Non-embeddable polar spaces

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Abstract

We provide an elementary but explicit description of the non-embeddable thick polar spaces introduced by Jacques Tits. These polar spaces are related to algebraic groups of absolute type E_7 and Tits index $\mathsf{E}_{7,3}^{28}$. Our approach includes all polar spaces of rank 3 related to a quadratic alternative division algebra.

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1 Introduction

In 1974, Jacques Tits [13] classified spherical buildings of rank at least 3, thereby treating all polar spaces as introduced and studied by Veldkamp [16]. Veldkamp also classified large classes of polar spaces, in particular, he classified all polar spaces of rank at least 3 for which the planes are Desarguesian and are constructed over a field of characteristic different from 2. In order to treat the missing cases, Jacques Tits introduced pseudo-quadratic forms, defined groups of mixed type, and proved the existence of the Tits index $\mathsf{E}_{7,3}^{28}$ in algebraic groups of type E_7 such that the field $\mathbb{E}$ of definition is included in a Cayley-Dickson division algebra over a subfield $\mathbb{K}$ of $\mathbb{E}$, and $\mathbb{E}$ is a quadratic Galois extension of $\mathbb{K}$. In the latter case, the associated polar space is then constructed using the parabolic subgroups of the algebraic group in question. This polar space cannot be embedded in a projective space, as its planes are non-Desarguesian Moufang planes. For this reason, these polar spaces are called *non-embeddable polar spaces*. There is also a class of non-embeddable non-thick polar spaces in rank 3, but these are well understood and we will not be concerned with these in the present paper (the non-thick polar spaces of rank 3 are the line Grassmannians of projective spaces of rank 3).

In 1987, Ronan and Tits [9] presented a general construction of buildings with no sub-diagram of type H_3, providing *all* spherical buildings. In particular, this applies to the non-embeddable polar spaces (see Condition (b) of Section 5 of [?]). Some more details of this construction can be found in Chapter 40 of Tits and Weiss [14], in particular Statements (40.54) and (40.55) treat the case of non-embeddable polar spaces stated in

(40.25)(iii). In 1990, Bernhard Mühlherr [8] constructed the thick non-embeddable polar spaces as fixed point sets of involutions in buildings of type E_7 (although this construction is also apparent on page 89 of [12]). In the present paper, we will provide a coordinate construction of the thick non-embeddable polar spaces, in the spirit of the coordinatization of the non-Desarguesian Moufang planes. An advantage of the latter over the former constructions is that it is the most explicit and allows for some applications that were apparently out of reach before. We mention two examples.

In [3], we explicitly construct an embedding of the corresponding dual polar space in a projective space of dimension 55, and we show that this embedding is the universal one. The universality of that embedding, which is the embedding deduced from the highest weight module for groups of type E_7, was an open question since the early 90's (see e.g. [11, p. 229]), and the introduced coordinatization plays a crucial role in the proof. Another application is given in [4], where the authors use the coordinates to show that the geometry in a non-embeddable polar space opposite a chamber is simply connected. The latter was open since 1996, see [1, p. 66].

We note that our approach allows to uniformly construct all polar spaces related to a quadratic alternative division algebra. That is exactly the way we will proceed. Note that, in [3], we also establish universality of a certain explicitly defined embedding of the corresponding dual polar space for arbitrary quadratic alternative division algebras.

The non-embeddable polar spaces are intimately related to the Cayley-Dickson division algebras. Hence, we will need to recall some basic results about such algebras. This will be done in Section 2 and in the beginning of Section 4. In Section 3, we introduce coordinates for some classical polar spaces, and we extend this coordinatization in Section 4 to obtain the non-embeddable (or non-classical) polar spaces of rank 3. Hence our approach is rather indirect: we do not start from a known description (using algebraic groups, for instance) and derive ours, but we simply construct from scratch a geometry (by analogy with the other polar spaces in the family) and prove it is a polar space either isomorphic to one of the classical examples of Section 3, or with non-Desarguesian Moufang planes; it then follows from the classification by Jacques Tits [13] that in the latter case the polar space in question is the unique non-embeddable one related to the given Cayley-Dickson division algebra.

Our treatment requires a lot of computations in Cayley-Dickson division algebras. We have written down the most intricate cases, leaving the easier ones to the interested reader. In fact, in most cases that we left out, the non-associativity is not a burden as it does not happen that one has to multiply three general elements (e.g., in the definition of planes, only Type VIII contains an expression which requires parentheses). Note that we did not perform any computation on a computer; everything has been checked only by hand.

2 Alternative division rings and Moufang planes

An *alternative division ring* is a set $\mathbb{D}$ of size at least 2 which is endowed with two binary operations, an addition $+$ and a multiplication $\cdot$, satisfying the following properties:

- the structure $(\mathbb{D}, +)$ is a commutative group;
- the multiplication is left- and right-distributive with respect to the addition;
- there exists a (necessarily unique) neutral element 1 for the multiplication;
- if 0 denotes the neutral element for the addition, then for every $a \in \mathbb{D} \setminus \{0\}$, there exists a (necessarily unique) element $a^{-1} \in \mathbb{D}$ such that $a^{-1} \cdot a = 1 = a \cdot a^{-1}$;
- for every $a \in \mathbb{D} \setminus \{0\}$ and every $b \in \mathbb{D}$, we have $a^{-1} \cdot (a \cdot b) = b = (b \cdot a) \cdot a^{-1}$.

It is a costume to denote the product $a \cdot b$ of two elements $a, b \in \mathbb{D}$ by ab. In the literature, one can find alternative but equivalent definitions for the notion of alternative division ring, see e.g. Tits and Weiss [14].

The alternative division rings with associative multiplication are precisely the skew fields. An important class of (non-associative) alternative division rings are the so-called Cayley-Dickson division algebras. Explicit constructions of such alternative division rings can be found in Jacobson [6, p. 426] (for characteristic distinct from 2), Schafer [10, p. 5] (for characteristic distinct from 2), Tits and Weiss [14, Section 9.8] and Van Maldeghem [15, Appendix B]. We describe the construction given in [15].

Suppose $\mathbb{K}$ is a field and $l_1, l_2, l_3 \in \mathbb{K}$ such that the equation $X_0^2 - l_1X_1^2 + X_0X_1 - l_2X_2^2 + l_1l_2X_4^2 - l_2X_2X_4 - l_3X_3^2 + l_1l_3X_7^2 - l_3X_3X_7 + l_2l_3X_5^2 - l_1l_2l_3X_6^2 + l_2l_3X_5X_6 = 0$ has no solutions for $(X_0, X_1, \ldots, X_7) \in \mathbb{K}^8$ distinct from $(0, 0, \ldots, 0)$. Then let $\mathbb{O}$ be an 8-dimensional vector space over $\mathbb{K}$ with basis $\{1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ such that $1 \in \mathbb{K}$. Then $\mathbb{O}$ can be given the structure of an alternative division ring if we define the multiplication in the following way:

$\cdot$	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$l_1 + e_1$	$e_2 - e_4$	$e_3 - e_7$	$-l_1e_2$	e_6	$l_1e_5 + e_6$	$-l_1e_3$
e_2	e_2	e_4	l_2	$-e_5$	l_2e_1	$-l_2e_3$	$-l_2e_7$	$-e_6$
e_3	e_3	e_7	e_5	l_3	e_6	l_3e_2	l_3e_4	l_3e_1
e_4	e_4	$l_1e_2 + e_4$	$l_2 - l_2e_1$	$-e_6$	$-l_1l_2$	$-l_2e_3 + l_2e_7$	$l_1l_2e_3$	$-l_1e_5 - e_6$
e_5	e_5	$e_5 - e_6$	l_2e_3	$-l_3e_2$	$l_2e_3 - l_2e_7$	$-l_2l_3$	$-l_2l_3 + l_2l_3e_1$	$-l_3e_2 + l_3e_4$
e_6	e_6	$-l_1e_5$	l_2e_7	$-l_3e_4$	$-l_1l_2e_3$	$-l_2l_3e_1$	$l_1l_2l_3$	$l_1l_3e_2$
e_7	e_7	$l_1e_3 + e_7$	e_6	$l_3 - l_3e_1$	$l_1e_5 + e_6$	$l_3e_2 - l_3e_4$	$-l_1l_3e_2$	$-l_1l_3$

