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Hjelmslev-Quadrangles of Level *n*

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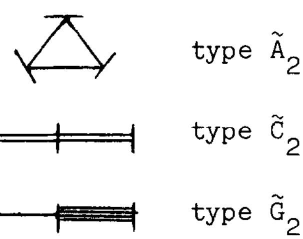
A Hjelmslev-quadrangle of level n is a rank 2 incidence structure having an ordinary generalized quadrangle as homomorphic image and satisfying some uniformity axioms. We show that certain infinite sequences of homomorphic Hjelmslev-quadrangles define an affine building of type $\tilde{C}_2$. This is similar to the fact that sequences of projective Hjelmslev planes give rise to affine buildings of type $\tilde{A}_2$. This analogy is emphasized by the fact that the inverse limit of the above sequences is proven to be an infinite generalized quadrangle.

INTRODUCTION AND MOTIVATION

By a celebrated result of J. Tits [12], all affine buildings of rank $\geqslant 4$ are classified and arise from algebraic groups over local fields. This leaves the question of the rank 3 affine buildings, i.e., the buildings with diagram



Such buildings are called *classical* if they arise from algebraic groups. In [1], it is proven that an infinite sequence of homomorphic level n projective Hjelmslev planes (for definition, see, e.g., [1]) defines an affine

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building of type $\tilde{A}_2$. In this way, non-classical affine buildings are obtained and the class of all buildings of type $\tilde{A}_2$ is characterized by putting a *valuation* on any *planar ternary ring* of the *spherical building at infinity* (see [14]). This spherical building at infinity is in fact a projective plane, the inverse limit of the above sequences. The present paper is the first in a sequence of four (besides the present paper, also [4, 15, 16]) characterizing the class of buildings of type $\tilde{C}_2$ by putting a valuation on any *quadratic quaternary ring* [3] of the spherical building at infinity, which is a generalized quadrangle. In [15], we use the present results to construct explicitly defined affine buildings of type $\tilde{C}_2$ which do not arise from algebraic groups. This construction is as explicit as the $\tilde{A}_2$-analogue hence one can, for example, investigate automorphism groups, etc.... .

The fact that the final result we are aiming at is similar to the characterization theorem of affine buildings of type $\tilde{A}_2$ may give the impression that also the techniques to prove it are similar and hence one might feel that the case $\tilde{C}_2$ is a *rewritten copy* of the case $\tilde{A}_2$. This is however not true. Of course, the *skeleton* of the proof is the same, i.e., we axiomatize the geometry of the vertices at a certain distance from a fixed vertex of the building and put a valuation on the algebraic coordinatizing structure of the building at infinity (viewed as a geometry) of the affine building. But the way to get there is totally different and much more tricky. We give three examples.

In the case $\tilde{A}_2$, the geometry of the vertices at a certain distance from a fixed vertex, is a projective Hjelmslev plane (see [5]). This is a well-known geometry and this makes things easier. In the case $\tilde{C}_2$, no such objects exist in the literature and hence we have to define them ourselves. This is done in the present paper. This is also motivated by the recent work of D. Keppens, who likewise treats geometries (*circle-geometries with neighbour relation, generalized quadrangles with neighbour relation*) in his thesis (unpublished) and shows some connections between them. Moreover, the definition of a Hjelmslev-quadrangle of level n will be by induction and this will cause difficulties in a lot of arguments (see [15]).

For the case $\tilde{A}_2$, one needs a coordinatization theory for projective planes. In the case $\tilde{C}_2$, one needs a coordinatization theory for generalized quadrangles, which was not available. We had to fill that gap. Needless to say that working with coordinates in generalized quadrangles is much more difficult and messy than in a projective plane.

The notion of a valuation on the coordinatizing ring of a generalized quadrangle cannot be guessed by generalizing the notion of valuations on fields, as it is the case for planar ternary rings with valuation. One can see in [15, 16] that a different approach is needed.

The paper is organized as follows. We give an axiomatic definition of *Hjelmslev-quadrangle of level n* in section 1. In Section 2, we show a series of properties culminating in the fact that the inverse limit of an infinite sequence of level n Hjelmslev-quadrangles is a customary infinite generalized quadrangle. In Section 3, we use such sequences to construct buildings of type $\tilde{C}_2$ in an explicit way. No examples are given, since we first need the notion of a *quadratic quaternary ring with valuation* in order to obtain non-classical examples. (See [16]). A classical example, however, is contained in the sequel of this paper when we show that every building of type $\tilde{C}_2$ arises in the way described in Section 3 (see [4]).

Prerequisites. Sections 1 and 2 are self-contained up to the notion of a point–line incidence geometry (see, e.g., [2]). For Sections 3 and 4, the reader should know about the diagram and residues of a rank 3 geometry (see [2] again) and the notion of a simplicial complex in connection with buildings (see, e.g., [10]). *Chamber systems* are needed only in a small argument and, therefore, we do not define them, but refer to [11, 8].

1. Definition of a Hjelmslev-Quadrangle of Level n

1.1. *Notation*

Suppose $X = (\mathscr{P}(X), \mathscr{L}(X), I)$ is a point–line incidence geometry with point set $\mathscr{P}(X)$ and line set $\mathscr{L}(X)$. We denote the set of points incident with a given line $\mathscr{L}$ by $\sigma(\mathscr{L})$ and call it the *shadow* (*of* $\mathscr{L}$) (see Buekenhout [2]). Suppose $\mathscr{L}_1$ and $\mathscr{L}_2$ are two distinct lines of X. If there is a point incident with both $\mathscr{L}_1$ and $\mathscr{L}_2$, then we call $\mathscr{L}_1$ and $\mathscr{L}_2$ *concurrent* and we denote "$\mathscr{L}_1 \perp \mathscr{L}_2$." Suppose $\mathscr{P}_1$ and $\mathscr{P}_2$ are two distinct points of X. If there is a line incident with both $\mathscr{P}_1$ and $\mathscr{P}_2$, then we call $\mathscr{P}_1$ and $\mathscr{P}_2$ *collinear* and we denote "$\mathscr{P}_1 \perp \mathscr{P}_2$." A *flag* in X is an incident point–line pair of X. The set of flags of X is denoted by $\mathscr{F}(X)$. A *morphism* from X to some other point–line incidence geometry $X' = (\mathscr{P}(X'), \mathscr{L}(X'), I)$ maps $\mathscr{P}(X)$ to $\mathscr{P}(X')$, $\mathscr{L}(X)$ to $\mathscr{L}(X')$ and the map induced on $\mathscr{F}(X)$ maps $\mathscr{F}(X)$ to $\mathscr{F}(X')$. An *epimorphism* is a morphism which is surjective on the set of flags. The geometry X is called *thick* if every line is incident with at least three points and every point is incident with at least three lines.

Suppose $\mathscr{A}$ is an arbitrary set and $\mathbb{P}_1(\mathscr{A})$ and $\mathbb{P}_2(\mathscr{A})$ are two arbitrary partitions of $\mathscr{A}$. Then we say that $\mathbb{P}_1(\mathscr{A})$ is *properly finer than* $\mathbb{P}_2(\mathscr{A})$ if every class of $\mathbb{P}_2(\mathscr{A})$ is the union of at least two classes of $\mathbb{P}_1(\mathscr{A})$. In that case, we denote

$$\mathbb{P}_2(\mathscr{A})/\mathbb{P}_1(\mathscr{A}) = \{\{\mathscr{C} \in \mathbb{P}_1(\mathscr{A}) \mid \mathscr{C} \subseteq \mathscr{D}\} \mid \mathscr{D} \in \mathbb{P}_2(\mathscr{A})\},$$

which is a partition of $\mathbb{P}_1(\mathscr{A})$. If $\mathscr{D} \in \mathbb{P}_2(\mathscr{A})$, then we call $\{\mathscr{C} \in \mathbb{P}_1(\mathscr{A}) \mid \mathscr{C} \subseteq \mathscr{D}\}$ the *canonical image of* $\mathscr{D}$ *in* $\mathbb{P}_2(\mathscr{A})/\mathbb{P}_1(\mathscr{A})$.

1.2. *Generalized Quadrangles*

Let $\mathcal{S} = (\mathcal{P}(\mathcal{S}), \mathcal{L}(\mathcal{S}), I)$ be point–line incidence geometry. Then we call $\mathcal{S}$ a *generalized quadrangle* if there exist positive integers $s \geqslant 1$ and $t \geqslant 1$ (s and/or t may also be infinite) such that the following axioms hold:

(Q1) Every point of $\mathcal{S}$ is incident with $1+t$ lines and two distinct points are incident with at most one line.

(Q2) Every line of $\mathcal{S}$ is incident with $1+s$ points and two distinct lines are incident with at most one point.

(Q3) If $\mathcal{P} \in \mathcal{P}(\mathcal{S})$ and $\mathcal{L} \in \mathcal{L}(\mathcal{S})$ are not incident, then there exists a unique flag $(\mathcal{Q}, \mathcal{M}) \in \mathcal{F}(\mathcal{S})$ such that $\mathcal{P}\, I\, \mathcal{M}\, I\, \mathcal{Q}\, I\, \mathcal{L}$.

Generalized quadrangles were introduced by J. Tits in his celebrated paper [9]. More information about generalized quadrangles can be found in, e.g., Payne and Thas [7] or in the survey paper of W. M. Kantor [6]. In this paper, we will always assume that every generalized quadrangle is thick. One can check that an axiom system for thick generalized quadrangles can be given as follows.

(QQ1) Every point is incident with at least two lines and there exists a point incident with at least three lines.

(QQ2) Every line is incident with at least two points and there exists a line incident with at least three points.

(QQ3) There exists a non-incident point–line pair.

(QQ4) If $\mathcal{P} \in \mathcal{P}(\mathcal{S})$ and $\mathcal{L} \in \mathcal{L}(\mathcal{S})$ are not incident, then there exists a unique flag $(\mathcal{Q}, \mathcal{M}) \in \mathcal{F}(\mathcal{S})$ such that $\mathcal{P}\, I\, \mathcal{M}\, I\, \mathcal{Q}\, I\, \mathcal{L}$.

The reason why we introduce this axiom system is because of the fact that (QQ1) up to (QQ4) is easier to check than (Q1) up to (Q3) (see Theorem (4.1)).

1.3. *Definition of a Hjelmslev-quadrangle of Level n*

Throughout, n, i, j, and k denote positive integers. We define a Hjelmslev-quadrangle of level n by induction on n. The induction will start with $n=1$. We give a separate definition for level 0. We abbreviate "Hjelmslev-quadrangle of level n" by "level n H-Q."

A *level* 0 H-Q is any trivial geometry $\mathcal{V}_0 = (\{*\}, \{*\}, =)$, where $*$ is any arbitrary (but twice the same) symbol.

A *level* 1 H-Q is any 6-tuple,

$$\mathcal{V}_1 = (\mathcal{P}(\mathcal{V}_1), \mathcal{L}(\mathcal{V}_1), I, (\mathbb{P}_i(\mathcal{V}_1))_{i \leqslant 1}, (\mathbb{L}_i(\mathcal{V}_1))_{i \leqslant 1},$$

$$(\mathcal{W}_0(\mathcal{V}_1, \{\mathcal{P}\}), \{\mathcal{P}\})_{\mathcal{P} \in \mathcal{P}(\mathcal{V}_1)}),$$

where $(\mathscr{P}(\mathscr{V}_1), \mathscr{L}(\mathscr{V}_1), I)$ is an arbitrary generalized quadrangle; $\mathbb{P}_0(\mathscr{V}_1)$ is the partition of $\mathscr{V}_1$ determined by: every class is a singleton; $\mathbb{P}_1(\mathscr{V}_1)$ is the partition of $\mathscr{P}(\mathscr{V}_1)$ consisting of one class; similar for $(\mathbb{L}_i(\mathscr{V}_n))_{i \leq 1}$, and for every $\mathscr{P} \in \mathscr{P}(\mathscr{V}_1)$, $\mathscr{W}_0(\mathscr{V}_1, \{\mathscr{P}\}) = (\{\mathscr{P}\}, \{\mathscr{P}\}, =)$. The last three elements of $\mathscr{V}_1$ do not add more structure to the generalized quadrangle, but they are necessary to start the induction. So, in fact, a level 1 H-Q "is" a generalized quadrangle.

Now suppose $n \geq 2$. Let $(\mathscr{P}(\mathscr{V}_n), \mathscr{L}(\mathscr{V}_n), I)$ be a thick incidence geometry. Suppose $(\mathbb{P}_i(\mathscr{V}_n))_{i \leq n}$ (resp. $(\mathbb{L}_i(\mathscr{V}_n))_{i \leq n}$) is a family of partitions of $\mathscr{P}(\mathscr{V}_n)$ (resp. $\mathscr{L}(\mathscr{V}_n)$) satisfying:

(PS1) $\mathbb{P}_0(\mathscr{V}_n) = \{\{\mathscr{P}\} \mid \mathscr{P} \in \mathscr{P}(\mathscr{V}_n)\}$; $\mathbb{P}_n(\mathscr{V}_n) = \{\mathscr{P}(\mathscr{V}_n)\}$,

(PS2) $\mathbb{L}_0(\mathscr{V}_n) = \{\{\mathscr{L}\} \mid \mathscr{L} \in \mathscr{L}(\mathscr{V}_n)\}$; $\mathbb{L}_n(\mathscr{V}_n) = \{\mathscr{L}(\mathscr{V}_n)\}$,

(PS3) $\mathbb{P}_i(\mathscr{V}_n)$ is properly finer than $\mathbb{P}_{i+1}(\mathscr{V}_n)$, for all $i < n$,

(PS4) $\mathbb{L}_i(\mathscr{V}_n)$ is properly finer than $\mathbb{L}_{i+1}(\mathscr{V}_n)$, for all $i < n$.

The elements of $\mathbb{P}_i(\mathscr{V}_n)$ (resp. $\mathbb{L}_i(\mathscr{V}_n)$) are called *i-point-neighbourhoods* (resp. *i-line-neighbourhoods*) (of their elements). An i-point-neighbourhood is also called a *point-neighbourhood*, an *i-neighbourhood*, or briefly a *neighbourhood*. Definitions for i-line-neighbourhoods are similar. If $\mathscr{P} \in \mathscr{P}(\mathscr{V}_n)$ and $\mathscr{L} \in \mathscr{L}(\mathscr{V}_n)$, then we denote by $\mathcal{O}^i(\mathscr{P})$ (resp. $\mathcal{O}^i(\mathscr{L})$) the unique i-point-neighbourhood of $\mathscr{P}$ (resp. i-line-neighbourhood of $\mathscr{L}$).

Suppose for every $\mathscr{C} \in \mathbb{P}_{n-1}(\mathscr{V}_n)$, we have a level $(n-1)$ H-Q, denoted by $\mathscr{W}_{n-1}(\mathscr{V}_n, \mathscr{C})$ (this is an element of a well-defined class of objects by induction) and select in every $\mathscr{W}_{n-1}(\mathscr{V}_n, \mathscr{C})$ an $(n-2)$-point-neighbourhood $\mathscr{N}_{\mathscr{C}}$. Then we call the 6-tuple

$$\mathscr{V}_n = (\mathscr{P}(\mathscr{V}_n), \mathscr{L}(\mathscr{V}_n), I, (\mathbb{P}_i(\mathscr{V}_n))_{i \leq n}, (\mathbb{L}_i(\mathscr{V}_n))_{i \leq n},$$
$$(\mathscr{W}_{n-1}(\mathscr{V}_n, \mathscr{C}), \mathscr{N}_C)_{\mathscr{C} \in \mathbb{P}_{n-1}(\mathscr{V}_n)})$$

a *level n* H-Q if $\mathscr{V}_n$ satisfies the axioms (IS), (GQ), and (NP) below. The geometry $(\mathscr{P}(\mathscr{V}_n), \mathscr{L}(\mathscr{V}_n), I)$ is called the *base geometry* of $\mathscr{V}_n$. Before stating the actual axioms, we need some preliminaries.

We first define the canonical $(n\text{-}1)$-image of $\mathscr{V}_n$ by induction on n. The *canonical* 0-*image* of a level 1 H-Q $\mathscr{V}_1$ is by definition the trivial geometry $(\{\mathscr{P}(\mathscr{V}_1)\}, \{\mathscr{P}(\mathscr{V}_1)\}, =)$. Now let $n \geq 2$. Define the geometry $(\mathbb{P}_1(\mathscr{V}_n)$, $(\mathbb{L}_1(\mathscr{V}_n), I)$ as follows. If $\mathscr{C} \in \mathbb{P}_1(\mathscr{V}_n)$ and $\mathscr{D} \in (\mathbb{L}_1(\mathscr{V}_n)$, then $\mathscr{C}\, I\, \mathscr{D}$ if and only if there exist $\mathscr{P} \in \mathscr{C}$ and $\mathscr{L} \in \mathscr{D}$ which are incident in $(\mathscr{P}(\mathscr{V}_n)\, \mathscr{L}(\mathscr{V}_n), I)$. Furthermore, denote by $\mathscr{W}_{n-2}(\mathscr{V}_n, \mathscr{C})$ the canonical $(n-2)$-image of $\mathscr{W}_{n-1}(\mathscr{V}_n, \mathscr{C})$ (well defined by the induction hypothesis). Denote by $\mathscr{N}'_{\mathscr{C}}$ the canonical image of $\mathscr{N}_{\mathscr{C}}$ in $\mathbb{P}_{n-2}(\mathscr{W}_{n-1}(\mathscr{V}_n, \mathscr{C}))/\mathbb{P}_1(\mathscr{W}_{n-1}(\mathscr{V}_n, \mathscr{C}))$ if $n > 2$ and $\mathscr{N}'_{\mathscr{C}} = \{\mathscr{P}(\mathscr{W}_1(\mathscr{V}_2, \mathscr{C}))\}$ if $n = 2$. Obviously, there is a bijective correspondence between $\mathbb{P}_{n-1}(\mathscr{V}_n)$ and $\mathbb{P}_{n-1}(\mathscr{V}_n)/\mathbb{P}_1(\mathscr{V}_n)$ and the unique element of $\mathbb{P}_{n-1}(\mathscr{V}_n)/\mathbb{P}_1(\mathscr{V}_n)$ corresponding with the element $\mathscr{C}$ of $\mathbb{P}_{n-1}(\mathscr{V}_n)$ is denoted by $\mathscr{C}^*$. In particular, all elements of $\mathbb{P}_{n-1}(\mathscr{V}_n)/\mathbb{P}_1(\mathscr{V}_n)$ are denoted with a *. We define the *canonical* $(n-1)$*-image of* $\mathscr{V}_n$ as the 6-tuple

$$\mathscr{V}_{n-1} = (\mathbb{P}_1(\mathscr{V}_n), (\mathbb{L}_1(\mathscr{V}_n), I, (\mathbb{P}_{i+1}(\mathscr{V}_n)/\mathbb{P}_1(\mathscr{V}_n))_{i \leqslant n-1}, \\ (\mathbb{L}_{i+1}(\mathscr{V}_n)/\mathbb{L}_1(\mathscr{V}_n))_{i \leqslant n-1}, (\mathscr{W}_{n-2}(\mathscr{V}_n, \mathscr{C}), \mathscr{N}'_{\mathscr{C}})_{\mathscr{C}^* \in \mathbb{P}_{n-1}(\mathscr{V}_n)/\mathbb{P}_1(\mathscr{V}_n)}).$$

We can now state the very natural axiom (IS):

(IS) *The canonical* $(n-1)$-image $\mathscr{V}_{n-1}$ *of* $\mathscr{V}_n$ *is a level* $n-1$ *H-Q.*

Using a similar notation for $\mathscr{V}_{n-1}$ as for $\mathscr{V}_n$, (IS) implies, e.g., $\mathbb{P}_i(\mathscr{V}_{n-1}) = \mathbb{P}_{i+1}(\mathscr{V}_n)/\mathbb{P}_1(\mathscr{V}_n)$ and similarly for the line partitions.

