing both C and D will contain C'.*

Let Σ be a twin apartment containing C and D, and let E be the unique germ opposite D in Σ. Then $\delta_\epsilon(C, E)$ equals w and Lemma 3 implies that $\delta_\epsilon(C', E) = vw$, which in its turn implies that C' lies in the unique twin apartment through E and D, which is Σ. □

A *twin root* α is a pair of roots (α_+, α_-) of the two components of a twin apartment Σ such that the roots $\mathrm{op}_\Sigma \alpha_+$ and α_- only have their wall in common while covering Σ_- (the dual condition is then also fulfilled). The properties of these twin roots will be discussed in more detail in the next section.

Let $C \in \mathcal{C}_+$ and $D \in \mathcal{C}_-$ be two germs lying in a certain twin apartment Σ. The intersection of all twin roots in Σ containing both germs will be called the *coconvex hull* of C and D (if there are no such twin roots then the intersection is considered to be Σ, which will be the unique twin apartment containing C and D due to Proposition 14). The next lemma shows that this is independent of the choice of Σ and so one has that each twin apartment containing both C and D also contains this coconvex hull.

Lemma 17. *The coconvex hull of two germs $C \in \mathcal{C}_+$ and $D \in \mathcal{C}_-$ is independent of the apartment in which it is determined.*

Proof. Expressing that the base point of a certain germ C' lies in the coconvex hull of C and D is equal to the condition on C' in the previous lemma, proving the independence. □

Similarly one can also define the coconvex hull of two points $\alpha \in \Lambda_+$ and $\beta \in \Lambda_-$, and for those an analogous result follows directly.

Corollary 18. *The coconvex hull of two points $\alpha \in \Lambda_+$ and $\beta \in \Lambda_-$ is independent of the apartment in which it is determined.*

Proof. By applying the above lemma to all the germs with base point α or β. □

A subset Ξ of $\Lambda_+ \cup \Lambda_-$ is said to be *coconvex* if the intersections of Ξ with the two components of the twinning are convex and for any two points $\alpha \in \Lambda_+$ and $\beta \in \Lambda_-$ the coconvex hull of these two points is contained in Ξ.

Corollary 19. *The intersection of a set of twin apartments is coconvex.*

Proof. Directly from the previous corollary. □

An interesting thing to note is that two germs are opposite if and only if their coconvex hull is a twin apartment, while two points are opposite if and only if their coconvex hull consists only of these two points.

4.5. Twin roots

In this section we take a closer look at twin roots.

Lemma 20. *Let $C \in \mathcal{C}_+$ and $D \in \mathcal{C}_-$ with $\delta^*(C, D) \in S$; then there is a unique twin root containing both.*

Proof. Let $s \in S$ be the codistance between C and D. Let Σ be a twin apartment containing both, and let α be a twin root in Σ containing C and D. As α_- has to contain D, but cannot contain the germ $\mathrm{op}_\Sigma C$, and these two germs are at distance s (by Lemma 11), α_- has to be the unique root of Σ_- containing C but not $\mathrm{op}_\Sigma C$. By Lemma 6, 11 and the proof of Proposition 14, the germs in that root are exactly the germs E in $\mathcal{C}_-$ such that if $w = \delta_-(D, E)$, then $sw > w$ and $\delta^*(C, E) = sw$. As this condition on E is not dependent on Σ, all twin apartments containing C and D share this root α_- (and also the twin root α by duality). This also implies uniqueness. □

Corollary 21. *Twin roots are the coconvex hulls of pairs of germs with $C \in \mathcal{C}_+$ and $D \in \mathcal{C}_-$ with $\delta^*(C, D) \in S$.* □

5. An alternative definition

One can also define twinnings of $\mathbb{R}$-buildings in terms of twin apartments; this is similar to a definition for (discrete) twin buildings by Abramenko and Ronan [4]. We just cite this equivalent definition; the equivalence in one direction is obvious by the results obtained above, and the proof of the other direction is analogous to that for the discrete case and will be omitted here.