The *Cayley-Dickson division algebras* are precisely the alternative division rings which can be obtained in the above-described way. The field $\mathbb{K}$ consists of those elements of $\mathbb{O}$ which commute with every element of $\mathbb{O}$. The Cayley-Dickson division algebra $\mathbb{O}$ has a so-called *standard involution* which maps $X_0 + X_1e_1 + X_2e_2 + X_3e_3 + X_4e_4 + X_5e_5 + X_6e_6 + X_7e_7$ to $X_0 + X_1 - X_1e_1 - X_2e_2 - X_3e_3 - X_4e_4 - X_5e_5 - X_6e_6 - X_7e_7$, $(X_0, X_1, \ldots, X_7) \in \mathbb{K}^8$.

The multiplication in a Cayley-Dickson division algebra is not associative. In fact, it is a result due to Bruck and Kleinfeld [2] and Kleinfeld [7] that the Cayley-Dickson division algebras are the only alternative division rings in which the multiplication is not associative. A proof of that result can also be found in Tits and Weiss [14, Chapter 20] and Van Maldeghem [15, Appendix B]. The proof given in [15] is attributed to Jacques Tits.

In this paper, we will also meet a class of alternative division rings in which the multiplication is associative, but not commutative. Suppose $\mathbb{K}$ is a field and $l_1, l_2 \in \mathbb{K}$ such that the equation $X_0^2 - l_1X_1^2 + X_0X_1 - l_2X_2^2 + l_1l_2X_3^2 - l_2X_2X_3 = 0$ has no solutions for $(X_0, X_1, X_2, X_3) \in \mathbb{K}^4$ distinct from $(0, 0, 0, 0)$. Let $\mathbb{H}$ be a fourdimensional vector space over $\mathbb{K}$ with basis $\{1, i, j, k\}$ such that $1 \in \mathbb{K}$. Then $\mathbb{H}$ can be given the structure of a skew field if we define the multiplication in the following way:

$\cdot$	1	i	j	k
1	1	i	j	k
i	i	l_2	k	l_2j
j	j	$i - k$	$l_1 + j$	$-l_1i$
k	k	$l_2 - l_2j$	$l_1i + k$	$-l_1l_2$

The *quaternion division algebras* are precisely the skew fields which can be obtained in the above described way. The field $\mathbb{K}$ consists of those elements of $\mathbb{H}$ which commute with every element of $\mathbb{H}$. The quaternion division algebra $\mathbb{H}$ has a unique involution which only fixes each element of $\mathbb{K}$. The involution is called the *standard involution* of $\mathbb{H}$ and maps $X_0 + X_1i + X_2j + X_3k$ to $X_0 + X_2 - X_1i - X_2j - X_3k$, $(X_0, X_1, X_2, X_3) \in \mathbb{K}^4$. Every Cayley-Dickson division algebra has subalgebras that are quaternion division algebras. In fact, it can be seen that the multiplication table above for the quaternion division algebras is a 'sub-table' of the multiplication table for the Cayley-Dickson division algebras by identifying i with e_2, j with e_1 and k with e_4.

With every alternative division ring $\mathbb{D}$, we can associate a point-line geometry $\pi_{\mathbb{D}}$ in the following way. There are three types of points:

- a symbol (∞), where $\infty \notin \mathbb{D}$;
- symbols (s), where $s \in \mathbb{D}$;
- symbols (a, b), where $a, b \in \mathbb{D}$.

There are also three types of lines:

- the set $[\infty] := \{(\infty)\} \cup \{(\lambda) : \lambda \in \mathbb{D}\}$;
- the sets $[k] := \{(\infty)\} \cup \{(k, \lambda) : \lambda \in \mathbb{D}\}$;

• the sets $[m,k] := \{(m)\} \cup \{(\lambda, m\lambda + k) \,:\, \lambda \in \mathbb{D}\}$.

It is well-known (and straightforward to verify) that $\pi_{\mathbb{D}}$ is a projective plane. In fact, $\pi_{\mathbb{D}}$ is a Moufang plane which means that every line is a so-called translation line. It is also known that every Moufang plane can be coordinatized by an alternative division ring in the above described way. More background information on the coordinatization of (Moufang) projective planes can be found in the monograph [5] by Hughes and Piper.

3 A common coordinatization of some families of polar spaces

In this section, we present a common coordinatization of some families of polar spaces of rank 3. In the following section, we will show that one other class of polar spaces (namely the thick non-embeddable polar spaces of rank 3) can be coordinatized in a similar way. We need the description below in order to identify the polar spaces we will construct merely using coordinates.

(i) Let $\mathbb{O} = \mathbb{K}$ be a field and let σ be the identical map on the set $\mathbb{O} = \mathbb{K}$. Let ζ be a symplectic polarity of $\mathsf{PG}(5,\mathbb{O})$ and let $W(5,\mathbb{O})$ denote the *symplectic polar space* associated with ζ. The points of $W(5,\mathbb{O})$ are the points of $\mathsf{PG}(5,\mathbb{O})$ and the singular subspaces of $W(5,\mathbb{O})$ are those subspaces α of $\mathsf{PG}(5,\mathbb{O})$ for which $\alpha \subseteq \alpha^{\zeta}$. We can choose a reference system in $\mathsf{PG}(5,\mathbb{O})$ such that two distinct points $(X_0, X_1, \ldots, X_5)$ and $(Y_0, Y_1, \ldots, Y_5)$ of $\mathsf{PG}(5,\mathbb{O})$ determine a singular line of $W(5,\mathbb{O})$ if and only if

$$X_0Y_5+X_1Y_4+X_2Y_3-X_3Y_2-X_4Y_1-X_5Y_0 = X_0^\sigma Y_5+X_1^\sigma Y_4+X_2^\sigma Y_3-X_3^\sigma Y_2-X_4^\sigma Y_1-X_5^\sigma Y_0 = 0.$$

A point $(X_0, X_1, \ldots, X_5)$ of $\mathsf{PG}(5,\mathbb{O})$ is a point of $W(5,\mathbb{O})$ if and only if

$$X_0^\sigma X_5 + X_1^\sigma X_4 + X_2^\sigma X_3 \in \mathbb{K}.$$

The last condition looks somewhat weird since it is always satisfied (also, every point of $\mathsf{PG}(5,\mathbb{O})$ is also a point of $W(5,\mathbb{O})$). It will however soon become clear why we have introduced this "superfluous condition".