Define inductively the *canonical* $(n-j)$*-image* of $\mathscr{V}_n$ $(0 < j \leqslant n)$ as the canonical $(n-j)$-image $\mathscr{V}_{n-j}$ of the canonical $(n-j+1)$-image $\mathscr{V}_{n-j+1}$ of $\mathscr{V}_n$, or as $\mathscr{V}_n$ (for $j=0$). Note that $\mathcal{O}^1$ defines a mapping from $\mathscr{P}(\mathscr{V}_n)$ to $\mathscr{P}(\mathscr{V}_{n-1})$ and from $\mathscr{L}(\mathscr{V}_n)$ to $\mathscr{L}(\mathscr{V}_{n-1})$. By the definition of the incidence relation in $\mathscr{V}_{n-1}$, we can see that this mapping is an epimorphism from $(\mathscr{P}(\mathscr{V}_n), \mathscr{L}(\mathscr{V}_n), I)$ onto $(\mathbb{P}_1(\mathscr{V}_n), (\mathbb{L}_1(\mathscr{V}_n), I)$. We denote this epimorphism by Π^n_{n-1}. By the induction hypothesis, a similar epimorphism exists from the base geometry of $\mathscr{V}_{n-j+1}$ onto the base geometry of $\mathscr{V}_{n-j}$ and we denote it by Π^{n-j+1}_{n-j}. By induction, we can put

$$\Pi^n_{n-j} = \Pi^{n-j+1}_{n-j} \circ \Pi^n_{n-j+1}.$$

From now on, we denote the canonical j-image $\mathscr{V}_j$ of $\mathscr{V}_n$ by

$$(\mathscr{P}_i(\mathscr{V}_j), \mathscr{L}_i(\mathscr{V}_j), I, (\mathbb{P}_i(\mathscr{V}_j))_{i<j}, (\mathbb{L}_i(\mathscr{V}_j))_{i<j}, (\mathscr{W}_{j-1}(\mathscr{V}_j, \mathscr{C}), \mathscr{N}_{\mathscr{C}})_{\mathscr{C} \in \mathbb{P}_{j-1}(\mathscr{V}_j)}),$$

for all j, $0 < j \leqslant n$. The epimorphism Π^n_j is called the *projection.* We define the *valuation map* $u: (\mathscr{P}(\mathscr{V}_n) \cup \mathscr{L}(\mathscr{V}_n)) \times (\mathscr{P}(\mathscr{V}_n) \cup \mathscr{L}(\mathscr{V}_n)) \to \mathbb{N}$ as follows. Let x, y be either both points or both lines of $\mathscr{V}_n$, then

$$u(x, y) = \sup\{j \leqslant n \mid \Pi^n_j(x) = \Pi^n_j(y)\}.$$

If $\mathscr{P} \in \mathscr{P}(\mathscr{V}_n)$ and $\mathscr{L} \in \mathscr{L}(\mathscr{V}_n)$, then

$$u(\mathscr{P}, \mathscr{L}) = u(\mathscr{L}, \mathscr{P}) = (u_1(\mathscr{P}, \mathscr{L}), u_2(\mathscr{P}, \mathscr{L}))$$

with

$$\begin{aligned} u_1(\mathscr{P}, \mathscr{L}) &= u_1(\mathscr{L}, \mathscr{P}) \\ &= \sup\{j \leqslant n \mid \exists Q I \mathscr{L} \text{ such that } \Pi^n_j(Q) = \Pi^n_j(\mathscr{P}), Q \in \mathscr{P}(\mathscr{V}_n)\} \end{aligned}$$

and

$$u_2(\mathscr{P}, \mathscr{L}) = u_2(\mathscr{L}, \mathscr{P})$$
$$= \sup\{j \leqslant n \mid \exists \mathscr{M} I \mathscr{P} \text{ such that } \Pi_j^n(\mathscr{M}) = \Pi_j^n(\mathscr{L}), \mathscr{M} \in \mathscr{L}(\mathscr{V}_n)\}.$$

We now write down the axiom (GQ), consisting of two statements (GQ1) and (GQ2):

(GQ1) If $\mathscr{P}, Q \in \mathscr{P}(\mathscr{V}_n)$, $\mathscr{L}, \mathscr{M} \in \mathscr{L}(\mathscr{V}_n)$, $Q\, I\, \mathscr{L}\, I\, \mathscr{P}\, I\, \mathscr{M}$, $u(\mathscr{P}, Q) = 0$, and $\mathscr{L} \neq \mathscr{M}$, then

$$\mathcal{O}^{n-j}(Q) \cap \sigma(\mathscr{M}) \neq \varnothing \Leftrightarrow 2 \cdot j \leqslant u(\mathscr{L}, \mathscr{M}).$$

(GQ2) If $\mathscr{P} \in \mathscr{P}(\mathscr{V}_n)$, $\mathscr{L} \in \mathscr{L}(\mathscr{V}_n)$, and $u(\mathscr{P}, \mathscr{L}) = (k, 2k)$ for some $k \leqslant n/2$, then there exists a unique $\mathscr{M} \in \mathscr{L}(\mathscr{V}_n)$ such that $\mathscr{P}\, I\, \mathscr{M} \perp \mathscr{L}$. Moreover, $u(\mathscr{L}, \mathscr{M}) = 2k$ and $u(\mathscr{P}, Q) = 0$, for all $Q \in \sigma(\mathscr{L}) \cap \sigma(\mathscr{M})$. If $k = 0$, then $u(Q_1, Q_2) \geqslant n/2$, for all $Q_1, Q_2 \in \sigma(\mathscr{L}) \cap \sigma(\mathscr{M})$.

We now define an affine structure on level j H-Qs. Suppose X_j is a level j H-Q, $0 < j < n$, with

$$X_j = (\mathscr{P}_i(X_j), \mathscr{L}_i(X_j), I, (\mathbb{P}_i(X_j))_{i<j}, (\mathbb{L}_i(X_j))_{i<j},$$
$$\times (\mathscr{W}_{j-1}(X_j, \mathscr{C}), \mathscr{N}_{\mathscr{C}})_{\mathscr{C} \in \mathbb{P}_{j-1}(X_j)}).$$

Let $X_1 = (\mathscr{P}(X_1), ...,)$ be its canonical 1-image. Let $\mathscr{D} \in \mathbb{P}_{j-1}(X_j)$ be arbitrary. We denote:

* $\mathscr{L}_{\mathscr{D}}^{\infty} = \{\mathscr{L} \in \mathscr{L}(X_j) \mid \sigma(\mathscr{L}) \cap \mathscr{D} \neq \varnothing\}$,
* $\mathscr{P}_{\mathscr{D}}^{\infty} = \{\mathscr{P} \in \mathscr{P}(X_j) \mid \exists \mathscr{L} \in \mathscr{L}_{\mathscr{D}}^{\infty} \text{ such that } \mathscr{P}\, I\, \mathscr{L}\}$,
* $\mathscr{A}\mathscr{P}(X_j, \mathscr{D}) = \mathscr{P}(X_j) - \mathscr{P}_{\mathscr{D}}^{\infty}$,
* $\mathscr{A}\mathscr{L}(X_j, \mathscr{D}) = \mathscr{L}(X_j) - \mathscr{L}_{\mathscr{D}}^{\infty}$.

We call the elements of $\mathscr{A}\mathscr{P}(X_j, \mathscr{D})$ the *affine points* (of $(X_j, \mathscr{D})$, if there is confusion possible) and the elements of $\mathscr{A}\mathscr{L}(X_j, \mathscr{D})$ the *affine lines* (of $(X_j, \mathscr{D})$). The elements of $\mathscr{P}_{\mathscr{D}}^{\infty} - \mathscr{D}$ (resp. of $\mathscr{L}_{\mathscr{D}}^{\infty}$) are called the *points* (resp. the *lines at infinity*) (of $(X_j, \mathscr{D})$). The elements of $\mathscr{D}$ are the *hyperpoints* (of $(X_j, \mathscr{D})$). The pair $(X_j, \mathscr{D})$ is called an *affine* $H-Q$ (of *level* n). Before going on, we prove a small lemma.

LEMMA (1.3). *Let $(X_j, \mathscr{D})$ be as above. Every element of the $(j-1)$-point-neighbourhood of any affine point is again an affine point. Hence every element of the $(j-1)$-point-neighbourhood of any point at infinity (resp. hyperpoint) is again a point at infinity (resp. hyperpoint).*

Proof. Clearly, every element of the $(j-1)$-point-neighbourhood of any hyperpoint is again a hyperpoint. Now suppose $\mathscr{P}_1, \mathscr{P}_2 \in \mathscr{P}(X_j)$, with $\mathscr{P}_1$ an affine point and $\mathscr{P}_2$ a point at infinity of $(X_j, \mathscr{D})$. Suppose $\mathcal{O}^{j-1}(\mathscr{P}_1) = \mathcal{O}^{j-1}(\mathscr{P}_2)$. Let, with the above standard notation, $\Pi_1^j(\mathscr{P}_i) = \mathscr{P} \in \mathscr{P}(X_1)$, $i=1, 2$ (one can indeed see easily that all points in the same k-point-neighbourhood have the same image under the action of Π_k^j, $0 \leqslant k \leqslant j$, because this is true for $k=j-1$; now use an inductive argument). Define $Q \in \mathscr{P}(X_1)$ as $\{Q\} = \Pi_1^j(\mathscr{D})$. Then $\mathscr{P}$ and Q are collinear in X_1, since $\mathscr{P}_2$ is collinear with a point of $\mathscr{D}$ in X_j. Since X_1 is a (thick) generalized quadrangle, there exists a line $\mathscr{L}$ in X_1 incident with Q and not incident with $\mathscr{P}$. Since Π_1^j is an epimorphism, there exists a line $\mathscr{M} \in \mathscr{L}(X_j)$ such that $\Pi_1^j(\mathscr{M}) = \mathscr{L}$. Note that $u(\mathscr{P}_1, \mathscr{M}) = (0, 0)$, otherwise $\Pi_1^j(\mathscr{P}_1)\ I\ \Pi_1^j(\mathscr{M})$, contradicting the condition on $\mathscr{M}$. So, by (GQ2), there exists a point $Q' \perp \mathscr{P}_1$ and with $Q'\ I\ \mathscr{M}$. Projecting down onto X_1, we see that, since X_1 contains no triangles, $\Pi_1^j(Q') = Q$ and hence $Q' \in \mathscr{D}$, contradicting $\mathscr{P}_1$ being affine. Q.E.D.

This lemma will give sense to axiom (NP) below.

We now introduce the notion of a "strip of width i" in an affine H-Q $(X_j, \mathscr{D})$. Suppose $\mathscr{P} \in \mathscr{P}(X_j)$ is a point at infinity of $(X_j, \mathscr{D})$ and $\mathscr{L} \in \mathscr{L}(X_j)$ is an affine line incident with $\mathscr{P}$. If $i<j$, then we call the set

$$\{Q \in \mathscr{A}\mathscr{P}(X_j, \mathscr{D}) \mid Q\ I\ \mathscr{M}\ I\ \mathscr{P} \text{ for some } \mathscr{M} \in \mathcal{O}^i(\mathscr{L})\}$$

a *strip of width* i (in $(X_j, \mathscr{D})$). If $i \geqslant j$, then the set

$$\{Q \in \mathscr{A}\mathscr{P}(X_j, \mathscr{D}) \mid Q \perp \mathscr{P}\}$$

is called a *strip of width* i (in $(X_j, \mathscr{D})$). In any case, we call $\mathscr{P}$ a *base point* (*of the strip*). It is not necessarily unique, even if the strip has width >0 (cp. Property (2.26) below).

We can now state the first part of (NP):

(NP1) If $\mathscr{C} \in \mathbb{P}_{n-1}(\mathscr{V}_n)$, then $\mathscr{A}\mathscr{P}(\mathscr{W}_{n-1}(\mathscr{V}_n, \mathscr{C}), \mathscr{N}_{\mathscr{C}}) = \mathscr{C}$. Moreover, the i-point-neighbourhood of any point $\mathscr{P} \in \mathscr{C}$ in $\mathscr{W}_{n-1}(\mathscr{V}_n, \mathscr{C})$ coincides with the i-point-neighbourhood of $\mathscr{P}$ in $\mathscr{V}_n$, for all $i \leqslant n-2$.

Suppose $\mathscr{C}_{n-j} \in \mathbb{P}_{n-j}(\mathscr{V}_n)$ and let $\mathscr{C}_{n-k}$ be the unique element of $\mathbb{P}_{n-k}(\mathscr{V}_n)$ containing $\mathscr{C}_{n-j}$ as a subset, $0 \leqslant k \leqslant j < n$. By (NP1),

$$\mathscr{C}_{n-2} \in \mathbb{P}_{n-2}(\mathscr{W}_{n-1}(\mathscr{V}_n, \mathscr{C}_{n-1}))$$

$$\mathscr{C}_{n-3} \in \mathbb{P}_{n-3}(\mathscr{W}_{n-2}(\mathscr{W}_{n-1}(\mathscr{V}_n, \mathscr{C}_{n-1}), \mathscr{C}_{n-2}))$$

etc. ... .

This way, we justify the following notation:

$$\mathscr{W}_{n-j}(\mathscr{V}_n, \mathscr{C}_{n-j}) = \mathscr{W}_{n-j}(\mathscr{W}_{n-j+1}(\cdots(\mathscr{W}_{n-1}(\mathscr{V}_n, \mathscr{C}_{n-1}), \ldots), \mathscr{C}_{n-j+1}), \mathscr{C}_{n-j}).$$

Moreover, $\mathscr{C}_{n-j} = \mathscr{A}\mathscr{P}(\mathscr{W}_{n-j}(\mathscr{V}_n, \mathscr{C}_{n-j}), \mathscr{N}_{\mathscr{C}_{n-j}})$.

The axiom (NP1) was about points of the point-neigbourhoods. The last axiom, (NP2), says something about the lines in the affine H-Qs corresponding to these neighbourhoods:

(NP2) If $\mathscr{P} \in \mathscr{P}(\mathscr{V}_n)$, $\mathscr{L} \in \mathscr{L}(\mathscr{V}_n)$, $0 < j < n$, and $\sigma(\mathscr{L}) \cap \mathscr{O}^{n-j}(\mathscr{P}) \neq \varnothing$, then the set

$$\mathbb{S}_j^n(\mathscr{P}, \mathscr{L}) = \sigma(\mathscr{L}) \cap \mathscr{O}^{n-j}(\mathscr{P})$$

is a strip of width j in $(\mathscr{W}_{n-j}(\mathscr{V}_n, \mathscr{O}^{n-j}(\mathscr{P})), \mathscr{N}_{\mathscr{O}^{n-j}(\mathscr{P})})$. Every strip of width 1 in $(\mathscr{W}_{n-1}(\mathscr{V}_n, \mathscr{O}^{n-1}(\mathscr{P})), \mathscr{N}_{\mathscr{O}^{n-1}(\mathscr{P})})$ can be obtained in this way (putting $j = 1$).

This completes our list of axioms for a level n H-Q.

We keep the same notation as above. Suppose $\mathscr{M}$ is an affine line of $(\mathscr{W}_{n-j}(\mathscr{V}_n, \mathscr{O}^{n-j}(\mathscr{P})), \mathscr{N}_{\mathscr{O}^{n-j}(\mathscr{P})})$ such that the set of affine points of $\mathscr{M}$ is a subset of $\sigma(\mathscr{L}) \cap \mathscr{O}^{n-j}(\mathscr{P})$ (with the notation of (NP2) above), then we call $\mathscr{M}$ a *component of* $\mathscr{L}$, or a *component of the strip* $\mathbb{S}_j^n(\mathscr{P}, \mathscr{L})$ and we denote $\mathscr{M} < \mathscr{L}$. The set of all components of $\mathbb{S}_j^n(\mathscr{P}, \mathscr{L})$ is denoted by $\mathbb{C}_j^n(\mathscr{P}, \mathscr{L})$. The set of affine points of $\mathscr{M}$ is called the *affine shadow of* $\mathscr{M}$. As an extension, we call every point of $\mathscr{V}_n$ incident with $\mathscr{L}$ a *component of* $\mathscr{L}$.

Now let $\mathscr{V}'_n = (\mathscr{P}(\mathscr{V}'_n), \mathscr{L}(\mathscr{V}'_n), \ldots)$ be a second level n H-Q and suppose

$$\Psi: (\mathscr{P}(\mathscr{V}_n), \mathscr{L}(\mathscr{V}_n), I) \to (\mathscr{P}(\mathscr{V}'_n), \mathscr{L}(\mathscr{V}'_n), I)$$

is an isomorphism of incidence geometries mapping the affine shadow of every component of any line $\mathscr{L}$ onto the affine shadow of a component of $\Psi(\mathscr{L})$ and mapping i-neighbourhoods onto i-neighbourhoods, for all i, $0 < i \leqslant n$, then we call $\mathscr{V}_n$ and $\mathscr{V}'_n$ *equivalent*. This way, we can extend Ψ to the set of all components of all lines of $\mathscr{V}_n$ and this extended map, which we also denote by Ψ, preserves "being component of." We call Ψ an *equivalence*.

We now define by induction the notion of an isomorphism between $\mathscr{V}_n$ and $\mathscr{V}'_n = (\mathscr{P}(\mathscr{V}'_n), \mathscr{L}(\mathscr{V}'_n), \ldots)$. If $n = 1$, then $\mathscr{V}_1$ and $\mathscr{V}'_1$ are called *isomorphic* if their base geometries are isomorphic generalized quadrangles. Now let $n \geqslant 2$; then we call $\mathscr{V}_n$ and $\mathscr{V}'_n$ *isomorphic* if they are equivalent (denote in that case the corresponding equivalence by Ψ) and if for all $\mathscr{C} \in \mathbb{P}_{n-1}(\mathscr{V}_n)$, $\mathscr{W}_{n-1}(\mathscr{V}_n, \mathscr{C})$ is isomorphic with $\mathscr{W}_{n-1}(\mathscr{V}'_n, \Psi(\mathscr{C}))$ and this isomorphism $\Psi_{\mathscr{C}}$ coincides with $\Psi/\mathscr{C}$ over $\mathscr{C}$. We can now extend Ψ with every $\Psi_{\mathscr{C}}$ and if we denote this extension still by Ψ, then we call Ψ an *isomorphism.* Obviously, isomorphic level n H-Qs are also equivalent.

Recall that Π^n_{n-1} is the projection mapping the base geometry of $\mathscr{V}_n$ onto the base geometry of the canonical $(n-1)$-image $\mathscr{V}_{n-1} = (\mathscr{P}(\mathscr{V}_{n-1}), ...)$. We can extend Π^n_{n-1} to all $\mathscr{W}_{n-j}(\mathscr{V}_n, \mathscr{C})$, $\mathscr{C} \in \mathbb{P}_{n-j}(\mathscr{V}_n)$ and $0<j<n$, with the projection of $\mathscr{W}_{n-j}(\mathscr{V}_n, \mathscr{C})$ onto $\mathscr{W}_{n-j-1}(\mathscr{V}_{n-1}, \Pi^n_{n-1}(\mathscr{C}))$. We denote that extension still by Π^n_{n-1}. Suppose now that $\mathscr{V}_{n-1}$ is isomorphic with some level $n-1$ H-Q X_{n-1} and call the corresponding isomorphism Ψ. Then we call $\Psi \circ \Pi^n_{n-1}$ a *HQ-epimorphism.* Suppose now that $(X_n, \nabla^{n+1}_n)_{n\in\mathbb{N}}$ is an infinite sequence with X_n a level n H-O and ∇^{n+1}_n an HQ-epimorphism from X_{n+1} onto X_n, then we call $(X_n, \nabla^{n+1}_n)_{n\in\mathbb{N}}$ an *HQ-Artmann-sequence.* This name is inspired by the work of Artmann [1], who studied similar sequences of level n Hjelmslev-planes, giving rise to affine buildings of type $\tilde{A}_2$ (by [13, 5]).

Suppose now that $(Y_n, \Gamma^{n+1}_n)_{n\in\mathbb{N}}$ is a second HQ-Artmann-sequence and suppose $\Psi_n : X_n \to Y_n$ is an equivalence, for all $n \in \mathbb{N}$. Then we call $(X_n, \nabla^{n+1}_n)_{n\in\mathbb{N}}$ and $(Y_n, \Gamma^{n+1}_n)_{n\in\mathbb{N}}$ *equivalent* if $\Gamma^{n+1}_n \circ \Psi_{n+1} = \Psi_n \circ \nabla^{n+1}_n$, for all $n \in \mathbb{N}$. If Z_n is the base geometry of X_n, then we call the sequence $(Z_n, \nabla^{n+1}_n/Z_{n+1})_{n\in\mathbb{N}}$ the *base sequence of* $(X_n, \nabla^{n+1}_n)_{n\in\mathbb{N}}$.

2. Properties of Hjelmslev-Quadrangles of Level n

In this section, $\mathscr{V}_n = (\mathscr{P}(\mathscr{V}_n), \mathscr{L}(\mathscr{V}_n), I, (\mathbb{P}_i(\mathscr{V}_n))_{i\leqslant n}, (\mathbb{L}_i(\mathscr{V}_n))_{i\leqslant n}, (\mathscr{W}_{n-1}(\mathscr{V}_n, \mathscr{C}), \mathscr{N}_{\mathscr{C}})_{\mathscr{C}\in\mathbb{P}_{n-1}(\mathscr{V}_n)})$ denotes an arbitrary but fixed level n H-Q, $n \in \mathbb{N}^*$. Its canonical j-image $(1 \leqslant j \leqslant n)$ is denoted by

$$\mathscr{V}_j = (\mathscr{P}_i(\mathscr{V}_j), \mathscr{L}_i(\mathscr{V}_j), I, (\mathbb{P}_i(\mathscr{V}_j))_{i\leqslant j}, (\mathbb{L}_i(\mathscr{V}_j))_{i\leqslant j}, (\mathscr{W}_{j-1}(\mathscr{V}_j, \mathscr{C}), \mathscr{N}_{\mathscr{C}})_{\mathscr{C}\in\mathbb{P}_{j-1}(\mathscr{V}_j)})$$

or by $\mathscr{V}_0$ $(j=0)$. The projections are denoted by

$$\Pi^k_j : \mathscr{V}_k \to \mathscr{V}_j, \qquad 0 \leqslant j \leqslant k \leqslant n.$$

The valuation map in $\mathscr{V}_j$ is denoted by $u[j, j]$ and we put $u = u[n, n]$. We denote the canonical k-image of $\mathscr{W}_j(\mathscr{V}_n, \mathscr{C})$ by $\mathscr{W}_k(\mathscr{V}_n, \mathscr{C})$, for all $\mathscr{C} \in \mathbb{P}_j(\mathscr{V}_n)$, $0 \leqslant k \leqslant j \leqslant n$. By (IS),

$$\mathscr{W}_{n-2}(\mathscr{V}_n, \mathscr{C}) = \mathscr{W}_{n-2}(\mathscr{V}_{n-1}, \Pi^n_{n-1}(\mathscr{C})) \qquad (*)$$

for all $\mathscr{C} \in \mathbb{P}_{n-1}(\mathscr{V}_n)$. Suppose $\mathscr{P} \in \mathscr{P}(\mathscr{V}_n)$. Denote $\mathcal{O}^j(\mathscr{P})$ briefly by $\mathscr{C}$. Then we denote the set of lines of $(\mathscr{W}_j(\mathscr{V}_n, \mathscr{C}), \mathscr{N}_{\mathscr{C}})$ by $\mathscr{L}^j(\mathscr{P})$. The projections are denoted by

$$\nabla^k_l : \mathscr{W}_k(\mathscr{V}_n, \mathscr{C}) \to \mathscr{W}_l(\mathscr{V}_n, \mathscr{C}), \qquad 0 \leqslant l \leqslant k \leqslant j \leqslant n-1,$$

still with $\mathcal{O}^j(\mathcal{P}) = \mathcal{C}$. The valuation map $\mathcal{W}_k(\mathcal{V}_j, \mathcal{O}^k(\Pi_j^n(\mathcal{P})))$ is denoted by $u[j, k]$. By Property (2.2), $u[j, k]$ will also be the valuation map in $\mathcal{W}_k(\mathcal{V}_m, \mathcal{O}^{k+m-j}(\Pi_m^n(\mathcal{P})))$, $0 < k < j \leqslant m \leqslant n$.

