Collineations of polar spaces with restricted displacements

B. Temmermans[*] J. A. Thas H. Van Maldeghem

Department of Mathematics, Ghent University,
Krijgslaan 281, S22, B–9000 Gent
btemmerm@cage.UGent.be, jat@cage.UGent.be, hvm@cage.UGent.be

Abstract

Let J be a set of types of subspaces of a polar space. A collineation (which is a type-preserving automorphism) of a polar space is called J-domestic if it maps no flag of type J to an opposite one.

In this paper we investigate certain J-domestic collineations of polar spaces. We describe in detail the fixed point structures of collineations that are i-domestic and at the same time $(i+1)$-domestic, for all suitable types i. We also show that {point, line}-domestic collineations are either point-domestic or line-domestic, and then we nail down the structure of the fixed elements of point-domestic collineations and of line-domestic collineations. We also show that $\{i, i+1\}$-domestic collineations are either i-domestic or $(i+1)$-domestic (under the assumption that $i+1$ is not the type of the maximal subspaces if i is even). For polar spaces of rank 3, we obtain a full classification of all chamber-domestic collineations. All our results hold in the general case (finite or infinite) and generalize the full classification of all domestic collineations of polar spaces of rank 2 performed in [8].

1 Introduction

In this paper, we investigate collineations of polar spaces that have "restricted displacement". That means that for some type set J, the collineation in question does not map any flag of type J onto an opposite one. Such a collineation is then called *J-domestic*. If J is the full type set, then we briefly talk about *domestic* collineations. The eventual goal would be to describe all domestic collineations. However, this goal is not attained here. Instead, we obtain results about collineations that are i-domestic and $(i+1)$-domestic

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at the same time. As we will see, this already generates some beautiful geometry. In particular, a kind of "geometric Tits diagram" turns up.

Where does this problem come from? In [2] it is shown that every automorphism of a non-spherical building has infinite displacement (with respect to chambers). This means that, in a (thick) non-spherical building, the identity is characterized among all automorphisms φ by the existence of a positive constant d such that the numerical distance between any chamber C and its image C^φ is at most d. For spherical buildings, where the displacement is automatically finite and bounded, it still makes sense to look for the maximal value of d, with d having the property stated in the previous sentence. In fact, one could think that $d = D - 1$, with D the diameter of the chamber graph of the building. But this is false in general, as [2] contains some counterexamples. However, we have taken a closer look at these counterexamples in [7] and [8] and could classify the automorphisms of projective spaces and generalized quadrangles which map no chamber to an opposite one. The bounds d then follow easily. An observation that can be made is that, if $d \neq D - 1$, then d diverges polynomially from D with growing rank. Related to this, it seems that domestic automorphisms have a rather large fixed element structure.

In the present paper, we want to extend this theory to polar spaces. However, a full classification of domestic collineations of polar spaces seems out of reach for now. We develop some methods to handle certain special cases, showing that the problem is interesting and rich. In particular, we solve the problem completely for polar spaces of rank 3. For general rank, we content ourselves with collineations that are point-domestic or line-domestic, and with collineations that are both i-domestic and $(i+1)$-domestic. For instance, in the latter case, the collineation does not only map no i-space to an opposite one, it always fixes at least one point in every i-space and so it maps any i-space to a non-disjoint one. This is in accordance with the general philosophy that, if a collineation does not map any flag of certain type to an opposite one, then the maximal distance between a flag of that type and its image is much smaller than opposition.

Let us remark that in any thick spherical building, there is always some flag that is mapped onto an opposite, i.e., no collineation is J-domestic for all J, except the identity. This is a general characterization of the identity in spherical buildings which follows from [4]. As a byproduct, we obtain a slightly stronger characterization of the identity in polar spaces: it is the only collineation that does not map any point to an opposite one and at the same time no line to any opposite one.

Also, in classifying point-domestic collineations, Tits-diagrams will turn up. Hence, certain fixed point buildings of non-quasi-split type relate to domestic automorphisms. This brings us to our second main motivation. When dealing with collineation groups, or, more generally, with permutation groups, it often helps to look at the fixed point structure. Here, we want to slightly wider that horizon: we take into account all distances and instead of asking ourselves which collineations have large fixed point structures, we ask ourselves which collineations do not map elements too far away. We hope our analysis will prove useful in the future, and that it will help to solve the problem of determining all domestic collineations of any polar space.

The paper is structured as follows. First we show that {point,line}-domestic collineations are either point-domestic or line-domestic. Also, a point-domestic collineation which is also line-domestic is necessarily the identity. Then we take a closer look at point-domestic collineations and show that these relate to Tits-diagrams in the non-symplectic case. For line-domestic collineations we show that they always fix a geometric hyperplane pointwise (and vice versa). Then, we consider i-domestic collineations which are also $(i+1)$-domestic, for $i > 1$, and we also show that, roughly, $\{i, i+1\}$-domestic collineations are either i-domestic or $(i+1)$-domestic. Finally we treat domesticity in full generality for polar spaces of rank 3.

We repeat that, with a *collineation* of a polar space we mean a collineation of the geometry. Hence, from the point of view of buildings, this includes non-type preserving automorphisms of buildings of type D_{n+1}. Indeed, we also allow for polar spaces whose B_{n+1}-diagram is thin on the last node. One can translate the results of the present paper to buildings of type D_{n+1} by reading $(n-1)$-domesticity in the polar space as $\{n_+, n_-\}$-domesticity in the oriflamme complex, and n-domesticity in the polar space as n_+- and n_--domesticity in the oriflamme complex (also considering dualities).

Throughout, Γ will denote a polar space of finite rank, furnished with all its projective subspaces. Usually we will assume that the rank of Γ is equal to $n+1$, so that the projective dimension of the maximal subspaces is n. This convention will exceptionally be interrupted in Section 4, where rank n gives better formulations of the results in the statements and in the proofs. In any case, the *type* of an element (a subspace) will always be its projective dimension.

Finally, we mention that we assume the rank of Γ to be larger than 2 since we have treated the generalized quadrangles in [8], where we basically show that all domestic collineations are either point-domestic or line-domestic, or one of three exceptional cases occuring in small quadrangles.

2 Preliminaries and main results

We begin with defining domesticity in the general context of spherical buildings, but we will only be concerned with a specific class of buildings, namely, polar spaces. Hence we do not define a building in full generality, but refer to the literature.

Let Ω be a spherical building, and let θ be an automorphism of Ω. We emphasize that θ need not be type-preserving. Then we call θ *domestic* if no chamber of Ω is mapped onto an opposite chamber. More in particular, for a subset J of the type set of Ω, we say that θ is *J-domestic*, if θ does not map any flag of type J onto an opposite one. The main result of Section 5 of [2], also proved earlier by Leeb [4], using entirely different methods, asserts that every automorphism of any (thick) spherical building is not J-domestic, for some type subset J. Hence being not J-domestic seems to be the rule, and so it is worthwhile to look at automorphisms which are J-domestic, for some J.

We now specialize to polar spaces.

A *polar space* Γ *of rank* $n+1$, $n \geq 1$, is a geometry of rank $n+1$ with type set $\{0, 1, \ldots, n\}$ satisfying the following axioms (see Chapter 7 of [10] or [12]), where we call the elements of type 0 *points*. In the following, a projective space of dimension -1 is the empty set, of dimension 0 is a singleton and of dimension 1 is a set of at least 3 points.

(PS1) The elements of type $< i$ incident with any element of type i form naturally a projective space of dimension i in which the type of an element in Γ is precisely the dimension of the corresponding subspace in that projective space.

(PS2) Every element of Γ is determined by the set of points incident with it and the point sets of two elements of Γ intersect in a subspace of each.

(PS3) For every point x and every element M of type n not incident with x, there exists a unique element M' through x of type n whose point set intersects the point set of M in the point set of an element of type $n-1$. Also, no element of type 1 is incident with x and a point of M unless it is incident with M' or coincides with M'.

(PS4) There exist two elements of type n not incident with any common point.

