GENERALIZED DESARGUES CONFIGURATIONS IN GENERALIZED QUADRANGLES

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To Professor J.Tits on the occasion of his 60th birthday

Abstract

We generalize the notion of *Desargues configuration* in projective planes to a similar notion in generalized quadrangles. In this way, we can generalize Baer's theorem and obtain new characterizations of many classes of generalized quadrangles.

1 Introduction

In the theory of projective planes, Baer's theorem (see e.g. [4]) connects a configurational property (being (V, L)-Desarguesian) and a collineation group property (being (V, L)-transitive). A successful attempt to obtain a similar result for generalized quadrangles was made by M.A.Ronan [6], who showed that a configurational property involving some line L is equivalent to some group transitivity property involving collineations fixing L pointwise and stabilizing all lines meeting L. Under these conditions, the line L is always regular (for the definition of regular line, see [5]). In the present paper, we generalize Ronan's configurational property in order to drop the regularity of L. At the same time, our configuration will more easily be seen as a generalization of the Desargues configuration in projective planes. The idea is the following. A classical projective plane is (V, L)-transitive for every point V and every line L incident with V. The group fixing L pointwise and fixing V linewise is in fact a *root group*. For generalized quadrangles, such root groups fix a line (resp. a point) pointwise (resp. linewise) and two distinct points on that line (resp. two distinct lines through that point) linewise (resp. pointwise). Hence our Desargues configuration, which we will call a *generalized Desargues configuration*, must involve a *root*, i.e. a triplet $\{P, L, P'\}$ where P and P' are two distinct points on a line L (or dually, a triplet $\{L, P, L'\}$ where L and L' are two distinct lines on a point P). In this way, we obtain the definition of *quadrilaterals in perspective from* $\{P, L, P'\}$ (or $\{L, P, L'\}$). "Enough" such generalized Desargues configurations will imply the existence of "enough" collineations in the sense of Baer's result. More exactly, we will show

Main result. *A generalized quadrangle $\mathcal{S}$ is $\{P, L, P'\}$-Desarguesian if and only if $\mathcal{S}$ is $\{P, L, P'\}$-transitive, P and P' being two distincts points on a line L of $\mathcal{S}$.*

The way to prove our main result differs from the way it is usually done for projective planes. Indeed, in our situation it is quite complicated to use only geometrical arguments. On the other

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hand, a purely algebraic proof would be too abstract. Hence we will combine geometrical and algebraic arguments. The algebra is brought in by coordinatization.

In section 4, we mention some immediate consequences and show how Ronan's result fits into our theory as an important corollary.

2 Definitions and notation

A *generalized quadrangle of order* (s,t) is a point-line incidence geometry $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ satisfying (GQ1), (GQ2) and (GQ3):

(GQ1) *Two distinct lines have at most one point in common and there are exactly* $1+s$ *points incident with every line.*

(GQ2) *Two distinct points lie on at most one common line and there are exactly* $1+t$ *lines incident with every point.*

(GQ3) *Given a point* P *and a line* L *not incident with* P, *then there exists a unique pair* $(Q,M) \in \mathcal{P} \times \mathcal{B}$ *such that* $P \mathrm{\ I\ } M \mathrm{\ I\ } Q \mathrm{\ I\ } L$.

Note that s and/or t can be infinite. Throughout this paper, we assume $s,t \geq 2$, that is, we assume that $\mathcal{S}$ is thick. In $\mathcal{S}$, two points P and Q are called *collinear* if they lie on a common line (notation $P \perp Q$), and two lines L and M are called *concurrent* if they share a common point (notation $L \perp M$). A *quadrilateral* of $\mathcal{S}$ is a subquadrangle (in the usual sense) of order $(1,1)$ of $\mathcal{S}$ (see [5]).

Let $(P,L,P') \in \mathcal{P} \times \mathcal{B} \times \mathcal{P}$ be such that $P \mathrm{\ I\ } L \mathrm{\ I\ } P'$ and $P \neq P'$. Denote $\mathcal{C} = \{P, L, P'\}$. Let $\mathcal{D}_1 = P_1 \mathrm{\ I\ } L_1 \mathrm{\ I\ } P_2 \mathrm{\ I\ } L_2 \mathrm{\ I\ } P_3 \mathrm{\ I\ } L_3 \mathrm{\ I\ } P_4 \mathrm{\ I\ } L_4 \mathrm{\ I\ } P_1$ and $\mathcal{D}_2 = P_1' \mathrm{\ I\ } L_1' \mathrm{\ I\ } P_2' \mathrm{\ I\ } L_2' \mathrm{\ I\ } P_3' \mathrm{\ I\ } L_3' \mathrm{\ I\ } P_4' \mathrm{\ I\ } L_4' \mathrm{\ I\ } P_1'$ be two quadrilaterals. Then we call $\mathcal{D}_1$ and $\mathcal{D}_2$ *in perspective from* $\mathcal{C}$ if for every $i \in \{1,2,3,4\}$, P_i and P_i' are collinear with a same point on L, and L_i and L_i' are concurrent with a same line through P, resp. P'. In such a case, we call $\{\mathcal{C}, \mathcal{D}_1, \mathcal{D}_2\}$ a *generalized Desargues configuration* in $\mathcal{S}$ and P_i, P_i' (resp. L_i, L_i') are called *corresponding elements* or *corresponding points* (resp. *lines*) of the configuration. A quadrilateral $\mathcal{D}_1 = P_1 \mathrm{\ I\ } L_1 \mathrm{\ I\ } P_2 \mathrm{\ I\ } L_2 \mathrm{\ I\ } P_3 \mathrm{\ I\ } L_3 \mathrm{\ I\ } P_4 \mathrm{\ I\ } L_4 \mathrm{\ I\ } P_1$ satisfying

($\mathcal{D}_1$) P_1 and P, or P' and P_1, are collinear,

($\mathcal{D}_2$) neither P nor P' is incident with any of the lines $L_i, i = 1,2,3,4$

is called a *$\mathcal{C}$-quadrilateral* and the line joining P and P_1, or P' and P_1, is called a *base-line* of $(\mathcal{C}, \mathcal{D}_1)$. Note that for a given $\mathcal{C}$-quadrilateral there can be at most two base-lines : one through P and one through P'.