Throughout, j, k, l, m are positive integers, all smaller than or equal to n and $\mathcal{P}, Q \in \mathcal{P}(\mathcal{V}_n)$, $\mathcal{L}$, $\mathcal{M} \in \mathcal{L}(\mathcal{V}_n)$, pre-assigned if not explicitly mentioned otherwise.

PROPERTY (2.1). *The partitions $(\mathbb{P}_i(\mathcal{V}_n))_{i \leqslant n}$, $(\mathbb{L}_i(\mathcal{V}_n))_{i \leqslant n}$, are completely determined by the valuation map u and also by all the projections Π_j^n. Moreover, $\mathcal{O}^{n-j}(\mathcal{P}) = (\Pi_j^n)^{-1}(\Pi_j^n(\mathcal{P}))$, and similarly for the lines; $\mathcal{O}^{n-j}(\mathcal{P}) = \{Q \in \mathcal{P}(\mathcal{V}_n) \mid u(\mathcal{P}, Q) \geqslant j\}$ and again similarly for the lines.*

Proof. We already remarked this in the proof of Lemma (1.3). By definition, the points of $\mathcal{V}_{n-1}$ are the elements of $\mathbb{P}_1(\mathcal{V}_n)$. By the definition of the partitions of $\mathcal{V}_{n-1}$, the points of $\mathcal{V}_{n-2}$ are the elements of $\mathbb{P}_2(\mathcal{V}_n)/\mathbb{P}_1(\mathcal{V}_n)$. So it is clear that $u(\mathcal{P}_1, \mathcal{P}_2) = n-2$ if and only if $\mathcal{P}_1$ and $\mathcal{P}_2$ are in the same partition class $\mathbb{P}_2(\mathcal{V}_n)$, but not in the same partition class of $\mathbb{P}_1(\mathcal{V}_n)$. An inductive argument and the analogue for the lines shows that the partitions of $\mathcal{V}_n$ are completely determined by u and proves the formula with u. The other assertion follows directly from the definition of u.

Q.E.D.

The symbol • in a formula means "anything that gives the formula any sense", e.g., $\Pi_{n-1}^n(\bullet) = \bullet_{n-1}$ means : for all $x \in \mathcal{P}(\mathcal{V}_n) \cup \mathcal{L}(\mathcal{V}_n)$ and all $x \subseteq \mathcal{P}(\mathcal{V}_n) \cup \mathcal{L}(\mathcal{V}_n)$, one has $\Pi_{n-1}^n(x) = x_{n-1}$.

PROPERTY (2.2). *If $\mathcal{C}_j \in \mathbb{P}_j(\mathcal{V}_n)$ and $k \leqslant j < n$, then*

$$\mathcal{W}_k(\mathcal{V}_n, \mathcal{C}_j) = \mathcal{W}_k(\mathcal{V}_{n-j+k}, \Pi_{n-j+k}^n(\mathcal{C}_j)).$$

Proof. We proceed by induction on $j-k$ for fixed j.

(1) If $j-k=0$ then the property is trivial.

(2) Suppose now $j-k>0$, then $j-(k+1) \geqslant 0$ and by the induction hypothesis,

$$\mathcal{W}_{k+1}(\mathcal{V}_n, \mathcal{C}_j) = \mathcal{W}_{k+1}(\mathcal{V}_{n-j+k+1}, \Pi_{n-j+k+1}^n(\mathcal{C}_j)).$$

Since $\Pi_{n-j+k+1}^n(\mathcal{C}_j) \in \mathbb{P}_{k+1}(\mathcal{V}_{n-j+k+1})$, we have by definition

$$\begin{aligned}\mathcal{W}_k(\mathcal{V}_n, \mathcal{C}_j) &= \mathcal{W}_k(\mathcal{V}_{n-j+k+1}, \Pi_{n-j+k+1}^n(\mathcal{C}_j)) \\ &= \mathcal{W}_k(\mathcal{W}_{k+2}(\mathcal{V}_{n-j+k+1}, \mathcal{C}'_{k+2}), \Pi_{n-j+k+1}^n(\mathcal{C}_j)),\end{aligned}$$

where $\mathscr{C}'_{k+2}$ is the the unique class of $\mathbb{P}_{k+2}(\mathscr{V}_{n-j+k+1})$ containing $\Pi^n_{n-j+k+1}(\mathscr{C}_j)$ as a subset. By the property $(*)$, we have

$$\begin{aligned}\mathscr{W}_k(\mathscr{V}_n, \mathscr{C}_j) &= \mathscr{W}_k(\mathscr{W}_{k+1}(\mathscr{V}_{n-j+k+1}, \mathscr{C}'_{k+2}), \Pi^n_{n-j+k}(\mathscr{C}_j)) \\ &= \mathscr{W}_k(\mathscr{W}_{k+1}(\mathscr{W}_{k+3}(\mathscr{V}_{n-j+k+1}, \mathscr{C}'_{k+3}), \mathscr{C}'_{k+2}), \Pi^n_{n-j+k}(\mathscr{C}_j)),\end{aligned}$$

where $\mathscr{C}'_{k+3}$ is the unique class of $\mathbb{P}_{k+3}(\mathscr{V}_{n-j+k+1})$ containing $\mathscr{C}'_{k+2}$. Hence again by $(*)$,

$$\begin{aligned}\mathscr{W}_k(\mathscr{V}_n, \mathscr{C}_j) = \mathscr{W}_k(\mathscr{W}_{k+1}(\mathscr{W}_{k+2}(\mathscr{V}_{n-j+k+1}, \mathscr{C}'_{k+3}), \\ \Pi^{n-j+k+1}_{n-j+k}(\mathscr{C}'_{k+2}), \Pi^n_{n-j+k}(\mathscr{C}_j)).\end{aligned}$$

Going on in this manner, we eventually end up, using the fact that $\mathscr{W}_{j-j+k-1}(\mathscr{V}_{n-j+k+1}, \bullet) = \mathscr{W}_{n-j+k-1}(\mathscr{V}_{n-j+k}, \Pi^{n-j+k+1}_{n-j+k}(\bullet))$, after $n-k$ steps, with

$$\mathscr{W}_k(\mathscr{V}_n, \mathscr{C}_j) = \mathscr{W}_k(\mathscr{V}_{n-j+k}, \Pi^n_{n-j+k}(\mathscr{C}_j)). \qquad \text{Q.E.D.}$$

PROPERTY (2.3). *If $\Pi^n_1(\mathscr{P})\, I\, \Pi^n_1(\mathscr{L})$ in $\mathscr{V}_1$, then $\sigma(\mathscr{L}) \cap \mathcal{O}^{n-1}(\mathscr{P}) \neq \emptyset$.*

Proof. Denote $\Pi^n_1(\mathscr{P}) = \mathscr{P}_1$, $\Pi^n_1(\mathscr{L}) = \mathscr{L}_1$ and choose $Q_1 \in \mathscr{P}(\mathscr{V}_1)$ such that $Q_1 \perp \mathscr{P}_1$ and not incident with $\mathscr{L}_1$, this is possible since $\mathscr{V}_1$ is a generalized quadrangle. Choose Q arbitrarily in $(\Pi^n_1)^{-1}(Q_1)$; then clearly $u(Q, \mathscr{L}) = (0, 0)$, otherwise $\mathscr{L}_1\, I\, Q_1$. By (GQ2), there exists a point $\mathscr{P}'$ incident with $\mathscr{L}$ and collinear with Q. If $\sigma(\mathscr{L}) \cap \mathcal{O}^{n-1}(\mathscr{P}) = \emptyset$, then $\mathscr{P}' \notin \mathcal{O}^{n-1}(\mathscr{P})$ and using Property (2.1), we see that $\{\Pi^n_1(\mathscr{P}'), \mathscr{P}_1, Q_3\}$ forms a triangle in $\mathscr{V}_1$, a contradiction. Q.E.D.

PROPERTY (2.4). *If $\Pi^n_1(\mathscr{P})\, I\, \Pi^n_1(\mathscr{L})$ in $\mathscr{V}_1$, then there exists a line $\mathscr{L}' \in \mathcal{O}^{n-1}(\mathscr{L})$ incident with $\mathscr{P}$.*

Proof. The dual argument of the proof of property (2.3) holds here. Q.E.D.

PROPERTY (2.5). * *There exist at least three points $\mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_3 \in \mathscr{P}(\mathscr{V}_n)$, all incident with $\mathscr{L}$ and having the property $u(\mathscr{P}_r, \mathscr{P}_s) = 0$, $r, s = 1, 2, 3$ and $r \neq s$.*

* *There exist at least three lines $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3 \in \mathscr{L}(\mathscr{V}_n)$, all incident with $\mathscr{P}$ and having the property $u(\mathscr{L}_r, \mathscr{L}_s) = 0$, $r, s = 1, 2, 3$, and $r \neq s$.*

Proof. This is an immediate consequence of the Properties (2.3) and (2.4) and the fact that $\mathscr{V}_1$ is a thick geometry. Q.E.D.

PROPERTY (2.6). $u_1(\mathscr{P}, \mathscr{L}) \leqslant u_2(\mathscr{P}, \mathscr{L}) \leqslant 2.\, u_1(\mathscr{P}, \mathscr{L})$.

Proof. Put $k = u_1(\mathscr{P}, \mathscr{L})$ and $l = u_2(\mathscr{P}, \mathscr{L})$. We show each inequality separately.

(1) $k \leqslant l$. Choose Q incident with $\mathscr{L}$ and such that $u(\mathscr{P}, Q) = 0$ (this is possible by Property (2.5)). Choose $\mathscr{M}$ incident with Q and such that $u(\mathscr{L}, \mathscr{M}) = 0$. By projecting onto $\mathscr{V}_1$, one can see that $u(\mathscr{P}, \mathscr{M}) = (0, 0)$. Denote $\bullet_k = \Pi^n_k(\bullet)$, then $u[k, k](\mathscr{P}_k, \mathscr{M}_k) = (0, 0)$. By (GQ2), thee exists a line $\mathscr{M}' \in \mathscr{L}(\mathscr{V}_n)$ incident with $\mathscr{P}$ and concurrent with $\mathscr{M}$. Hence $\mathscr{P}_k \, I \, \mathscr{M}'_k \perp \mathscr{M}_k$. But since $u_1(\mathscr{P}, \mathscr{L}) = k$, we also have $\mathscr{P}_k \, I \, \mathscr{L}_k \perp \mathscr{M}_k$. So by (GQ2), $\mathscr{L}_k = \mathscr{M}'_k$, hence $u(\mathscr{L}, \mathscr{M}') \geqslant k$. Since $\mathscr{P} \, I \, \mathscr{M}'$, $l \geqslant k$ by definition.

(2) $l \geqslant 2 \cdot k$. Choose $Q' \in \mathscr{P}(\mathscr{V}_n)$ collinear with $\mathscr{P}$ and such that $u(\mathscr{P}, Q') = 0$ and $u(Q', \mathscr{L}) = (0, 0)$ (this is the dual of part (1)). We again denote $\bullet_l = \Pi^n_l(\bullet)$. Then $u[l, l](Q'_l, \mathscr{L}_l) = (0, 0)$. Choose $L' \in \mathscr{L}(\mathscr{V}_n)$ incident with Q' and concurrent with $\mathscr{L}$ and $\mathscr{M}' \in \mathscr{L}(\mathscr{V}_n)$ incident with $\mathscr{P}$ and Q'. Note that $P_l \, I \, \mathscr{L}_l$ by definition of u_2. So by (GO2), $\mathscr{L}'_l = \mathscr{M}'_l \perp \mathscr{L}_l$. Now choose $Q'' \in \sigma(\mathscr{L}) \cap \sigma(\mathscr{L}')$ arbitrarily, then $Q''_l \in \sigma(\mathscr{L}_l) \cap \sigma(\mathscr{L}'_l)$. But also $\mathscr{P}_l \in \sigma(\mathscr{L}_l) \cap \sigma(\mathscr{L}'_l)$. By (GQ2), $u[l, l](\mathscr{P}_l, Q''_l) \geqslant l/2$. Hence $u(\mathscr{P}, Q'') \geqslant l/2$. But since $Q'' \, I \, \mathscr{L}$, we have $u_1(\mathscr{P}, \mathscr{L}) \geqslant u(\mathscr{P}, Q'') \geqslant l/2$. Q.E.D.

PROPERTY (2.7). *If $\mathscr{L} \perp \mathscr{M}$ and $\mathscr{L} \neq \mathscr{M}$, then $u(\mathscr{L}, \mathscr{M})$ is even.*

Proof. Suppose $u(\mathscr{L}, \mathscr{M}) = 2k + 1$ and let $\mathscr{P}$ be incident with $\mathscr{L}$ and $\mathscr{M}$. By Property (2.5), there is a point Q incident with $\mathscr{L}$ such that $u(\mathscr{P}, Q) = 0$. Now, there are two possibilities:

(1) $\mathcal{O}^{n-k-1}(Q) \cap \sigma(\mathscr{M}) = \varnothing$. In this case, $u_1(Q, \mathscr{M}) \leqslant k$. But $u_2(Q, \mathscr{M}) \geqslant 2k + 1$, since $Q \, I \, \mathscr{L}$ and $u(\mathscr{L}, \mathscr{M}) = 2k + 1$. This contradicts Property (2.6).

(2) $\mathcal{O}^{n-k-1}(Q) \cap \sigma(\mathscr{M}) \neq \varnothing$. By (GQ1), $2k + 2 \leqslant 2k + 1$, again a contradiction. Q.E.D.

PROPERTY (2.8). *If $u(\mathscr{P}, Q) \geqslant n - j$, then $j - u[n, j](\mathscr{P}, Q) = n - u(\mathscr{P}, Q)$.*

Proof. This is an immediate consequence of (NP1) and Property (2.1). Q.E.D.

PROPERTY (2.9). *If $Q \, I \, \mathscr{L} \, I \, \mathscr{P} \, I \, \mathscr{M}$, $u(\mathscr{P}, Q) = 0$, and $\mathscr{L} \neq \mathscr{M}$, then*

$$u(Q, \mathscr{M}) = \left(\frac{u(\mathscr{L}, \mathscr{M})}{2}, u(\mathscr{L}, \mathscr{M}) \right).$$

Proof. Let $u(Q, \mathscr{M}) = k$, then $\mathcal{O}^{n-k}(Q) \cap \sigma(\mathscr{M}) \neq \varnothing$ and by (GQ1),

$2k \leqslant u(\mathscr{L}, \mathscr{M})$. But $u_2(Q, \mathscr{M}) \geqslant u(\mathscr{L}, \mathscr{M})$, so combining these inequalities with Property (2.6), we obtain

$$u(\mathscr{L}, \mathscr{M}) \leqslant u_2(Q, \mathscr{M}) \leqslant 2 \cdot u_1(Q, \mathscr{M}) \leqslant u(\mathscr{L}, \mathscr{M});$$

hence equality holds everywhere. Q.E.D.

PROPERTY (2.10). *Suppose $\mathscr{C} \in \mathbb{P}_{n-1}(\mathscr{V}_n)$ and $n-j$ odd, then every strip of width j in $(\mathscr{V}_n, \mathscr{C})$ is also a strip of width $j-1$, for all $j \neq 0$.*

Proof. This follows from the definition of strip of width j and Property (2.7). Q.E.D.

PROPERTY (2.11). *If $\mathscr{P}$ and Q are both incident with both $\mathscr{L}$ and $\mathscr{M}$, and $u(\mathscr{P}, Q) = 0$, then $\mathscr{L} = \mathscr{M}$.*

Proof. This is a direct consequence of (GQ1) applied for $j = n$. Q.E.D.

PROPERTY (2.12). *The line $\mathscr{L}$ is completely determined by its shadow $\sigma(\mathscr{L})$.*

Proof. Follows from Properties (2.5) and (2.11). Q.E.D.

Remark (2.13). Although $\mathscr{L}$ is completely determined by $\sigma(\mathscr{L})$, the set $\sigma(\Pi_j^n(\mathscr{L}))$ is not necessarily equal to the set $\Pi_j^n(\sigma(\mathscr{L}))$. However, by Properties (2.5) and (2.11), $\Pi_j^n(\sigma(\mathscr{L}))$ determines $\Pi_j^n(\mathscr{L})$ and hence also $\sigma(\Pi_j^n(\mathscr{L}))$.

PROPERTY (2.14). *Suppose $\mathscr{C} \in \mathbb{P}_{n-1}(\mathscr{V}_n)$, $\mathscr{P}$ is an affine point of $(\mathscr{V}_n, \mathscr{C})$ and Q is a point at infinity of $(\mathscr{V}_n, \mathscr{C})$. Then $u(\mathscr{P}, Q) = 0$.*

Proof. This follows directly from Lemma (1.3). Q.E.D.

PROPERTY (2.15). *Suppose $\mathscr{C} \in \mathbb{P}_{n-1}(\mathscr{V}_n)$. Every affine line $\mathscr{L}$ of $(\mathscr{V}_n, \mathscr{C})$ is incident with at least two affine points $\mathscr{P}_1, \mathscr{P}_2$ and at least one point at infinity Q of $(\mathscr{V}_n, \mathscr{C})$. Moreover, one can choose $\mathscr{P}_1$ and $\mathscr{P}_2$ such that $u(\mathscr{P}_1, \mathscr{P}_2) = 0$ and $\mathscr{O}^{n-1}(Q)$ is the unique $(n-1)$-point-neighbourhood of points at infinity containing elements incident with $\mathscr{L}$.*

Proof. Choose $\mathscr{P} \in \mathscr{C}$ arbitrarily. Since $\mathscr{L}$ is affine, $u_1(\mathscr{P}, \mathscr{L}) = 0$ by definition, and so by Property (2.6), $u(\mathscr{P}, \mathscr{L}) = (0, 0)$. Applying (GQ2), we obtain a point Q collinear with $\mathscr{P}$ and incident with $\mathscr{L}$. This point Q is by definition a point at infinity. Suppose $\mathscr{L}$ is incident with a (different) point at infinity Q' and assume $\mathscr{O}^{n-1}(Q) \neq \mathscr{O}^{n-1}(Q')$. We denote $\Pi_1^n(\bullet) = \bullet_1$. By Property (2.3), $\mathscr{P}_1$ is not incident with $\mathscr{L}_1$ and hence $\{\mathscr{P}_1, Q_1', Q_1\}$ forms a triangle in $\mathscr{V}_1$, a contradiction; hence $\mathscr{O}^{n-1}(Q)$ is unique with the above-mentioned property. Applying Property (2.5) on $\mathscr{L}$, we obtain three points $\mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_3$, all incident with $\mathscr{L}$ and two by two in distinct $(n-1)$-point-neighbourhoods. By the first part of the proof, at most one can be a point at infinity of $(\mathscr{V}_n, \mathscr{C})$, e.g., $\mathscr{P}_3$. Then $\mathscr{P}_1, \mathscr{P}_2$ are two affine points such that $u(\mathscr{P}_1, \mathscr{P}_2)=0$. Q.E.D.

PROPERTY (2.16). *If $\sigma(\mathscr{L}) \cap \mathcal{O}^j(\mathscr{P}) \neq \emptyset$, then every component of $\mathscr{L}$ in $\mathscr{W}_j(\mathscr{V}_n, \mathcal{O}^j(\mathscr{P}))$ is determined by any affine point incident with $\mathscr{L}$, i.o.w. no two elements of $\mathbb{C}_j^n(\mathscr{P}, \mathscr{L})$ are incident with a common affine point. Moreover, every element of $\mathbb{C}_j^n(\mathscr{P}, \mathscr{L})$ is incident with every base point of $\mathbb{S}_j^n(\mathscr{P}, \mathscr{L})$, and every line of $\mathscr{W}_j(\mathscr{V}_n, \mathcal{O}^j(\mathscr{P}))$ incident with some points $Q_1, Q_2 \in \mathbb{S}_j^n(\mathscr{P}, \mathscr{L})$ such that $u[n,j](Q_1, Q_2)=0$, belongs to $\mathbb{C}_j^n(\mathscr{P}, \mathscr{L})$.*

Proof. The first assertion will follow from the second and properties (2.11), (2.14), and (2.15). So suppose $\mathscr{L}' \in \mathbb{C}_j^n(\mathscr{P}, \mathscr{L})$. Let $\mathscr{P}_1, \mathscr{P}_2$ be two affine points of $\mathscr{W}_j(\mathscr{V}_n, \mathcal{O}^j(\mathscr{P}))$ incident with $\mathscr{L}'$ and such that $u[n,j](\mathscr{P}_1, \mathscr{P}_2)=0$ (this is possible by Property (2.15)). If $\mathscr{P}_1$ and $\mathscr{P}_2$ are incident with a common element $\mathscr{L}''$ of $\mathbb{C}_j^n(\mathscr{P}, \mathscr{L})$ which is incident with all base points, then by Property (2.11), $\mathscr{L}''=\mathscr{L}' \in \mathbb{C}_j^n(\mathscr{P}, \mathscr{L})$. So suppose $\mathscr{P}_i \, I \, \mathscr{L}_i \in \mathbb{C}_j^n(\mathscr{P}, \mathscr{L})$, $i=1,2$, with $\mathscr{L}_1 \neq \mathscr{L}_2$ and both incident with all base points of $\mathbb{S}_j^n(\mathscr{P}, \mathscr{L})$. Let $\mathscr{T}$ be such a base point, not incident with $\mathscr{L}'$, then by Property (2.14), $u[n,j](\mathscr{T}, \mathscr{P}_i)=0$, $i=1,2$, and from Property (2.9) follows $2u[n,j]_1(\mathscr{T}, \mathscr{L}')=u[n,j]_2(\mathscr{T}, \mathscr{L}')$; hence by (GQ2), $\mathscr{L}_1=\mathscr{L}_2$, a contradiction, showing the second assertion. The last assertion clearly follows from the same argument, putting $\mathscr{P}_1=Q_1$ and $\mathscr{P}_2=Q_2$. Q.E.D.