Axiom (PS1) justifies the following terminology: we call elements of type i *i-dimensional subspaces*. Also, 1-dimensional subspaces are simply called *lines*, 2-dimensional ones *planes* and n-dimensional ones *maximal subspaces*. The *codimension* of an i-dimensional subspace is by definition equal to $n-1-i$. Two points that are incident with a unique common line will be called *collinear*, and we will thus also use the notation $x^\perp$ for the set of points collinear to the point x completed with x itself. In such a way we can consider a polar space as a point-line geometry by "forgetting" the i-dimensional subspaces for $i \geq 2$. These can always be reconstructed merely using the points and lines. In this setting, it is natural to see the subspaces as sets of points, and we will indeed take this point of view. This way we can talk about the intersection of subspaces.

We call Γ *thick* if every subspace of type $n-1$ is contained in at least three subspaces of type n.

If x is a point and L is a line of a polar space Γ, x not on L, then considering a maximal subspace M incident with L (we also say *through* L), and applying Axiom (PS3), we see that

(BS) either all points on L are collinear with x, or exactly one point on L is collinear with x.

A major result of Beukenhout & Shult [3] is that this observation—known as the *Bueken-hout-Shult one-or-all axiom*—along with some nondegeneracy conditions such as (1) every line contains at least three points, (2) no point is collinear with all other points, and under a suitable condition that bounds the rank, characterizes the class of polar spaces. The simplicity of Axiom (BS) played a major role in the success of studying polar spaces and, in fact, we will also use that axiom as a central property of polar spaces. Moreover, the above motivates the notion of a *degenerate polar space* as a point-line geometry in which Axiom (BS) holds, but which is not the restriction to points and lines of a polar space. With "polar space" we will never include the degenerate ones, except when explicitly mentioned (for example, we sometimes say "possibly degenerate polar space").

Polar spaces of rank 2 are just thick generalized quadrangles or generalized quadrangles with 2 lines through every point (the so-called *grids*). As soon as the rank is at least 3, then there is a classification due to Tits [10]. Roughly, this classification says that a polar space of rank at least 4 arises from a bilinear, sesquilinear or pseudo-quadratic form in some vector space. In the rank 3 case there is one other class of (thick, i.e., there are at least three planes through every line) polar spaces parametrized by octonion division rings (here, the planes of the polar space are projective planes over alternative division rings).

A *chamber* of a polar space is a set of nested projective subspaces of dimension 0 up to n. A *flag* is a subset of a chamber. The *type* of a flag is the set of types of its elements.

A *geometric subspace* of a polar space is a set of points such that, if two collinear points x and y belong to that set, then all points of the line xy belong to it. We often view geometric subspaces as substructures endowed with all subspaces completely contained in it. In order to explicitly distinguish between geometric subspaces and ordinary subspaces, we sometimes call the latter *projective subspaces*. A *geometric hyperplane* is a geometric subspace with the property that every line contains at least one point of it, and at least one line contains exactly one point of it. It is easy to show that geometric subspaces are (possibly degenerate) polar spaces. The *corank* of a geometric subspace equals i if every i-dimensional (projective) subspace meets it in at least one point, and there exists an i-dimensional subspace meeting it in exactly one point. Hence, geometric hyperpanes are the geometric subspaces of corank 1.

If we understand with *distance between two elements* the graph-theoretical distance between them in the incidence graph, then this notion does not fully cover all the possible *mutual positions* of two elements (by which we mean the isomorphism classes of the substructures induced by all shortest paths between them in the incidence graph). For instance, for two lines, there are six possible mutual positions given by (1) equality, (2) being contained in a common plane, (3) intersecting in a unique point but not contained in a plane, (4) being disjoint but some plane contains one of them and intersects the other in a point, (5) being disjoint and no plane containing one of them intersects the other in a point, (6) both contained in a common projective subspace, but not in a plane. Clearly, in the cases (2), (3) and (6) the lines are at distance two from each other. But for points, it does. Two points can have only three possible mutual positions, given by the distances $0, 2, 4$ in the incidence graph. A special mutual position is *opposition* given by the maximum distance between two elements of the same type. It is characterized as follows: two subspaces U and U' of dimension i are opposite if and only if no point of U is collinear with all points of U'. It follows that this relation is symmetric.

We now define the notion of "projection" in polar spaces. Our definition is in conformity with the definition of projection in buildings, where the projection of a flag F onto another flag F' is the intersection of all chambers appearing as last chamber in a minimal gallery connecting F with F' (i.e., the first chamber contains F and the last one F'). Since F' is, however, always in that final chamber, one usually does not mention it. Let U and V be two projective subspaces of a polar space Γ, and suppose that they are neither opposite nor incident (otherwise the projection onto U is empty or U, respectively). Then $\mathrm{proj}_U V$ is the set (a flag) of the following subspaces: the intersection $V \cap U$, if not empty; the set of points of U collinear with all points of V, if not empty and if it does not coincide with U; the unique minimal subspace containing all points of U and all points of V that are collinear with all points of U, if it does not coincide with U. At least one of these subspaces is well defined. In the generic case, all these subspaces are distinct and hence $\mathrm{proj}_U V$ is a set of three subspaces: two contained in U and one containing U. We will make the following agreement: with $\mathrm{proj}_{\subseteq U} V$ we mean the set of points of U collinear with all points of V (and this time it could coincide with U, with $V \cap U$ or with the empty set) and with $\mathrm{proj}_{\supseteq U} V$ we shall denote the subspace generated by U and the set of points of V that are collinear with all points of U (and this time, this could also coincide with U).

For a subspace U of Γ, the notation $\mathrm{Res}_\Gamma(U)$ denotes the polar space obtained from Γ and U by considering all subspaces properly containing U. It is called the *residue* of U. Strictly speaking, the residue is the direct product of the aforementioned polar space, with the projective space defined by U (considering all subspaces properly contained in U), but we will not need this.

A *collineation* is a type preserving permutation that preserves incidence. According to the terminology introduced above for spherical buildings, we will call a collineation which maps no flag of type J to an opposite one *J-domestic*. We sometimes substitute the type by the name, e.g., point-domestic for 0-domestic. If $J = \{0, 1, \ldots, n\}$, then we talk about *domestic* collineations.

Our main results could be briefly stated as follows.

- *A point- and line-domestic collineation is the identity.*
- *A line-domestic collineation fixes a hyperplane pointwise.*
- *A point-domestic collineation in a polar space different from a symplectic one admits a Tits-diagram*
- *An i- and $(i+1)$-domestic collineation which is not $(i-1)$-domestic fixes a geometric subspace of codimension i pointwise.*
- *A domestic collineation of a thick polar space of rank 3 is either an involution fixing a Baer sub polar space, or a collineation that fixes pointwise a geometric subspace of corank at most 2.*

For the definition of Baer sub polar space of a polar space of rank 3, we refer to Section 7. More detailed statements are given in the next sections.

We have chosen not to interrupt our exposition with examples. We content ourselves with mentioning that for all situations we will encounter, there exist examples of appropriate J-domestic collineations. But to describe these examples in detail, one needs to introduce additional notions, and this would make the paper too long, without adding essential knowledge. However, we will briefly mention how to construct examples, without going into details.

3 {point,line}-Domestic collineations

The following lemma will turn out to be very useful. We provide two proofs. One proof uses the result of Leeb [4], the other is independent and somewhat longer but introduces a technique that we shall use later.

Lemma 3.1 *Suppose that Γ is a polar space of rank $n+1$ and θ is a point-domestic and line-domestic collineation, then θ is the identity.*

Proof We will first prove that there are no i-dimensional spaces which are mapped to an opposite i-dimensional space, $2 \leq i \leq n$. Suppose, by way of contradiction, that there exists such a space Ω. Take an arbitrary point x in Ω; this point is mapped to the point x^θ in Ω^θ which is not opposite x. Consider the projection $H_x = \text{proj}_{\subseteq\Omega} x^\theta$ of x^θ into Ω; this is an $(i-1)$-dimensional space containing x. The mapping $x \mapsto H_x$ is clearly a duality of Ω (since it is the composition of the collineation $\Omega \to \Omega^\theta : x \mapsto x^\theta$ and the duality $\Omega^\theta \to \Omega : y \mapsto y^\perp \cap \Omega$), and since $x \in H_x$, it is a domestic duality. By the main result of [7], this duality is a symplectic polarity. Hence, if i is even, we obtain a contradiction. If i is odd (with $i > 1$), then there exists a non-isotropic line L in Ω which is mapped to an opposite $(i-2)$-dimensional space of Ω under the symplectic polarity. It follows that L is mapped to an opposite line under θ, again a contradiction.