Let $\mathcal{C}$ be as above, then we call $\mathcal{S}$ *$\mathcal{C}$-Desarguesian* if for every $\mathcal{C}$-quadrilateral $\mathcal{D}_1$ and every point P_1', $P_1' \neq P$, P', on any base-line, there exists a $\mathcal{C}$-quadrilateral $\mathcal{D}_2$ containing P_1' which is in perspective with $\mathcal{D}_1$ from $\mathcal{C}$.

An automorphism θ of $\mathcal{S}$ is called a *$\mathcal{C}$-elation*, $\mathcal{C}$ as above, if it stabilizes each line through P, each line through P', and fixes each point on L. Let M be any line through P distinct from L and denote by A the set of points incident with M and distinct from P. One shows that every $\mathcal{C}$-elation fixing at least one element of A must be the identity [5]. Hence the group of all $\mathcal{C}$-elations acts semiregularly on A. If this group is transitive on A, then we call $\mathcal{S}$ *$\mathcal{C}$-transitive* [3]. If $\mathcal{S}$ is $\{P, L, P'\}$-transitive for every distinct P and P' on L, then, with the notation of [5], we say that $\mathcal{S}$ *satisfies* $(\hat{M})_L$ (a local Moufang condition). If $\mathcal{S}$ satisfies $(\hat{M})_L$ for all $L \in \mathcal{B}$, then we say that $\mathcal{S}$ *satisfies* $(\hat{M})$. Is $\mathcal{S}$ satisfies $(\hat{M})$ and the dual property (M), then $\mathcal{S}$ is called *Moufang.*

All of the above definitions can of course be dualized.

We now show how $\mathcal{S}$ can be coordinatized by a *quadratic quaternary ring* (R_1, R_2, Q_1, Q_2), where R_1 is an arbitrary set of s elements all distinct from the symbol ∞, but containing the symbols 0 and 1, and R_2 is an arbitrary set of t elements, not containing ∞ but also containing 0 and 1. We choose an arbitrary quadrilateral in $\mathcal{S}$ and coordinatize its elements by (∞) I $[\infty]$ I (0) I $[0,0]$ I $(0,0,0)$ I $[0,0,0]$ I $(0,0)$ I $[0]$ I (∞), where round brackets are used for points and square brackets for lines. The coordinatization of the lines is the dual of the coordinatization of the points, hence we only describe the latter explicitly. Choose bijectively a coordinate (a), $a \in R_1$, for each point of $[\infty]$ distinct from (∞), such that 0 corresponds to (0). We do the same for points of $[0,0]$ distinct from (0), which are given coordinates $(0,0,a)$, $a \in R_1$. A point P collinear with (∞) but not lying on $[\infty]$ has coordinates $(k,a) \in R_2 \times R_1$ if P lies on $[k]$ and is collinear with $(0,0,a)$. Finally, a point not collinear with (∞) is assigned the coordinate $(a,l,a') \in R_1 \times R_2 \times R_1$ if it lies on $[a,l]$ and is collinear with $(0,a')$. We now define two quaternary operations Q_1 and Q_2 as follows :

$$\begin{aligned} Q_1(k,a,l,a') &= b \iff (k,b) \perp (a,l,a'),\\ Q_2(a,k,b,k') &= l \iff [a,l] \perp [k,b,k'], \end{aligned}$$

where $a,b,a' \in R_1$ and $k,l,k' \in R_2$. The fact that $\mathcal{S}$ is a generalized quadrangle turns the quadruple (R_1, R_2, Q_1, Q_2) into a *quadratic quaternary ring.* For the precise definition of quadratic quaternary rings and their properties, we refer to [2]. We also define the quaternary operations Q_1^* and Q_2^* as follows :

$$Q_1^*(a,k,b,k') = a' \iff (a) \text{ and } (0,0,a') \text{ are collinear with the same point on } [k,b,k'],$$

$$Q_2^*(k,a,l,a') = k' \iff [k] \text{ and } [0,0,k'] \text{ are concurrent with the same line through } (a,l,a'),$$

where $a,b,a' \in R_1$ and $k,l,k' \in R_2$. Hence,

$$(a,l,a') \text{ I } [k,b,k'] \iff Q_1^*(a,k,b,k') = a' \text{ and } Q_2(a,k,b,k') = l,$$

where $a,b,a' \in R_1$ and $k,l,k' \in R_2$.

Furthermore, we require that the bijections from R_1 to the points of $[\infty]$ distinct from (∞), and from R_1 to the points of $[0,0]$ distinct from (0), have the property that (a) and $(0,a)$ are collinear with the same point on the line $[1,0,0]$. This is called *$*$-normalization* in [3]. From that paper, we also recall the following (straightforward) identities :

$$\begin{array}{rrcccl}
\text{(I)} & Q_1(k,0,0,a') & = & a' & = & Q_1(0,a,k,a'),\\
\text{(II)} & Q_2(a,0,0,k') & = & k' & = & Q_2(0,k,b,k'),\\
\text{(III)} & Q_1^*(0,k,b,0) & = & b & = & Q_1^*(a,0,b,k'),\\
\text{(IV)} & Q_2^*(0,a,l,0) & = & l & = & Q_2^*(k,0,l,a'),\\
\text{(V)} & Q_1^*(a,1,0,0) & = & a, & & \\
\text{(VI)} & Q_2^*(k,1,0,0) & = & k, & &
\end{array}$$

for all $a,b,a' \in R_1$ and $k,l,k' \in R_2$.

We define an addition in R_1 by

$$a+b = Q_1^*(a,1,b,0) \qquad a,b \in R_1,$$

and a multiplication on $R_2 \times R_1$ with values in R_1 by

$$k.a = Q_1^*(a,k,0,0) \qquad k \in R_2 \text{ and } a \in R_1.$$

Note that by the properties (I) through (VI) above, $0 + a = a = a + 0, 1.a = a$ and $0.a = k.0 = 0$ for all $a \in R_1, k \in R_2$. More details can be found in [3].