Remark (2.17). Suppose $n \geqslant 2$. Assume $\mathcal{O}^{n-1}(\mathscr{P}) \cap \sigma(\mathscr{L}) \neq \emptyset$ and let $\mathscr{L}' \in \mathbb{C}_{n-1}^n(\mathscr{P}, \mathscr{L})$. By Property (2.15), there exist two points $\mathscr{P}_1, \mathscr{P}_2 \in \mathcal{O}^{n-1}(\mathscr{P})$ incident with $\mathscr{L}'$ in $\mathscr{W}_{n-1}(\mathscr{V}_n, \mathcal{O}^{n-1}(\mathscr{P}))$ and such that $u[n, n-1](\mathscr{P}_1, \mathscr{P}_2)=0$. By Property (2.8), $u(\mathscr{P}_1, \mathscr{P}_2)=1$ and, since $\mathscr{P}_1, \mathscr{P}_2 \in \sigma(\mathscr{L})$, they are collinear. So we see that the dual of Property (2.7) is false for $n \geqslant 2$. Hence, there is no point–line duality for level n H-Qs, although the dual of *some* properties is true.

PROPERTY (2.18). *Suppose $\mathscr{C}_j \in \mathbb{P}_j(\mathscr{V}_n)$, Every affine line of $(\mathscr{W}_j(\mathscr{V}_n, \mathscr{C}_j), \mathscr{N}_{\mathscr{C}_j})$ is a component of at least one line in $\mathscr{V}_n$.*

Proof. Let $\mathscr{L}'$ be an affine line of $(\mathscr{W}_j(\mathscr{V}_n, \mathscr{C}_j), \mathscr{N}_{\mathscr{C}_j})$ and let $\mathscr{C}_{j+1}$ be the unique element of $\mathbb{P}_{j+1}(\mathscr{V}_n)$ containing $\mathscr{C}_j$ as a subset. Note that $\mathscr{N}_{\mathscr{C}_j}$ is the unique $(j-1)$-point-neighbourhood in $\mathscr{W}_j(\mathscr{V}_n, \mathscr{C}_j)$ for which the set of all affine points of $(\mathscr{W}_j(\mathscr{V}_n, \mathscr{C}_j), \mathscr{N}_{\mathscr{C}_j})$ is exactly $\mathscr{C}_j$. We will use that notation frequently. By Property (2.15), $\mathscr{L}'$ is incident with a point at infinity $\mathscr{T}$ of $(\mathscr{W}_j(\mathscr{V}_n, \mathscr{C}_j), \mathscr{N}_{\mathscr{C}_j})$. Now we take $\mathscr{T}$ as base point of a strip of width 1 in $(\mathscr{W}_j(\mathscr{V}_n, \mathscr{C}_j), \mathscr{N}_{\mathscr{C}_j})$ containing the affine shadow of $\mathscr{L}'$. By (NP2), there exists a line $\mathscr{L}''$ of $\mathscr{W}_{j+1}(\mathscr{V}_n, \mathscr{C}_{j+1})$ such that $\mathscr{L}' < \mathscr{L}''$. The line $\mathscr{L}''$ is clearly an affine line of $(\mathscr{W}_{j+1}(\mathscr{V}_n, \mathscr{C}_{j+1}), \mathscr{N}_{\mathscr{C}_{j+1}})$. Now we let $\mathscr{L}''$ play the rôle of $\mathscr{L}'$ and by an inductive argument, we obtain eventually after $n-j$ steps, a line $\mathscr{L} \in \mathscr{V}_n$ whose shadow contains the affine shadow of $\mathscr{L}'$. By poperty (2.16), $\mathscr{L}' < \mathscr{L}$. Q.E.D.

PROPERTY (2.19). *If $n=2$ and $\mathscr{O}^1(\mathscr{P}) \cap \sigma(\mathscr{L}) \neq \emptyset$, then $\mathscr{L}$ is completely determined by any component in $\mathscr{W}_1(\mathscr{V}_2, \mathscr{O}^1(\mathscr{P}))$, i.e., every element of $\mathscr{L}^1(\mathscr{P})$ is a component of exactly one line of $\mathscr{V}_2$.*

Proof. Suppose $\mathscr{M} \neq \mathscr{L}$ and $\mathscr{M}$ and $\mathscr{L}$ share a component $\mathscr{L}'$ in $\mathscr{W}_1(\mathscr{V}_2, \mathscr{O}^1(\mathscr{P}))$. By Property (2.7), $u(\mathscr{L}, \mathscr{M}) = 0$. Now $\mathscr{W}_1(\mathscr{V}_2, \mathscr{O}^1(\mathscr{P}))$ is a (thick) generalized quadrangle and it is easy to see that two strips of same width, which share the affine shadow of an affine line, coincide. Hence, $\mathscr{O}^1(\mathscr{P}) \cap \sigma(\mathscr{L}) = \mathscr{O}^1(\mathscr{P}) \cap \sigma(\mathscr{M})$. Choose Q such that $Q\, I\, \mathscr{L}$ and $u(Q, \mathscr{P}) = 0$. By (NP2), $\mathscr{O}^1(Q) \cap \sigma(\mathscr{L}) \neq \mathscr{O}^1(Q)$ and we can choose a point $Q' \in \mathscr{O}^1(Q)$ not incident with $\mathscr{L}$. By projecting onto $\mathscr{V}_1$, we see that $u_1(Q', \mathscr{M}) = 0$ (otherwise $u(\mathscr{L}, \mathscr{M}) \geqslant 1$). Hence, by Property (2.6), $u(Q', \mathscr{M}) = (0, 0)$. So by (GQ2), there exists a unique line $\mathscr{M}'\, I\, Q'$ and concurrent with $\mathscr{M}$. By projecting onto $\mathscr{V}_1$, we obtain $u(\mathscr{L}, \mathscr{M}') = 1$. But $\mathscr{M}$ and $\mathscr{M}'$ must meet in $\mathscr{O}^1(\mathscr{P})$ by Property (2.11). Hence $\mathscr{M}'$ and $\mathscr{L}$ also meet there, contradicting Property (2.7), hence $\mathscr{L}$ and $\mathscr{M}$ cannot have any component in common. Q.E.D.

PROPERTY (2.20). *If $\sigma(\mathscr{L}) \cap \mathscr{O}^j(\mathscr{P}) \neq \emptyset$ and $\mathscr{L}' < \mathscr{L}$, $\mathscr{L}' \in \mathscr{L}^j(\mathscr{P})$, then $\nabla^j_{j-k}(\mathscr{L}') < \Pi^n_{n-k}(\mathscr{L})$, $\forall k \leqslant j$.*

Proof. By an inductive argument, we only need to show the property for $k=1$. So suppose $k=1$. We choose two points $Q_1, Q_2 \in \mathscr{O}^j(\mathscr{P})$ incident with $\mathscr{L}'$ and such that $u[n, j](Q_1, Q_2) = 0$ (this is possible by Property (2.15)). We denote briefly $\Pi^n_{n-1}(\bullet) = \bullet^*$. Now Q_1^* and Q_2^* are both incident with $\mathscr{L}^*$, but also with $\nabla^j_{j-1}(\mathscr{L}')$. From the third assertion of Property (2.16) it follows that $\nabla^j_{j-1}(\mathscr{L}')$ is a component of $\mathscr{L}^*$. Q.E.D.

PROPERTY (2.21). *If $\sigma(\mathscr{L}) \cap \mathscr{O}^j(\mathscr{P}) \neq \emptyset$ and $k \leqslant j$. then*

$$\nabla^j_k(\mathbb{C}^n_{n-j}(\mathscr{P}, \mathscr{L}) \subseteq \mathbb{C}^{n-j+k}_{n-j}(\Pi^n_{n-j+k}(\mathscr{P}), \Pi^n_{n-j+k}(\mathscr{L})).$$

Proof. This is an immediate consequence of the previous property. Q.E.D.

PROPERTY (2.22). *If $\mathscr{P}$ and Q are both incident with $\mathscr{L}$ and $\mathscr{M}$, and $u(\mathscr{L}, \mathscr{M}) = 0$, then $u(\mathscr{P}, Q) \geqslant n/2$.*

Proof. We can choose a point $\mathscr{P}' \, I \, \mathscr{L}$ such that $u(\mathscr{P}', \mathscr{M}) = (0, 0)$ (use Properties (2.5) and (2.6)). The assertion now follows from (GQ2) applied on $\mathscr{P}'$ and $\mathscr{M}$. Q.E.D.

PROPERTY (2.23). *If $u(\mathscr{L}, \mathscr{M}) \geqslant n-1$, $\mathscr{L} \, I \, \mathscr{P} \, I \, \mathscr{M}$ and n is odd, then $\mathbb{C}_1^n(\mathscr{P}, \mathscr{L}) = \mathbb{C}_1^n(\mathscr{P}, \mathscr{M})$ and $\mathbb{S}_1^n(\mathscr{P}, \mathscr{L}) = \mathbb{S}_1^n(\mathscr{P}, \mathscr{M})$.*

Proof. By Property (2.17), $\mathbb{C}_1^n(\mathscr{P}, \mathscr{L})$ and $\mathbb{C}_1^n(\mathscr{P}, \mathscr{M})$ are both singletons with respective elements, say $\mathscr{L}'$ and $\mathscr{M}'$. By Property (2.20), $\nabla_{n-2}^{n-1}(\mathscr{L}')$ and $\nabla_{n-2}^{n-1}(\mathscr{M}')$ are components of $\Pi_{n-1}^n(\mathscr{L}) = \Pi_{n-1}^n(\mathscr{M})$. But $\Pi_{n-1}^n(\mathscr{P})$ is incident with both $\nabla_{n-2}^{n-1}(\mathscr{L}')$ and $\nabla_{n-2}^{n-1}(\mathscr{M}')$ and hence by Property (2.16), $\nabla_{n-2}^{n-1}(\mathscr{L}')$ and $\nabla_{n-2}^{n-1}(\mathscr{M}')$ coincide. But since $n-2$ is odd, $\mathscr{L}' = \mathscr{M}'$ by Property (2.7). Q.E.D.

PROPERTY (2.24). *If $\sigma(\mathscr{L}) \cap \mathcal{O}^j(\mathscr{P}) \neq \varnothing$, then there exists $\mathscr{M} \in \mathcal{O}^j(\mathscr{L})$ incident with $\mathscr{P}$.*

Proof. By Property (2.1), $\sigma(\mathscr{L}) \cap \mathcal{O}^j(\mathscr{P}) \neq \varnothing$ if and only if $u_1(\mathscr{P}, \mathscr{L}) \geqslant n-j$, hence $u_2(\mathscr{P}, \mathscr{L}) \geqslant n-j$ (Property (2.6)). But this is by definition equivalent with the existence of a line $\mathscr{M} \in \mathcal{O}^j(\mathscr{L})$ incident with $\mathscr{P}$. Q.E.D.

PROPERTY (2.25). *If $\Pi_j^n(\mathscr{P}) \, I \, \Pi_j^n(\mathscr{L})$, then there exists $\mathscr{M} \in \mathcal{O}^{n-1}(\mathscr{L})$ incident with $\mathscr{P}$.*

Proof. Choose $Q \in \mathcal{O}^{n-j}(\mathscr{P})$ and $\mathscr{M}' \in \mathcal{O}^{n-j}(\mathscr{L})$ such that $Q \, I \, \mathscr{M}'$ (this is possible since Π_j^n is an epimorphism). Hence $\sigma(\mathscr{M}') \cap \mathcal{O}^{n-j}(Q) \neq \varnothing$. Now $\mathcal{O}^{n-j}(\mathscr{P}) = \mathcal{O}^{n-j}(Q)$ and so the assertion follows from Property (2.24). Q.E.D.

PROPERTY (2.26). *If $\mathscr{C} \in \mathbb{P}_{n-1}(\mathscr{V}_n)$ and n is odd, then every strip of width 1 in $(\mathscr{V}_n, \mathscr{C})$ is completely determined by any component.*

Proof. Let $\mathscr{S}$ be a strip of width 1 in $(\mathscr{V}_n, \mathscr{C})$ and $\mathscr{L} \in \mathscr{S}$. By Property (2.15), $\mathscr{L}$ meets a unique $(n-1)$-point-neighbourhood at infinity, say $\mathcal{O}^{n-1}(\mathscr{P}_\infty)$. But if $\mathscr{M} \in \mathscr{S}$, then $\varnothing \neq \sigma(\mathscr{L}) \cap \sigma(\mathscr{M}) \subseteq \mathcal{O}^{n-1}(\mathscr{P}_\infty)$. By Property (2.23), $\sigma(\mathscr{L}) \cap \mathcal{O}^{n-1}(\mathscr{P}_\infty) = \sigma(\mathscr{M}) \cap \mathcal{O}^{n-1}(\mathscr{P}_\infty)$. Hence every element of $\sigma(\mathscr{L}) \cap \mathcal{O}^{n-1}(\mathscr{P}_\infty)$ can serve as a base point. Q.E.D.

PROPERTY (2.27). *If $\sigma(\mathscr{L}) \cap \mathcal{O}^{n-1}(\mathscr{P}) \neq \varnothing$ and n is even, then $\mathscr{L}$ is completely determined by any component in $\mathscr{W}_{n-1}(\mathscr{V}_n, \mathcal{O}^{n-1}(\mathscr{P}))$.*

Proof. We proceed by means of induction on $n \in 2 \cdot \mathbb{N}$.

(1) For $n = 2$, this is Property (2.19).

(2) Let $n>2$. By the previous property, any component of $\mathscr{L}$ in $\mathscr{W}_{n-1}(\mathscr{V}_n, \mathcal{O}^{n-1}(\mathscr{P}))$ determines the strip completely. Hence we can assume that there is line $\mathscr{M} \neq \mathscr{L}$ such that $\mathbb{C}_1^n(\mathscr{P}, \mathscr{L}) = \mathbb{C}_1^n(\mathscr{P}, \mathscr{M})$. We seek a contradiction. By the induction hypothesis and Property (2.20), we have $u(\mathscr{L}, \mathscr{M}) = n-2$, because $u(\mathscr{L}, \mathscr{M})$ must be even. Choose now $Q\, I\, \mathscr{L}$ such that $u(\mathscr{P}, Q) = 0$. By Property (2.9), $u(Q, \mathscr{M}) = (n/2-1, n-2)$. Choose now $Q' \in \mathcal{O}^1(Q)$ not incident with $\mathscr{L}$ (this is possible since $\mathcal{O}^1(Q)$ meets $\mathscr{L}$ only in a strip). By Properties (2.6) and (2.25), $u(Q', \mathscr{M}) = (n/2-1, n-2)$ and consequently by Property (2.9), $u[n-1, n-1](\Pi_{n-1}^n(Q'), \Pi_{n-1}^n(\mathscr{M})) = (n/2-1, n-2)$. Applying (GQ2) in respectively $\mathscr{V}_n$ and $\mathscr{V}_{n-1}$, we obtain a line $\mathscr{L}' \in \mathscr{L}(\mathscr{V}_n)$ incident with Q' and concurrent with $\mathscr{M}$ and such that $u(\mathscr{L}, \mathscr{L}') = n-1$. If $\mathscr{L}'$ and $\mathscr{M}$ would meet outside $\mathcal{O}^{n-1}(\mathscr{P})$, then by Property (2.11), $\Pi_{n-1}^n(\mathscr{L}') = \Pi_{n-1}^n(\mathscr{M})$, because $\Pi_{n-1}^n(\mathscr{L}') = \Pi_{n-1}^n(\mathscr{L})$ and both lines also have $\Pi_{n-1}^n(\mathscr{P})$ in common. Hence $\mathscr{L}'$ and $\mathscr{M}$ meet in $\mathcal{O}^{n-1}(\mathscr{P})$, where $\mathscr{M}$ and $\mathscr{L}$ have the same shadow. Hence $\mathscr{L}$ meets $\mathscr{L}'$, contradicting $u(\mathscr{L}, \mathscr{L}') = n-1$, $n-1$ odd and Property (2.7). Q.E.D.

PROPERTY (2.28). *If $\sigma(\mathscr{L}) \cap \sigma(\mathscr{M}) \cap \mathcal{O}^{n-1}(\mathscr{P}) \neq \varnothing$ and $\mathbb{C}_1^n(\mathscr{P}, \mathscr{L}) \cap \mathbb{C}_1^n(\mathscr{P}, \mathscr{M}) \neq \varnothing$, then $u(\mathscr{L}, \mathscr{M}) \geqslant n-1$ and if n is even, then equality does not occur.*

Proof. If n is even, then the assertion is completely equivalent with Property (2.27). So suppose n odd. Consider $\nabla_{n-2}^{n-1}(\mathbb{C}_1^n(\mathscr{P}, \mathscr{M}))$. Its unique element is a component of $\Pi_{n-1}^n(\mathscr{L})$ and also of $\Pi_{n-1}^n(\mathscr{M})$, by Property (2.20). The result now follows from Property (2.27) applied in $\mathscr{V}_{n-1}$.

Q.E.D.

PROPERTY (2.29). *Suppose $\mathscr{P}\, I\, \mathscr{L}$ and $k<j$. If $\mathscr{L}_j$* (resp. $\mathscr{L}_k$) *is the unique component of $\mathscr{L}$ in $\mathscr{W}_j(\mathscr{V}_n, \mathcal{O}^j(\mathscr{P}))$* (resp. $\mathscr{W}_k(\mathscr{V}_n, \mathcal{O}^k(\mathscr{P}))$) *incident with $\mathscr{P}$, then $\mathscr{L}_j < \mathscr{L}_k$. In particular, $\mathscr{L}_k$ is the unique component of $\mathscr{L}$ in $\mathscr{W}_k(\mathscr{V}_n, \mathcal{O}^k(\mathscr{P}))$ containing $\mathscr{L}_j$ as a component.*

Proof. Since $\mathscr{L}_k < \mathscr{L}$, we have $\mathbb{S}_{k-j}^k(\mathscr{P}, \mathscr{L}_k) \subseteq \mathbb{S}_{n-j}^n(\mathscr{P}, \mathscr{L})$. The result now follows from Property (2.16). Q.E.D.

PROPERTY (2.30). *If $\sigma(\mathscr{L}) \cap \sigma(\mathscr{M}) \cap \mathcal{O}^j(\mathscr{P}) \neq \varnothing$ and $\mathbb{C}_{n-j}^n(\mathscr{P}, \mathscr{L}) \cap \mathbb{C}_{n-j}^n(\mathscr{P}, \mathscr{M}) \neq \varnothing$, then $u(\mathscr{L}, \mathscr{M}) \geqslant j$ and if j is odd, then equality does not occur.*

Proof. We proceed by means of induction on $n \geqslant j+1$ for fixed j.

(1) For $n=j+1$, this is Property (2.28).

(2) Suppose now $n \geqslant j+2$. Let $\mathscr{K}$ be an arbitrary component of both $\mathscr{L}$ and $\mathscr{M}$ in $\mathscr{W}_j(\mathscr{V}_n, \mathcal{O}^j(\mathscr{P}))$ and let $\mathscr{L}'$ (resp. $\mathscr{M}'$) be the unique compo-

nent of $\mathscr{L}$ (resp. $\mathscr{M}$) in $\mathscr{W}_{j+1}(\mathscr{V}_n, \mathcal{O}^{j+1}(\mathscr{P}))$ having $\mathscr{K}$ as a component (cp. Property (2.29)). By Property (2.28), $\nabla_j^{j+1}(\mathscr{M}') = \nabla_j^{j+1}(\mathscr{L}')$. Hence $\Pi_{n-1}^n(\mathscr{L})$ and $\Pi_{n-1}^n(\mathscr{M})$ share a component in $\mathscr{W}_j(\mathscr{V}_{n-1}, \mathcal{O}^j(\Pi_{n-1}^n(\mathscr{P}))$. By the induction hypothesis, $u[n-1, n-1](\Pi_{n-1}^n(\mathscr{P}), \Pi_{n-1}^n(\mathscr{L})) \geqslant j$. But then $u(\mathscr{P}, \mathscr{L}) \geqslant j$ obviously. If j is odd, then the result follows from Property (2.7). Q.E.D.

For any real number r, we denote by $[r]$ the biggest integer smaller than or equal to r.