First proof. We will now prove that every n-dimensional space is fixed. Suppose, by way of contradiction, that there is an n-dimensional space Ω' which is not fixed. Then the intersection of Ω' with Ω'^θ is an i-dimensional space, with $0 \leq i \leq n-1$. Take an $(n-i-1)$-dimensional subspace U of Ω' disjoint from $\Omega' \cap \Omega'^\theta$ and also disjoint from the pre-image (under θ) of $\Omega' \cap \Omega'^\theta$. Because of the previous part of this proof U cannot be mapped to an opposite subspace. Hence there exists a point $u \in U$ which is collinear with every point of U^θ. But u is also collinear with every point of $\Omega' \cap \Omega'^\theta$. Since U^θ is disjoint from $\Omega' \cap \Omega'^\theta$ by the above conditions on U, we see that U^θ and $\Omega' \cap \Omega'^\theta$ generate Ω'^θ, and it follows that u is collinear with all points of Ω'^θ. Hence $u \in \Omega'^\theta$, contradicting $u \in U$ and U disjoint from Ω'^θ.

Second proof. We have proved above that θ is i-domestic, for every type i. This means that no flag whatsoever can be mapped onto an opposite flag, which contradicts the result of Klein & Leeb [4] if θ is not the identity, see also Abramenko & Brown [1]. $\square$

Theorem 3.2 *Suppose that Γ is a polar space of rank $n+1 > 2$ and θ is a {point, line}-domestic collineation. Then θ is either point-domestic or line-domestic.*

Proof Suppose θ is not point-domestic and consider a point x which is mapped to an opposite point x^θ. Take a line L through x and consider the unique line L^φ through x intersecting L^θ in a point. Because θ is {point, line}-domestic, L and L^θ can not be opposite lines. Hence there exists a point y on L^θ which is collinear to all points of L (here collinearity also includes equality). This point should be the intersection of L^θ and L^φ, because it is the only point on L^θ collinear with x. Hence L and L^φ are not opposite in $\mathrm{Res}_\Gamma(x)$, which means that the collineation in the residue of x corresponding with φ is point-domestic.

Consider a plane α of Γ through x. Suppose that α and α^θ are opposite. Take a flag $\{y, M\}$ in α. This flag can not be opposite its image $\{y^\theta, M^\theta\}$. Consider the projection $\{\mathrm{proj}_{\subseteq\alpha} M^\theta, \mathrm{proj}_{\subseteq\alpha} y^\theta\}$ of $\{y^\theta, M^\theta\}$ into α, this is a flag which is not opposite in α to the flag $\{y, M\}$. Hence we obtain a {point, line}-domestic duality in α, a contradiction to Theorem 3.1 in [7]. Hence α and α^θ are not opposite. This means that there exists a point z in α^θ which is collinear to all points of α. Similarly as above, the point z should be in the intersection of α^θ and the unique plane α^φ through x intersecting α^θ in a line. Hence α and α^φ are not opposite in the residue of x, which means that the collineation in the residue of x corresponding with φ is line-domestic. Because of Lemma 3.1, this collineation is the identity.

Let z be an arbitrary point in $x^\perp \cap (x^\theta)^\perp$ and let π be an arbitrary plane containing x and z. Also, let x' be a point of π not on the line xz and not collinear with x^θ. By the foregoing, the line xz is mapped under φ to itself, which means that θ maps xz to $x^\theta z$. Our choice of x' implies that x' is opposite x'^θ, and hence z also belongs to $x'^\perp \cap (x'^\theta)^\perp$. Consequently, letting x' play the role of x above, we also have that θ maps $x'z$ to $x'^\theta z$. Hence the intersection z of xz and $x'z$ is mapped to the intersection z of $x^\theta z$ and $x'^\theta z$. We have shown that z is fixed under θ. Hence θ fixes $x \cap x^\theta$ pointwise.

Now consider an arbitrary line K. If K intersects $x^\perp \cap (x^\theta)^\perp$, then it contains at least one fixed point and hence is not mapped onto an opposite line. If K does not intersect $x^\perp \cap (x^\theta)^\perp$, then there exists a line N through x which intersects K in a point y. Since xy is mapped to $x^\theta y'$, with $y' = \mathrm{proj}_{\subseteq xy} x^\theta$, we see that y and y^θ are opposite. Hence we can let y play the role of x above and conclude that K has a fixed point and consequently cannot be mapped onto an opposite line. So θ is line-domestic. $\square$

4 Point-domestic collineations

In this section we assume that θ is a point-domestic collineation of the polar space Γ of rank n, with $n > 2$.

Lemma 4.1 *The orbit of a point x under the collineation θ is contained in a projective subspace of Γ.*

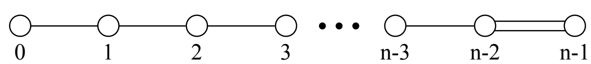
Proof We first show by induction on ℓ that the set $\{x, x^\theta, x^{\theta^2}, \ldots, x^{\theta^\ell}\}$ is contained in a subspace. For $\ell = 1$, this is by definition of point-domestic. Now suppose that $\{x, x^\theta, x^{\theta^2}, \ldots, x^{\theta^{\ell-1}}\}$ is contained in some subspace X, and we may assume that X is generated by $x, x^\theta, x^{\theta^2}, \ldots, x^{\theta^{\ell-1}}$. Applying θ, we see that also $\{x^\theta, x^{\theta^2}, \ldots, x^{\theta^\ell}\}$ is contained in some subspace, namely, X^θ. Consider the line $L := xx^{\theta^{\ell-1}}$, which is mapped onto the line $L^\theta = x^\theta x^{\theta^\ell}$. Consider a point z on L distinct from x and from $x^{\theta^{\ell-1}}$. Then z is collinear to z^θ on L^θ. Since z is also collinear with x^θ (as both points belong to X), we see that z is collinear to x^{θ^ℓ}. Since x^{θ^ℓ} is also collinear to $x^{\theta^{\ell-1}}$, it is collinear with all points of L and hence also with x. This shows that X and X^θ are contained in a common subspace.

This already proves the lemma for θ of finite order. Now suppose the order of θ is infinite. Since the rank of Γ is finite, there exists some natural number k such that x^{θ^k} is contained in the subspace Y generated by $x, x^\theta, \ldots, x^{\theta^{k-1}}$. It is now clear that θ stabilizes Y, as $x^\theta, x^{\theta^2}, \ldots, x^{\theta^k}$ generates a subspace Y^θ contained in Y and of the same dimension as Y (hence coinciding with Y). Consequently the orbit of x generates Y. □

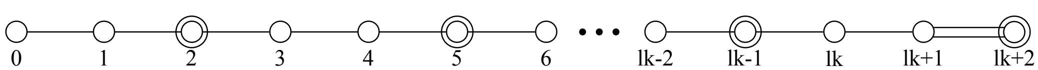
We have now reduced the problem to a geometric one: classify closed configurations of polar spaces whose union is the whole point set. With *closed configuration*, we here understand a set of subspaces closed under projection. Note that this implies closedness of intersection (of intersecting subspaces) and generation (of two subspaces contained in a common subspace).

Usually, such configurations can be rather wild (for instance, one can take any flag), unless every member has an opposite in the configuration, in which case the configuration forms a building itself. In this case, there is a *Tits diagram*, and so the types of the elements of the configuration behave rather well. But if some member has no opposite in the configuration, then there is no reason to believe that these types follow certain rules (think of the example of an arbitrary flag). However, the extra condition that every point is contained in some member of the configuration forces the types of elements to obey the same rules as the *Tits diagrams*, at least when Γ is not a symplectic polar space, i.e. when Γ is not of type C_n.

Before stating an proving the theorem, we explain our remarks above about the *Tits diagrams*. We only mention the geometric relevant part. The diagram of a polar space Γ of rank n has type B_n, i.e.



Now, if we have a closed configuration Ω, then we can encircle the nodes on the diagram that correspond to the types of the members contained in Ω. In our case, it will turn out that the last node is always encircled, in which case the general rules of the Tits diagrams (see [9]) say that there is a natural number k such that the encircled nodes occur precisely every k nodes. For example, for $k = 3$, we obtain the diagram (where l nodes are encircled)



The existence of a Tits diagram for closed configurations for polar spaces not of symplectic type—a result that is clearly worthwhile in its own right—is precisely the content of the next theorem.