Let $(\Lambda_+, \mathcal{F}_+)$ and $(\Lambda_-, \mathcal{F}_-)$ be two $\mathbb{R}$-buildings of type (W, S), with an opposition relation $\mathrm{op} \subset (\Lambda_+ \times \Lambda_-) \cup (\Lambda_- \times \Lambda_+)$ and a subset $\mathcal{A}$ of the pairs $\{(\Sigma_+, \Sigma_-) | \Sigma_\epsilon \text{ is an apartment of } \Lambda_\epsilon\}$ called the twin apartments; then there is a twinning if the following four conditions are fulfilled:

(TA1) For every $\Sigma = (\Sigma_+, \Sigma_-) \in \mathcal{A}$, the opposition relation induces an isometry op_Σ between affine real spaces.

(TA2) For all germs $C \in \mathcal{C}_+$ and $D \in \mathcal{C}_-$ there is a twin apartment containing both.
(TA3) For all $\Sigma, \Sigma' \in \mathcal{A}$ containing $C \in \mathcal{C}_+$ and $D \in \mathcal{C}_-$, there exists an isometry $\alpha : \Sigma \to \Sigma'$ preserving the opposition relation and mapping C and D to themselves.
(TA4) The intersection of two twin apartments is coconvex.

(Coconvexity in the last statement is considered only within both of the two twin apartments, as for a general notion of coconvexity one would need the independence of the coconvex hull.)

Corollary 22. *The metric realisation of a discrete affine twin building forms a twin $\mathbb{R}$-building.*

Proof. The above alternative definition for twin $\mathbb{R}$-buildings is satisfied by the alternative definition for (discrete) twin buildings by Abramenko and Ronan [4]. □

6. Twinnings and the Moufang property

An *automorphism* of a twin $\mathbb{R}$-building is a pair of automorphisms of the two components preserving codistance.

We say that a series of automorphisms g_n ($n \in \mathbb{N}$) of an $\mathbb{R}$-building *induces* an automorphism of the $\mathbb{R}$-building if for a point p (and hence each point) of the $\mathbb{R}$-building and each real number d, there exists an $n_0 \in \mathbb{N}$ such that g_n ($n \geq n_0$) coincides with g_{n_0} for points at distance less than d from p. One can then define a limit map g of this series. The definition then implies that g will be an automorphism of the $\mathbb{R}$-building.

Given a twin root α of the twin $\mathbb{R}$-building, the *root group* U_α is defined to be the set of automorphisms of the twin $\mathbb{R}$-building such that the germs containing a germ of the sector-panels in α, but not completely on the wall of α, are fixed. The elements of the root group U_α are referred to as the *root elations* of α.

Lemma 23. *If p_ϵ and $p_{-\epsilon}$ are opposite points such that $p_{-\epsilon}$ and the germs with base point p_ϵ are fixed by some automorphism g of the twin $\mathbb{R}$-building, then the germs with base point $p_{-\epsilon}$ are also fixed.*

Proof. This follows from the fact that $[\Lambda_+]_{p_+}$ and $[\Lambda_-]_{p_-}$ can be considered to be a spherical twin building. □

Lemma 24. *If a twin apartment Σ containing the twin root α is fixed by some $g \in U_\alpha$, then g is the identity.*

Proof. Each germ with base point in Σ will be fixed due to the previous lemma and basic properties of spherical buildings. If some point p_ϵ of Λ_ϵ were to be fixed, then the germs based at this point are fixed by Corollary 8 and the previous one. This implies that the fixed structure of g is open in both components. The fixed structure of an automorphism is however also closed, so connectedness implies that g is the identity. □

If the group U_α acts transitively (and hence regularly by the above lemma) on the twin apartments containing α, we say that α is a *Moufang twin root*. If every twin root of the twin $\mathbb{R}$-building is Moufang, then we say that the twin $\mathbb{R}$ building is *Moufang*.