(ii) Suppose $\mathbb{O}$ and $\mathbb{K}$ are two fields such that $\mathbb{O}$ is a quadratic separable extension of $\mathbb{K}$. Let σ denote the unique nontrivial automorphism of $\mathbb{O}$ fixing each element of $\mathbb{K}$. Let Ω be a nonsingular Hermitian variety of $\mathsf{PG}(5,\mathbb{O})$ whose equation with respect to a suitable reference system is given by $X_0^\sigma X_5 - X_5^\sigma X_0 + X_1^\sigma X_4 - X_4^\sigma X_1 + X_2^\sigma X_3 - X_3^\sigma X_2 = 0$. This equation is equivalent with the following condition:

$$X_0^\sigma X_5 + X_1^\sigma X_4 + X_2^\sigma X_3 \in \mathbb{K}.$$

Two distinct points $(X_0, X_1, \ldots, X_5)$ and $(Y_0, Y_1, \ldots, Y_5)$ of Ω are contained in a line of $\mathsf{PG}(5, \mathbb{O})$ which is completely contained in Ω if and only if

$$X_0^\sigma Y_5 + X_1^\sigma Y_4 + X_2^\sigma Y_3 - X_3^\sigma Y_2 - X_4^\sigma Y_1 - X_5^\sigma Y_0 = 0.$$

The points and subspaces of $\mathsf{PG}(5, \mathbb{O})$ which are contained in Ω define a *Hermitian polar space* $\mathcal{P}_\Omega$.

(iii) Suppose $\mathbb{O}$ and $\mathbb{K}$ are two fields such that $\mathbb{K} \subseteq \mathbb{O}$, $\mathsf{char}\,\mathbb{K} = \mathsf{char}\,\mathbb{O} = 2$ and $\mathbb{O}^2 := \{\lambda^2 : \lambda \in \mathbb{O}\} \subseteq \mathbb{K}$. Let σ be the identity map of $\mathbb{O}$. Let Ω denote the set of all points $(X_0, X_1, \ldots, X_5)$ of $\mathsf{PG}(5, \mathbb{O})$ for which

$$X_0 X_5 + X_1 X_4 + X_2 X_3 = X_0^\sigma X_5 + X_1^\sigma X_4 + X_2^\sigma X_3 \in \mathbb{K}.$$

Two distinct points $(X_0, X_1, \ldots, X_5)$ and $(Y_0, Y_1, \ldots, Y_5)$ of Ω are contained in a line of $\mathsf{PG}(5, \mathbb{O})$ which is completely contained in Ω if and only if

$$X_0Y_5+X_1Y_4+X_2Y_3-X_3Y_2-X_4Y_1-X_5Y_0 = X_0^\sigma Y_5+X_1^\sigma Y_4+X_2^\sigma Y_3-X_3^\sigma Y_2-X_4^\sigma Y_1-X_5^\sigma Y_0 = 0.$$

The points and subspaces of $\mathsf{PG}(5, \mathbb{O})$ which are contained in Ω define a polar space $\mathcal{P}_\Omega$. Observe that $\mathcal{P}_\Omega$ is a subspace of $W(5, \mathbb{O})$. If $\mathbb{K} = \mathbb{O}$, then $\mathcal{P}_\Omega \cong W(5, \mathbb{O})$. If $\mathbb{K} = \mathbb{O}^2$, then $\mathcal{P}_\Omega$ is isomorphic to the polar space $Q(6, \mathbb{O})$ of rank 3 associated to a nonsingular quadric of Witt index 3 of $\mathsf{PG}(6, \mathbb{O})$. If $\mathbb{K} \neq \mathbb{O}$, then we call $\mathcal{P}_\Omega$ the *polar space of rank 3 of mixed type* associated with $(\mathbb{O}, \mathbb{K})$. Polar spaces, and more generally, spherical buildings of mixed type were introduced in Chapter 10 of [13] through the notion of 'groups of mixed type'.

(iv) Suppose that $\mathbb{O}$ is a quaternion division algebra, that $\mathbb{K}$ is the center of $\mathbb{O}$ and that σ is the standard involution of $\mathbb{O}$. Let U be a 6-dimensional right vector space over $\mathbb{O}$ and let $\mathsf{PG}(5, \mathbb{O})$ denote the 5-dimensional projective space associated with U. Suppose we have fixed a basis of U. Then the points of $\mathsf{PG}(5, \mathbb{O})$ can be represented by 6-tuples $(X_0, X_1, \ldots, X_5)$, where $X_0, X_1, \ldots, X_5 \in \mathbb{O}$. Let Ω denote the set of all points $(X_0, X_1, \ldots, X_5)$ of $\mathsf{PG}(5, \mathbb{O})$ for which

$$X_0^\sigma X_5 + X_1^\sigma X_4 + X_2^\sigma X_3 \in \mathbb{K}.$$

Two distinct points $(X_0, X_1, \ldots, X_5)$ and $(Y_0, Y_1, \ldots, Y_5)$ of Ω are contained in a line of $\mathsf{PG}(5, \mathbb{O})$ which is completely contained in Ω if and only if

$$X_0^\sigma Y_5 + X_1^\sigma Y_4 + X_2^\sigma Y_3 - X_3^\sigma Y_2 - X_4^\sigma Y_1 - X_5^\sigma Y_0 = 0.$$

The points and subspaces of $\mathsf{PG}(5, \mathbb{O})$ which are completely contained in Ω define a polar space $\mathcal{P}_\Omega$ which we call a *quaternionic polar space*. Observe also that the map q from U to the quotient group $\mathbb{O}/\mathbb{K}$ which maps the vector with coordinates $(X_0, X_1, \ldots, X_5)$ to $(X_0^\sigma X_5 + X_1^\sigma X_4 + X_2^\sigma X_4) + \mathbb{K}$ is a $(\sigma, -1)$-pseudo-quadratic form and that $\mathcal{P}_\Omega$ is the polar space associated with this pseudo-quadratic form.

Now, let $(\mathbb{O}, \mathbb{K}, \sigma)$ be as in (i), (ii), (iii) or (iv) above, and let $\mathcal{P}$ be the polar space of rank 3 associated with $(\mathbb{O}, \mathbb{K}, \sigma)$. A point $(X_0, X_1, \ldots, X_5)$ of $\mathsf{PG}(5, \mathbb{O})$ is a point of $\mathcal{P}$ if and only if $X_0^\sigma X_5 + X_1^\sigma X_4 + X_2^\sigma X_3 \in \mathbb{K}$. This condition allows us to give explicit coordinates to the points of $\mathcal{P}$. We can divide the points of $\mathcal{P}$ into the following six classes.

- Type 0: The point $(\infty) := (0, 0, 0, 0, 0, 1)$.
- Type 1: The points $(x) := (0, 0, 0, 0, 1, x)$ where $x \in \mathbb{O}$.
- Type 2: The points $(x_1, x_2) := (0, 0, 0, 1, x_1, x_2)$, where $x_1, x_2 \in \mathbb{O}$.
- Type 3: The points $(x_1, x_2; k) := (0, 0, 1, k, x_1, x_2)$, where $x_1, x_2 \in \mathbb{O}$ and $k \in \mathbb{K}$.
- Type 4: The points $(x_1, x_2, x_3; k) := (0, 1, x_1, x_2, k - x_1^\sigma x_2, x_3)$, where $x_1, x_2, x_3 \in \mathbb{O}$ and $k \in \mathbb{K}$.
- Type 5: The points $(x_1, x_2, x_3, x_4; k) := (1, x_1, x_2, x_3, x_4, k - x_1^\sigma x_4 - x_2^\sigma x_3)$, where $x_1, x_2, x_3, x_4 \in \mathbb{O}$ and $k \in \mathbb{K}$.

Two points $(X_0, X_1, \ldots, X_5)$ and $(Y_0, Y_1, \ldots, Y_5)$ of $\mathcal{P}$ are collinear (as points of $\mathcal{P}$) if and only if $X_0^\sigma Y_5 + X_1^\sigma Y_4 + X_2^\sigma Y_3 - X_3^\sigma Y_2 - X_4^\sigma Y_1 - X_5^\sigma Y_0 = 0$. This condition easily allows us to verify the following proposition.