3 Proof of the main theorem

In this section, we fix $\mathcal{C} = \{P, L, P'\}, P, P' \in \mathcal{P}, P \neq P', L \in \mathcal{B}, P \mathrm{~I~} L \mathrm{~I~} P'$. We show that $\mathcal{S}$ is $\mathcal{C}$-transitive if and only if $\mathcal{S}$ is $\mathcal{C}$-Desarguesian.

Suppose first that $\mathcal{S}$ is $\mathcal{C}$-transitive and consider an arbitrary $\mathcal{C}$-quadrilateral $\mathcal{D}_1$ as well as an arbitrary point P_1' on any base-line M of $(\mathcal{C}, \mathcal{D}_1)$, with P_1' distinct from P and P'. If P_1 denotes the point of $\mathcal{D}_1$ on M, then the unique $\mathcal{C}$-elation mapping P_1 onto P_1' maps $\mathcal{D}_1$ onto a $\mathcal{C}$-quadrilateral $\mathcal{D}_2$ which is in perspective with $\mathcal{D}_1$ from $\mathcal{C}$ and which contains P_1'. Hence $\mathcal{S}$ is $\mathcal{C}$-Desarguesian.

Throughout the remainder of this section, we suppose that $\mathcal{S}$ is $\mathcal{C}$-Desarguesian. We show that $\mathcal{S}$ is also $\mathcal{C}$-transitive.

We coordinatize $\mathcal{S}$ as in the previous section where $P = (\infty), P' = (0)$ and $L = [\infty]$. Then, by [3, theorem (3.3)], $\mathcal{S}$ is $\mathcal{C}$-transitive if and only if

$$Q_1^*(a, k, b + B, k') = Q_1^*(a, k, b, k') + B$$

and

$$Q_2(a, k, b + B, k') = Q_2(a, 0, B, Q_2(a, k, b, k')),$$

for all $a, b, B \in R_1$ and $k, k' \in R_2$.

If $t = 2$, then the theorem is trivial since $\mathcal{S}$ must then be classical [5]. So we can assume $t > 2$ throughout. We show the result in six steps.

Notation. We abbreviate the sequence $P_1 \mathrm{~I~} L_1 \mathrm{~I~} P_2 \mathrm{~I~} L_2$, for $P_1, P_2 \in \mathcal{P}, L_1, L_2 \in \mathcal{B}$ and P_1 not incident with L_2, by $P_1 \sim L_2$.

Remark. By the special choices for (∞), (0) and $[\infty]$, two corresponding elements of two $\mathcal{C}$-quadrilaterals that are in perspective from $\mathcal{C}$ have a common first coordinate and if they are lines, also their last coordinates coincide.

STEP 1. In this section, we prove some useful lemmas.

Lemma 1. *For all $a, B \in R_1$ and all $k \in R_2$, we have*

$$Q_1^*(a, k, B, 0) = k.a + B.$$

Proof. The lemma is trivial for $k = 0$ or $k = 1$. So suppose $k \neq 0$ and $k \neq 1$. Consider the quadrilateral $\mathcal{D}_1 = (0,0,0) \mathrm{~I~} [k,0,0] \mathrm{~I~} (a, Q_2(a,k,0,0), Q_1^*(a,k,0,0)) \mathrm{~I~} [0, Q_1^*(a,k,0,0), k_1'] \mathrm{~I~} (0, Q_1^*(a,k,0,0)) \mathrm{~I~} [0, Q_1^*(a,k,0,0), k_2'] \mathrm{~I~} (Q_1^*(a,k,0,0), Q_2(Q_1^*(a,k,0,0),1,0,0), Q_1^*(a,k,0,0)) \mathrm{~I~} [1,0,0] \mathrm{~I~} (0,0,0)$, where k_1' and k_2' (both in R_2) are well defined. Now consider the quadrilateral $\mathcal{D}'_1$ in perspective with $\mathcal{D}_1$ from $\mathcal{C}$ and containing $(0,0,B)$ (which corresponds to $(0,0,0)$). By the remark above, $\mathcal{D}'_1$ must be necessarily written as $(0,0,B) \mathrm{~I~} [k,B,0] \mathrm{~I~} (a, Q_2(a,k,B,0), Q_1^*(a,k,B,0)) \mathrm{~I~} [0, Q_1^*(a,k,B,0), k_1'] \mathrm{~I~} (0, Q_1^*(a,k,B,0)) \mathrm{~I~} [0, Q_1^*(a,k,B,0), k_2'] \mathrm{~I~} (Q_1^*(a,k,0,0), Q_2(Q_1^*(a,k,0,0),1,B,0), Q_1^*(Q_1^*(a,k,0,0),1,B,0)) \mathrm{~I~} [1,B,0] \mathrm{~I~} (0,0,B)$. Hence by the sixth incidence in this chain, we conclude

$$\begin{aligned} k.a + B &= Q_1^*(Q_1^*(a,k,0,0),1,B,0) \\ &= Q_1^*(Q_1^*(a,k,0,0),0,Q_1^*(a,k,B,0),k_2') \qquad (1) \\ &= Q_1^*(a,k,B,0), \end{aligned}$$

by identity (III) of the previous section. Q.E.D.

Lemma 2. *The equation $a+x=b$ has a unique solution in the unknown x, for all $a,b \in R_1$.*

Proof. We first prove that there is at most one solution. Therefore, suppose $a+x_1 = b = a+x_2$. Let L_i be the unique line through $(0,b)$ meeting $[1,x_i,0], i=1,2$. Consider the $\mathcal{C}$-quadrilateral $\mathcal{D}_2 = (1,x_2)$ I $[1,x_2,0] \sim (0,b)$ I $L_1 \sim (1,x_2)$ and let $\mathcal{D}_2'$ be the $\mathcal{C}$-quadrilateral in perspective with $\mathcal{D}_2$ from $\mathcal{C}$ which contains $(1,x_1)$ as the corresponding point to $(1,x_2)$. Then the line $[1,x_1,0]$ of $\mathcal{D}_2$ corresponds to $[1,x_2,0]$ of $\mathcal{D}'_2$. Suppose $x_1 \neq x_2$. The point $(a,\ldots,b)$ of $\mathcal{D}_2$ corresponds to a point $(a,\ldots,b)$ of $\mathcal{D}'_2$, hence L_1 corresponds to L_2. But then, since the last coordinates of these lines coincide, there exists a line through (0) meeting both L_1 and L_2, a contradiction. Hence $x_1 = x_2$.