If an affine twin building is Moufang, one can show that the components of it are Moufang in a classical sense for affine buildings; for a definition see for instance [14]. In this paper one also proves that the root elations of the affine building of dimension ≥ 2 induce root elations of the building at infinity of the affine building.

For the non-discrete case, not every root elation at infinity is induced by a given set of root elations (for a suitable generalisation of the definition of root elation as in [14]) of the $\mathbb{R}$-building. An example of such behaviour is given by $\mathbb{R}$-buildings defined over certain Hahn–Mal'cev–Neumann series. (Let $\mathbb{K}$ be a field; then the formal power series $f = \sum_{i\in\mathbb{R}} a_i f^i$, where $a_i \in \mathbb{K}$ for each i, and the support ($\{i \in \mathbb{R} | a_i \neq 0\}$) is well-ordered (each subset has a least element), forms a field (using ordinary polynomial-like addition and multiplication).) One can define a set of root elations using monomials, but a series cannot be approached by combining monomials if the support of this series has more than one limit point. However if we start with the root elations arising from a twin $\mathbb{R}$-building, we will prove that this is still true.

Theorem 25. *The building at infinity of a component of a Moufang twin $\mathbb{R}$-building of dimension ≥ 2 is Moufang, and the root elations are induced by the group generated by the root elations of the twin $\mathbb{R}$-building.*

Proof. Choose a twin apartment $\Sigma = (\Sigma_+, \Sigma_-)$. Let α be any twin root in Σ and Σ'_+ an apartment containing α_+. Let B be some germ in α_+.

The first step is to show that there are a series of automorphisms of the twin $\mathbb{R}$-building inducing an automorphism that fixes α_+ and maps Σ_+ to Σ'_+. Let α^1_+ be the root defined by the intersection Σ_+ and Σ'_+. As it lies in the twin apartment Σ it can be considered part of a twin root α^1. Using Proposition 14 and applying Lemma 20, one can find a twin apartment Σ^1 containing the root α^1 such that the intersection of Σ^1_+ with Σ'_+ is more than just α^1_+. Because the twin $\mathbb{R}$-building is Moufang, there exists a root elation g_1 mapping Σ to Σ^1. We now can repeat this procedure, finding a root elation g_2 mapping Σ^1 to Σ^2, etc.

We claim that $h_1 := g_1, h_2 := g_1 g_2, \ldots,$ is the desired series. The second step of the proof is to show that it is impossible that there exist points of Σ_+ which cannot be mapped to Σ'_+ by any element of this series. So suppose that this is the case. Let γ_+ be the supremum of the roots in Σ_+ containing α_+ that can be mapped to Σ'_+ by an element of the series (ordered by containment). This root is part of a twin root γ of Σ. Let ζ_+ be the corresponding root of Σ'_+ (using the canonical isometry from Σ_+ to Σ'_+ preserving the intersection). Let C be a germ in γ_+, with base point on the wall of γ_+. Let D be the image of C under the opposition isometry of Σ, and E the image of C under the canonical isometry from Σ_+ to Σ'_+ preserving the intersection.

Note that $\delta^*(B, D) = \delta_+(B, C) = \delta_+(B, E)$ (by Lemma 11 and the construction of the germ E). Lemma 7 now implies that the germs D and E are opposite. The unique twin apartment Σ'' containing both also contains B by the proof of Proposition 14. Because B was chosen freely in α_+, it follows that Σ'' contains this root. Convexity then implies that this twin apartment contains ζ_+.

Let β be the complementary twin root of γ in Σ. As Σ'' and β_- both contain the germ D, the intersection also has to contain a germ F of which the base point does not lie on the wall of β_-. Let G be the opposite germ of F in Σ''.

As γ_+ was defined as a supremum, there is a minimal $n \in \mathbb{N}$ such that Σ^n contains the germ G. By minimality of n and the definition of root elations it follows that Σ^n also contains the opposite germ F. Since also the twin apartment Σ'' contains both, uniqueness implies that $\Sigma'' = \Sigma^n$. We now have a contradiction to γ_+ being the supremum of the roots in Σ_+ containing α_+ that can be mapped into Σ'_+ by an element of the series. This is because Σ_+^{n+1} has a larger intersection with Σ'_+. This finishes the second step of the proof.