Proposition 3.1 *Let $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4 \in \mathbb{O}$ and $k, l \in \mathbb{K}$.*

• *The point (∞) is collinear with all points of Type 1, all points of Type 2, all points of Type 3 and all points of Type 4. The point (∞) is collinear with no point of Type 5.*

• *The point (x_1) is collinear with all points of Type 1, all points of Type 2 and all points of Type 3. The point (x_1) is collinear with no point of Type 4. The point (x_1) is collinear with the point $(y_1, y_2, y_3, y_4; l)$ of Type 5 if and only if $x_1 + y_1^\sigma = 0$.*

• *The point (x_1, x_2) is collinear with all points of Type 2 and no point of Type 3. The point (x_1, x_2) is collinear with the point $(y_1, y_2, y_3; l)$ if and only if $x_1 + y_1^\sigma = 0$. The point (x_1, x_2) is collinear with the point $(y_1, y_2, y_3, y_4; l)$ if and only if $y_2 + x_1^\sigma y_1 + x_2^\sigma = 0$.*

• *The point $(x_1, x_2; k)$ is collinear with the point $(y_1, y_2; l)$ if and only if $k = l$. The point $(x_1, x_2; k)$ is collinear with the point $(y_1, y_2, y_3; l)$ if and only if $y_2 - k y_1 - x_1^\sigma = 0$. The point $(x_1, x_2; k)$ is collinear with the point $(y_1, y_2, y_3, y_4; l)$ if and only if $y_3 - k y_2 - x_1^\sigma y_1 - x_2^\sigma = 0$.*

• *The point $(x_1, x_2, x_3; k)$ is collinear with the point $(y_1, y_2, y_3; l)$ if and only if $l - k = y_1^\sigma y_2 + x_2^\sigma y_1 - x_1^\sigma y_2 - x_2^\sigma x_1$. The point $(x_1, x_2, x_3; k)$ is collinear with the point $(y_1, y_2, y_3, y_4; l)$ if and only if $y_4 + x_1^\sigma y_3 - x_2^\sigma y_2 - (k - x_2^\sigma x_1) y_1 - x_3^\sigma = 0$.*

• *The points $(x_1, x_2, x_3, x_4; k)$ and $(y_1, y_2, y_3, y_4; l)$ are collinear if and only if $l - k = y_1^\sigma y_4 + y_2^\sigma y_3 + x_3^\sigma y_2 + x_4^\sigma y_1 - x_1^\sigma y_4 - x_2^\sigma y_3 - x_4^\sigma x_1 - x_3^\sigma x_2$.*

Since a polar space of finite rank is completely determined by its point set and the collinearity relation defined on this point set, Proposition 3.1 can be used to give explicit descriptions of the lines and planes of $\mathcal{P}$. We have done this and these explicit descriptions can be found in the next section.

The primary goal of this paper was to give a coordinatization of the nonclassical polar spaces which are associated with Cayley-Dickson division algebras. We tried to achieve this goal by altering the above-alluded descriptions of the lines and planes so that they also would give rise to a polar space in the case that $\mathbb{O}$ is a Cayley-Dickson division algebra, $\mathbb{K}$ is the center of $\mathbb{O}$ and σ is the standard involution of $\mathbb{O}$. An important obstacle toward that goal was the fact that the multiplication in a Cayley-Dickson division algebra is not commutative nor associative, implying that in the descriptions of the lines and planes, the order in which the various multiplications should be carried out needs to be explicitly indicated. We were successful in doing that. As we will see, with the descriptions of the lines and planes given in the next section we also obtain a polar space if one starts with a Cayley-Dickson division algebra.

In the following section, we will not repeat the explicit computations that allowed us to obtain explicit expressions for the lines and planes. We will rather follow another path[1]. We will give the descriptions for the lines right from the start. Then we will prove that this structure is a polar space, which we will be able to identify in case our alternative division ring is associative. If not, then we need to prove some extra properties such as the fact that the planes are projective planes over our non-associative alternative division ring. There are several ways to do this, and here we choose not to do it in the most economical way, but to present coordinates for all planes. The advantage of this approach is that it provides an alternative way of defining the polar spaces with coordinates, by giving only the coordinates of the points and the planes. In fact, the lines can easily be deduced from the description of the planes. Indeed, the parameters s, a and b that are mentioned in the description of each plane also occur in the description of the Moufang plane at the end of Section 2, and identifying the corresponding points (those having the same values for s, a and b) in both descriptions gives rise to an explicit isomorphism between the two planes (see proof of Proposition 4.15). The explicit description of the planes shall also be used in the applications in [3] and [4].

4 A common construction of some families of polar spaces of rank 3

Throughout this section, $\mathbb{O}$ is an alternative division ring. The *center* $Z(\mathbb{O})$ of $\mathbb{O}$ is defined to be the set of all $a \in \mathbb{O}$ such that $ab = ba$, $a(bc) = (ab)c$, $(ba)c = b(ac)$ and $(bc)a = b(ca)$

[1] It seems we have to follow this alternative path anyway in case we deal with a Cayley-Dickson division algebra. In that case, the associated polar space is not embeddable in a projective space and so there seems to be no natural way to attribute homogeneous coordinates to its points as it was the case in each of the four above-discussed cases.

for all $b, c \in \mathbb{O}$. Clearly, $Z(\mathbb{O})$ is a field and $\mathbb{O}$ can be regarded as an algebra over $Z(\mathbb{O})$.

4.1 Quadratic alternative division rings

Suppose $\mathbb{F}$ is a subfield of $Z(\mathbb{O})$. We say that $\mathbb{O}$ *is quadratic over* $\mathbb{F}$ if there exist (necessarily unique) functions $T : \mathbb{O} \to \mathbb{F}$ and $N : \mathbb{O} \to \mathbb{F}$ such that:

- $a^2 - T(a)a + N(a) = 0$ for any $a \in \mathbb{O}$;
- $T(a) = 2a$ and $N(a) = a^2$ for any $a \in \mathbb{F}$.

The following proposition is precisely Theorem 20.3 of Tits and Weiss [14].

Proposition 4.1 ([14]) *Suppose $\mathbb{O}$ is an alternative division ring which is quadratic over some subfield $\mathbb{K}$ of its center $Z(\mathbb{O})$. Let $T : \mathbb{O} \to \mathbb{K}$ and $N : \mathbb{O} \to \mathbb{K}$ be the unique functions as defined above and put $a^\sigma := T(a) - a$ for all $a \in \mathbb{O}$. Then exactly one of the following holds:*

(a) $\mathbb{O} = \mathbb{K}$ is a field and $\sigma = 1$;

(b) $\mathbb{O}$ and $\mathbb{K}$ are fields, $\mathbb{O}$ is a separable quadratic extension of $\mathbb{K}$ and σ is the nontrivial element of the Galois group $Gal(\mathbb{O}/\mathbb{K})$;

(c) $\mathbb{O}$ is a field of characteristic 2, $\sigma = 1$ and $\mathbb{O}^2 \subseteq \mathbb{K} \neq \mathbb{O}$;

(d) $\mathbb{O}$ is a quaternion division algebra, $\mathbb{K} = Z(\mathbb{O})$ and σ is the standard involution of $\mathbb{O}$;

(e) $\mathbb{O}$ is a Cayley-Dickson division algebra over $\mathbb{K} = Z(\mathbb{O})$ and σ is the standard involution of $\mathbb{O}$.

In each case, σ is an involution of $\mathbb{O}$ and $N(a) = a^\sigma a \in \mathbb{K}$ for all $a \in \mathbb{O}$.