Now we show that there is at least one solution. Consider any $\mathcal{C}$-quadrilateral $\mathcal{D}_3 = (0,0,0)$ I $[1,0,0]$ I $(a, Q_2(1,a,0,0),a)$ I $[0,a,k']$ I $(0,a) \sim [k,0,0]$ I $(0,0,0)$, where k' is well-defined and $k \in R_2$ is arbitrary but distinct from 0 and 1 (existing since $t \geq 3$). Denote by $\mathcal{D}_3'$ the $\mathcal{C}$-quadrilateral in perspective with $\mathcal{D}_3$ from $\mathcal{C}$ and containing the point $(0,b)$ which corresponds to $(0,a)$. Then the line of $\mathcal{D}'_3$ corresponding to $[1,0,0]$ has coordinates of the form $[1,x,0], x \in R_1$. Clearly $a+x=b$. Q.E.D.

Lemma 3. *Let $\mathcal{D}$ be an arbitrary $\mathcal{C}$-quadrilateral containing the point P^* collinear with (∞). Let L^* be a line of $\mathcal{D}$ not incident with P^*. Then for every line M^* not concurrent with $[\infty]$ and sharing its first and last coordinate with L^*, there exists a (unique) $\mathcal{C}$-quadrilateral $\mathcal{D}'$ in perspective with $\mathcal{D}$ from $\mathcal{C}$ and containing M^* as corresponding line to L^*.*

Proof. One can easily check that we can recoordinatize in such a way that $L^* = [1,0,0]$ and $P^* = (0,a)$ for some $a \in R_1$. Note that, if M^* satisfies the assumptions of the lemma, then it has new coordinates of the form $[1,B,0], B \in R_1$. Denote by $\mathcal{D}'$ the unique $\mathcal{C}$-quadrilateral in perspective with $\mathcal{D}$ from $\mathcal{C}$ and containing the point $(0,a+B)$. But then the chain $(0,a)$ I $[0,a,k']$ I $(a,\ldots,a)$ I $[1,0,0]$ of $\mathcal{D}$ corresponds to $(0,a+B)$ I $[0,a+B,k']$ I $(a,\ldots,a+C)$ I $[1,C,0]$ of $\mathcal{D}'$, for some $C \in R_1$. By property (III), it follows that $a+B = a+C$ or $B=C$ by the previous lemma, and the result follows. Q.E.D.

Lemma 4. *Let $\mathcal{D} = P_1$ I L_1 I P_2 I L_2 I P_3 I L_3 I P_4 I L_4 I P_1 be a quadrilateral and suppose that L_1 I (∞) and that L_i does not meet $[\infty], i=2,3,4$. Then for every point P_1' I L_1 distinct from (∞), there exists a unique quadrilateral P_1' I L_1 I P_2' I $\ldots$ I P_1' in perspective with $\mathcal{D}$ from $\mathcal{C}$.*

Proof. Consider an arbitrary line L_5 through P_2, distinct from L_1 and L_2. Let P_5 be the unique point on L_5 collinear with the same point on $[\infty]$ as P_3. The unique line L_6 through P_5 meeting L_3 (in, say, P_6) does not meet $[\infty]$. Let L_7 be the unique line through P_1 meeting L_6 in, say, P_7. Clearly $L_7 \neq L_1$. Let P_1' I L_4' I P_4' I L_3' I P_6' I L_6' I P_7' I L_7' I P_1' be the $\mathcal{C}$-quadrilateral in perspective with P_1 I L_4 I P_4 I L_3 I P_6 I L_6 I P_7 I L_7 I P_1 from $\mathcal{C}$. Now consider the $\mathcal{C}$-quadrilateral $\mathcal{D}_4 = P_2$ I L_2 I P_3 I L_3 I P_6 I L_6 I P_5 I L_5 I P_2. The unique $\mathcal{C}$-quadrilateral $\mathcal{D}'_4$ in perspective with $\mathcal{D}_4$ from $\mathcal{C}$ containing L_3' as corresponding line to L_3 (such a quadrilateral exists by the previous lemma) must contain P_3' (since it is the *unique* point on L_3' collinear with the same point on $[\infty]$ as P_3). Hence $\mathcal{D}'_4$ looks like P_2' I L_2' I P_3' I L_3' I...I P_2', where P_2' I L_1. But now it is easy to check that P_1' I L_1 I P_2' I L_2' I P_3' I L_3' I P_4' I L_4' I P_1' is a quadrilateral which is in perspective with $\mathcal{D}$ from $\mathcal{C}$. Q.E.D.

STEP 2. Let (a,l,a') be a point of $\mathcal{S}$ not collinear with $(0,0,0)$. Since $t \geq 3$, there exists $k \in R_2 - \{0\}$ such that the line $[a,l]$ does not meet $[k,0,0]$; in other words, the unique line $[p,x,p']$ through (a,l,a') meeting $[k,0,0]$ does not meet $[\infty]$ and hence has indeed three coordinates. Suppose first $p \neq 0$. We consider the $\mathcal{C}$-quadrilateral $\mathcal{D}_5 = [k,0,0] \perp [p,x,p'] \text{ I } (a,l,a') \text{ I } [0,a',k_1'] \text{ I } (0,a') \text{ I } [0,a',k_2'] \text{ I } (b,m,a') \text{ I } [k,0,0]$, where k_1', k_2', b and m are well-defined. We have $m = Q_2(b,k,0,0)$ and $a' = Q_1^*(b,k,0,0) = k \cdot b$. Consider the $\mathcal{C}$-quadrilateral $\mathcal{D}'_5$ containing $[k,B,0], B \in R_1$, and which is in perspective with $\mathcal{D}_5$ from $\mathcal{C}$ (which exists by lemma 3). Then, by lemma 1, $\mathcal{D}'_5$ contains the chain $[k,B,0] \text{ I } (b,m',a'+B) \perp (0,a'+B) \perp (a,l',a'+B)$ for some $l', m' \in R_2$. *Hence the unique $\mathcal{C}$-quadrilateral containing $[k,B,0]$ (for any $B \in R_1$) and which is in perspective from $\mathcal{C}$ with any $\mathcal{C}$-quadrilateral containing the chain $[k,0,0] \sim (a,l,a')$ (with the above restrictions) contains the chain $[k,B,0] \sim (a,l',a'+B)$, for some $l' \in R_2$.* This property is immediate if $p = 0$. If we assume $l \neq 0$ and we consider the $\mathcal{C}$-quadrilateral $\mathcal{D}_6 = [k,0,0] \sim (a,l,a') \sim [0,0,0] \text{ I } (0,0,0) \text{ I } [k,0,0]$, then, by considering the corresponding $\mathcal{C}$-quadrilateral in perspective with $\mathcal{D}_6$ from $\mathcal{C}$ and containing the line $[0,B,0]$, we see that the above property also holds for $k = 0$ (in which case l' must be 0). *Hence the property holds for every chain $[k,0,0] \sim (a,l,a')$ not containing a line concurrent with $[\infty]$.*