Theorem 4.2 *Let Ω be a set of subspaces of a polar space Γ closed under projection and such that every point is contained in some member of Ω. Assume that Γ is not symplectic.*

Then there exists a unique natural number i such that the type of each member of Ω is equal to $mi-1$, for some integer m, with i a divisor of n, and m ranging from 1 to n/i (included). Also, for every m with $1 \leq m \leq n/i$, there exists at least one subspace of type $mi-1$ belonging to Ω, and for every member U of Ω, say of type $ti-1$, and for every m, $1 \leq m \leq n/i$, there exists a subspace of type $mi-1$ belonging to Ω and incident with U.

Proof We prove the assertions by induction on n. Despite the fact that we assume that Γ has rank at least 3, we can include the case $n = 2$ and start the induction with $n = 2$. For $n = 2$, all assertions follow from the fact that we are dealing with a dual geometric hyperplane as follows from [8].

So we may assume $n > 2$. We define i as the smallest positive integer for which there exists a member of Ω of type $i-1$, and we let $U \in \Omega$ be of type $i-1$. If $i = n$, then our assumptions readily imply that Ω consists of a spread and all assertions follow. So we may assume from now on that $i \leq n-1$. Now let Ω_U be the set of all members of Ω containing U. Then clearly Ω_U is closed under projection. We now show that every point of $\mathrm{Res}_\Gamma(U)$ is contained in a member of Ω_U. Hence let W be a subspace of type i containing U. Pick a point x in $W \setminus U$. By assumption, there is some member U' of Ω containing x. Then the subspace $\mathrm{proj}_{\supseteq U} U'$ belongs to Ω and contains both U and x, hence W.

Note that the same argument can be applied to any element of Ω, and in particular, by repeated application, it proves that every element of Ω is contained in a maximal subspace belonging to Ω.

Consequently, for $i \leq n-2$, we can apply induction in $\mathrm{Res}_\Gamma(U)$ and obtain a natural number j such that the type of each member of Ω_U (in Γ) is equal to $i+mj-1$, for some integer m, with j a divisor of $n-i$, and m ranging from 1 to $(n-i)/j$ (included). Also, for every m with $1 \leq m \leq (n-i)/j$, there exists at least one subspace of type $i+mj-1$ belonging to Ω_U, and for every member W of Ω_U, say of type $i+tj-1$, and for every m, $1 \leq m \leq (n-i)/j$, there exists a subspace of type $i+mj-1$ belonging to Ω_U and incident with W.

If $i = n-1$, then the first assertion in the previous paragraph still holds setting $j = 1$. The second assertion is trivially true. Hence, for now, we do not need to consider the case $i = n-1$ separately.

Consider a point x of Γ not collinear to at least one point of U, and let U_x be a member of Ω containing x (guaranteed to exist by assumption on Ω). By a previous note above, we may assume that U_x has dimension $n-1$. We note that U_x cannot contain U, as x is not collinear to all points of U. Also, U_x is disjoint from U by minimality of i. Hence, as $i \leq n-1$, the subspace $\mathrm{proj}_{\subseteq U_x} U$ is a proper nonempty subspace of U_x disjoint from U and belonging to Ω (nonempty, because the dimension of U is strictly smaller then the dimension of U_x). So the subspace $U' := \mathrm{proj}_{\supseteq U} U_x$ belongs to Ω_U. The induction hypothesis implies that there exists some subspace U'' of (minimal) type $i+j-1$ belonging to Ω_U and incident with (hence contained in) U'. The intersection $V := U'' \cap \mathrm{proj}_{\subseteq U_x} U$ has minimal dimension $j-1$ and belongs to Ω. Minimal here means that every subspace of Ω, all of whose points are collinear with all points of U, has dimension at least $j-1$.

1. First we assume $i + j < n$. Note that the minimal dimension of a subspace of Ω, all of whose points are collinear to all points of U'', is $i - 1$. Indeed, U is such a subspace and the minimality follows from the minimality of i. Consequently, if we interchange the roles of U and V, we also interchange the roles of i and j (the minimality of i is responsible for the fact that the previous paragraphs also hold for j).Hence, looking from both points of view, the subspace of minimal dimension at least $i + j$ (which exists due to $i + j < n$) containing both U and V must have dimension $i + 2j - 1$ and at the same time $j + 2i - 1$. This implies $i = j$ and there only remains to prove the last assertion.

   We first show that any element W of Ω contains a member of Ω of dimension $i-1$. Indeed, let U be the above member of Ω of type $i - 1$. We may suppose that U is not contained in W and so U and W are disjoint, by minimality of i. We may also assume that the dimension of W is larger than $i - 1$. Then $W' = \mathrm{proj}_{\subseteq W} U$ is nonempty (again because the dimension of U is strictly smaller than the dimension of W) and belongs to Ω. It suffices to show that W' contains an element of type $i - 1$ of Ω. But this now follows from the induction hypothesis by considering a subspace of dimension $2i - 1$ of Ω_U contained in the subspace generated by U and W'.

   Next we show that every subspace of dimension $i-1$ belonging to Ω is contained in a subspace of dimension $2i - 1$ belonging to Ω. Indeed, let $U' \in \Omega$ be a subspace of dimension $i - 1$, distinct from U (and hence disjoint from it, too). If U and U' are contained in a common subspace, then $\langle U, U' \rangle$ meets the requirement. Otherwise, let $H \in \Omega$ be a maximal subspace through U'. Then the induction hypothesis ensures that there exists a subspace $W \in \Omega_U$ of dimension $2i - 1$ contained in the subspace $\mathrm{proj}_{\supseteq U} H$. The intersection $W \cap H$ has dimension $i-1$ and so $\langle U', W \cap H \rangle$ meets our requirement (indeed, U' is not contained in W because otherwise U and U' would be contained in a common subspace; U' is disjoint from W by minimality of i).

   It now follows that we can interchange the roles of U with any member V of Ω of type $i - 1$. In particular, Ω_V satisfies the assumptions of our theorem and contains elements of type $mi - 1$, for every $m \in \{2, 3, \ldots, n/i\}$. Hence the last assertion of the theorem follows from first constructing a member $V \in \Omega$ of type $i - 1$ inside a given member W of Ω, and then apply the induction hypothesis to Ω_V to obtain a member of any dimension $mi - 1$, $m \in \{2, 3, \ldots, n/i\}$, incident with both V and W, but in particular W.

2. Next we suppose that $i + j = n$ (and so $U' = U''$ is a maximal subspace). Then $\mathrm{proj}_{\subseteq U_x} U$ already has dimension $j - 1$, and so, by minimality of i, we have $i \leq j$. If $i = j$, there is nothing left to prove. If $i < j$, then consider a point z not collinear to all points of U, and not collinear to all points of V (z can be obtained by using any hyperplane of U'' that does neither contain U nor V). As before, we know that there is a maximal subspace H of Ω containing z. Since by our choice of z, the subspace H can neither contain U nor V, it is disjoint from both U and V. We claim that H is disjoint from U'. Indeed, suppose H meets U' in some subspace S.

By minimality of j, S is disjoint from V and has dimension at least $j-1$. But since $j > n-j$, this is a contradiction. Our claim follows. Projecting U and V into H, we obtain two complementary subspaces U_H and V_H in H of dimension $j-1$ and $i-1$, respectively, belonging to Ω. It is clear that these are the only proper subspaces of H that belong to Ω, as otherwise the projection (with the operator $\mathrm{proj}_{\subseteq U'}$) into U' produces a contradiction just like in the proof of our last claim above. Now let H' be any maximal subspace of Γ which belongs to Ω. Then H' meets U' and/or H either in one of the proper subspaces of U and H belonging to Ω, or H' coincides with one of H or U', or it is disjoint from both. If H' is disjoint from one of U' or H, then it contains exactly two proper subspaces belonging to Ω, and they have again dimensions $i-1$ and $j-1$. If H' meets U' in U, and if it meets H nontrivially, then it must meet H in U_H (it can clearly not meet H in V_H because $2j > n$). Also, if H' meets U' in V, then it must meet H in V_H (granted it meets H nontrivially) because no point of U_H is collinear to all points of V. So in any case, H' properly contains two members of Ω, of dimensions $i-1$ and $j-1$. Moreover, the above arguments also show that any member of Ω of dimension $i-1$ and any member of Ω of dimension $j-1$ lie together in a joined maximal subspace of Γ.