In the sequel of this section, we suppose that $\mathbb{O}$ is an alternative division ring which is quadratic over some subfield $\mathbb{K}$ of its center $Z(\mathbb{O})$. By Proposition 4.1, there are five possibilities for the pair $\mathcal{T} := (\mathbb{O}, \mathbb{K})$. Let σ be the involution of $\mathbb{O}$ as defined in Proposition 4.1. For each $a \in \mathbb{O}$, the elements $a + a^\sigma$ and $a^{\sigma+1} := a^\sigma a = aa^\sigma$ belong to $\mathbb{K}$. If $a \in \mathbb{K}$, then $a^\sigma = a$. If $a \neq 0$, then, since $a^\sigma = a^{\sigma+1} \cdot a^{-1}$ with $a^{\sigma+1} \in \mathbb{K}$, we have $(a^\sigma)^{-1} = (a^{-1})^\sigma = \frac{a}{a^{\sigma+1}}$. We denote $(a^\sigma)^{-1} = (a^{-1})^\sigma$ also by $a^{-\sigma}$.

We prove in this section that with the pair $\mathcal{T}$ there is associated a polar space $\mathcal{P}_\mathcal{T}$. We also determine which kind of polar space $\mathcal{P}_\mathcal{T}$ is. In several proofs, we will invoke some properties of alternative division rings. In Propositions 4.2 and 4.3 below, we state some results which we will need later.

For all $a, b, c \in \mathbb{O}$, we define the *commutator* $[a, b]$ of a and b as the number $ab - ba$ and the *associator* $[a, b, c]$ of a, b and c as the number $(ab)c - a(bc)$. Since $\mathbb{O}$ is an alternative division ring, we have $[a, b] = 0$ for all $a, b \in \mathbb{O}$ for which $\{a, b\} \cap \mathbb{K} \neq \emptyset$, $[a, b, c] = 0$ for all $a, b, c \in \mathbb{O}$ for which $\{a, b, c\} \cap \mathbb{K} \neq \emptyset$ and $[a^{-1}, a, b] = [b, a, a^{-1}] = 0$ for all $a, b \in \mathbb{O}$ for which $a \neq 0$. The commutator can be regarded as a map from $\mathbb{O}^2$ to $\mathbb{O}$ and the associator can be regarded as a map from $\mathbb{O}^3$ to $\mathbb{O}$. These maps are $\mathbb{K}$-linear in each of their components.

The following properties of alternative division rings are well-known, see e.g. Bruck and Kleinfeld [2], Tits and Weiss [14, Chapter 9] and Van Maldeghem [15, Appendix B].

Proposition 4.2 (1) *If $a, b, c \in \mathbb{O}$, then $[a, b, c] = 0$ if a, b and c are not mutually distinct*[2].

(2) *We have $[b, a] = -[a, b]$ for all $a, b \in \mathbb{O}$.*

(3) *If $a_1, a_2, a_3 \in \mathbb{O}$, then $[a_{\pi(1)}, a_{\pi(2)}, a_{\pi(3)}] = sgn(\pi) \cdot [a_1, a_2, a_3]$ for any permutation π of $\{1, 2, 3\}$.*

(4) *The* Moufang identities *hold in $\mathbb{O}$. This means that $a(b(ac)) = (aba)c$, $((ab)c)b = a(bcb)$ and $(ac)(ba) = a(cb)a$ for all $a, b, c \in \mathbb{O}$.*

(5) *For all $a, b, c \in \mathbb{O}$, we have $a \cdot [a, b, c] = [a, ba, c] = [a, b, ca]$ and $[a, b, c] \cdot a = [a, ab, c] = [a, b, ac]$.*

(6) *The subring generated by two distinct elements of $\mathbb{O}$ is associative.*

The following properties can be derived from Proposition 4.2.

Proposition 4.3 (1) *For all $a, b, c \in \mathbb{O}$, we have $a^\sigma \cdot [a, b, c] = [a, ba^\sigma, c] = [a, b, ca^\sigma]$ and $[a, b, c] \cdot a^\sigma = [a, a^\sigma b, c] = [a, b, a^\sigma c]$.*

(2) *For all $a, b, c \in \mathbb{O}$ with $a \neq 0$, we have $a^{-1} \cdot [a, b, c] = [a, ba^{-1}, c] = [a, b, ca^{-1}]$ and $[a, b, c] \cdot a^{-1} = [a, a^{-1}b, c] = [a, b, a^{-1}c]$.*

(3) *For all $a, b, c \in \mathbb{O}$, we have $[a^\sigma, b] = [a, b^\sigma] = -[a, b]$ and $[a^\sigma, b, c] = [a, b^\sigma, c] = [a, b, c^\sigma] = -[a, b, c]$.*

(4) *For all $a, b, c \in \mathbb{O}$, we have $[a, b]^\sigma = -[a, b]$ and $[a, b, c]^\sigma = -[a, b, c]$.*

(5) *For all $a, b \in \mathbb{O}$, we have $(ab)^{\sigma+1} = a^{\sigma+1} b^{\sigma+1}$.*

(6) *Let $a, b, c \in \mathbb{O}$. Then $T(ab) = T(ba)$ and $T((ab)c) = T(a(bc))$. Hence, $T(a(bc)) = T((ab)c) = T(b(ca)) = T((bc)a) = T(c(ab)) = T((ca)b)$.*

(7) *For all $a, b, c \in \mathbb{O}$, we have $a^{\sigma+1}(b^\sigma c + c^\sigma b) = (a^\sigma b^\sigma)(ca) + (a^\sigma c^\sigma)(ba) = (b^\sigma a^\sigma)(ac) + (c^\sigma a^\sigma)(ab)$.*

(8) *For all $a, b, c, d \in \mathbb{O}$, we have $a^\sigma((bc)d) + b^\sigma((ac)d) = (c(da^\sigma))b + (c(db^\sigma))a$.*

[2] So, $(ab)a = a(ba)$ for all $a, b \in \mathbb{O}$. We denote this number also by aba.

Proof. (1)+(2) Claim (2) follows from Proposition 4.2(5). Claim (1) follows from (2) and the fact that $a^\sigma = a^{\sigma+1}a^{-1}$ with $a^{\sigma+1} \in \mathbb{K}$. Alternatively, Claim (1) follows from Proposition 4.2(5) if one takes into account that $a^\sigma = T(a) - a$ with $T(a) \in \mathbb{K}$ and Claim (2) follows from (1) and the fact that $a^{-1} = \frac{a^\sigma}{a^{\sigma+1}}$ with $a^{\sigma+1} \in \mathbb{K}$.

(3) We have $[a^\sigma, b] = [T(a) - a, b] = [T(a), b] - [a, b] = -[a, b]$ and $[a^\sigma, b, c] = [T(a) - a, b, c] = [T(a), b, c] - [a, b, c] = -[a, b, c]$. The other claims are proved in a similar way.

(4) We have $[a, b]^\sigma = (ab - ba)^\sigma = -a^\sigma b^\sigma + b^\sigma a^\sigma = -[a^\sigma, b^\sigma] = -[a, b]$ and $[a, b, c]^\sigma = ((ab)c - a(bc))^\sigma = -(c^\sigma b^\sigma)a^\sigma + c^\sigma(b^\sigma a^\sigma) = -[c^\sigma, b^\sigma, a^\sigma] = [c, b, a] = -[a, b, c]$.

(5) This follows from Proposition 4.2(6).

(6) We have $T(ab) = ab + b^\sigma a^\sigma = ba + a^\sigma b^\sigma + [a, b] + [b^\sigma, a^\sigma] = T(ba) + [a, b] + [b, a] = T(ba)$ and $T((ab)c) = (ab)c + c^\sigma(b^\sigma a^\sigma) = a(bc) + (c^\sigma b^\sigma)a^\sigma + [a, b, c] - [c^\sigma, b^\sigma, a^\sigma] = T(a(bc)) + [a, b, c] + [c, b, a] = T(a(bc))$.