Suppose now $(a,l,a') \perp (0,0,0)$ but still not incident with $[0,0]$. Then $(a,l,a') \text{ I } [k,0,0]$ for some $k \in R_2$ and $k \cdot a = a'$. If $B \in R_1$, then the last coordinate of the point $(a,\ldots)$ incident with $[k,B,0]$ is by definition $Q_1^*(a,k,B,0) = k \cdot a + B = a' + B$. *So we can generalize the property mentioned above to chains of the form $[k,0,0] \text{ I } (a,l,a'), k \neq 0$.*

STEP 3. Let (a,l,a') and $[k,0,0]$ be again as in the very beginning of step 3 (so without the condition $p \neq 0$). Suppose moreover $l \neq 0$. We consider the $\mathcal{C}$-quadrilaterals $\mathcal{D}_7 = [0,0,0] \sim (a,l,a') \sim [k,0,0] \text{ I } (0,0,0) \text{ I } [0,0,0]$ and $\mathcal{D}_8 = [0,0,0] \sim (a,l,a') \text{ I } [a,l] \text{ I } (a,l,0) \text{ I } [0,0,l] \text{ I } (0,0) \text{ I } [0,0,0]$. The $\mathcal{C}$-quadrilaterals in perspective with $\mathcal{D}_7$ and $\mathcal{D}_8$ from $\mathcal{C}$ and containing $[0,B,0], B \in R_1$, are resp. $\mathcal{D}_7' = [0,B,0] \sim (a,l',a'+B) \sim [k,B,0] \text{ I } (0,0,B) \text{ I } [0,B,0]$ for some $l' \in R_2$ and $\mathcal{D}_8' = [0,B,0] \sim (a,l',a'+B) \text{ I } [a,Q_2(a,0,B,l)] \text{ I } (a,Q_2(a,0,B,l),B) \text{ I } [0,B,l] \text{ I } (0,B) \text{ I } [0,B,0]$. Hence $l' = Q_2(a,0,B,l)$. Now suppose $l = 0$. The quadrilateral $\mathcal{D}_7$ above is still well-defined (unlike $\mathcal{D}_8$) and so is $\mathcal{D}'_7$, which must contain the chain $[0,B,0] \text{ I } (a,Q_2(a,0,B,0),B) \perp (a,l',a'+B)$, for some $l' \in R_2$. But since $l = 0$, the line joining $(a,Q_2(a,0,B,0),B)$ and $(a,l',a'+B)$ meets $[\infty]$, hence $l' = Q_2(a,0,B,0)$. *Consequently, if a $\mathcal{C}$-quadrilateral $\mathcal{D}$ contains the chain $[k,0,0] \sim (a,l,a')$, then the $\mathcal{C}$-quadrilateral in perspective with $\mathcal{D}$ from $\mathcal{C}$ and containing $[k,B,0], B \in R_1$, contains the point $(a,Q_2(a,0,B,l),a'+B)$ as the point corresponding to (a,l,a').* Suppose again $l \neq 0$, then by considering the $\mathcal{C}$-quadrilaterals $\mathcal{D}_7$ and $\mathcal{D}'_7$, *we see that this property also holds for $k = 0$.*

Now suppose (a,l,a') is collinear with $(0,0,0)$, but not incident with $[0,0]$. So $(a,l,a') \text{ I } [k,0,0]$ for some $k \in R_2$. We compute $l' \in R_2$ in $(a,l',a'+B) \text{ I } [k,B,0]$ (the third coordinate $a'+B$ is calculated in the last part of step 2). If $k = 0$, then $a' = l = 0$ and $[0,B,0] \text{ I } (a,Q_2(a,0,B,0),B)$, hence $l' = Q_2(a,0,B,0) = Q_2(a,0,B,l)$. Suppose now $k \neq 0$. Consider the $\mathcal{C}$-quadrilateral $\mathcal{D}_9 = (0,0,0) \text{ I } [k,0,0] \text{ I } (a,l,a') \text{ I } [a,l] \text{ I } (a,l,0) \text{ I } [0,0,l] \text{ I } (0,0) \text{ I } [0,0,0] \text{ I } (0,0,0)$. The $\mathcal{C}$-quadrilateral $\mathcal{D}'_9$ in perspective with $\mathcal{D}_9$ from $\mathcal{C}$ contains the chain $(0,0,B) \text{ I } [0,B,0] \text{ I } (0,B) \text{ I } [0,B,l] \text{ I } (a,Q_2(a,0,B,l),B) \text{ I } [a,Q_2(a,0,B,l)] \text{ I } (a,l',a'+B)$, hence $l' = Q_2(a,0,B,l)$. *So one can extend the above property to $\mathcal{C}$-quadrilaterals containing any chain $[k,0,0]$ I (a,l,a').*