Now choose a subspace X in U' of dimension $n-3$ and intersecting U in a subspace of dimension $i-2$ (remember that dimension -1 means the empty subspace) and intersecting V in a subspace of dimension $j-2$ (this is never -1). If we now consider the residue $Q := \mathrm{Res}_\Gamma(X)$, which is a generalized quadrangle, then we see that the projection $\mathrm{proj}_{\supseteq X}\Omega$ of Ω onto X is a dual grid in Q, with the extra property that every point of that generalized quadrangle is incident with some line of the dual grid. The latter implies that there are two opposite points x, y in Q with the property that for all points z in Q opposite x the sets $x^\perp \cap y^\perp$ and $x^\perp \cap z^\perp$ have exactly one point in common. Since Q is a Moufang quadrangle; and in particular has the BN-pair property, we see that this property holds for *all* opposite points x and y. Hence, by [6], Q is a symplectic quadrangle and so Γ is a symplectic polar space. This contradicts our assumptions.

The proof of the theorem is complete. □

In the symplectic case there are plenty of counterexamples to the above theorem. An obvious counterexample is the situation at the end of the proof of the previous theorem, when we take $i = 1$, $j = 3$ and hence $n = 4$.

Note that in the symplectic case we can use the structure of the underlying projective space when looking for the fixed structure of a collineation. In particular an inductive process can be used if two non-maximal opposite subspaces of Γ are fixed: the subspace of the surrounding projective space generated by these two subspaces induces a nondegenerate symplectic polar space of lower rank which is also fixed. Nevertheless, this extra tool seems not to be enough to explicitly classify all possibilities, or to at least give a general and uniform description, as in the non-symplectic case.

Examples can be found using the theory of Galois descent in algebraic groups of type B_n, see [9].

5 Line-domestic collineations

In this section we assume that $n \geq 2$ and we will prove the following theorem:

Theorem 5.1 *Suppose that Γ is a polar space of rank $n+1$ and θ is a nontrivial line-domestic collineation, then θ fixes pointwise a geometric hyperplane.*

For $n = 1$ this is included in [8]. We will prove this theorem by using the following lemmas.

Lemma 5.2 *Suppose that Γ is a polar space of rank $n+1$ and θ is a line-domestic collineation which is not point-domestic. Suppose that the point x is mapped to an opposite point x^θ. Then $x^\perp \cap (x^\theta)^\perp$ is fixed pointwise.*

Proof Consider the mapping φ which maps a line L through x to the unique line L^φ through x intersecting L^θ in a point. This mapping is the composition of the restriction to $\mathrm{Res}_\Gamma(x)$ of θ and the projection from $\mathrm{Res}_\Gamma(x^\theta)$ to $\mathrm{Res}_\Gamma(x)$ using the operator $\mathrm{proj}_{\supseteq x}$. So, φ can be conceived as a collineation of the polar space $\mathrm{Res}_\Gamma(x)$ of rank n. If some line L through x were opposite L^φ in $\mathrm{Res}_\Gamma(x)$, then clearly L would be opposite L^θ, contradicting the fact that θ is line-domestic. Hence L and L^φ are not opposite in $\mathrm{Res}_\Gamma(x)$, which means that the collineation in the residue of x corresponding with φ is point-domestic.

Now take a plane α through x and consider again the collineation φ which maps α to the unique plane α^φ through x intersecting α^θ in a line. Suppose that α and α^θ are opposite. Then the duality of α which maps a line L in α to the projection $\mathrm{proj}_{\subseteq\alpha} L^\theta$ is point-domestic. This is a contradiction, since by [7] the only such dualities are symplectic polarities and there are no such polarities in a projective plane. Hence the planes α and α^θ are not opposite and hence there exists a point y in α^θ which is collinear to all points of α. This point should be in $\alpha^\varphi \cap \alpha^\theta$. Hence the collineation corresponding to φ in the residue of x is line-domestic.

Analogously to the third paragraph of the proof of Lemma 3.2, we can now prove that $x^\perp \cap (x^\theta)^\perp$ is fixed pointwise (but we leave the details to the interested reader).

This completes the proof of Lemma 5.2. □

Lemma 5.3 *Suppose that Γ is a polar space of rank $n+1$ and θ is a line-domestic collineation. Then every line of Γ contains at least one fixed point.*

Proof If θ is point-domestic, then the assertion trivially follows from Lemma 3.1, so we can assume that θ is not point-domestic. Take a point x which is mapped to an opposite point x^θ. If a line intersects $x^\perp \cap (x^\theta)^\perp$, then because of Lemma 5.2, it contains at least one fixed point. Hence we consider a line L which does not intersect $x^\perp \cap (x^\theta)^\perp$. There exists a line M through x which intersects L in a point y. Since by Lemma 5.2 xy is mapped to $x^\theta y'$, with $y' = \mathrm{proj}_{\subseteq xy} x^\theta$, we see that y and y^θ are opposite. Hence they can play the same role as x and x^θ and so the lines L and L^θ intersect each other in a fixed point. □

Lemma 5.4 *Suppose that Γ is a polar space of rank $n+1$ and θ is a line-domestic collineation. If an i-dimensional subspace Ω in Γ, with $0 \leq i \leq n$, is fixed by θ, then Ω is fixed pointwise.*

Proof Suppose first that Ω is n-dimensional and fixed. If all $(n-1)$-spaces in Ω are fixed, then Ω is fixed pointwise. Hence we may assume that there exists an $(n-1)$-dimensional space H in Ω which is not fixed. Take an n-dimensional space Ω' through H different from Ω and consider a line L in Ω', but not in Ω, intersecting H in a point of $H \setminus (H^\theta \cup H^{\theta^{-1}})$. Because θ is line-domestic, there exists a point x on L^θ which is collinear to all points of L (and note that x obviously does not belong to Ω). But this point is also collinear to all points of the $(n-2)$-dimensional space $H \cap H^\theta$. Since L and $H \cap H^\theta$ are skew, they generate Ω', and so $x \in \Omega' \setminus \Omega$. Hence $L^\theta \subseteq \Omega'$ and so L intersects H in a point of $H \cap H^{\theta^{-1}}$, a contradiction.

Secondly, suppose Ω is $(n-1)$-dimensional and fixed. If Ω is contained in a fixed n-dimensional space, we are already done because of the first paragraph of this proof. So we may assume that there does not exist any fixed n-dimensional space containing Ω. Consider an n-dimensional space Σ containing Ω and take a point x in $\Omega \setminus \Sigma$. The point x is mapped to a point opposite x under θ. The $(n-1)$-dimensional space Ω is contained in $x^\perp \cap (x^\theta)^\perp$. Hence, by Lemma 5.2, Ω is fixed pointwise.

Now in general take an i-dimensional space Ω, with $i < n-1$, which is fixed by θ. Consider an $(i+2)$-dimensional space through Ω. Take a line in this space skew to Ω to have a line in $\mathrm{Res}_\Gamma(\Omega)$. This line is not opposite its image, hence the corresponding line in $\mathrm{Res}_\Gamma(\Omega)$ cannot be mapped to an opposite line. Hence the collineation in the residue of Ω which corresponds to θ is line-domestic. By Lemma 5.3 and the corresponding result for generalized quadrangles (see [8]), it follows that there exists at least one $(i+1)$-dimensional space containing Ω which is fixed. We can go on like this until we obtain a fixed $(n-1)$-dimensional space Σ. By the foregoing paragraph, Σ, and hence also Ω, is fixed pointwise.

This completes the proof of the lemma. $\square$

Theorem 5.1 now follows from the last two lemmas; indeed, Lemma 5.4 says that the set of fixed points is a geometric subspace while Lemma 5.3 implies that this subspace is a geometric hyperplane.

Examples of line-domestic collineations can be constructed for embeddable polar spaces by constructing collineations of the surrounding projective space stabilizing the polar space and fixing a hyperplane (for instance, central collineations).