(7) Using (6), we have $(a^\sigma b^\sigma)(ca) + (a^\sigma c^\sigma)(ba) = T((a^\sigma b^\sigma)(ca)) = T((ca)(a^\sigma b^\sigma)) = T(c(a(a^\sigma b^\sigma))) = T(c(a^{\sigma+1}b^\sigma)) = a^{\sigma+1} \cdot T(cb^\sigma) = a^{\sigma+1} \cdot T(b^\sigma c) = a^{\sigma+1}(b^\sigma c + c^\sigma b)$ and $(b^\sigma a^\sigma)(ac) + (c^\sigma a^\sigma)(ab) = T((b^\sigma a^\sigma)(ac)) = T(b^\sigma(a^\sigma(ac))) = T(b^\sigma(a^{\sigma+1}c)) = a^{\sigma+1} \cdot T(b^\sigma c) = a^{\sigma+1}(b^\sigma c + c^\sigma b)$.

(8) It suffices to show that $a^\sigma((ac)d) = (c(da^\sigma))a$, because then the result will follow by substituting $a + b$ for a. Now we have $a^\sigma((ac)d) = a^\sigma([a, c, d] + a(cd)) = [a, ca^\sigma, d] + a^{\sigma+1}(cd)$, where we have used (1). By (1) and (3), this equals $[a^\sigma, ac^\sigma, d] + (cd)a^{\sigma+1} = -[a^\sigma, c, d]a + ((cd)a^\sigma)a = (-[c, d, a^\sigma] + (cd)a^\sigma)a = (c(da^\sigma))a$. ∎

4.2 A point-line approach

We are now ready to describe our polar space. Let ∞ be a symbol not belonging to $\mathbb{O}$ and let Ω be the following set:

$$\{(\infty), (x_1), (x_1, x_2), (x_1, x_2; k), (x_1, x_2, x_3; k), (x_1, x_2, x_3, x_4; k) \,:\, x_1, x_2, x_3, x_4 \in \mathbb{O},\ k \in \mathbb{K}\}.$$

We call the elements of Ω *points*. The point (∞) is called the *point of Type 0*. If $x_1, x_2, x_3, x_4 \in \mathbb{O}$ and $k \in \mathbb{K}$, then (x_1) is called a *point of Type 1*, (x_1, x_2) is called a *point of Type 2*, $(x_1, x_2; k)$ is called a *point of Type 3*, $(x_1, x_2, x_3; k)$ is called a *point of Type 4* and $(x_1, x_2, x_3, x_4; k)$ is called a *point of Type 5*. We now define twelve families of subsets of Ω which we call *lines*.

(A) Let L_1 be the following set of points:

$$\{(\infty)\} \cup \{(\lambda) \,:\, \lambda \in \mathbb{O}\}.$$

We call L_1 the *line of Type A*.

(B) For every $x \in \mathbb{O}$, let $L_2(x)$ denote the following set of points:

$$\{(\infty)\} \cup \{(x, \lambda) \,:\, \lambda \in \mathbb{O}\}.$$

We call $L_2(x)$ a *line of Type B*.

(C) For every $x \in \mathbb{O}$ and every $k \in \mathbb{K}$, let $L_3(x, k)$ denote the following set of points:

$$\{(\infty)\} \cup \{(x, \lambda; k) \,:\, \lambda \in \mathbb{O}\}.$$

We call $L_3(x, k)$ a *line of Type C*.

(D) For all $x, y \in \mathbb{O}$ and every $k \in \mathbb{K}$, let $L_4(x, y, k)$ denote the following set of points:

$$\{(\infty)\} \cup \{(x, y, \lambda; k) \,:\, \lambda \in \mathbb{O}\}.$$

We call $L_4(x, y, k)$ a *line of Type D*.

(E) For all $x, y, z \in \mathbb{O}$, let $L_5(x, y, z)$ denote the following set of points:

$$\{(x)\} \cup \{(\lambda, z + x(\lambda - y)) \,:\, \lambda \in \mathbb{O}\}.$$

We call $L_5(x, y, z)$ a *line of Type E*. The set $L_5(x, y, z)$ contains the points (x) and (y, z).

(F) For all $x, y, z \in \mathbb{O}$ and every $k \in \mathbb{K}$, let $L_6(x, y, z, k)$ be the following set of points:

$$\{(x)\} \cup \{(\lambda, z + x(\lambda - y); k) \,:\, \lambda \in \mathbb{O}\}.$$

We call $L_6(x, y, z, k)$ a *line of Type F*. The set $L_6(x, y, z, k)$ contains the points (x) and $(y, z; k)$.

(G) For all $x, y, z, u \in \mathbb{O}$ and every $k \in \mathbb{K}$ satisfying $x = -y^\sigma$, let $L_7(x, y, z, u, k)$ denote the following set of points:

$$\{(x)\} \cup \{(y, z, u, \lambda; k) \,:\, \lambda \in \mathbb{O}\}.$$

We call $L_7(x, y, z, u, k)$ a *line of Type G*.

(H) For all $x, y, u, v, w \in \mathbb{O}$ and every $k \in \mathbb{K}$ satisfying $u = -x^\sigma$, let $L_8(x, y, u, v, w, k)$ be the following set of points:

$$\{(x, y)\} \cup \{(u, \lambda, w + y(\lambda - v); k) \,:\, \lambda \in \mathbb{O}\}.$$

We call $L_8(x, y, u, v, w, k)$ a *line of Type H*. The set $L_8(x, y, u, v, w, k)$ contains the points (x, y) and $(u, v, w; k)$.

(I) For all $x, y, z, u, v, w \in \mathbb{O}$ and every $k \in \mathbb{K}$ satisfying $y = -u^\sigma - z^\sigma x$, let $L_9(x, y, z, u, v, w, k)$ be the following set of points:

$$\{(x, y)\} \cup \{(z, u, \lambda, w + x(\lambda - v); k) \,:\, \lambda \in \mathbb{O}\}.$$

We call $L_9(x,y,z,u,v,w,k)$ *a line of Type I.* The set $L_9(x,y,z,u,v,w,k)$ contains the points (x,y) and $(z,u,v,w;k)$.

(J) For all $k_1,k_2 \in \mathbb{K}$ and all $x,y,u,v,w \in \mathbb{O}$ satisfying $v = x^\sigma + k_1 u$, let $L_{10}(x,y,u,v,w,k_1,k_2)$ be the following set of points:

$$\{(x,y;k_1)\} \cup \{(\lambda, v+k_1(\lambda-u), w+y(\lambda-u); k_2+x(\lambda-u)+(\lambda-u)^\sigma x^\sigma + k_1(\lambda^{\sigma+1}-u^{\sigma+1})) : \lambda \in \mathbb{O}\}$$

$$= \{(x,y;k_1)\} \cup \{(\lambda, v+k_1(\lambda-u), w+y(\lambda-u); k_2+v^\sigma(\lambda-u)+(\lambda-u)^\sigma v + k_1(\lambda-u)^{\sigma+1}) : \lambda \in \mathbb{O}\}.$$

We call $L_{10}(x,y,u,v,w,k_1,k_2)$ a *line of Type J.* The set $L_{10}(x,y,u,v,w,k_1,k_2)$ contains the points $(x,y;k_1)$ and $(u,v,w;k_2)$.