STEP 4. In this section, (k,b) denotes a point collinear with (∞) with $k \neq 0$ and $b \neq 0$. Since $t \geq 3$, there exists $k^* \in R_2 - \{0\}$. We define $P_0, L_0, P_1, \ldots, P_5, L_5$ as follows: $P_0 = (k,b), L_0 = [k^*,0,0], P_0 \text{ I } L_1 \text{ I } P_1 \text{ I } L_0, L_2 = [k,b,0], P_2 = (0,0,b), L_3 = [0,b,0], P_3 = (0,b), P_3 \text{ I } L_4 \text{ I } P_4 \text{ I } L_0$ and $P_0 \text{ I } L_5 \text{ I } P_5 \text{ I } L_4$. Suppose first $P_1 \neq P_4$. Then we can consider the quadrilateral $\mathcal{D}_9 = (0,0,0) \text{ I } [0,0] \text{ I } P_2 \text{ I } L_3 \text{ I } P_3 \text{ I } L_4 \text{ I } P_4 \text{ I } L_0 \text{ I } (0,0,0)$. By lemma 4, there exists a quadrilateral $\mathcal{D}'_9$ in perspective with $\mathcal{D}_9$ from $\mathcal{C}$ containing $(0,0,B), B \in R_1$. Then $\mathcal{D}'_9 = (0,0,B) \text{ I } L_0' = [k^*,B,0] \text{ I } P_4' = (\ldots,b+B) \text{ I } L_4' \text{ I } P_3' = (0,b+B) \text{ I } L_3' = [0,b+B,0] \text{ I } P_2' = (0,0,b+B) \text{ I } [0,0] \text{ I } (0,0,B)$. Consider the $\mathcal{C}$-quadrilaterals $\mathcal{D}_{10} = L_0 \text{ I } P_4 \text{ I } L_4 \text{ I } P_5 \text{ I } L_5 \text{ I } P_0 \text{ I } L_1 \text{ I } P_1 \text{ I } L_0$ and $\mathcal{D}'_{10}$ in perspective with $\mathcal{D}_{10}$ from $\mathcal{C}$ and containing L_0'. By the construction of $\mathcal{D}'_9$ and $\mathcal{D}'_{10}$, we have : $\mathcal{D}'_{10} = L_0' \text{ I } P_4' \text{ I } L_4' \text{ I } P_5' \text{ I } L_5' \text{ I } P_0' \text{ I } L_1' \text{ I } P_1' \text{ I } L_0'$, where

P_5', L_5', P_0', L_1' and P_1' are well-defined. Finally consider the $\mathcal{C}$-quadrilateral $\mathcal{D}_{11} = L_4$ I P_3 I L_3 I P_2 I L_2 I P_0 I L_5 I P_5 I L_4 and the $\mathcal{C}$-quadrilateral $\mathcal{D}'_{11}$ in perspective with $\mathcal{D}_{11}$ from $\mathcal{C}$ and containing L_4'. By construction, the chains L_4' I P_3' I L_3' I P_2' and L_4' I P_5' I L_5' I P_0' are contained in $\mathcal{D}'_{11}$. Hence $P_0' \perp P_2'$, so $P_0' = (k, b+B)$. *This shows that, if $\mathcal{D}$ is any quadrilateral containing the chain $[k^*, 0, 0] \sim (k, b)$ (with the above restrictions), then the quadrilateral in perspective with $\mathcal{D}$ from $\mathcal{C}$ and containing $[k^*, B, 0]$ contains $(k, b+B)$. This property also holds for $k = 0$* as the coordinates of P_3' show us.

Suppose now $P_4 = P_1$. Then $L_5 = L_1$ and $P_5 = P_4$. The quadrilaterals $\mathcal{D}_9$ and $\mathcal{D}_{11}$ are still well-defined and hence so are $\mathcal{D}'_9$ and $\mathcal{D}'_{11}$. We obtain, with the same notation as above and in a similar way, $P_2' = (0, 0, b+B) \perp P_0' = (k, b+B) \sim L_0'$. *So the above stated property still holds.* Applying this property to the $\mathcal{C}$-quadrilateral $[0,0,0] \sim (k, b) \sim [k^*, 0, 0]$ I $(0,0,0)$ I $[0,0,0]$, *we see that this property extends to quadrilaterals containing the chain* $[0,0,0] \sim (k, b), k \neq 0, b \neq 0$.

STEP 5. Suppose $[k, b, k']$ I $(a, l, a') \perp (0,0,0), a, b, a' \in R_1$ and $k, l, k' \in R_2$. If $a \neq 0$, then there exists a unique $k^* \in R_2$ such that (a, l, a') I $[k^*, 0, 0]$. We consider five cases.

•*Case 1* : $a \neq 0$ and $k = k^* \neq 0$. Clearly $b = 0$ and $k' = 0$. By step 3, we have, for every $B \in R_1, [k, b+B, k'] = [k, B, 0]$ I $(a, Q_2(a, 0, B, l), a' + B)$.

•*Case 2* : $a \neq 0$ and $k = k^* = 0$. Clearly $a' = b = 0$ and $k' = l = 0$. By definition, $(a, Q_2(a, 0, B, l), a' + B) = (a, Q_2(a, 0, B, 0), B)$ I $[0, B, 0] = [k, b+B, k']$.

•*Case 3* : $a \neq 0$ and $k \neq k^* \neq 0$. Consider the chain $[k^*, 0, 0]$ I (a, l, a') I $[k, b, k']$ I (k, b) and any $\mathcal{C}$-quadrilateral $\mathcal{D}$ containing it. The $\mathcal{C}$-quadrilateral $\mathcal{D}'$ in perspective with $\mathcal{D}$ from $\mathcal{C}$ containing $[k^*, B, 0]$ contains $(a, Q_2(a, 0, B, l), a' + B)$ (by step 3) and $(k, b+B)$ (by step 4). Hence $\mathcal{D}'$ contains the chain $(a, Q_2(a, 0, B, l), a' + B)$ I $[k, b+B, k']$.