6 Collineations that are i-domestic and $(i+1)$-domestic

In general, it seems difficult to nail down the fixed point structure of an i-domestic collineation of a polar space, with $i \geq 2$ even. For example, we claim that every point-domestic collineation is i-domestic for all even i. Indeed, if a space U of even positive dimension were mapped onto an opposite one, then the duality of U obtained by first applying θ and then $\mathrm{proj}_{\subseteq U}$ is point-domestic. Lemma 3.2 in [7] implies that this is a symplectic polarity, contradicting the fact that the dimension of U is even. Hence, in this case, in view of Theorem 4.2, there should not even be a fixed point! But if i is odd, and θ is an i-domestic collineation, then it is automatically also $(i+1)$-domestic. Indeed, this follows similarly to the second paragraph of the proof of Theorem 3.2. Hence, in reality, we classified in the previous section the collineations which are both line-domestic and plane-domestic! In this section, we will generalize this to collineations which are both i-domestic and $(i+1)$-domestic, for $i \geq 2$. It does not matter whether i is odd or even, but we will assume that i is minimal with respect to the property of θ being both i- and $(i+1)$-domestic. Note that we also already treated this question for $i = 0$. Indeed, this is Lemma 3.1.

We have the following theorem, which is somehow the counterpart of Theorem 4.3 in [7] for polar spaces.

Theorem 6.1 *Suppose that Γ is a polar space of rank $n+1$ and suppose that θ is an i-domestic and $(i+1)$-domestic collineation, with $n > i \geq 0$, which is not $(i-1)$-domestic if $i > 0$. Then θ fixes pointwise a geometric subspace of corank i. In particular, every i-dimensional space contains at least one fixed point.*

Proof We will prove this by induction on i. For $i = 0, 1$ we already proved this in Lemma 3.1 and Theorem 5.1. Hence we may assume from now on that $i > 1$. In particular, $n+1 > 3$ and so Γ is an *embeddable* polar space (meaning, it arises from a form in a vector space and so it can be viewed as a substructure of a projective space). Since by assumption θ is not $(i-1)$-domestic, there exists a projective subspace X of type $i-1$ which is opposite its image X^θ. Consider an i-dimensional space U through X and consider the mapping φ which maps the i-dimensional space U to the unique i-dimensional space U^φ through X which is the projection $\mathrm{proj}_{\supseteq X} U^\theta$ of U^θ onto X. Because θ is i-domestic, it follows that U and U^θ are not opposite and one verifies easily that this implies that U and U^φ are not opposite in $\mathrm{Res}_\Gamma(X)$. This means that the collineation—which we also denote by φ—in the residue of X corresponding with φ is point-domestic. Similarly, φ is also line-domestic. By induction, or just by Lemma 3.1, it follows that φ is the identity. Hence, with U as above, we know that U and U^θ meet in a point x.

We now claim that x is fixed under θ. Indeed, consider an $(i+1)$-dimensional subspace V through U; then V^θ intersects V in a line L through x. Suppose, by way of contradiction, that x is not fixed. Then x^θ is contained in $U^\theta \setminus U$. It is easy to find an i-dimensional subspace U' in V containing x but neither containing L nor $x^{\theta^{-1}}$. Then U' and its image are clearly disjoint. Let y be the intersection of U' with $L^{\theta^{-1}}$. Choose an $(i-1)$-dimensional subspace Y in U' not through x and not through y. Then Y^θ has no point in common with L (and notice that this is also true for Y). If some point z of Y^θ were collinear to all points of Y, then, since it is also collinear with all points of L it would be collinear with all points of V. Since $z \notin L$, this implies that all points of X are collinear to all points of the plane spanned by L and z, and hence to at least one point of X^θ, contradicting the fact that X and X^θ are opposite. This contradiction shows that Y and Y^θ are opposite.

Replacing X by Y in the first paragraph of this proof, we deduce that U' meets its image in a point, a contradiction. This now proves our claim.

Hence we have shown that $X^\perp \cap (X^\theta)^\perp$ is fixed pointwise.

Let M be any maximal subspace of Γ incident with X and consider an arbitrary $(i-1)$-space Z contained in M and not incident with any point of M^θ. The foregoing shows that $M \cap M^\theta$ has dimension $n-i$ in M. Hence Z is complementary in M with respect to that intersection. It follows that no point of Z^θ is collinear with every point of Z, as otherwise that point would be collinear with all points of M, contradicting the fact that it is not contained in M. So we have shown that Z is opposite its image Z^θ. We can now play the same game with Z and obtain that $Z^\perp \cap (Z^\theta)^\perp$ is fixed pointwise. We claim that the set $S := (X^\perp \cap (X^\theta)^\perp) \cup (Z^\perp \cap (Z^\theta)^\perp)$ is connected (meaning that, in the incidence graph, one can walk from any vertex corresponding to a point of this set S to any another such vertex only using subspaces all of whose points belong to S). Indeed, both $X^\perp \cap (X^\theta)^\perp$ and $Z^\perp \cap (Z^\theta)^\perp$ are connected, and their intersection contains $M \cap M^\theta$. The claim follows. This implies the following. We know that the rank of Γ is at least 4, so Γ is embeddable. Let Γ live in the projective space Σ (of possibly infinite dimension). Then θ can be extended to Σ and the subspace of Σ spanned by S is pointwise fixed under this extension of θ.

Let $\mathfrak{X}$ be the set of al $(i-1)$-dimensional subspaces of Γ which can be obtained from X by a finite number of steps, where in each step the next subspace is contained in a common maximal subspace with the previous one, and is mapped onto an opposite subspace under θ. It then follows from the previous paragraph that the projective subspace $\mathfrak{S}$ of Σ generated by $\mathfrak{X}$ is pointwise fixed under θ. Hence the intersection $\mathfrak{G}$ of $\mathfrak{S}$ with Γ is a geometric subspace. Since clearly X is disjoint from $\mathfrak{G}$, the corank of $\mathfrak{G}$ is at most i.

Left to prove is the assertion that every i-space has at least one point in common with $\mathfrak{G}$. Let W be any i-dimensional subspace of Γ. For every $Z \in \mathfrak{X}$, define k_Z to be the dimension of $\mathrm{proj}_{\subseteq W} Z$. Let k be the maximum of all k_Z, with Z running through $\mathfrak{X}$. If $i-k=0$, the assertion is clear since, if $Z \in \mathfrak{X}$, and W and Z are contained in a common subspace M, then $M \cap M^\theta$ has codimension $i-1$. Now suppose that $i-k>0$ and let $Z \in \mathfrak{X}$ be such that $k_Z = k$. Define $\overline{Z} = \mathrm{proj}_{\supseteq Z} W$. Suppose that W does not meet $Z^* := \overline{Z}^\perp \cap (\overline{Z}^\theta)^\perp$. Note that every $(i-1)$-dimensional subspace of $\overline{Z}$ that does not meet Z^* belongs to $\mathfrak{X}$. Indeed, otherwise some point of $\overline{Z}^\theta \setminus Z^*$ would be collinear with all points of Z, and so all points of Z would be collinear to at least one point of Z^θ (using the fact that Z^* and Z^θ are complementary subspaces of $\overline{Z}^\theta$), contradicting the fact that Z and Z^θ are opposite. Now choose $Z' \in \mathfrak{X}$ such that it is contained in $\overline{Z}$, it contains $\mathrm{proj}_{\subseteq W} Z$ and it is disjoint from Z^* (this is easy). Then $Z' \cap W$ is a k-space, and since the dimension of Z' is smaller than the dimension of W, there are points in W outside Z' collinear to all points of Z'. In other words, $k_{Z'} > k$. This contradicts the maximality of k.

Hence W meets Z^* in at least a point, and the assertion follows. $\square$

This now has the following interesting corollaries.

Corollary 6.2 *Suppose that Γ is a polar space of rank $n+1$ with an underlying skewfield which is not commutative. Then a collineation θ is i-domestic for some i, with $0 \leq i < n$, if and only if θ pointwise fixes some geometric subspace of corank at most $n-1$.*

Proof Let θ be i-domestic, with $i < n$. If a k-subspace U, with $k > i$, is mapped onto an opposite k-space, then the composition of the restriction to U of θ with the projection (using the operator $\mathrm{proj}_{\subset U}$) onto U is an i-domestic duality of a k-dimensional projective space, hence a symplectic polarity by Theorem 3.1 in [7], contradicting our assumption on the underlying skew field. Hence θ is in particular $(i+1)$-domestic. The assertion now follows from Theorem 6.1. $\square$

Corollary 6.3 *Suppose that Γ is a polar space of rank $n+1$. Then a collineation θ is i-domestic for some odd i, with $0 \leq i < n$, if and only if θ pointwise fixes some geometric subspace of corank at most $n-1$.*

Proof The proof is totally analogous to the proof of Corollary 6.2, noting that no projective space of even dimension $i+1$ admits a symplectic polarity. $\square$

Despite the fact that we are not able to handle the cases of i-domestic collineations for even $i > 0$, we mention the following reduction, which is the analogue of Theorem 3.2.