In what follows we denote the object at infinity, if it exists, of a given object O by O^∞.

The third step is to study the fixed structure of the automorphism h_i. It will suffice however to just consider $h_1 = g_1$. Recall that α_+^1 is the root defined by the intersection of Σ_+ and $\Sigma_+^{g_1}$, and that this root defines a twin root α^1 of Σ. We denote the wall defined by α_+^1 as M_+. Let S_+^∞ be a chamber at infinity having only an interior panel P_+^∞ in common with the root at infinity $(\alpha_+^1)^\infty$. We want to show that S_+^∞ is fixed by g_1. Choose a point p_+ on the wall M_+. Let S_+ be the sector with base point p_+ and direction S_+^∞. Let T_+ be the unique sector-facet of S_+ with direction the rank 1 facet of S_+^∞ not in P_+^∞. So T_+ is a half-line starting in p_+. If this sector-facet is fixed by g_1 then S_+ will also be fixed. Suppose that this is not the case.

Let q_+ be the last point on this half-line T_+ which is fixed (starting from p_+). Let C be the germ of the sector with base point q_+ and direction S_+^∞. If we can show that this germ is fixed, then we have found a contradiction. In the way we constructed the series h_i, we can now construct an automorphism f_j using another twin root of the twin apartment Σ fixing the intersection of S_+ with α_+^1 and mapping q_+ to a point of α_+^1 (we take the least j such that these conditions are fulfilled). The apartment Σ_+ and the facet P_+^∞ define a parallel set walls in Σ_+. Take the unique such wall N_+ containing $q_+^{f_j}$. Let N_- be the corresponding wall of Σ_- (using the opposition isometry).

Let r_+ be a point in the intersection of the walls M_+ and N_+, and r_- the corresponding opposite point in Σ_-. Denote the germ of the sector with base point r_+ and containing C^{f_j} as D. Let E be the corresponding germ based at r_- using the isomorphism between the spherical residues of r_+ and r_- (see Corollary 10).

Because of the way we constructed f_j and chose the least j such that the conditions were fulfilled, r_- will be fixed by both g_1 and f_j. The germs E and E^{f_j} are fixed by g_1, and so also by the commutator $[g_1, f_j]$. The point r_+ is also fixed by this commutator, implying that D and, finally, also C are fixed, contradicting the definition of q_+.

The last step is to verify that for each point x of the $\mathbb{R}$-building and positive real number d, there exists an $n_0 \in \mathbb{N}$ such that if $n \in \mathbb{N}$ is larger than n_0 then g_n fixes each point at distance less that d. This follows from the previous step. □

7. An example using direct limits

We end this paper by constructing an easy example of a twin $\mathbb{R}$-building which is not discrete.

Consider any affine irreducible Coxeter system (W, S). Given a field $\mathbb{K}$, there exists a natural affine (discrete) twin building of type (W, S) over the ring Laurent polynomials $\mathbb{K}[t, t^{-1}]$. Replacing the dummy variable t by $t^{1/n}$, one obtains affine twin buildings over the rings $\mathbb{K}[t^{1/n}, t^{-1/n}]$ with $n \in \mathbb{N} \setminus \{0\}$. The metric realisation of the affine twin building over $\mathbb{K}[t^{1/n}, t^{-1/n}]$ with $n \in \mathbb{N} \setminus \{0\}$ can be embedded naturally in the metric realisation of the affine twin building over $\mathbb{K}[t^{1/m}, t^{-1/m}]$ with $m \in \mathbb{N} \setminus \{0\}$ when n divides m (with appropriate normalisation of the distance functions). By taking the direct limit of a tower of such embeddings one obtains a twin $\mathbb{R}$-building.

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