PROPERTY (2.31). *If $u(\mathscr{P}, \mathscr{L}) = (k, l)$, $2k \leqslant n$, and $\mathscr{L}'$ is an arbitrary component of $\mathscr{L}$, $\mathscr{L}' \in \mathscr{L}^{n-2k+l}(\mathscr{P})$, then $u[n, n-2k+l](\mathscr{P}, \mathscr{L}') = (l-k, 2(l-k))$.*

Proof. Let us denote $u[n, n-2k+l]$ briefly by u' and $n-2k+l$ by i. Since $u(\mathscr{P}, \mathscr{L}) = k$, at least one component $\mathscr{L}_1$ of $\mathscr{L}$ in $\mathscr{W}_{n-i}(\mathscr{V}_n, \mathcal{O}^{n-i}(\mathscr{P}))$ will meet the set $\mathcal{O}^{n-k}(\mathscr{P})$. We denote $\mathfrak{B} = \mathbb{C}_i^n(\mathscr{P}, \mathscr{L})$. By (NP2), $u'(\mathscr{L}_1, \mathscr{L}') \geqslant n-2i$. Now $\mathscr{L}_1$ and $\mathscr{L}'$ meet in a base point and by (GQ1) and Property (2.14), they meet all the same $[(n+1)/2]$-point-neighbourhoods. Since $n-k \geqslant [(n+1)/2]$, $\mathscr{L}'$ meets $\mathcal{O}^{n-k}(\mathscr{P})$. Since $\mathscr{L}$ does not meet $\mathcal{O}^{n-k-1}(\mathscr{P})$, neither does $\mathscr{L}'$ and hence $u_1(\mathscr{P}, \mathscr{L}') = i-(n-k) = l-k$. Since $u(\mathscr{P}, \mathscr{L}) = l$, we find $\mathscr{M}$ in $\mathscr{V}_n$ such that $u(\mathscr{L}, \mathscr{M}) = l$ and $\mathscr{P} \, I \, \mathscr{M}$. We project this situation onto $\mathscr{V}_l$ and denote $\Pi_l^n(\bullet) = \bullet_l$. So $\mathscr{W}_{n-i}(\mathscr{V}_n, \mathcal{O}^{n-i}(\mathscr{P}))$ is projected onto $\mathscr{W}_{l-i}(\mathscr{V}_n, \mathcal{O}^{n-i}(\mathscr{P})) = \mathscr{W}_{l-i}(\mathscr{V}_l, \mathcal{O}^{l-i}(\mathscr{P}_l))$. One has $\mathscr{L}_l = \mathscr{M}_l$ and the strip $\mathbb{S}_{l-1}^l(\mathscr{P}_l, \mathscr{L}_l)$ is of width i. Denote $\mathfrak{B}' = \mathbb{C}_{l-1}^l(\mathscr{P}_l, \mathscr{L}_l)$. Let $\mathscr{M}_1$ be the unique element of $\mathfrak{B}'$ incident with $\mathscr{P}_l$ (cp. Property (2.16)) and let $\mathscr{M}_2$ be the unique element of $\nabla_{l-1}^{n-1}(\mathfrak{B})$ (the latter is indeed a singleton since $\mathfrak{B}$ is a strip of width $i \leqslant (n-i)-(l-i)$). By Property (2.20), $\mathscr{M}_2 \in \mathfrak{B}'$. And $\mathscr{M}_2$ meets $\nabla_{l-i}^{n-i}(\mathcal{O}^{n-k}(\mathscr{P})) = \mathcal{O}^{l-k}(\mathscr{P}_l)$. Since $\mathscr{M}_1$ and $\mathscr{M}_2$ meet in a base point, we have by (GQ1), $u'(\mathscr{M}_1, \mathscr{M}_2) \geqslant l-i = 2l-2k$, but this means that $\mathscr{M}_1 = \mathscr{M}_2$; hence $\nabla_{l-i}^{n-i}(\mathscr{L}') = \mathscr{M}_2 = \mathscr{M}_1 \, I \, \nabla_{l-1}^{n-1}(\mathscr{P}) = \mathscr{P}_l$. By Property (2.25), $u_2'(\mathscr{P}, \mathscr{L}') \geqslant 2l-2k$ and the equality follows straight from Property (2.6). Q.E.D.

PROPERTY (2.32). *If $u(\mathscr{L}, \mathscr{M}) \geqslant 2j-1$, then*

$$\sigma(\mathscr{L}) \cap \mathcal{O}^{n-j}(\mathscr{P}) \neq \varnothing \Leftrightarrow \sigma(\mathscr{M}) \cap \mathcal{O}^{n-j}(\mathscr{P}) \neq \varnothing.$$

Proof. The property is trivial for $j=0$ and $2j>n$. So we can assume $0<2j \leqslant n$. Suppose $\sigma(\mathscr{L}) \cap \mathcal{O}^{n-j}(\mathscr{P}) \neq \varnothing$. Without loss of generality, we can also assume that $\mathscr{P} \, I \, \mathscr{L}$. Choose a line $\mathscr{K} \in \mathscr{L}(\mathscr{V}_n)$ incident with $\mathscr{P}$ such that $u(\mathscr{L}, \mathscr{K}) = 0$, and choose a point $Q \, I \, \mathscr{K}$ such that $u(\mathscr{P}, Q) = 0$. The $(n-1)$-point-neighbourhood of Q does not meet $\mathscr{L}$; otherwise $u(\mathscr{L}, \mathscr{K}) > 0$. Hence $\mathcal{O}^{n-1}(Q)$ does not meet $\mathscr{M}$ and by Property (2.6),

$u(Q, \mathcal{M})=(0,0)$. We apply (GQ2) and obtain a line $\mathcal{K}' \in \mathcal{L}(\mathcal{V}_n)$ incident with Q and concurrent with $\mathcal{M}$. We put $i=2j-1$ and denote again $\Pi_i^n(\bullet)=\bullet_i$. A similar argument as above shows $u[i,i](Q_i, \mathcal{M}_i)=(0,0)$ and from $\mathcal{L}_i=\mathcal{M}_i$ follows by (GQ2), $\mathcal{K}_i=\mathcal{K}_i'$. By Property (2.22), all common points of $\mathcal{K}_i'$ and $\mathcal{M}_i$ lie in $\mathcal{O}^{j-1}(\mathcal{P}_i)=(\mathcal{O}^{n-j}(\mathcal{P}))_i$ and hence, all common points of $\mathcal{K}'$ and $\mathcal{M}$ lie in $\mathcal{O}^{n-j}(\mathcal{P})$. Since there is at least one such point, we have $\sigma(\mathcal{M}) \cap \mathcal{O}^{n-j}(\mathcal{P}) \neq \varnothing$. Q.E.D.

PROPERTY (2.33). *If* $2u[j,j]_1(\Pi_j^n(\mathcal{P}), \Pi_j^n(\mathcal{L})) \leqslant j \leqslant n$, *then* $u[j,j](\Pi_j^n(\mathcal{P}), \Pi_j^n(\mathcal{L}))=u(\mathcal{P}, \mathcal{L})$.

Proof. Let $u[j,j](\Pi_j^n(\mathcal{P}), \Pi_j^n(\mathcal{L})) = (k,l)$ and $u(\mathcal{P}, \mathcal{L}) = (k', l')$. Clearly, $k \geqslant k'$ and $l \geqslant l'$ and if $u(\mathcal{L}, \mathcal{M}) \leqslant j$, for arbitrary $\mathcal{M}$, then $u(\mathcal{L}, \mathcal{M})=u[j,j](\Pi_j^n(\mathcal{P}), \Pi_j^n(\mathcal{L}))$. Denote $\Pi_j^n(\bullet)=\bullet_j$ and, choose $\mathcal{M}$ such that $\mathcal{M}_j I \mathcal{P}_j$ and $u[j,j](\mathcal{M}_j, \mathcal{L}_j)=l \leqslant 2k \leqslant j$ (this is possible by the definition of $u[j,j]$ and the fact that Π_j^n in surjective). By Property (2.25), we can even assume that $\mathcal{M} I \mathcal{P}$ and $u(\mathcal{L}, \mathcal{M}) \geqslant l$ (because $l \leqslant j$). Hence $l' \geqslant l$ and consequently $l=l'$. Now choose Q such that $Q_j I \mathcal{L}_j$ and $u[j,j](\mathcal{P}_j, Q_j)=k$. Since Π_j^n is surjective on the set of flags, there exists a $\mathcal{L}' \in \mathcal{L}(\mathcal{V}_n)$ incident with Q and such that $u(\mathcal{L}, \mathcal{L}') \geqslant j \geqslant 2k > 2k-1$. So by Property (2.32), there exists a point $Q' \in \mathcal{P}(\mathcal{V}_n)$ incident with $\mathcal{L}$ and such that $u(Q, Q') \geqslant k$. We also have $u(Q, \mathcal{P}) \geqslant k$, hence $u(Q', \mathcal{P}) \geqslant k$. This shows $k' \geqslant k$. Q.E.D.

PROPERTY (2.34). *If* $2u_1(\mathcal{P}, \mathcal{L}) < j \leqslant n$, *then* $u[j,j](\Pi_j^n(\mathcal{P}), \Pi_j^n(\mathcal{L}))=u(\mathcal{P}, \mathcal{L})$.

Proof. Again denote $\Pi_j^n(\bullet)=\bullet_j$ and put $u[j,j]_1(\mathcal{P}_j, \mathcal{L}_j)=k$, $u(\mathcal{P}, \mathcal{L})=k'$, and suppose $2k>j$. Choose Q such that $Q_j I \mathcal{L}_j$ and $u(Q_j, \mathcal{P}_j)=k$. Choose $\mathcal{M} I Q$ and such that $u(\mathcal{L}, \mathcal{M}) \geqslant j$ (cp. Property (2.25)). By Property (2.32), $\mathcal{L}$ meets $\mathcal{O}^{n-i}(Q)$, where $i=[(j+1)/2]$. Since $k \geqslant i$, we have $u(\mathcal{P}, Q') \geqslant i$ for all $Q' \in \sigma'(\mathcal{L}) \cap \mathcal{O}^{n-1}(Q)$. Hence $k' \geqslant i$ and thus $2k' \geqslant j$, contradicting our assumptions. Hence $2k \leqslant j$ and the result follows from the previous property. Q.E.D.

PROPERTY (2.35). $u_2(\mathcal{P}, \mathcal{L})=\sup\{i \leqslant n \mid \Pi_i^n(\mathcal{P}) I \Pi_i^n(\mathcal{L})\}$.

Proof. (1) $u_2(\mathcal{P}, \mathcal{L}) \leqslant \sup\{i \leqslant n \mid \Pi_i^n(\mathcal{P}) I \Pi_i^n(\mathcal{L})\}$ by the definition of $u_2(\mathcal{P}, \mathcal{L})$.

(2) $u_2(\mathcal{P}, \mathcal{L}) \geqslant \sup\{i \leqslant n \mid \Pi_i^n(\mathcal{P}) I \Pi_i^n(\mathcal{L})\}$ by Property (2.25). Q.E.D.

PROPERTY (2.36). *If* $u(\mathcal{P}, \mathcal{L})=(k,l)$, $2k \leqslant n$, *and* $\mathcal{L}' \in \mathcal{L}^{n-i}(\mathcal{P})$ *is a component of* $\mathcal{L}$ *in* $\mathcal{W}_{n-i}(\mathcal{V}_n, \mathcal{O}^{n-i}(\mathcal{P}))$, $i \leqslant 2k-l$, *then* $u[n, n-i](\mathcal{P}, \mathcal{L}')=(k-i, l-i)$.

Proof. Similar to the proof of Property (2.31), one shows here that $u'(\mathscr{P}, \mathscr{L}') = k - i$ and $u'(\mathscr{P}, \mathscr{L}') = l' - i \geq l - i$, where $u' = u[n, n-i]$. By Property (2.35), $\nabla_{l'-i}^{n-i}(\mathscr{P}) \, I \, \nabla_{l'-i}^{n-i}(\mathscr{L}')$. By Property (2.20), this implies $\Pi_{l'}^{n}(\mathscr{P}) \, I \, \Pi_{l'}^{n}(\mathscr{L})$. Consequently $l' \leq l$ by Property (2.35). Q.E.D.

PROPERTY (2.37). *If $u(\mathscr{L}, \mathscr{M}) = j$ and $\mathscr{L} \, I \, \mathscr{P} \, I \, \mathscr{M}$, then $\mathscr{L}$ and $\mathscr{M}$ share a common component in $\mathscr{W}_j(\mathscr{V}_n, \mathcal{O}^j(\mathscr{P}))$ incident with $\mathscr{P}$.*

Proof. By Property (2.7), j is even. We now fix $j \in 2\mathbb{N}$ and proceed by means of induction on $n > j$.

(1) For $n = j + 1$, the property is equivalent to Property (2.23).

(2) Suppose $n > j + 1$. Again denote $\Pi_{n-1}^{n}(\bullet) = \bullet_{n-1}$. By the induction hypothesis, $\mathscr{L}_{n-1}$ and $\mathscr{M}_{n-1}$ share a component $\mathscr{K}$ incident with $\mathscr{P}_{n-1}$ in $\mathscr{W}_j(\mathscr{V}_{n-1}, \mathcal{O}^j(\mathscr{P}_{n-1})) = \mathscr{W}_j(\mathscr{V}_n, \mathcal{O}^{j+1}(\mathscr{P}))$. Now let $\mathscr{L}'$ (resp. $\mathscr{M}'$) the component of $\mathscr{L}$ (resp. $\mathscr{M}$) in $\mathscr{W}_{j+1}(\mathscr{V}_n, \mathcal{O}^{j+1}(\mathscr{P}))$ incident with $\mathscr{P}$. By Properties (2.16) and (2.20), $\nabla_j^{j+1}(\mathscr{L}') = \nabla_j^{j+1}(\mathscr{M}') = \mathscr{K}$. The result now follows from Property (2.23). Q.E.D.

Remark (2.38). In the above property, we can substitute $u(\mathscr{L}, \mathscr{M}) \geq j$ for $u(\mathscr{L}, \mathscr{M}) = j$ and the same conclusion remains valid (by Property (2.29)).

PROPERTY (2.39). *Suppose $u(\mathscr{P}, \mathscr{L}) = (k, l)$ with $l < 2 \cdot k \leq n$. Denote $\mathcal{O}^{n-2k+l}(\mathscr{P}) = \mathscr{C}$ and let $\mathscr{L}'$ be an arbitrary component of $\mathscr{L}$ in $\mathscr{W}_{n-2k+1}(\mathscr{V}_n, \mathscr{C})$. Then there exists a unique line $\mathscr{M}'$ in $\mathscr{W}_{n-2k+1}(\mathscr{V}_n, \mathscr{C})$ such that $\mathscr{P} \, I \, \mathscr{M}' \perp \mathscr{L}'$. Moreover, all elements of $\sigma(\mathscr{M}') \cap \sigma(\mathscr{L}')$ are affine points of $(\mathscr{W}_{n-2k+1}(\mathscr{V}_n, \mathscr{C}), \mathscr{N}_{\mathscr{C}})$.*

Proof. The existence and the uniqueness of $\mathscr{M}'$ follows from Property (2.31) and (GQ2). We now show the second assertion. By Property (2.36), we can assume without loss of generality that $l = 2k - 1$. Hence $\mathscr{L}', \mathscr{M}' \in \mathscr{L}^{n-1}(\mathscr{P})$ and $u[n, n-1](\mathscr{P}, \mathscr{L}') = (k-1, 2(k-1))$. Suppose that $\mathscr{L}'$ and $\mathscr{M}'$ are incident with a common point at infinity. From $u[n, n-1](\mathscr{L}', \mathscr{M}') = 2k - 2$, we deduce $u[2k, 2k-1](\nabla_{2k-1}^{n-1}(\mathscr{L}'), \nabla_{2k-1}^{n-1}(\mathscr{M}')) = 2k - 2$ and hence by (NP2), there exists a line $\mathscr{M} \in \mathscr{L}(\mathscr{V}_n)$ such that $\nabla_{2k-1}^{n-1}(\mathscr{L}')$ and $\nabla_{2k-1}^{n-1}(\mathscr{M}')$ are components of $\Pi_{2k}^{n}(\mathscr{M})$. By Property (2.25), we can choose $\mathscr{M}$ incident with $\mathscr{P}$. By Property (2.20), $\nabla_{2k-1}^{n-1}(\mathscr{L}') < \Pi_{2k}^{n}(\mathscr{L})$. As a consequence of Property (2.28), $\Pi_{2k}^{n}(\mathscr{L}) = \Pi_{2k}^{n}(\mathscr{M})$ and this implies $l \geq 2k$, a contradiction. Q.E.D.

PROPERTY (2.40). *If $u(\mathscr{P}, \mathscr{L}) = (k, l)$, $2k < n$, and $\mathscr{P} \, I \, \mathscr{M} \, I \, Q \, I \, \mathscr{L}$, then $u(\mathscr{P}, Q) \geq 2k - l$.*

Proof. This is trivial for $l = 2k$, so assume $2k > l$. Suppose $u(\mathscr{P}, Q) =$

$j<2k-l$. We seek a contradiction. Let $\mathcal{M}_{\mathcal{P}}$ (resp. $\mathcal{M}_Q$) be the component of $\mathcal{M}$ in $\mathcal{W}_{n-j}(\mathcal{V}_n, \mathcal{O}^{n-j}(\mathcal{P}))$ incident with $\mathcal{P}$ (resp. Q). We know $u[n, n-j](\mathcal{M}_{\mathcal{P}}, \mathcal{M}_Q) \geq n-2j$. Let $\mathcal{L}_Q$ be the component of $\mathcal{L}$ in $\mathcal{W}_{n-j}(\mathcal{V}_n, \mathcal{O}^{n-j}(\mathcal{P}))$ incident with Q. By Property (2.36), $u[n, n-j](\mathcal{P}, \mathcal{L}_Q)=(k-j, l-j)$. Since $2(k-j)<(n-j)-j$, we can apply Property (2.34) and obtain $u[n-j, n-2j](\nabla^{n-j}_{n-2j}(\mathcal{P}), \nabla^{n-j}_{n-2j}(\mathcal{L}_Q))=(k-j, l-j)$. But $\nabla^{n-j}_{n-2j}(\mathcal{M}_{\mathcal{P}})=\nabla^{n-j}_{n-2j}(\mathcal{M}_Q)$ and hence $\nabla^{n-j}_{n-2j}(\mathcal{L}_Q)\, I\, \nabla^{n-j}_{n-2j}(Q)\, I\, \nabla^{n-j}_{n-2j}(\mathcal{M}_Q)\, I\, \nabla^{n-j}_{n-2j}(\mathcal{P})$. But $u[n-j, n-2j](\nabla^{n-j}_{n-2j}(\mathcal{P}), \nabla^{n-j}_{n-2j}(Q))=0$ by Property (2.8), and we conclude with Property (2.9), $2(k-j)=l-j$, a contradiction. Q.E.D.

PROPERTY (2.41). *If $u(\mathcal{P}, \mathcal{L})=(k, l)$ with $2k<n$. then there exists a line $\mathcal{M}$ incident with $\mathcal{P}$ and concurrent with $\mathcal{L}$.*

Proof. For $l=2k$, this follows from (GQ2), so suppose $l<2k$. Choose an arbitrary component $\mathcal{L}'$ of $\mathcal{L}$ in $\mathcal{W}_j(\mathcal{V}_n, \mathcal{O}^j(\mathcal{P}))$, with $j=n-2k+l$. Then, by Property (2.31), $u[n, j](\mathcal{P}, \mathcal{L}')=(l-k, 2l-2k)$. Choose $\mathcal{M}' \in \mathcal{L}^j(\mathcal{P})$ such that $\mathcal{P}\, I\, \mathcal{M}' \perp \mathcal{L}'$, then $\mathcal{L}'$ and $\mathcal{M}'$ meet in affine points, i.e., in points of $\mathcal{O}^j(\mathcal{P})$ by Property (2.39). It suffices now to choose $\mathcal{M}$ such that $\mathcal{M}'<\mathcal{M}$, which is possible by Property (2.18). Q.E.D.

PROPERTY (2.42). *If $u(\mathcal{P}, \mathcal{L})=(k, l)$, $2k<n$, $\mathcal{M}_1$, $\mathcal{M}_2 \in \mathcal{L}(\mathcal{V}_n)$, and $\mathcal{P}\, I\, \mathcal{M}_i \perp \mathcal{L}$, $i=1, 2$, then $u(\mathcal{M}_1, \mathcal{M}_2) \geq n-4k+2l$.*

Proof. For $l=2k$, this follows from (GQ2), so suppose $l<2k$. The strips $\mathbb{S}^n_{2k-1}(\mathcal{P}, \mathcal{M}_i)$ and $\mathbb{S}^n_{2k-1}(\mathcal{P}, \mathcal{L})$, $i=1, 2$, have both width $2k-l$. So we can put $\{\mathcal{L}^*\}=\nabla^{n-2k+1}_{n-4k+2l}(\mathbb{C}^n_{2k-1}(\mathcal{P}, \mathcal{L}))$ and $\{\mathcal{M}_i^*\}=\nabla^{n-2k+1}_{n-4k+2l}(\mathbb{C}^n_{2k-1}(\mathcal{P}, \mathcal{M}_i))$, $i=1$, 2. By Property (2.40), $\mathcal{P}\, I\, \mathcal{M}_i^* \perp \mathcal{L}^*$, $i=1, 2$. But by Properties (2.31) and (2.34) and (GQ2), $\mathcal{M}_1^*=\mathcal{M}_2^*$. Hence by Properties (2.20) and (2.30), we have $u[j, j](\Pi^n_j(\mathcal{M}_1), \Pi^n_j(\mathcal{M}_2)) \geq n-4k+2l$ and thus $u(\mathcal{M}_1, \mathcal{M}_2) \geq n-4k+2l$. Q.E.D.