Corollary 6.4 *Let θ be an $\{i, i+1\}$-domestic collineation of a polar space Γ of rank $n+1$, with $0 \leq i < n$, with $i < n-1$ if i is even. Then θ is either i-domestic or $(i+1)$-domestic. In particular, if i is odd, then it is $(i+1)$-domestic, and if i is even, then θ is either i-domestic, or $(i+1)$-domestic, but always $(i+2)$-domestic.*

Proof Suppose first that i is odd. Let U be a subspace of dimension $i+1$ and assume that U is mapped onto an opposite subspace. Then our assumption implies that the composition of the restriction to U of θ with the projection (using $\mathrm{proj}_{\supseteq U}$) onto U is an i-domestic duality, hence a symplectic polarity, contradicting $i+1$ even. So θ is $(i+1)$-domestic.

Suppose now that i is even. Then $i < n-1$ and so we can consider $(i+2)$-dimensional subspaces. If such a subspace U were mapped onto an opposite one, then, as in the previous paragraph, we would have a symplectic polarity in U, contradicting $i+2$ is even. Now, if θ is not i-domestic, we can consider an i-space X mapped onto an opposite one. Completely similar as in the proof of Theorem 6.1, one shows that every $(i+1)$-dimensional subspace contains a fixed point, hence cannot be mapped onto an opposite one, and so θ is $(i+1)$-domestic. $\square$

We are still far from a complete understanding of all chamber-domestic collineations, but the above is, in our opinion, a good start.

Examples of i-domestic collineations, which are also $(i+1)$-domestic can be found in embeddable polar spaces using collineations of the surrounding projective space stabilizing the polar space and fixing a projective subspace of the appropriate dimension.

7 Polar spaces of rank 3

In this section we classify all domestic collineations of polar spaces of rank 3. Before we can state the main result, we introduce some additional notions.

Let Γ be a polar space of rank 3, all of whose planes are nondegenerate. If Γ is not thick, then it arises from a 3-dimensional projective space $\mathsf{PG}(3,\mathbb{K})$, with $\mathbb{K}$ a not necessarily commutative field, by taking as points the lines of $\mathsf{PG}(3,\mathbb{K})$, and collinearity corresponds with intersecting lines. This projective space is called the *thick frame* of the polar space Γ, see e.g. [5]. It is clear that every type-preserving collineation of $\mathsf{PG}(3,\mathbb{K})$ induces a domestic collineation of Γ, since opposite planes of Γ have different type.

A *Baer subplane* of a projective plane $\mathfrak{P}$ is a (non-degenerate) sub projective plane $\mathfrak{P}'$ with the property that every line of $\mathfrak{P}$ is incident with at least one point of $\mathfrak{P}'$ and every point of $\mathfrak{P}$ is on some line of $\mathfrak{P}'$.

Let Γ' be a rank 3 sub polar space of Γ with the following properties.

(Pr1) Every plane of Γ' is a Baer subplane of some plane of Γ.

(Pr2) Every plane of Γ containing a line of Γ' belongs to Γ'.

Then we call Γ' an *ideal Baer sub polar space* of Γ. The adjective "ideal" refers to (Pr2); the name "Baer" to (Pr1) of course. An involution that fixes an ideal Baer sub polar space of Γ and nothing more is called an *ideal Baer involution*.

We will now show that, with the above notation, every plane of Γ meets Γ' in at least one point.

Lemma 7.1 *If Γ' is an ideal Baer sub polar space of Γ, then every plane of Γ has a point in common with Γ'.*

Proof Let π be an arbitrary plane of Γ, and let x be an arbitrary point of Γ'. We may assume that $x \notin \pi$. Consider a line L of Γ' through x. Then $\alpha := \mathrm{proj}_{\supseteq L}\pi$ is a plane of Γ through L. Since all planes through L belong to Γ', we see that α belongs to Γ', and it meets π nontrivially (within Γ). Since the points of Γ' in α form a Baer subplane, there is a line M in α belonging to Γ' and intersecting π (intersection point in Γ). Now, since all planes through M belong to Γ', there is at least one such plane meeting π in a line of Γ. But since planes of Γ' are Baer subplanes of planes of Γ, that line contains a point of Γ' and the proof is complete. $\square$

We already defined a geometric hyperplane. Here, we will call a subspace a *geometric subhyperplane* if it is a geometric hyperplane of a geometric hyperplane. It follows that every plane of Γ meets a geometric subhyperplane in at least one point. Hence a geometric subhyperplane is a particular example of a subspace of corank 2.

We will show the following theorem.

Theorem 7.2 *Let Γ be a polar space of rank 3 and let θ be a domestic collineation. Then one of the following holds.*

Case 1 Γ *is not thick and then either θ is an arbitrary type-preserving collineation in the thick frame* $\mathsf{PG}(3,\mathbb{K})$ *of* Γ *(with $\mathbb{K}$ a not necessarily commutative field), or θ induces a symplectic polarity in* $\mathsf{PG}(3,\mathbb{K})$ *(and hence $\mathbb{K}$ is commutative);*

Case 2 Γ *is thick and θ is an ideal Baer involution;*

Case 3 Γ *is thick, at least one plane is fixed, and the fixed points that are incident with pointwise fixed lines form a polar subspace which is either a geometric hyperplane, or a geometric subhyperplane.*

Case 4 Γ *is thick, no plane is fixed, and the fixed points form a subspace of corank* 1 *or* 2*.*

Examples for Case 1 are obvious. An example for Case 2 includes every thick non-embeddable polar space where the involution is induced by a non-standard involution of the octonion division ring. Finally, Cases 3 and 4 also arose in the previous sections.

For the rest of this section, we assume that Γ is a polar space of rank 3 and θ is a domestic collineation of Γ. We start by noting that this is equivalent with requiring that θ is plane-domestic. Indeed, plane-domesticity clearly implies chamber-domesticity. Conversely, if a plane π were mapped onto an opposite plane π^θ, then the duality of π that maps a point x of π onto the line $\pi \cap (x^\theta)^\perp$ must be domestic and hence would be a symplectic polarity, using once again Lemma 3.2 of [7]. But there are no symplecic polarities in a plane, so we conclude that θ is indeed plane-domestic.

Let us first settle the non-thick case. Let $\mathsf{PG}(3,\mathbb{K})$ be the thick frame of Γ. We have to show that no duality of $\mathsf{PG}(3,\mathbb{K})$ gives rise to a domestic collineation θ of Γ, except for a symplectic duality. Since θ is automatically plane-domestic, we see that each point of $\mathsf{PG}(3,\mathbb{K})$ is mapped onto an incident plane. Hence, as a duality of $\mathsf{PG}(3,\mathbb{K})$, θ is domestic, and hence is a symplectic polarity, by Lemma 3.2 of [7]. This gives us Case 1 of Theorem 7.2.

Hence from now on we may assume that Γ is thick. First we prove a lemma that basically shows that every plane contains at least one fixed point (heading for geometric subhyperplanes or ideal Baer subspaces, which both have the property that every plane contains at least one point of that structure).

Lemma 7.3 *If π is a plane of Γ not fixed under θ, then every point of $\pi \cap \pi^\theta$ is fixed.*

Proof Assume that π is not fixed, and that some point x of $\pi \cap \pi^\theta$ is not fixed. Then we can choose a line L in π through x such that L^θ is not incident with x. There is at most one plane through L which meets L^θ (exactly one if $\pi \cap \pi^\theta$ is a point, none if it is a line). By thickness, we can choose a plane α through L not meeting L^θ and distinct from π. By domesticity, α and α^θ share some point z which, by assumption, does not lie in

π^θ. Then z is collinear with x and all points of L^θ, implying that it is collinear with all points of π^θ. This contradicts $z \notin \pi^\theta$.

The lemma is proved. $\square$

We can now deduce Case 4 of Theorem 7.2.

Proposition 7.4 *If θ does not fix any plane, then the set of fixed points forms a subspace of corank at most* 2.