•*Case 4* : $a \neq 0$ and $k \neq k^* = 0$. As in case 2, $a' = 0$ and $l = 0$. Since $t \geq 3$, we can choose $k^{**} \in R_2 - \{0\}, k^{**} \neq k$. Consider the $\mathcal{C}$-quadrilaterals $\mathcal{D}_{12} = (0,0,0)$ I $[0,0,0]$ I $(a, 0, 0)$ I $[k, b, k']$ I $(k, b) \sim [k^{**}, 0, 0]$ I $(0,0,0)$ and $\mathcal{D}'_{12}$ in perspective with $\mathcal{D}_{12}$ from $\mathcal{C}$ and containing $[0, B, 0]$. By case 3 above (substituting k^{**}, for k^*), $[k, b, k']$ corresponds to $[k, b+B, k']$ in $\mathcal{D}'_{12}$. Furthermore, if $(a, 0, 0)$ corresponds to (a, l', a''), then $l' = Q_2(a, 0, B, 0)$ and $a'' = B$ since (a, l', a'') is incident with $[0, B, 0]$ (which corresponds to $[0,0,0]$). Hence, again, $(a, Q_2(a, 0, B, l), a' + B) = (a, Q_2(a, 0, B, 0), B)$ I $[k, b+B, k']$.

•*Case 5* : $a = 0$. Clearly $(a, l, a') = (0, 0, a')$ I $[0, 0]$ and $[k, b, k'] = [k, a', 0]$. In this case, trivially $(a, Q_2(a, 0, B, l), a' + B) = (0, 0, a' + B)$ I $[k, a' + B, 0] = [k, b+B, k']$.

So if $(0,0,0) \perp (a, l, a')$ I $[k, b, k']$, then for every $B \in R_1$, we have

$$(a, Q_2(a, 0, B, l), a' + B) \text{ I } [k, b+B, k'].$$

STEP 6. Suppose now that $(0,0,0)$ is not collinear with the point (a, l, a'), which is incident with the line $[k, b, k']$. We can choose $k^* \in R_2$ such that $[a, l]$ does not meet $[k^*, 0, 0]$ $(t \geq 3)$. We consider two cases.

•*Case 1* : $b \neq 0$. Consider the $\mathcal{C}$-quadrilateral $\mathcal{D}_{13} = [k^*, 0, 0] \sim (k, b)$ I $[k, b, k']$ I $(a, l, a') \sim [k^*, 0, 0]$. By lemma 3, there exists a unique $\mathcal{C}$-quadrilateral $\mathcal{D}'_{13}$ in perspective with $\mathcal{D}_{13}$ from $\mathcal{C}$ and containing $[k^*, B, 0]$. By the properties stated in steps 2, 3 and 4, $\mathcal{D}'_{13}$ contains the chain $(a, Q_2(a, 0, B, l), a' + B)$ I $[k, b+B, k']$.

•*Case 2* : $b = 0$. Consider the chain $[k, 0, 0]$ I $(k, 0)$ I $[k, 0, k']$ I (a, l, a'). By steps 2 and 3, one has $[k, B, 0]$ I (k, B) I $[k, B, k']$ I $(a, Q_2(a, 0, B, l), a' + B)$.

So we have shown that, whenever (a, l, a') I $[k, b, k']$, then

$$(a, Q_2(a, 0, B, l), a' + B) \text{ I } [k, b+B, k']. \tag{1}$$

In coordinates, this means that, whenever $Q_2(a,k,b,k') = l$ and $Q_1^*(a,k,b,k') = a'$, then $Q_2(a,0,B,l) = Q_2(a,k,b+B,k')$ and $a'+B = Q_1^*(a,k,b+B,k')$. Substituting the values for l and a' in the latter equations, we obtain for all arbitrary $a,b,B \in R_1$ and $k,k' \in R_2$,

$$Q_1^*(a,k,b+B,k') = Q_1^*(a,k,b,k')+B \quad (2)$$

and

$$Q_2(a,k,b+B,k') = Q_2(a,0,B,Q_2(a,k,b,k')), \quad (3)$$

i.e. $\mathcal{S}$ is $\mathcal{C}$-transitive. This completes the proof of our main result. Q.E.D.

4 Corollaries

We keep the notation $\mathcal{S},\mathcal{C},\ldots$ of the previous section. A line M of $\mathcal{S}$ is said to be *regular* if for every line M' not concurrent with M, the set of lines meeting two distinct lines M_1 and M_2, $M \perp M_i \perp M', i=1,2$, is independent of the choice of M_1 and M_2 (see [5]). By [2], the line $[\infty]$ is regular if and only if Q_2 is independent of the third argument. Now suppose $\mathcal{S}$ is $\mathcal{C}$-Desarguesian (or equivalently $\mathcal{C}$-transitive). Suppose we add the following extra-condition :

(R) Corresponding points of any two $\mathcal{C}$-quadrilaterals in perspective from $\mathcal{C}$ are collinear.

In coordinates this means that corresponding points with three coordinates have the same second coordinate. In that case, for $t \geq 3$, (1) becomes

$$(a,l,a'+B) \quad \mathrm{I} \quad [k,b+B,k'], \quad (4)$$

with

$$(a,l,a') \quad \mathrm{I} \quad [k,b,k'].$$

Consequently, $l = Q_2(a,k,b+B,k') = Q_2(a,k,b,k')$ for all $a,b,B \in R_1$ and all $k,k' \in R_2$. Hence, for $t \geq 3$, Q_2 is independent of its third argument and $[\infty]$ is a regular line.

Consider now the following condition $(\mathcal{D}3)$ on a quadrilateral $\mathcal{D}$.

$(\mathcal{D}3)$ No line of $\mathcal{D}$ meets L.

A $\mathcal{C}$-quadrilateral $\mathcal{D}$ satisfying $(\mathcal{D}3)$ is called *opposite* $\mathcal{C}$. Call $\mathcal{S}$ *weakly* $\mathcal{C}$*-Desarguesian* if for every quadrilateral $\mathcal{D}_1$ opposite $\mathcal{C}$ and every point P_1' (distinct from both P and P') on any base-line, there exists a quadrilateral $\mathcal{D}_2$ opposite $\mathcal{C}$ containing P_1' and which is in perspective with $\mathcal{D}_1$ from $\mathcal{C}$.