PROPERTY (2.43). *If $u(\mathcal{P}, \mathcal{L})=(k, l)$, $2k<n$, and $\mathcal{P}\, I\, \mathcal{M} \perp \mathcal{L}$, then $u(\mathcal{L}, \mathcal{M})=2l-2k$.*

Proof. Since $2l-2k<n-4k+2l$, we can assume that $\mathcal{M}$ is constructed as in the proof of Property (2.41) (by the previous property). Throughout this proof we use the same notation as in the proof of Property (2.41). We have $u[n, j](\mathcal{L}', \mathcal{M}')=2l-2k$. Choose $Q\, I\, \mathcal{L}', \mathcal{M}'$, then we have already shown that Q is an affine point (cp. Property (2.39)), i.e., $Q \in \mathcal{P}(\mathcal{V}_n)$. By Property (2.37), $\mathcal{L}'$ and $\mathcal{M}'$ possess the same component incident with Q in $\mathcal{W}_{2l-2k}(V_n, \mathcal{O}^{2l-2k}(Q))$. So by Property (2.29), $\mathcal{L}$ and $\mathcal{M}$ share a component in $\mathcal{W}_{2l-2k}(V_n, \mathcal{O}^{2l-2k}(Q))$. Hence $u(\mathcal{L}, \mathcal{M}) \geq 2l-2k$ (by Property (2.30)). By Remark (2.38), the proper inequality is contradicting $u[n, j](\mathcal{L}', \mathcal{M}')=2l-2k$ and Properties (2.29) and (2.30). Q.E.D.

PROPERTY (2.44). *If* $u(\mathscr{P}, \mathscr{L})$, $2k<n$, *and* $\mathscr{P}\ I\ \mathscr{M}\ I\ Q\ I\ \mathscr{L}$, *then* $u(\mathscr{P}, Q)=2k-l$ *and* $\mathscr{M}$ *is constructed as in the proof of Property* (2.41).

Proof. By Property (2.40), $u(\mathscr{P}, Q)\geqslant 2k-l$ and by Property (2.43), $u(\mathscr{L}, \mathscr{M})=2l-2k$. Let $n-2k+l=j$ and let $\mathscr{M}_Q$ (resp. $\mathscr{L}_Q$) be the component of $\mathscr{M}$ (resp. $\mathscr{L}$) in $\mathscr{W}_j(\mathscr{V}_n, \mathcal{O}^j(\mathscr{P}))$ incident with Q, and let $\mathscr{M}_{\mathscr{P}}$ be the component of $\mathscr{M}$ in $\mathscr{W}_j(\mathscr{V}_n, \mathcal{O}^j(\mathscr{P}))$ incident with $\mathscr{P}$. Similarly as in the proof of Property (2.43), we have here $u[n,j](\mathscr{L}_Q, \mathscr{M}_Q)=2l-2k$. Now let $\mathscr{P}_{\mathscr{L}}$ be a base point of $\mathbb{S}^n_{n-j}(\mathscr{P}, \mathscr{L})$; then by Properties (2.9) and (2.14), $u[n,j](\mathscr{P}_{\mathscr{L}}, \mathscr{M}_Q)=(l-k, 2l-2k)$. From Properties (2.33) and (2.34), we deduce (using $\nabla^{n-2k+l}_{n-4k+2l}$) $u[n,j](\mathscr{P}_{\mathscr{L}}, \mathscr{M}_{\mathscr{P}})=(l-k, 2l-2k)$. As a consequence of Property (2.34) and (GQ2), the unique line $\mathscr{L}_{\mathscr{P}}\in\mathscr{L}^j(\mathscr{P})$ such that $\mathscr{P}_{\mathscr{L}}\ I\ \mathscr{L}_{\mathscr{P}}\perp\mathscr{M}_{\mathscr{P}}$, is a component of $\mathscr{L}$ (again using $\nabla^{n-2k+l}_{n-4k+2l}$). Hence $\mathscr{M}$ is constructed as in the proof of Property (2.41). We now show $u(\mathscr{P}, Q)=2k-l$. We will write briefly ∇ for $\nabla^{n-2k+l}_{n-4k+2l}$. So $\nabla(Q)\ I\nabla(\mathscr{L}_{\mathscr{P}})$, $\nabla(\mathscr{M}_{\mathscr{P}})$ (because $\nabla(\mathfrak{G})$ is a singleton, for the set $\mathfrak{G}$ of components of any strip of width j). But similarly as above, we have $u[j,j-2k+l](\nabla(\mathscr{L}_{\mathscr{P}}), \nabla(\mathscr{M}_{\mathscr{P}}))=2l-2k<n-4k+2l$. Hence $\nabla(\mathscr{L}_{\mathscr{P}})\neq\nabla(\mathscr{M}_{\mathscr{P}})$. So for any $Q'\in\sigma(\mathscr{M}_{\mathscr{P}})\cap\sigma(\mathscr{L}_{\mathscr{P}})$, $u[j,j-2k+l](\nabla(Q),\nabla(Q'))>0$ (Property (2.11)), and hence $u[n,j](Q, Q')>0$. But by Property (2.31), $u[n,j](\mathscr{P}, \mathscr{L}_{\mathscr{P}})=(l-k, 2l-2k)$ and by (GQ2), $u[n,j](\mathscr{P}, Q')=0$. Consequently, $u[n,j](\mathscr{P}, Q)=0$ and we conclude with Property (2.8), $u(\mathscr{P}, Q)=2k-l$.

Q.E.D.

PROPERTY (2.45). *If* $\mathscr{P}, Q\ I\ \mathscr{L}, \mathscr{M}$, *then* $2u(\mathscr{P}, Q)+u(\mathscr{L}, \mathscr{M})\geqslant n$.

Proof. Let $u(\mathscr{P}, Q)=k$ and consider $\mathscr{W}_{n-k}(\mathscr{V}_n, \mathcal{O}^{n-k}(\mathscr{P}))$. The strips $\mathbb{S}^n_k(\mathscr{P}, \mathscr{L})$ and $\mathbb{S}^n_k(\mathscr{P}, \mathscr{M})$ have both width k and thus $\nabla^{n-k}_{n-2k}(\mathbb{C}^n_k(\mathscr{P}, \mathscr{L}))$ and $\nabla^{n-k}_{n-2k}(\mathbb{C}^n_k(\mathscr{P}, \mathscr{M}))$ are singletons. By assumption, $u[n, n-k](\mathscr{P}, Q)=0$ (Property (2.8)) and both $\nabla^{n-k}_{n-2k}(\mathscr{P})$ and $\nabla^{n-k}_{n-2k}(Q)$ are incident with the two unique elements of the above mentioned singletons, which coincide by Property (2.11). So Properties (2.20) and (2.30) imply $u[j,j](\Pi^n_{n-k}(\mathscr{L}), \Pi^n_{n-k}(\mathscr{M}))\geqslant n-2k$. Hence the result. Q.E.D.

We now show a generalization of (GQ2).

PROPERTY (2.46). *If* $u(\mathscr{P}, \mathscr{L})=(k, l)$, $2k<n$, *then there exists a line* $\mathscr{M}$ *and a point* Q *such that* $\mathscr{P}\ I\mathscr{M}\ I\ Q\ I\ \mathscr{L}$. *Moreover, there holds*:

(1) $u(\mathscr{L}, \mathscr{M})=2l-2k$,

(2) $u(\mathscr{P}, Q)=2k-l$,

(3) *If* $\mathscr{M}'\in\mathscr{L}(\mathscr{V}_n)$, $Q'\in\mathscr{P}(\mathscr{V}_n)$, *and* $\mathscr{P}\ I\ \mathscr{M}'\ I\ Q'\ I\ \mathscr{L}$, *then*

(3a) $u(\mathscr{M}, \mathscr{M}')\geqslant n-4k+2l$,

(3b) $u(Q, Q')\geqslant n/2+k-l$.

Proof. The existence of $\mathcal{M}$ and Q with the desired properties and the claims (1), (2), and (3a) follows directly from Properties (2.41), (2.42), (2.43), and (2.44). We show (3b). We use the same notation as in the proof of Property (2.44). So we have $\nabla(Q), \nabla(Q')\ I\ \nabla(\mathcal{L}_{\mathcal{P}}),\ \nabla(\mathcal{M}_{\mathcal{P}})$. By Property (2.45), $u[j, j-2k+l](\nabla(Q), \nabla(Q')) \geqslant (n-4k+2l)/2-(2l-2k) = n/2-k$. Hence $u[n, j](Q, Q') \geqslant n/2-k$. The result now follows from Property (2.8). Q.E.D.

PROPERTY (2.47). *Suppose $\mathcal{P}\ I\ \mathcal{M} \perp \mathcal{L}$, n odd, $\mathcal{L}' \in \mathbb{C}_1^n(\mathcal{P}, \mathcal{L}) \neq \emptyset$, $\Pi_{n-1}^n(\mathcal{P})\ I\ \Pi_{n-1}^n(\mathcal{L})$, $\mathcal{L}' < \mathcal{L}$, and $u(\mathcal{M}, \mathcal{L}) = 0$. Then $\nabla_{n-2}^{n-1}(\mathcal{L}')$ is the unique component of $\Pi_{n-1}^n(\mathcal{L})$ incident with $\Pi_{n-1}^n(\mathcal{P})$.*

Proof. Let $\mathcal{M}' < \mathcal{M}$ with $\mathcal{M}' \in \mathbb{C}_1^n(\mathcal{P}, \mathcal{M})$. Since $\mathcal{M}'$ is unique (n is odd), $\mathcal{M}'$ is incident with $\mathcal{P}$ in $\mathcal{W}_{n-1}(\mathcal{V}_n, \mathcal{O}^{n-1}(\mathcal{P}))$. So $\nabla_{n-2}^{n-1}(\mathcal{M}')$ is the unique component of $\Pi_{n-1}^n(\mathcal{M})$ incident with $\Pi_{n-1}^n(\mathcal{P})$. Since $\mathcal{L}' \perp \mathcal{M}'$, we also have $\nabla_{n-2}^{n-1}(\mathcal{L}') \perp \nabla_{n-2}^{n-1}(\mathcal{M}')$, but only one component of $\Pi_{n-1}^n(\mathcal{L})$ can meet the line $\nabla_{n-2}^{n-1}(\mathcal{M}')$ (otherwise in contradiction with (GQ2) and Property (2.14); cp. also (2.49)) and that must be the one incident with $\Pi_{n-1}^n(\mathcal{P})$. Q.E.D.

PROPERTY (2.48). *Suppose $\mathcal{L}\ I\ \mathcal{P}\ I\ \mathcal{M}$, n odd, $u(\mathcal{P}, Q) = 0$, $\mathcal{L}', \mathcal{M}' \in \mathcal{L}^{n-1}(Q)$, $\mathcal{L}' < \mathcal{L}$. $\mathcal{M}' < \mathcal{M}$ and $u[n, n-1](\mathcal{L}', M') \geqslant n-2$. Then $\mathcal{L}' = \mathcal{M}'$ and $\mathcal{L} = \mathcal{M}$.*

Proof. Since $\nabla_{n-2}^{n-1}(\mathcal{L}') = \nabla_{n-2}^{n-1}(\mathcal{M}')$ is a component of both $\Pi_{n-1}^n(\mathcal{L})$ and $\Pi_{n-1}^n(\mathcal{M})$, we already have $u(\mathcal{L}, \mathcal{M}) \geqslant n-1$. Suppose now $u(\mathcal{L}, \mathcal{M}) = n-1$. Let $Q' \in \mathcal{O}^{n-1}(Q)$ be arbitrary but incident with $\mathcal{M}$, then Q' is not incident with $\mathcal{L}$ by Property (2.11), but $u[n, n-1]_2\,(Q', \mathcal{L}') = n-2$. Since $\mathcal{L}'$ is the unique component of $\mathcal{L}$ in $\mathcal{W}_{n-1}(\mathcal{V}_n, \mathcal{O}^{n-1}(Q))$, it follows from Property (2.8) that $u[n, n-1]_1\,(Q', \mathcal{L}') + 1 = u_1(Q', \mathcal{L})$. But by Property (2.10), $u(Q', \mathcal{L}) = ((n-1)/2, n-1)$. So $u[n, n-1](Q', \mathcal{L}') = ((n-3)/2, n-2)$, contradicting Property (2.6). Q.E.D.

PROPERTY (2.49). *Suppose $\sigma(\mathcal{L}) \cap \mathcal{O}^{n-1}(\mathcal{P}) \neq \emptyset \neq \sigma(\mathcal{M}) \cap \mathcal{O}^{n-1}(\mathcal{P})$, $u(\mathcal{L}, \mathcal{M}) = n-2$, $\nabla_{n-2}^{n-1}(\mathbb{C}_1^n(\mathcal{P}, \mathcal{L})) = \nabla_{n-2}^{n-1}(\mathbb{C}_1^n(\mathcal{P}, \mathcal{M}))$, and n even. Then for every $\mathcal{L}' \in \mathbb{C}_1^n(\mathcal{P}, \mathcal{L})$, there exists a unique $\mathcal{M}' \in \mathbb{C}_1^n(\mathcal{P}, \mathcal{M})$ concurrent with $\mathcal{L}'$.*

Proof. If $n = 2$, then the assertion is clear, since $\mathcal{W}_1(\mathcal{V}_2, \mathcal{O}^1(\mathcal{P}))$ is a generalized quadrangle. So suppose $n > 2$. We first show that there is a pair $(\mathcal{L}'', \mathcal{M}'') \in \mathbb{C}_1^n(\mathcal{P}, \mathcal{L}) \times \mathbb{C}_1^n(\mathcal{P}, \mathcal{M})$ with the property that $\mathcal{L}'' \perp \mathcal{M}''$ in $\mathcal{W}_{n-1}(\mathcal{V}_n, \mathcal{O}^{n-1}(\mathcal{P}))$. Let Q be arbitrary but incident with $\mathcal{L}$ and not lying in $\mathcal{O}^{n-1}(\mathcal{P})$. We again denote $\Pi_{n-1}^n(\bullet) = \bullet_{n-1}$. By Property (2.9) and the fact that $\mathcal{L}_{n-1} \perp \mathcal{M}_{n-1}$, we have $u[n-1, n-1](Q_{n-1}, \mathcal{M}_{n-1}) =$

$((n-2)/2, n-2)$, and so by Property (2.33), $u(Q, \mathscr{M}) = ((n-2)/2, n-2)$. Hence there exists by (GQ2) a unique line $\mathscr{K} \in \mathscr{L}(\mathscr{V}_n)$ incident with Q and concurrent with $\mathscr{M}$. But (GQ2) applied in $\mathscr{V}_{n-1}$ implies that $\mathscr{K}_{n-1} = \mathscr{L}_{n-1}$ and, since Q is incident with both $\mathscr{K}$ and $\mathscr{L}$ and $n-1$ is odd, it follows from Property (2.7) that $\mathscr{K} = \mathscr{L}$. By Property (2.11) applied in $\mathscr{V}_{n-1}$, all points incident with both $\mathscr{L}$ and $\mathscr{M}$ must be elements of $\mathcal{O}^{n-1}(\mathscr{P})$. So at least one $\mathscr{L}'' \in \mathbb{C}_1^n(\mathscr{P}, \mathscr{L})$ meets at least one $\mathscr{M}'' \in \mathbb{C}_1^n(\mathscr{P}, \mathscr{M})$. Let $\mathscr{P}_{\mathscr{L}}$ (resp. $\mathscr{P}_{\mathscr{M}}$) be a base point of $\mathbb{S}_1^n(\mathscr{P}, \mathscr{L})$ (resp. $\mathbb{S}_1^n(\mathscr{P}, \mathscr{M})$). By Properties (2.15) and (2.32), $\mathcal{O}^{n-2}(\mathscr{P}_{\mathscr{L}}) = \mathcal{O}^{n-2}(\mathscr{P}_{\mathscr{M}})$. By Property (2.23), the shadow of every element of $\mathbb{C}_1^n(\mathscr{P}, \mathscr{L})$ (resp. $\mathbb{C}_1^n(\mathscr{P}, \mathscr{M})$) meets $\mathcal{O}^{n-2}(\mathscr{P}_{\mathscr{L}}) = \mathcal{O}^{n-2}(\mathscr{P}_{\mathscr{M}})$ in the same subset ($n-1$ is odd!). Since $\mathscr{L}''$ and $\mathscr{M}''$ only have affine points in common, it follows from Properties (2.9) and (2.14) that $u[n, n-1](\mathscr{P}_{\mathscr{M}}, \mathscr{L}'') = ((n-2)/2, n-2)$ (taking into account the assumption $u[n, n-1](\mathscr{L}'', \mathscr{M}'') = n-2$). Similar to an argument in the foregoing proof, one shows that $u[n, n-1](\mathscr{P}_{\mathscr{M}}, \mathscr{L}') = ((n-2)/2, n-2)$, for every $\mathscr{L}' \in \mathbb{C}_1^n(\mathscr{P}, \mathscr{L})$. The result now follows directly from (GQ2) and the definition of strip. Q.E.D.

PROPERTY (2.50). *If $\Pi_{n-2}^n(\mathscr{P})\, I\, \Pi_{n-2}^n(\mathscr{L})$ for $n \geq 2$, then there exist distinct lines $\mathscr{L}_1, \mathscr{L}_2 \in \mathcal{O}^2(\mathscr{L})$ incident with $\mathscr{P}$.*

Proof. We proceed by means of induction on $n \in \mathbb{N}$, $n \geq 2$.

(1) $n = 2$. This is trivial since $\mathscr{V}_2$ is a thick geometry.

(2) $n > 2$ and n odd. By Property (2.25), we can put $\mathscr{L} = \mathscr{L}_1$. Now let $\mathscr{M}$ be such that $\mathscr{M} \perp \mathscr{L}$, $u(\mathscr{L}, \mathscr{M}) = 0$, and $u(\mathscr{P}, \mathscr{M}) = (0, 0)$ (cp. Property (2.6)). Let $Q \in \sigma(\mathscr{L}) \cap \sigma(\mathscr{M})$ be arbitrary and let $\mathscr{M}'$ be the unique component of $\mathscr{M}$ in $\mathscr{W}_{n-1}(\mathscr{V}_n, \mathcal{O}^{n-1}(Q))$. Let $\mathscr{P}_\infty$ be an arbitrary point at infinity of $(\mathscr{W}_{n-1}(\mathscr{V}_n, \mathcal{O}^{n-1}(Q)), \mathscr{N}_{\mathcal{O}^{n-1}(Q)})$ incident with $\mathscr{M}'$ in $\mathscr{W}_{n-1}(\mathscr{V}_n, \mathcal{O}^{n-1}(Q))$. Let $\mathscr{K}' \in \mathcal{O}^2(\mathscr{M}')$ be a line of $\mathscr{W}_{n-1}(\mathscr{V}_n, \mathcal{O}^{n-1}(Q))$ incident with $\mathscr{P}_\infty$ ($\mathscr{K}'$ exists by the induction hypothesis). Let $\mathscr{K} \in \mathscr{L}(\mathscr{V}_n)$ be such that $\mathscr{K}' < \mathscr{K}$. Since a strip of width 1 is completely determined by any of its components, it follows from Properties (2.20) and (2.28) by projecting onto $\mathscr{V}_{n-1}$ that $u(\mathscr{M}, \mathscr{K}) = n-1$. With Properties (2.35) and (2.6), we deduce $u(\mathscr{P}, \mathscr{K}) = (0, 0)$. Now define $\mathscr{L}_2 \in \mathscr{L}(\mathscr{V}_n)$ as the unique line for which $\mathscr{P}\, I\, \mathscr{L}_2 \perp \mathscr{K}$ (cp.(GQ2)). Let $\mathscr{L}'$ be the component of $\mathscr{L}$ in $\mathscr{W}_{n-1}(\mathscr{V}_n, \mathcal{O}^{n-1}(Q))$, then $u[n, n-1](\mathscr{P}_\infty, \mathscr{L}') = (0, 0)$ and so by uniqueness of $\mathscr{M}'$ (cp.(GQ2)), $\mathscr{L}'$ does not meet $\mathscr{K}'$. But if $\sigma(\mathscr{L}) \cap \sigma(\mathscr{K}) \neq \varnothing$, then it must be a subset of $\mathcal{O}^{n-1}(Q)$ (project down onto $\mathscr{V}_{n-2}$ to see that), and so $\mathscr{L}'$ must meet $\mathscr{K}'$. Hence $\mathscr{L}$ and $\mathscr{K}$ does not meet and $\mathscr{L}_2$ is distinct from $\mathscr{L}$! By projecting down onto $\mathscr{V}_{n-1}$ and by the uniqueness part of (GQ2), we conclude (since $u(\mathscr{K}, \mathscr{M}) = n-1$) that $\mathscr{L}_2 \in \mathcal{O}^1(\mathscr{L}) \subseteq \mathcal{O}^2(\mathscr{L})$.

(3) $n>2$ and n even. Obviously $\Pi_{n-3}^{n-1}(\Pi_{n-1}^{n}(\mathscr{P}))\ I\ \Pi_{n-3}^{n-1}(\Pi_{n-1}^{n}(\mathscr{L}))$ with $n-1$ odd. So by the induction hypothesis, there exist lines $\Pi_{n-1}^{n}(\mathscr{L}_1)$ and $\Pi_{n-1}^{n}(\mathscr{L}_2)$, both incident with $\Pi_{n-1}^{n}(\mathscr{P})$ and both belonging to $\mathcal{O}^2(\Pi_{n-1}^{n}(\mathscr{L}))$. By Property (2.7) and the fact that $n-2$ is even, $\Pi_{n-2}^{n}(\mathscr{L})=\Pi_{n-2}^{n}(\mathscr{L}_1)=\Pi_{n-2}^{n}(\mathscr{L}_2)$ (after all, these three lines are all incident with $\Pi_{n-2}^{n}(\mathscr{P})$). By Property (2.25), we can choose $\mathscr{L}_1$ and $\mathscr{L}_2$ incident with $\mathscr{P}$. Q.E.D.