Proof If two collinear points x and y are fixed by θ, then an arbitrary plane π containing x and y is mapped onto a different plane π^θ intersecting π in the line xy. Lemma 7.3 implies that every point of xy is fixed. Hence the set of fixed points is a subspace. Since every plane contains at least one fixed point, by our assumption of domesticity, by our assumption that no plane is fixed, and by Lemma 7.3, we see that the subspace of fixed points has corank at most 2. $\square$

We now look at planes that are fixed under θ.

Lemma 7.5 *Let π be a plane of Γ with $\pi^\theta = \pi$. Then the fixed point structure of θ in π is one of the following.*

(*i*) *Every point of π.*

(*ii*) *A Baer subplane of π.*

(*iii*) *All points of a certain line of π.*

(*iv*) *All points of a certain line of π, and one additional point.*

Proof First we remark that the set of fixed elements in π is a closed configuration $\mathfrak{C}$ in π. If a line L of π is not fixed by θ, then, by considering a plane α through L, with $\alpha \neq \pi$, we see that $L \cap L^\theta$ is fixed. Hence, if $\mathfrak{C}$ is a thick subplane, then it must be either the whole plane, or a Baer subplane. Now assume $\mathfrak{C}$ is not thick. Then an easy exercise shows that either (*iii*) or (*iv*) occurs. $\square$

We now consider each possibility in turn. We start with the case of a Baer subplane.

Proposition 7.6 *If θ fixes some plane in which the fixed point structure is a Baer subplane, then θ is an ideal Baer involution.*

Proof Let π be a fixed plane and assume that the set of fixed points in π forms a Baer subplane of π. Let x be any fixed point, not contained in π. Then the projection of x onto π is a line L, which must be fixed, and hence the plane π' containing x and L is fixed, and since the fixed points on L form a Baer subline, the set of fixed points in π' forms a Baer subplane. Now every plane through every fixed line in π and in π' is fixed and the fixed points form a Baer subplane. It is now easy to see that, continuing with another line in

π', considering a plane through it, and doing this one more time, we obtain a fixed plane opposite π. It now follows that the set of fixed points forms a Baer sub polar space.

Left to prove is that θ^2 is the identity. Let z be any point of Γ not fixed under θ. If z lies in π, then it is fixed under θ^2, as θ induces a Baer involution in each fixed plane. If z is not in π, then it is collinear to some fixed point x of π. The fixed lines and planes through x form an ideal subquadrangle of the residue of x. But it is also clear that the fixed element quadrangle of θ^2 of the residue at x is a full and ideal subquadrangle, and hence coincides with the residue itself. So the line xz is fixed under θ^2. Considering another fixed point x' with which z is collinear (obtained by looking at a fixed plane disjoint from π), we see that z must be fixed under θ^2 (whether or not x and x' are collinear). $\square$

We now treat the case where in some fixed plane a line is fixed pointwise, and an additional point is fixed. Hence, θ induces a non-trivial homology in such a plane.

Proposition 7.7 *If θ induces a non-trivial homology in some fixed plane, then the set of fixed points that are incident with a pointwise fixed line forms a generalized quadrangle in the perp of some (other) fixed point x in such a way that every line through x contains a unique point of that quadrangle.*

Proof Suppose some plane π is fixed under θ such that the fixed points in π form a line L plus an additional point x. Note that this automatically implies that each plane has order > 2. Lemma 7.3 implies that all planes through x intersecting L are fixed. Let α be such a plane. Then α contains two fixed points x and y, with y on L, such that the line xy contains no further fixed points. Hence, in view of Lemma 7.5, there is a line M in α fixed pointwise, and either $x \in M$ or $y \in M$. If $x \in M$, then, on the one hand we have that θ does not induce the identity in $\mathrm{Res}_\Gamma(x)$ since the lines in α through x and distinct from M and from xy are not fixed, but on the other hand, since $\mathrm{Res}_\Gamma(x)$ is a thick generalized quadrangle, and θ fixes all lines intersecting a given line in $\mathrm{Res}_\Gamma(x)$, plus an additional point, Theorem 4.4.2(v) of [11] implies that θ induces the identity in $\mathrm{Res}_\Gamma(x)$. This contradiction shows that M contains y. But then θ fixes all lines though x in α and is the identity in $\mathrm{Res}_\Gamma(x)$ after all.

Hence the situation is that all lines through x are fixed, but none is fixed pointwise. Each line through x contains exactly two fixed points. Let $\mathfrak{F}$ be the set of fixed points collinear with x, but x not included. Then each point of $\mathfrak{F}$ is contained in a pointwise fixed line. Also, $\mathfrak{F}$ is a geometric hyperplane of $x^\perp$, which is a geometric hyperpane of Γ. Hence, if there are no further fixed points in Γ besides x and those in $\mathfrak{F}$, then we are in Case 3 of Theorem 7.2.

Suppose now that there is a fixed point y not in $x^\perp$. Then $x^\perp \cap y^\perp$ is fixed pointwise. Hence $x^\perp \cap y^\perp = \mathfrak{F}$. If y were contained in a pointwise fixed line of the polar space, then for every point $u \notin x^\perp$ on that line, we would have $u^\perp \cap x^\perp = \mathfrak{F}$, a contradiction. Hence no fixed point outside $\mathfrak{F}$ is contained in a pointwise fixed line and the proposition is proved. $\square$

So we again obtain Case 3 of Theorem 7.2.

Proposition 7.8 *If θ induces an elation in some fixed plane, then the set of fixed points is contained in the perp of some point x, for every fixed point y the line xy is fixed pointwise, and the pointwise fixed lines form an ovoid in* $\mathrm{Res}_\Gamma(x)$.

Proof Suppose θ fixes the plane π and induces an elation in π with center x and axis L. Then all planes through x not containing L, but meeting π in a line, are fixed by Lemma 7.3. Since every such plane contains a fixed line with only one fixed point (namely, x), we deduce from Lemma 7.5 that θ induces an elation with center x in every such plane. Hence θ fixes all lines through x which are contained in a plane meeting π in a line distinct from L. It easily follows that θ fixes $\mathrm{Res}_\Gamma(x)$ pointwise, and that in every plane through x, an elation with center x is induced by θ. Since every elation has an axis, the proposition follows, if we just show that there are no further fixed points, i.e., no point opposite x is fixed. But if some point y opposite x were fixed, then $x^\perp \cap y^\perp$ would be fixed, and this contradicts the fact that not all lines through x are fixed pointwise. $\square$

So from now one we may assume that every fixed plane is fixed pointwise, and that there is at least one such plane.

Let π be a pointwise fixed plane. Let α be an arbitrary plane disjoint from π. Then $\alpha \cap \alpha^\theta$ is fixed pointwise, and hence we have a fixed point x not in π. The unique plane π_x through x meeting π in a line, say L, is fixed pointwise. If all fixed points are collinear with all points of L, then all planes through L must be fixed (indeed, for each plane through L, we can find a plane β meeting it in exactly one point not on L, and then this point must be fixed as it is the only point of β collinear with all points of L and hence coincide with the fixed point $\beta \cap \beta^\theta$). Hence we obtain the geometric hyperplane $L^\perp$ in the geometric hyperplane $z^\perp$, with $z \in L$ arbitrary.

Now suppose that some fixed point y is not collinear to all points of L. Then y is collinear with a unique point z of L. Suppose first that all fixed points are collinear with z. Then the set of fixed lines and planes through z forms a full subquadrangle in $\mathrm{Res}_\Gamma(z)$ which is also a geometric hyperplane of $\mathrm{Res}_\Gamma(z)$. It follows easily that the fixed points form a geometric hyperplane of $z^\perp$.

Finally, suppose that not all fixed points are collinear with z. Then we find an apartment all of whose planes are fixed pointwise, and so the fixed point structure is a non-degenerate full polar subspace Γ' of rank 3. Noting that it follows from Proposition 5.9.6 of [11] that the thick non-embeddable polar spaces do not admit full polar subspaces of rank 3, we may assume that Γ is embeddable. It now follows easily that Γ' is obtained from either a hyperplane section, or a section of a subspace of codimension 2 of the ambient projective space. In the second case, considering a hyperplane containing that codimension 2 space, we again obtain Case 3 of Theorem 7.2.

Now Theorem 7.2 is completely proved.

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