(K) For all $x,y,z,u,v,w \in \mathbb{O}$ and all $k_1,k_2 \in \mathbb{K}$ satisfying $v = x^\sigma z + y^\sigma + k_1 u$, let $L_{11}(x,y,z,u,v,w,k_1,k_2)$ be the following set of points:

$$\{(x,y;k_1)\} \cup \{(z,\lambda, v+k_1(\lambda-u), w+x(\lambda-u); k_2+(y+z^\sigma x)(\lambda-u)+(\lambda-u)^\sigma(y^\sigma+x^\sigma z)+ k_1(\lambda^{\sigma+1}-u^{\sigma+1})) \,:\, \lambda \in \mathbb{O}\}$$

$$= \{(x,y;k_1)\} \cup \{(z,\lambda, v+k_1(\lambda-u), w+x(\lambda-u); k_2+v^\sigma(\lambda-u)+(\lambda-u)^\sigma v + k_1(\lambda-u)^{\sigma+1}) : \lambda \in \mathbb{O}\}.$$

We call $L_{11}(x,y,z,u,v,w,k_1,k_2)$ a *line of Type K.* The set $L_{11}(x,y,z,u,v,w,k_1,k_2)$ contains the points $(x,y;k_1)$ and $(z,u,v,w;k_2)$.

(L) For all $x,y,z,u,v,w,r \in \mathbb{O}$ and all $k_1,k_2 \in \mathbb{K}$ satisfying $r = z^\sigma - y^\sigma(xu-v)+k_1u-x^\sigma w$, let $L_{12}(x,y,z,u,v,w,r,k_1,k_2)$ be the following set of points:

$$\{(x,y,z;k_1)\} \cup \{(\lambda, v+x(\lambda-u), w+y(\lambda-u), r+k_1(\lambda-u)-x^\sigma(y(\lambda-u)); k_2+(z-(xu-v)^\sigma y)(\lambda-u) +(\lambda-u)^\sigma(z^\sigma - y^\sigma(xu-v)) + k_1(\lambda^{\sigma+1}-u^{\sigma+1})) \,:\, \lambda \in \mathbb{O}\}$$

$$= \{(x,y,z;k_1)\} \cup \{(\lambda, v+x(\lambda-u), w+y(\lambda-u), r+k_1(\lambda-u)-x^\sigma(y(\lambda-u)); k_2+(r^\sigma+w^\sigma x)(\lambda-u) +(\lambda-u)^\sigma(r+x^\sigma w) + k_1(\lambda-u)^{\sigma+1}) \,:\, \lambda \in \mathbb{O}\}.$$

We call $L_{12}(x,y,z,u,v,w,r,k_1,k_2)$ a *line of Type L.* The set $L_{12}(x,y,z,u,v,w,r,k_1,k_2)$ contains the points $(x,y,z;k_1)$ and $(u,v,w,r;k_2)$.

Two (not necessarily distinct) points are said to be X-*collinear*, $X \in \{\mathrm{A},\mathrm{B},\ldots,\mathrm{L}\}$, if they are contained in some line of Type X. Two (not necessarily distinct) points are said to be *collinear* if they are X-collinear for some $X \in \{\mathrm{A},\mathrm{B},\ldots,\mathrm{L}\}$. With each $X \in \{\mathrm{A},\mathrm{B},\ldots,\mathrm{L}\}$, we associate the parameters i_X and j_X as in Table 1.

Figure 1 pictures the incidence of the different types of points, lines and also planes (to be defined in Subsection 4.3) on an octahedron, which is an apartment in the corresponding building.

X	i_X	j_X	X	i_X	j_X
A	0	1	G	1	5
B	0	2	H	2	4
C	0	3	I	2	5
D	0	4	J	3	4
E	1	2	K	3	5
F	1	3	L	4	5

Table 1: The parameters i_X and j_X

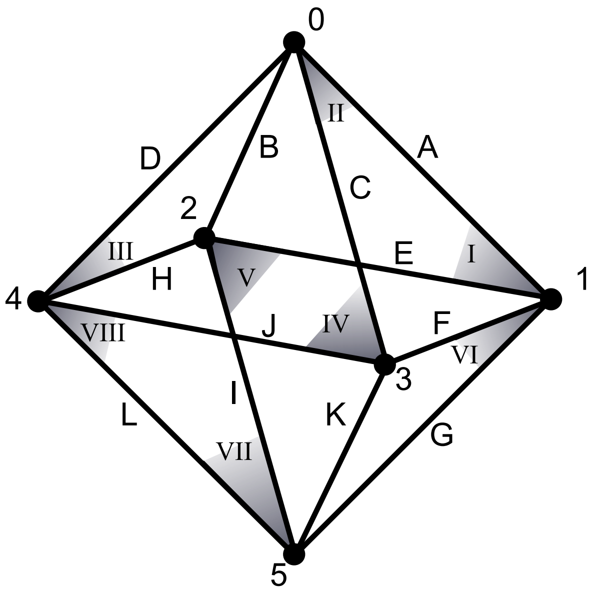

Figure 1: Incidence for types of points, lines and planes

Proposition 4.4 *Let $X \in \{\mathrm{A}, \mathrm{B}, \ldots, \mathrm{L}\}$.*

(1) *Let L be a line of Type X. Then L contains a unique point of Type i_X and all the remaining points of L have Type $j_X > i_X$.*

(2) *If a point of Type i_X and a point of Type j_X are X-collinear, then they are contained in a unique line of Type X.*

(3) *If p and p' are two distinct points of Type j_X which are contained in some line L of Type X, then the unique point of Type i_X of L is uniquely determined by p and p'. As a consequence, two distinct points of Type j_X are contained in at most one line of Type X.*

Proof. Obviously, Claim (1) holds.

As for Claim (2), we will only give a sketch in the case $X = \mathrm{L}$. The other cases are similar. Consider the line $M = L_{12}(x, y, z, u, v, w, r, k_1, k_2)$ as described above. We can regard M as the line of Type L defined by a point $(x, y, z; k_1)$ of Type 4 and a point $(u, v, w, r; k_2)$

of Type 5 which satisfy the compatibility condition $r = z^\sigma - y^\sigma(xu - v) + k_1 u - x^\sigma w$. If $(u', v', w', r'; k_2') = (\lambda, v + x(\lambda - u), w + y(\lambda - u), r + k_1(\lambda - u) - x^\sigma(y(\lambda - u)); k_2 + (z - (xu - v)^\sigma y)(\lambda - u) + (\lambda - u)^\sigma(z^\sigma - y^\sigma(xu - v)) + k_1(\lambda^{\sigma+1} - u^{\sigma+1}))$ is another point of Type 5 of M, then one can easily verify that also the points $(x, y, z; k_1)$ and $(u', v', w', r'; k_2')$ satisfy the compatibility condition. Moreover, the line of Type L defined by $(x, y, z; k_1)$ and $(u', v', w', r'; k_2')$ coincides with M. This information is sufficient to conclude that a point of Type 4 and a point of Type 5 are contained in at most one line of Type L.

As for Claim (3), we only treat the case $X = \mathrm{L}$. The other cases are similar (and even easier). We must show that $(x, y, z; k_1)$ is uniquely determined by $(u_1, v_1, w_1, r_1; l_1) = (\lambda_1, v + x(\lambda_1 - u), w + y(\lambda_1 - u), r + k_1(\lambda_1 - u) - x^\sigma(y(\lambda_1 - u)); k_2 + (r^\sigma + w^\sigma x)(\lambda_1 - u) + (\lambda_1 - u)^\sigma(r + x^\sigma w) + k_1(\lambda_1 - u)^{\sigma+1})$ and $(u_2, v_2, w_2, r_2; l_2) = (\lambda_2, v + x(\lambda_2 - u), w + y(\lambda_2 - u), r + k_1(\lambda_2 - u) - x^\sigma(y(\lambda_2 - u)); k_2 + (r^\sigma + w^\sigma x)(\lambda_2 - u) + (\lambda_2 - u)^\sigma(r + x^\sigma w) + k_1(\lambda_2 - u)^{\sigma+1})$. Here, $x, y, z, u, v, w, r, \lambda_1, \lambda_2$ are elements of $\mathbb{O}$ and k_1, k_2 are elements of $\mathbb{K}$ such that $\lambda_1 \neq \lambda_2$ and $r = z^\sigma - y^\sigma(xu - v) + k_1 u - x^\sigma w$. We have

$$\begin{aligned}
x &= (v_2 - v_1)(u_2 - u_1)^{-1}, \\
y &= (w_2 - w_1)(u_2 - u_1)^{-1}, \\
k_1 &= (r_2 - r_1 + x^\sigma(w_2 - w_1))(u_2 - u_1)^{-1}, \\
r + x^\sigma w - k_1 u &= r_1 + x^\sigma w_1 - k_1 u_1, \\
xu - v &= xu_1 - v_1, \\
z &= (r + x^\sigma w - k_1 u + y^\sigma(xu - v))^\sigma.
\end{aligned}$$