PROPERTY (2.51). *If* $n\geq 2$, $j\leq n/2$ *and* $\sigma(\Pi_{n-2}^{n}(\mathscr{L}))\cap\mathcal{O}^{j-1}(\Pi_{n-2}^{n}(\mathscr{P}))\neq\emptyset$, *then there exist two points* $\mathscr{P}_1,\mathscr{P}_2\in\mathcal{O}^{j+1}(\mathscr{P})$ *satisfying*:

(1) $\sigma(\mathscr{L})\cap\mathcal{O}^{j}(\mathscr{P}_i)\neq\emptyset$, $i=1,2$,

(2) $\mathcal{O}^{j}(\mathscr{P}_1)\neq\mathcal{O}^{j}(\mathscr{P}_2)$.

Proof. Without loss of generality, we can assume that $\Pi_{n-2}^{n}(\mathscr{L})\ I\ \Pi_{n-2}^{n}(\mathscr{P})$. There exists at least one line $\mathscr{M}\ I\ \mathscr{P}$ such that $u(\mathscr{L},\mathscr{M})\geq n-2$ (by Property (2.25)). Now $n-2\geq 2j-1$ and so by Property (2.32), $\sigma(\mathscr{L})\cap\mathcal{O}^{j+1}(\mathscr{P})\neq\emptyset$. The result now follows from Property (2.15) by considering an arbitrary component of $\mathscr{L}$ in $\mathscr{W}_{j+1}(\mathscr{V}_n,\mathcal{O}^{j+1}(\mathscr{P}))$. Q.E.D.

Remark (2.52). Note that all above properties remain true if we replace the projection Π_j^k by HQ-epimorphisms (obviously). In view of that remark, we can prove the next property.

PROPERTY (2.53). *The inverse limit of the base sequence of any HQ-Artmann-sequence is a generalized quadrangle of order* (s, t), *where both* s *and* t *are infinite, but set-theoretically not necessarily equal.*

Proof. We now leave our usual notation and let $(\mathscr{Y}_n,\Omega_n^{n+1})_{n\in\mathbb{N}}$ (with $\mathscr{Y}_n=(\mathscr{P}(\mathscr{Y}_n),\mathscr{L}(\mathscr{Y}_n),I))$ be the base sequence of an arbitrary HQ-Artmann-sequence. We denote the inverse limit of this sequence by $\mathscr{Y}_\infty=(\mathscr{P}(\mathscr{Y}_\infty),\mathscr{L}(\mathscr{Y}_\infty),I)$. Note that by Property (2.50), there are infinitely many lines incident with any point and dually, by Property (2.51), there are infinitely many points incident with any line in $\mathscr{Y}_\infty$. Hence (QQ1) and (QQ2) hold. By surjectivity of Ω_n^{n+1} and the fact that $\mathscr{Y}_1$ is a generalized quadrangle, also (QQ3) holds. We now check (QQ4). So let $\mathscr{P}_\infty\in\mathscr{P}(\mathscr{Y}_\infty)$, $\mathscr{L}_\infty\in\mathscr{L}(\mathscr{Y}_\infty)$ with $\mathscr{P}_\infty=(\mathscr{P}_j)_{j\in\mathbb{N}}$, $\mathscr{L}_\infty=(\mathscr{L}_j)_{j\in\mathbb{N}}$, $\mathscr{P}_j\in\mathscr{P}(\mathscr{Y}_j)$, $\mathscr{L}_j\in\mathscr{L}(\mathscr{Y}_j)$, and $\mathscr{P}_\infty$ not incident with $\mathscr{L}_\infty$ in $\mathscr{Y}_\infty$. This means : for certain $n\in\mathbb{N}$, $\mathscr{P}_n$ is not incident with $\mathscr{L}_n$. Hence $\mathscr{P}_{2n}$ is not incident with $\mathscr{L}_{2n}$ in $\mathscr{Y}_{2n}$. By Properties (2.6) and (2.35), $u[2n,2n]_1(\mathscr{P}_{2n},\mathscr{L}_{2n})<n$. Now choose $m\geq 2n$ arbitrarily. By Property (2.33), $u[m,m](\mathscr{P}_m,\mathscr{L}_m)=(k,l)$ does not depend on m. By Property (2.46), there exists a line $\mathscr{M}_m\in\mathscr{L}(\mathscr{V}_m)$ such that $\mathscr{P}_m\ I\ \mathscr{M}_m\perp\mathscr{L}_m$. We define $\mathscr{K}_i^m=\Omega_i^m(\mathscr{M}_m)$, $i\leq m-4k+2l$. By Property (2.46)(3a), $\mathscr{K}_i^m$ does not depend on $\mathscr{M}_m$ for fixed m. Note that, if $m'<m$, we can choose

$\mathcal{M}_{m'} = \Omega^m_{m'}(\mathcal{M}_m)$. Hence $\mathcal{K}^m_i$ does not depend on the choice of $m \geqslant i + 4k - 2l$ and we can denote it by $\mathcal{K}_i$. So $\mathcal{K}_\infty = (\mathcal{K}_j)_{j \in \mathbb{N}} \in \mathcal{L}(\mathcal{Y}_\infty)$ is well defined and $\mathcal{P}_\infty \, I \, \mathcal{K}_\infty \perp \mathcal{L}_\infty$. Clearly $\mathcal{K}_\infty$ is unique (by applying Property (2.46) (3a) again). Dually, one shows completely analogously (now using Property (2.46)(3b), of course) that there is a unique point $Q_\infty \in \mathcal{P}(\mathcal{Y}_\infty)$ incident with $\mathcal{L}_\infty$ and collinear with $\mathcal{P}_\infty$. Hence $\mathcal{P}_\infty \, I \, \mathcal{K}_\infty \, I \, Q_\infty \, I \, \mathcal{L}_\infty$ with $(\mathcal{K}_\infty, Q_\infty)$ unique.

Q.E.D.

3. Definition of the Building Corresponding to an HQ-Artmann-Sequence

In this section, we are given an HQ-Artmann-sequence $(\mathcal{V}_n, \Pi^{n+1}_n)_{n \in \mathbb{N}}$, with $\mathcal{V}_n$ as in Section 2. We also keep the notation ∇^i_j of the previous section. We now define a simplicial complex $\Delta(X, \mathcal{S})$ with X de set of vertices and $\mathcal{S}$ the set of simplices. The dimension of $\Delta(X, \mathcal{S})$ will be 2, i.e., the cardinality of the maximal simplices is 3. We denote:

$$\mathbb{B}^j_n = \{\mathcal{L} \in \mathcal{L}^j(\mathcal{P}) \mid \mathcal{P} \in \mathcal{P}(\mathcal{V}_n)\}, \qquad 0 < j < n,$$

$$\mathbb{B}^0_n = \mathcal{P}(\mathcal{V}_n),$$

$$\mathbb{B}^n_n = \mathcal{L}(\mathcal{V}_n).$$

In all other cases $\mathbb{B}^j_n$ is the empty set (j and n integers). We can define:

$$X = \bigcap \{\mathbb{B}^j_n \mid 0 \leqslant j \leqslant n \in \mathbb{N}\}.$$

We now define $\mathcal{S}$. let $x \in \mathbb{B}^j_n$ and $y \in X$, then $\{x, y\} \in \mathcal{S}$ if one of the following conditions are satisfied:

(A1) $y \in \mathbb{B}^{j+1}_n$ and $x < y$,

(A1′) $y \in \mathbb{B}^{j-1}_n$ and $y < x$,

(A2) $y \in \mathbb{B}^{j-1}_{n-1}$ and $\nabla^j_{j-1}(x) = y$,

(A2′) $y \in \mathbb{B}^{j+1}_{n+1}$ and $\nabla^{j+1}_j(y) = x$,

(A3) $y \in \mathbb{B}^j_{n-1}$ and $\nabla^j_{j-1}(x) < y$ and j is even,

(A3′) $y \in \mathbb{B}^j_{n+1}$ and $\nabla^j_{j-1}(y) < x$ and j is even,

(A4) $y \in \mathbb{B}^{j-2}_{n-1}$ and $y < \nabla^j_{j-1}(x)$ and j is even,

(A4′) $y \in \mathbb{B}^{j+2}_{n+1}$ and $x < \nabla^{j+2}_{j+1}(y)$ and j is even.

Here, "$\mathcal{P} \, I \, \mathcal{L}$" is for the sake of brevity denoted by "$\mathcal{P} < \mathcal{L}$" and "$\Pi^{\bullet}_{\bullet}$" by "$\nabla^{\bullet}_{\bullet}$." The 2-dimensional simplices are by definition the 3-sets $\{x, y, z\}$,

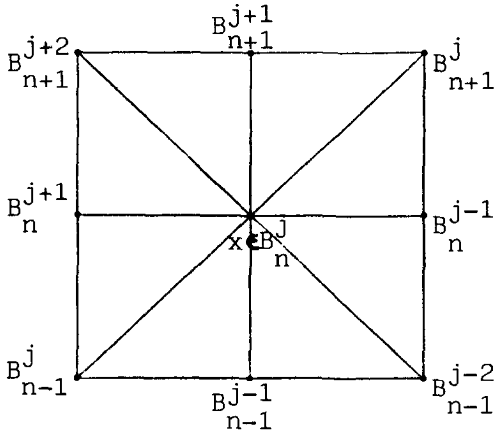


FIGURE 1

where every 2-subset is a 1-dimensional simplex. We now define a type-map *typ* on the set X of vertices:

$$typ\colon X \to [\{0\}, \{1\}, \{0, 1\}] : x \in \mathbb{B}_n^j \to \{n (\mathrm{mod}\ 2), n - j (\mathrm{mod}\ 2)\}.$$

All possible kinds of "adjacencies" with a fixed vertex $x \in \mathbb{B}_n^j$ for j even can be pictured as in Fig. 1.

In Fig. 1, two sets are connected if they contain respective elements which are adjacent in $\Delta(X, \mathscr{S})$ (this is easy to see using the properties of Section 2).

Now let j again be arbitrary and $x \in \mathbb{B}_n^j$. We denote the *residue* of x by $\mathscr{R}(x) = (\mathscr{P}(x), \mathscr{L}(x), I_x)$, where

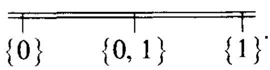
$$\begin{aligned} \text{(R1)} \quad \mathscr{P}(x) &= \{ y \in \mathbb{B}_{n+1}^{j+1} \cup \mathbb{B}_n^{j-1} \cup \mathbb{B}_n^{j+1} \cup \mathbb{B}_{n-2}^{j-1} \mid \{x, y\} \in \mathscr{S} \}, \\ \mathscr{L}(x) &= \{ z \in \mathbb{B}_{n+1}^{j} \cup \mathbb{B}_{n-1}^{j} \cup \mathbb{B}_{n-1}^{j-2} \cup \mathbb{B}_{n+1}^{j+2} \mid \{x, z\} \in \mathscr{S} \}, \\ I_x &= \{ (y, z), (z, y) \mid y \in \mathscr{P}(x), z \in \mathscr{L}(x), \\ &\qquad \text{and} \quad \{y, z\} \in \mathscr{S} \}, j \text{ even}; \end{aligned}$$

$$\begin{aligned} \text{(R2)} \quad \mathscr{P}(x) &= \{ y \in \mathbb{B}_n^{j-1} \cup \mathbb{B}_n^{j+1} \mid \{x, y\} \in \mathscr{S} \}, \\ \mathscr{L}(x) &= \{ z \in \mathbb{B}_{n-1}^{j-1} \cup \mathbb{B}_{n+1}^{j+1} \mid \{x, z\} \in \mathscr{S} \}, \\ I_x &= \{ (y, z), (z, y) \mid y \in \mathscr{P}(x), z \in \mathscr{L}(x), \\ &\qquad \text{and} \quad \{y, z\} \in \mathscr{S} \}, j \text{ odd}. \end{aligned}$$

By introducing a type-map, we turned $\Delta(X, \mathscr{S})$ into an incidence structure of rank 3. We denote it from now on briefly by Δ. In the next section, we show that Δ is an affine building of $\tilde{C}_2$.

4. Δ Is a Building of Type $\tilde{C}_2$

THEOREM (4.1). *The rank* 3 *incidence structure* Δ, *as defined above, is a thick geometry of type* $\tilde{C}_2$ *with diagram*



Proof. Note that Δ is connected since every vertex is joined to the unique element of $\mathbb{B}_0^0$ by consecutive projections. Now let $x \in \mathbb{B}_0^j$. There are two distinct possibilities.

(1) j is odd. Then $typ(x) = \{0, 1\}$ and we must show that $\mathscr{R}(x)$ is a thick generalized digon. So let $(y, z) \in \mathscr{P}(x) \times \mathscr{L}(x)$. We show $y\, I_x\, z$, i.e., $\{y, z\} \in \mathscr{S}$. As an example, we do this for the case $y \in \mathbb{B}_n^{j+1}$ and $z \in \mathbb{B}_{n-1}^{j-1}$, $j < n$. All other cases are similar (including $j = n$). So, by definition, $x < y$ and $\nabla_{j-1}^{j}(x) = z$. Hence $z = \nabla_{j-1}^{j}(x) < \nabla_{j-1}^{j}(y)$ by Property (2.20). Since $j + 1$ is even, we have by definition $\{y, z\} \in \mathscr{S}$.

To show that $\mathscr{R}(x)$ is thick , it suffices to find three points and three lines in $\mathscr{R}(x)$. Obviously, $\mathscr{P}(x) \cap \mathbb{B}_n^{j+1} \neq \varnothing$ (by (NP2)) and $\mathscr{L}(x) \cap \mathbb{B}_{n-1}^{j-1} \neq \varnothing$ (it contains $\nabla_{j-1}^{j}(x)$). By Property (2.15), we also have $|\mathscr{P}(x) \cap \mathbb{B}_n^{j-1}| \geqslant 2$ and so $|\mathscr{P}(x)| \geqslant 3$. By (PS4), we have $|\mathscr{L}(x) \cap \mathbb{B}_{n+1}^{j+1}| \geqslant 2$ and so $|\mathscr{L}(x)| \geqslant 2$ and so $|\mathscr{L}(x)| \geqslant 3$. Hence $\mathscr{R}(x)$ is thick.

(2) j is even. Now we have to show that $\mathscr{R}(x)$ is a thick generalized quadrangle. We will assume $n \neq j \neq 0$, the cases $j = 0$ and $j = n$ being special cases of this with either a similar proof (here and there a subcase less) or a very simple one (e.g., for $j = 0$, everything happens in a level 2 H-Q). Note that the case $n = j = 0$ is trivial since $\mathscr{V}_1$ is the residue. Before starting the actual proof, we introduce some shorter notation:

$$\nabla_2 = \nabla_{j+1}^{j+2} : \mathscr{W}_{j+2}(\mathscr{V}_{n+1}, \mathscr{C}) \to \mathscr{W}_{j+1}(\mathscr{V}_{n+1}, \mathscr{C}), \qquad \mathscr{C} \in \mathbb{P}_{j+2}(\mathscr{V}_{n+1}),$$

$$\nabla_1 = \nabla_{j}^{j+1} : \mathscr{W}_{j+1}(\mathscr{V}_{n+1}, \mathscr{C}) \to \mathscr{W}_{j}(\mathscr{V}_{n+1}, \mathscr{C}), \qquad \mathscr{C} \in \mathbb{P}_{j+1}(\mathscr{V}_{n+1}),$$

$$\nabla_0 = \nabla_{j-1}^{j} : \mathscr{W}_{j}(\mathscr{V}_{n+1}, \mathscr{C}) \to \mathscr{W}_{j-1}(\mathscr{V}_{n+1}, \mathscr{C}), \qquad \mathscr{C} \in \mathbb{P}_{j}(\mathscr{V}_{n+1}),$$

$$\Delta_1 = \nabla_{j}^{j+1} : \mathscr{W}_{j+1}(\mathscr{V}_{n}, \mathscr{C}) \to \mathscr{W}_{j}(\mathscr{V}_{n}, \mathscr{C}), \qquad \mathscr{C} \in \mathbb{P}_{j+1}(\mathscr{V}_{n}),$$

$$\Delta_2 = \nabla_{j-2}^{j-1} : \mathscr{W}_{j-1}(\mathscr{V}_{n}, \mathscr{C}) \to \mathscr{W}_{j-2}(\mathscr{V}_{n}, \mathscr{C}), \qquad \mathscr{C} \in \mathbb{P}_{j-1}(\mathscr{V}_{n}).$$

By Property (2.27), there exists for every $y \in \mathbb{B}_{n+1}^{j+1} \cap \mathscr{P}(x)$ a unique $z \in \mathbb{B}_{n+1}^{j+2} \cap \mathscr{L}(x)$ such that $y \, I_x \, z$. We denote z by $\Lambda(y)$. Recall that x is a fixed line in a fixed $\mathscr{W}_j(\mathscr{V}_n, \mathscr{C}_j)$, $\mathscr{C}_j \in \mathbb{P}_j(\mathscr{V}_n)$ also fixed from now on. We denote by $\mathscr{C}_{j+1}$ the unique element of $\mathbb{P}_{j+1}(\mathscr{V}_n)$ containing $\mathscr{C}_j$ as a subset. Furthermore, we put $\mathscr{D}_{j+1} = (\Pi_n^{n+1})^{-1}(\mathscr{C}_j)$ and $\mathscr{D}_{j+2} = (\Pi_n^{n+1})^{-1}(\mathscr{C}_{j+1})$.

We now verify the axioms (QQ1), (QQ2), (QQ3), and (QQ4).

(QQ1) From Fig. 1, it follows clearly that every point is incident with at least two lines. Now take a fixed $y \in \mathbb{B}_{n-1}^{j-1} \cap \mathscr{P}(x)$. Note first that y is the unique element of $\mathbb{B}_{n-1}^{j-1} \cap \mathscr{P}(x)$. Note also that $\mathbb{B}_{n-1}^{j} \cap \mathscr{L}(x)$ is a singleton and its unique element is the unique line in $\mathscr{W}_j(\mathscr{V}_{n-1}, \Pi_{n-1}^{n}(\mathscr{C}_{j+1}))$ having y as a component. As a consequence of Property (2.15), there are at least two elements of $\mathbb{B}_{n-1}^{j-1}$ adjacent to y and they belong to $\mathscr{L}(x)$; hence y is incident with at least three lines.

(QQ2) From Fig. 1, it follows again that every line of $\mathscr{R}(x)$ is incident with at least two points. Now let z be the unique element of $\mathbb{B}_{n-1}^{j} \cap \mathscr{L}(x)$. The unique element of $\mathbb{B}_{n-1}^{j-1} \cap \mathscr{P}(x)$ is already incident with z. All lines in $\mathscr{W}_{j+1}(\mathscr{V}_n, \mathscr{C}_{j+1})$ having x as a component are by definition points of $\mathscr{R}(x)$, incident in $\mathscr{R}(x)$ with z. We now show that there are at least two such. Let $\mathscr{P}$ be a point of $\mathscr{W}_j(\mathscr{V}_n, \mathscr{C}_j)$ incident with x and let $\mathscr{L}$ be a line of $\mathscr{W}_{j+1}(\mathscr{V}_n, \mathscr{C}_{j+1})$ with $x < \mathscr{L}$. By Property (2.50), there exists a second line $\mathscr{L}'$ of $\mathscr{W}_{j+1}(\mathscr{V}_n, \mathscr{C}_{j+1})$ incident with $\mathscr{P}$ and such that $u[n, j+1](\mathscr{L}, \mathscr{L}') \geq j-1$. Since $j-1$ is odd and both lines are incident with $\mathscr{P}$, Property (2.7) implies $u[n, j+1](\mathscr{L}, \mathscr{L}') = j$. By Property (2.23), $x < \mathscr{L}'$. Hence, z is incident with at least three points.

(QQ3) Also this axiom follows directly from Fig. 1 by considering f.e. $y \in \mathbb{B}_n^{j-1} \cap \mathscr{P}(x)$ and $z \in \mathbb{B}_{n-1}^{j} \cap \mathscr{L}(x)$.

(QQ4) Let $y \in \mathscr{P}(x)$ and $z \in \mathscr{L}(x)$ not be incident in $\mathscr{R}(x)$. In view of Fig. 1, there are $4 \times 4 = 16$ distinct cases for (y, z), according to which set $\mathbb{B}_{\bullet}^{\bullet}$ they belong. Note that an element α of $\mathscr{P}(x) \cup \mathscr{L}(x)$ is incident with *exactly one* element of the set of Fig. 1 lying vertically right below the set where a belongs (if there is one such set) (use the projections). We already remarked that the sets $\mathbb{B}_{n-1}^{j-1} \cap \mathscr{P}(x)$ and $\mathbb{B}_{n-1}^{j} \cap \mathscr{L}(x)$ are singletons. In view of these observations, the proof of the following situations is nearly trivial (see Fig. 2).

In the last case of Fig. 2 we must use Property (2.27) to show uniqueness, more exactly, to show that if $z \, I_x \, a \, I_x \, b \, I_x \, y$, then (a, b) does

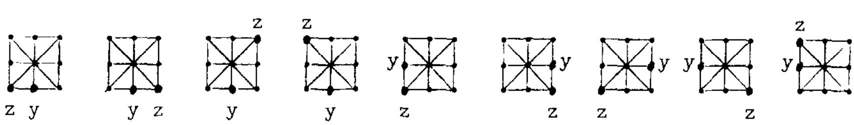


FIGURE 2

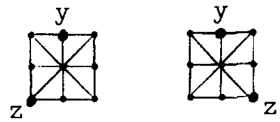


FIGURE 3

not belong to $\mathbb{B}_{n+1}^{j+1}\times\mathbb{B}_{n+1}^{j+2}$. The next two cases (see Fig. 3) are an immediate consequence of Property (2.27) (resp. Property (2.11)) (of course, always remembering *natural* properties like Property (2.20), for instance).