If $\mathcal{S}$ is weakly $\mathcal{C}$-Desarguesian and if we add at the same time the condition (R), then it is an easy exercise to show that $\mathcal{S}$ is $\mathcal{C}$-Desarguesian. So $\mathcal{S}$ is $\mathcal{C}$-transitive. Moreover, for $t \geq 3$, L is regular and consequently L is an axis of symmetry (with the terminology of [5]). This is more or less what M.A.Ronan shows in [6]. Compared to the version of Ronan's theorem in [5], we use here a somewhat weaker hypothesis : indeed, we assume our quadrilaterals to be opposite $\mathcal{C}$, which is weaker than being *opposite* L (a quadrilateral is *opposite* L if it is opposite $\{P_1,L,P_2\}$ for at least one pair of distinct points P_1 and P_2 on L). In [5], it is also shown that for $t=2$ all conditions are satisfied, but that L is not necessarily an axis of symmetry.

In [7], it is shown that, if a finite generalized quadrangle is $\{P,L,P'\}$-transitive for all roots $\{P,L,P'\}$, then it is $\{L,P,L'\}$-transitive for all roots $\{L,P,L'\}$ (with P,P' points and L,L' lines). Hence $\mathcal{S}$ is Moufang, and then J. Tits [8] shows that from a theorem of P. Fong and G. M. Seitz [1] it is straightforward that the quadrangle is classical or dual classical.

Corollary 1. *If a finite generalized quadrangle $\mathcal{S}$ is $\{P, L, P'\}$-Desarguesian for all roots $\{P, L, P'\}$, then $\mathcal{S}$ is classical or dual classical.*

For the next results, we need the definition of a TGQ. By [5], 9.2.2 (ii), $\mathcal{S}^{(P)}$ is a *translation generalized quadrangle*, $P \in \mathcal{P}$, if each line through P is regular and if $\mathcal{S}$ is $\{P, L, P'\}$-transitive for all lines L through P and some point P' distinct from P on L. It follows that $\mathcal{S}$ is $\{P', L, P''\}$-transitive for all lines L through P and all pairs of distinct points P', P'' on L. Hence

Corollary 2. *Let P be a point of the generalized quadrangle $\mathcal{S}$. Then the following conditions are equivalent.*

(C.1) $\mathcal{S}^{(P)}$ is a translation generalized quadrangle.

(C.2) All lines through P are regular and $\mathcal{S}$ is $\{P, L, P'\}$-Desarguesian for all lines L through P and some point P' distinct from P on L.

(C.3) All lines through P are regular and $\mathcal{S}$ is $\{P', L, P''\}$-Desarguesian for all lines L through P and all pairs of distinct points P', P'' on L.

For the following results, we need some definitions. A *triad* is a set of three pairwise non collinear points. A *center* of a given triad is a point collinear with all elements of the triad. The *span* of two distinct points P_1 and P_2, denoted by $\{P_1, P_2\}^{\perp\perp}$, is the set of points of $\mathcal{S}$ collinear with all points that are collinear with both P_1 and P_2. The definition of a *regular* point is dual to the definition of a regular line above.

The following results are immediate consequences of our main result, combined with results of S. E. Payne and/or J. A. Thas (see [5,ch.9]) :

Corollary 3. *If $\mathcal{S}^{(P)}$ is a translation generalized quadrangle, then $\mathcal{S}$ is $\{L_1, P, L_2\}$-Desarguesian for all pairs of distinct lines L_1, L_2 through P.*

Corollary 4. *Let $\mathcal{S}$ be a finite generalized quadrangle of order (s,t). Suppose $\mathcal{S}$ is $\{L_1, P, L_2\}$-Desarguesian for all pairs of distinct lines L_1, L_2 through some fixed point P. Then we have :*

(i) If all lines through P are regular and t is odd, then $\mathcal{S}^{(P)}$ is a translation generalized quadrangle.

(ii) If each triad containing P has at least two centers, then $\mathcal{S}^{(P)}$ is a translation generalized quadrangle and $t = s^2$.

(iii) If P does not belong to a triad having exactly one center, then $\mathcal{S}^{(P)}$ is a translation generalized quadrangle; if t is even, then $t = s^2$.

(iv) If $t = s^2$, then $\mathcal{S}^{(P)}$ is a translation generalized quadrangle.

Combining corollaries 2 and 3, we obtain Ronan's theorem, which we state using our notation.

Corollary 5. (M. A. Ronan [6]). *Let $\mathcal{S}$ be weakly $\{P, L, P'\}$-Desarguesian for all lines L of $\mathcal{S}$ and some pair of points P, P' on L, and suppose $\mathcal{S}$ satisfies the condition* (R) *for all $\mathcal{C} = \{P, L, P'\}$. If $t \geq 3$, then $\mathcal{S}$ is Moufang and all lines of $\mathcal{S}$ are regular (in particular, if $\mathcal{S}$ is finite, then it arises from a nonsingular quadric of Witt index 2 in $PG(4, s)$ or $PG(5, s)$).*

Proof. By the preceding discussions all lines of $\mathcal{S}$ are regular and $\mathcal{S}$ is $\{P, L, P'\}$-Desargue sian for all lines L and all pairs of distinct points P, P' on L. Hence, by corollary 2, $\mathcal{S}^{(P)}$ is a translation generalized quadrangle for all points P of $\mathcal{S}$. By corollary 3, $\mathcal{S}$ is $\{L, P, L'\}$-Desarguesian for all points P and all pairs of distinct lines L, L' through P. Hence $\mathcal{S}$ is Moufang. If $\mathcal{S}$ is finite, then, since all lines are regular, it consists of all points and all lines of a nonsingular quadric of Witt index 2 in $PG(4, s)$ or $PG(5, s)$ (cfr. [5]). Q.E.D.

Combining our main result with a fundamental result of J. Tits [9], stating that every Moufang generalized quadrangle is known, we finally have

Corollary 6. *If a generalized quadrangle $\mathcal{S}$ is $\{P, L, P'\}$-Desarguesian and $\{L, P, L'\}$-Desarguesian for all roots $\{P, L, P'\}$ and all roots $\{L, P, L'\}$ (with P, P' points and L, L' lines), then $\mathcal{S}$ is known.*

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