We now investigate the remaining cases in detail. So we are given $y\in\mathscr{P}(x)$ and $z\in\mathscr{L}(x)$ and we seek $a\in\mathscr{P}(x)$ and $b\in\mathscr{L}(x)$ such that $z\ I_x\ a\ I_x\ b\ I_x\ y$ with (a, b) unique. In order not to overburden the proof, we leave it to the reader to check that all the a's and b's we will define are indeed in $\mathscr{R}(x)$ (in most cases, this will be trivial). The five cases in question are shown in Fig. 4.

Case (I). The assumptions are : $z\in\mathscr{L}^j(\mathscr{P})$ for certain $\mathscr{P}\in\mathscr{P}(\mathscr{V}_{n+1})$ and $y\in\mathscr{L}(\mathscr{W}_{j+1}(\mathscr{V}_n,\mathscr{C}_{j+1}))$. Since y and z belong to $\mathscr{R}(x)$, $\nabla_0(z)<x<y$. Clearly, we must find (a, b) in $\mathbb{B}_{n+1}^{j+1}\times\mathbb{B}_{n+1}^{j+2}$ (this is the only possibility in view of Fig. 4). Without loss of generality, we can assume $\mathscr{P}\ I\ z$ in $\mathscr{W}_j(\mathscr{V}_{n+1},\mathscr{O}^j(\mathscr{P}))$; hence $\nabla_0(\mathscr{P})\ I\ y$ in $\mathscr{W}_{j+1}(\mathscr{V}_n,\mathscr{C}_{j+1})$. Note that $\mathscr{O}^{j+2}(\mathscr{P})=\mathscr{D}_{j+2}$. By Properties (2.7) and (2.24) and the fact that $j+2$ is even, there exists a unique $b\in(\nabla_2)^{-1}(y)$ incident with $\mathscr{P}$ in $\mathscr{W}_{j+2}(\mathscr{V}_{n+1},\mathscr{D}_{j+2})$. Let b' be the unique component of b in $\mathscr{W}_j(\mathscr{V}_{n+1},\mathscr{O}^j(\mathscr{P}))$ incident with $\mathscr{P}$. By Property (2.16), $\nabla_0(z)=\nabla_0(b')$ because they are both incident with $\nabla_0(\mathscr{P})$ and they are both a component of y. But since $j-1$ is odd, by Property (2.7), $z=b'$. Now let a be the unique component of b in $\mathscr{W}_{j+1}(\mathscr{V}_{n+1},\mathscr{D}_{j+1})$ having z as a component (cp. Property (2.29)); then we have $z\ I_x\ a\ I_x\ b\ I_x\ y$. Since b does not depend on $\mathscr{P}$ (after all, all points of z are incident with b), b is unique (again see Property (2.7)) and so is (a, b).

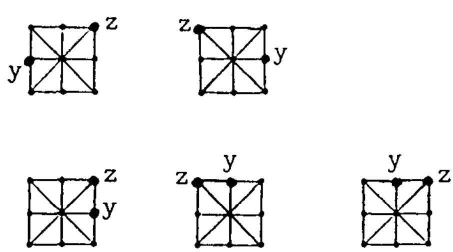


FIGURE 4

Case (II). The assumptions are now: $y \in \mathscr{L}^{j-1}(\mathscr{P})$ for certain $\mathscr{P} \in \mathscr{C}_j$ and $z \in \mathscr{L}(\mathscr{W}_{j+2}(\mathscr{V}_{n+1}, \mathscr{D}_{j+2}))$. Without loss of generality, we can again assume $\mathscr{P}\, I\, y$. Expressing $y, z \in \mathscr{R}(x)$ gives : $y < x < \nabla_2(z)$. In view of Fig. 4, we must find (a, b) in $\mathbb{B}_{n+1}^{j+1} \times \mathbb{B}_{n+1}^{j}$. Let $\mathscr{D}_j = (\Pi_n^{n+1})^{-1}(\mathscr{O}^{j-1}(\mathscr{P})) \in \mathbb{P}_j(\mathscr{V}_{n+1})$. By Property (2.3), the shadow of every component of z in $\mathscr{W}_{j+1}(\mathscr{V}_{n+1}, \mathscr{D}_{j+1})$ meets the set $\mathscr{D}_j$. Let $Q \in \mathscr{D}_j$, then $\mathbb{S}_2^{j+2}(Q, z)$ is well defined and has width 2. From the assumptions, it follows that $y \in \mathbb{C}_2^{j+1}(\nabla_2(Q), \nabla_2(z))$. But $\mathbb{S}_2^{j+1}(\nabla_2(Q), \nabla_2(z))$ has width 1 (cp. Property (2.10)), so by Property (2.20), $\nabla_0(\mathbb{C}_2^{j+2}(Q, z)) \subseteq \mathbb{C}_2^{j+1}(\nabla_2(Q), \nabla_2(z))$. But applying Property (2.25) on a base point of $\mathbb{S}_2^{j+2}(Q, z)$, we easily see $\nabla_0(\mathbb{C}_2^{j+2}(Q, z)) = \mathbb{C}_2^{j+1}(\nabla_2(Q), \nabla_2(z))$. Since j is even, Property (2.7) implies that ∇_0 induces a bijection from $\mathbb{C}_2^{j+2}(Q, z)$ to $\mathbb{C}_2^{j+1}(\nabla_2(Q), \nabla_2(z))$. Hence, there exists a unique $b \in \mathbb{C}_2^{j+2}(Q, z)$ such that $\nabla_0(b) = y$. Let a be the unique component of z in $\mathscr{W}_{j+1}(\mathscr{V}_{n+1}, \mathscr{D}_{j+1})$ having b as a component, then $z\, I_x\, a\, I_x\, b\, I_x\, y$ and, from the reasoning above, it follows that (a, b) is unique.

Case (III). The assumptions are now: $y \in \mathscr{L}(\mathscr{W}_{j-1}(\mathscr{V}_n, \mathscr{C}_{j-1}))$ for certain $\mathscr{C}_{j-1} \in \mathbb{P}_{j-1}(\mathscr{V}_n)$ and $z \in \mathscr{L}(\mathscr{W}_j(\mathscr{V}_{n+1}, \mathscr{D}_j))$ for certain $\mathscr{D}_j \in \mathbb{P}_j(\mathscr{V}_{n+1})$. Expressing that y and z are in $\mathscr{R}(x)$ and that they are not incident, gives us: $y < x$; $\nabla_0(z) < x$ and $y \neq \nabla_0(z)$. Note that it is needless to try to find (a, b) in $\mathbb{B}_n^{j-1} \times \mathbb{B}_{n+1}^{j}$ because this would imply $\nabla_0(b) = a = y$. There are two possibilities:

(IIIa) $\Pi_n^{n+1}(\mathscr{D}_j) = \mathscr{C}_{j-1}$. Then a and y are both components of x, belonging to a strip of width 1. So we can put $b = \Lambda_{-1}(a) = \Lambda_{-1}(y)$ and $b\, I_x\, a\, I_x\, b\, I_x\, y$. Since every line of $\mathscr{W}_{j+1}(\mathscr{V}_{n+1}, \mathscr{D}_{j+1})$, having z as a component, has no other line of $\mathscr{W}_j(\mathscr{V}_{n+1}, \mathscr{D}_j)$ as a component (j even), we cannot have $z\, I_x\, a'\, I_x\, b'\, I_x\, y$ with $a' \in \mathbb{B}_{n+1}^{j+1}$ (for any b'). Hence (a, b) is unique.

(IIIb) $\Pi_n^{n+1}(\mathscr{D}_j) \neq \mathscr{C}_{j-1}$. This time, we need not look for a in $\mathbb{B}_n^{j-1}$, because $\Lambda_{-1}(\nabla_0(z)) \neq \Lambda_{-1}(y)$. So we try to find a in the set $\mathbb{B}_{n+1}^{j+1}$. Let $\mathscr{P} \in \mathscr{C}_{j-1}$ be incident with y in $\mathscr{W}_{j-1}(\mathscr{V}_n, \mathscr{C}_{j-1})$ and let $\mathscr{L} \in \mathscr{L}(\mathscr{W}_j(\mathscr{V}_n, \mathscr{C}_j))$ be incident with $\mathscr{P}$ such that $u[n, j](\mathscr{L}, x) = 0$. Let Q be an arbitrary point in $\mathscr{D}_{j+1}$ such that $\nabla_1(Q) = \mathscr{P}$ and let $\mathscr{M}$ be a line in $\mathscr{W}_{j+1}(\mathscr{V}_{n+1}, \mathscr{D}_{j+1})$ such that $\nabla_0(\mathscr{M}) = \mathscr{L}$ and $\mathscr{M}\, I\, Q$. Let $\mathscr{P}'$ be an affine point of z, i.e., $\mathscr{P}' \in \mathscr{D}_j \cap \sigma(z)$. By the assumption of (IIIb) and Property (2.35), we have $u[n+1, j+1](\mathscr{P}', \mathscr{M}) = (0, 0)$. We now apply axiom (GQ2) and get a line $a \in \mathscr{L}(\mathscr{W}_{j+1}(\mathscr{V}_{n+1}, D_{j+1}))$ incident with $\mathscr{P}'$ and concurrent with $\mathscr{M}$ in $\mathscr{W}_{j+1}(\mathscr{V}_{n+1}, \mathscr{D}_{j+1})$. By projecting and using the uniqueness part of (GQ2), one sees that $\nabla_1(a) = x$. Now let b be the unique component of a in $\mathscr{W}_j(\mathscr{V}_{n+1}, \mathscr{O}^j(Q))$, then $\nabla_0(b) < x$ by Property (2.20). By Property (2.47), $\nabla_0(b) = y$. Let z' be the unique component of a in $\mathscr{W}_j(\mathscr{V}_{n+1}, \mathscr{D}_j)$, then $\nabla_0(z)$ and $\nabla_0(z')$ are both components of x incident with $\nabla_0(\mathscr{P}')$. By Property

(2.16), they coincide. But since j is even and $\mathscr{P}'$ is incident with both z and z', z and z' also coincide. Hence $z\, I_x\, a\, I_x\, b\, I_x y$. By Property (2.48), (a, b) is unique.

Case (IV). Here the assumptions are: $y \in \mathscr{L}(\mathscr{W}_{j+1}(\mathscr{V}_{n+1}, \mathscr{D}_{j+1}))$ and $z \in \mathscr{L}(\mathscr{W}_{j+2}(\mathscr{V}_{n+1}, \mathscr{D}_{j+2}))$. Expressing y, $z \in \mathscr{R}(x)$ gives: $x = \nabla_1(y) < \nabla_2(z)$. Since y and z are not incident in $\mathscr{R}(x)$, y is not a component of z. Since $j+2$ is even, we will not be able to find (a, b) in $\mathbb{B}_{n+1}^{j+1} \times \mathbb{B}_{n+1}^{j+2}$ by property (2.27). Let $\mathscr{L} = \Lambda(y)$. By property (2.30), $u[n+1, j+2](\mathscr{L}, z) \geqslant j$. So there are two possibilities.

(IVa) $u[n+1, j+2](\mathscr{L}, z) = j+1$. Since $j+1$ is odd, $\mathscr{L}$ and z are not concurrent by Property (2.7) and hence they share no component. So we need not look for b in $\mathbb{B}_{n+1}^{j}$. The only possibility is $b = \mathscr{L}$, $a = \nabla_2(z) = \nabla_2(b)$, and thus $z\, I_x\, a\, I_x\, b\, I_x\, y$ with (a, b) unique.

(IVb) $u[n+1, j+2](\mathscr{L}, z) = j$. This time, b can only be found in $\mathbb{B}_{n+1}^{j}$ (if it exists). Let $\mathscr{P}$ be an arbitrary element of $\mathscr{D}_{j+1}$, then $\mathbb{C}_1^{j+2}(\mathscr{P}, z)$ and $\mathbb{C}_1^{j+2}(\mathscr{P}, \mathscr{L})$ are non-empty by Property (2.3) and they are both mapped on x by ∇_1. By Property (2.49), there exists a unique element $a \in \mathbb{C}_1^{j+2}(\mathscr{P}, z)$ concurrent with $y \in \mathbb{C}_1^{j+2}(\mathscr{P}, \mathscr{L})$. By Property (2.23), y and a share a unique component b in $\mathscr{W}_j(\mathscr{V}_{n+1}, \mathscr{D}_j)$ for a certain $\mathscr{D}_j \in \mathbb{P}_j(\mathscr{V}_{n+1})$. Hence $z\, I_x\, a\, I_x\, b\, I_x\, y$ and it also follows from this reasoning that (a, b) is unique.

Case (V). The assumptions are now: $y \in \mathscr{L}(\mathscr{W}_{j+1}(\mathscr{V}_{n+1}, \mathscr{D}_{j+1}))$ and $z \in \mathscr{L}(\mathscr{W}_j(\mathscr{V}_{n+1}, \mathscr{D}_j))$ for a certain $\mathscr{D}_j \in \mathbb{P}_j(\mathscr{V}_{n+1})$ with $\mathscr{D}_j \subseteq \mathscr{D}_{j+1}$. Expressing y, $z \in \mathscr{R}(x)$ gives: $\nabla_0(z) < \nabla_1(y) = x$. Though, z is not a component of y, since y is not incident with z in $\mathscr{R}(x)$. Note that x is incident with elements of $\Pi_n^{n+1}(\mathscr{D}_j)$ and hence, by Property (2.3), there are elements of $\mathscr{D}_j$ incident with y. So there exists a $\mathscr{L}' < y$ with $\mathscr{L}' \in \mathscr{L}(\mathscr{W}_j(\mathscr{V}_{n+1}, \mathscr{D}_j))$. Since $\nabla_0(\mathscr{L}')$ and $\nabla_0(z)$ are both components of x, we have $u[n+1, j](\mathscr{L}', z) \geqslant j-2$. So there are two possibilities.

(Va) $u[n+1, j](\mathscr{L}', z) = j-1$. Put $a = \nabla_0(z) = \nabla_0(\mathscr{L}')$ and $b = \mathscr{L}'$. Then $z\, I_x\, a\, I_x\, b\, I_x\, y$. It is clear that (a, b) is unique in $\mathbb{B}_n^{j-1} \times \mathbb{B}_{n+1}^{j}$. Suppose $z\, I_x\, a'\, I_x\, b'\, I_x\, y$, then, by Property (2.48), $(a', b') \notin \mathbb{B}_{n+1}^{j+1} \times \mathbb{B}_{n+1}^{j}$ (this would lead to $z < y$). Suppose now $(a', b') \in \mathbb{B}_{n+1}^{j+1} \times \mathbb{B}_{n+1}^{j+2}$, then both a' and y are components of b' and so they are incident with a common point at infinity. In view of Property (2.14), we can use Property (2.48) to see $a' = y$.

(Vb) $u[n+1, j](\mathscr{L}', z) = j-2$. Let $Q \in \mathscr{D}_j$ be incident with z in $\mathscr{W}_j(\mathscr{V}_{n+1}, \mathscr{D}_j)$. Since $\nabla_0(z)$ and $\nabla_0(\mathscr{L}')$ are both components of x, Properties (2.14) and (2.33) and axiom (GQ1) imply $u[n+1, j](Q, \mathscr{L}') = ((j-2)/2, j-2)$. By Property (2.35) and the uniqueness of the component $\mathscr{L}'$ of y, one can verify that this implies $u[n+1, j+1](Q, y) = (j/2, j)$. By

(GQ2), there exists a unique line $a \in \mathscr{L}(\mathscr{W}_{j+1}(\mathscr{V}_{n+1}, \mathscr{D}_{j+1}))$ such that $Q\, I\, a \perp y$ in $\mathscr{W}_{j+1}(\mathscr{V}_{n+1}, \mathscr{D}_{j+1})$ and we have $u[n+1, j+1](a, y) = j$. Note that $z < a$ (same proof as at the end of case (IIIb) above). Hence a does not depend on $Q \in \mathscr{D}_j \cap \sigma(z)$ and is unique with respect to the property $z < a \perp y$ in $\mathscr{W}_{j+1}(\mathscr{V}_{n+1}, \mathscr{D}_{j+1})$. Again, there are two possibilities.

(Vb.1) a and y are incident with common points at infinity of $(\mathscr{W}_{j+1}(\mathscr{V}_{n+1}, \mathscr{D}_{j+1}), \mathscr{N}_{\mathscr{D}_{j+1}})$. Then a and y are components of a unique line $b \in \mathscr{L}(\mathscr{W}_{j+2}(\mathscr{V}_{n+1}, \mathscr{D}_{j+2}))$. Hence $z\, I_x\, a\, I_x\, b\, I_x\, y$ with (a, b) unique (the construction of (Vb.2) is not possible here).

(Vb.2). a and y are inicdent with common affine points of $(\mathscr{W}_{j+1}(\mathscr{V}_{n+1}, \mathscr{D}_{j+1}), \mathscr{N}_{\mathscr{D}_{j+1}})$. By Property (2.23), a and y share a component b in $\mathscr{W}_j(\mathscr{V}_{n+1}, \mathscr{D}'_j)$ for a certain but unique $\mathscr{D}'_j \in \mathbb{P}_j(\mathscr{V}_{n+1})$ with $\mathscr{D}'_j \subseteq \mathscr{D}_{j+1}$ and $\mathscr{D}_j \neq \mathscr{D}'_j$. We have $z\, I_x\, a\, I_x\, b\, I_x\, y$ with (a, b) unique (this time, the solution of (Vb.1) is not possible).

This completes the proof of (QQ4) and the theorem. Q.E.D.

The proof of the next theorem is very similar to [13, 6.2]. For this reason, we will only sketch it. But one will be able to fill in every missing detail. First, recall the following standard definitions concerning *homotopy* in simplicial complices. A *path* is a sequence of vertices $(v_0, v_1, v_2, ..., v_n)$ with the condition that $\{v_i, v_{i+1}\}$ is a simplex for all $i \in \{0, 1, 2, ..., n-1\}$. Two paths $\mathscr{G}^i = (v_0^i, v_1^i, ..., v_{n_i}^i)$, $i = 1, 2$, are called *elementary homotopic* if $n_1 = n_2 + 1$ and if there exists $j \in \{0, 1, 2, ..., n_2 - 1\}$ such that $v_k^1 = v_k^2$ for all $k \leqslant j$; $v_{k+1}^1 = v_k^2$ for all $k \in \{j+1, ..., n_2\}$ and $\{v_j^1, v_{j+1}^1, v_{j+2}^1\}$ is a simplex. Two paths $\mathscr{G}^1$ and $\mathscr{G}^2$ are called *homotopic* if they can be joined by a sequence of consecutively elementary homotopic paths. A path is called *trivial* if it consists of a unique vertex. A path $\mathscr{G} = (v_0, ..., v_n)$ is *closed* if $v_0 = v_n$. In this case, we also say that $\mathscr{G}$ is *based* at v_0.

THEOREM (4.2). *The rank* 3 *incidence structure* Δ, *as defined above, is a building of type* $\tilde{C}_2$.

Proof. By the previous theorem, Δ is a connected Tits-geometry of type $\tilde{C}_2$ (with all residues thick). Suppose every closed path based at the unique element β of $\mathbb{B}_0^0$ is homotopic with the trivial path (β). Then by Ronan [8, Corollary 2.3], Δ is 2-simply-connected as a *chamber system*. Tits [11] implies that Δ is an affine building of type $\tilde{C}_2$. So let $\mathscr{G} = (v_0, v_1, ..., v_n)$ be a closed path based in $\beta = v_0 = v_n$. Define

$$\mathscr{N}(\mathscr{G}) = \sup\{k \in \mathbb{N} \mid \exists i \in \{0, 1, ..., n\} \text{ such that } v_i \in \mathbb{B}_k^j \text{ for some } j \leqslant n\}.$$

Note that $\mathscr{N}(\mathscr{G})$ is well defined, since $\mathscr{G}$ is finite (in fact, $\mathscr{N}(\mathscr{G}) < n/2$). Let $\mathscr{G}'$ be the path obtained from $\mathscr{G}$ by copying all vertices $v \in \mathbb{B}_k^j$ for which

$0 \leqslant j \leqslant k < \mathcal{N}(\mathcal{G})$ and replacing the vertices $w \in \mathbb{B}^j_{\mathcal{N}(\mathcal{G})}$, $0 \leqslant j \leqslant \mathcal{N}(\mathcal{G})$, by $\nabla^j_{j-1}(w)$. One can check that $\mathcal{G}'$ is indeed a path and that it is, moreover, homotopic to $\mathcal{G}$ (cf. [13, 6.2]). Clearly is $\mathcal{N}(\mathcal{G}') = \mathcal{N}(\mathcal{G}) - 1$. With an inductive argument, we conclude that $\mathcal{G}$ is homotopic to the path $(\beta, \beta, ..., \beta)$ (n elements) which is homotopic to (β). Q.E.D.

In view of the definition of equivalent HQ-Artmann-sequences, the next theorem is logical.

THEOREM (4.3). *Equivalent HQ-Artmann-sequences define isomorphic buildings of type* $\tilde{C}_2$.

Proof. This is obvious, because equivalent H-Qs coincide over the affine part of all $\mathcal{W}_j$'s, and the definition of the corresponding building of type $\tilde{C}_2$ only uses these affine parts. Q.E.D.

Remark (4.4). As alluded to in the Introduction, we shall prove in a forthcoming paper [4] that all buildings of type $\tilde{C}_2$ arise in this way. Then we shall also deal with examples.

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