So, $(x, y, z; k_1)$ is indeed uniquely determined by $(u_1, v_1, w_1, r_1; l_1)$ and $(u_2, v_2, w_2, r_2; l_2)$. ■

The following proposition gives necessary and sufficient conditions for two distinct points to be X-collinear ($X \in \{\mathrm{A}, \mathrm{B}, \ldots, \mathrm{L}\}$).

Proposition 4.5 *Let $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4 \in \mathbb{O}$ and $k, l \in \mathbb{K}$.*

- *Let p be a point of Type $i \in \{0, 1, \ldots, 5\}$ and $p' \neq p$ a point of Type $i' \in \{0, 1, \ldots, 5\}$. Let $X \in \{\mathrm{A}, \mathrm{B}, \ldots, \mathrm{L}\}$. If $(i, i') \notin \{(i_X, j_X), (j_X, j_X)\}$, then p and p' are not X-collinear.*

- *The point (∞) is* A*-collinear with all points of Type* 1, B*-collinear with all points of Type* 2, C*-collinear with all points of Type* 3 *and* D*-collinear with all points of Type* 4.

- *The point (x_1) is* A*-collinear with all points of Type* 1, E*-collinear with all points of Type* 2 *and* F*-collinear with all points of Type* 3. *The point (x_1) is* G*-collinear with the point $(y_1, y_2, y_3, y_4; l)$ if and only if $x_1 + y_1^\sigma = 0$.*

- *The point (x_1, x_2) is* B*-collinear with the point (y_1, y_2) if and only if $x_1 = y_1$. The point (x_1, x_2) is* E*-collinear with the point $(y_1, y_2) \neq (x_1, x_2)$ if and only if $x_1 \neq y_1$. The point (x_1, x_2) is* H*-collinear with the point $(y_1, y_2, y_3; l)$ if and only if $x_1 + y_1^\sigma = 0$. The point (x_1, x_2) is* I*-collinear with the point $(y_1, y_2, y_3, y_4; l)$ if and only if $y_2 + x_1^\sigma y_1 + x_2^\sigma = 0$.*

- *The point $(x_1, x_2; k)$ is* C*-collinear with the point $(y_1, y_2; l)$ if and only if $(x_1, k) = (y_1, l)$. The point $(x_1, x_2; k)$ is* F*-collinear with the point $(y_1, y_2; l) \neq (x_1, x_2; k)$ if and only if*

$x_1 \neq y_1$ and $k = l$. The point $(x_1, x_2; k)$ is J*-collinear with the point $(y_1, y_2, y_3; l)$ if and only if $y_2 - ky_1 - x_1^\sigma = 0$. The point $(x_1, x_2; k)$ is* K*-collinear with the point $(y_1, y_2, y_3, y_4; l)$ if and only if $y_3 - ky_2 - x_1^\sigma y_1 - x_2^\sigma = 0$.*

- *The point $(x_1, x_2, x_3; k)$ is* D*-collinear with the point $(y_1, y_2, y_3; l)$ if and only if $(x_1, x_2, k) = (y_1, y_2, l)$. The point $(x_1, x_2, x_3; k)$ is* H*-collinear with the point $(y_1, y_2, y_3; l) \neq (x_1, x_2, x_3; k)$ if and only if $(x_1, k) = (y_1, l)$ and $x_2 \neq y_2$. The point $(x_1, x_2, x_3; k)$ is* J*-collinear with the point $(y_1, y_2, y_3; l) \neq (x_1, x_2, x_3; k)$ if and only if $x_1 \neq y_1$ and $l - k = y_1^\sigma y_2 + x_2^\sigma y_1 - x_1^\sigma y_2 - x_2^\sigma x_1$. The point $(x_1, x_2, x_3; k)$ is* L*-collinear with the point $(y_1, y_2, y_3, y_4; l)$ if and only if $y_4 + x_1^\sigma y_3 - x_2^\sigma y_2 - ky_1 + x_2^\sigma(x_1 y_1) - x_3^\sigma = 0$.*

- *Suppose $(x_1, x_2, x_3, x_4; k) \neq (y_1, y_2, y_3, y_4; l)$. The points $(x_1, x_2, x_3, x_4; k)$ and $(y_1, y_2, y_3, y_4; l)$ are* G*-collinear if and only if $(x_1, x_2, x_3, k) = (y_1, y_2, y_3, l)$. The points $(x_1, x_2, x_3, x_4; k)$ and $(y_1, y_2, y_3, y_4; l)$ are* I*-collinear if and only if $(x_1, x_2, k) = (y_1, y_2, l)$ and $x_3 \neq y_3$. The points $(x_1, x_2, x_3, x_4; k)$ and $(y_1, y_2, y_3, y_4; l)$ are* K*-collinear if and only if $x_1 = y_1$, $x_2 \neq y_2$ and $l - k = y_2^\sigma y_3 + x_3^\sigma y_2 - x_2^\sigma y_3 - x_3^\sigma x_2$. The points $(x_1, x_2, x_3, x_4; k)$ and $(y_1, y_2, y_3, y_4; l)$ are* L*-collinear if and only if $x_1 \neq y_1$ and $l - k = y_1^\sigma y_4 + y_2^\sigma y_3 + x_3^\sigma y_2 + x_4^\sigma y_1 - x_1^\sigma y_4 - x_2^\sigma y_3 - x_4^\sigma x_1 - x_3^\sigma x_2 - [y_1 - x_1, y_2 - x_2, y_3 - x_3] \cdot (y_1 - x_1)^{-\sigma}$.*

Proof. The verification of these conditions is straightforward, except (perhaps) in the three cases discussed below.

(i) Two points $(x_1, x_2, x_3; k)$ and $(y_1, y_2, y_3; l) \neq (x_1, x_2, x_3; k)$ are J-collinear if and only if there exist $x, y, u, v, w, \lambda \in \mathbb{O}$ and $k_1, k_2 \in \mathbb{K}$ such that $v = x^\sigma + k_1 u$, $(u, v, w; k_2) = (x_1, x_2, x_3; k)$ and $(\lambda, v + k_1(\lambda - u), w + y(\lambda - u); k_2 + v^\sigma(\lambda - u) + (\lambda - u)^\sigma v + k_1(\lambda - u)^{\sigma+1}) = (y_1, y_2, y_3; l)$. The condition $(y_1, y_2, y_3; l) \neq (x_1, x_2, x_3; l)$ is equivalent with $y_1 \neq x_1$. The above conditions yield $u = x_1$, $v = x_2$, $w = x_3$, $k_2 = k$, $\lambda = y_1$,

$$\begin{aligned}
k_1 &= (y_2 - x_2)(y_1 - x_1)^{-1}, \\
y &= (y_3 - x_3)(y_1 - x_1)^{-1}, \\
x &= (v - k_1 u)^\sigma = (x_2 - ((y_2 - x_2)(y_1 - x_1)^{-1})x_1)^\sigma, \\
l - k &= v^\sigma(\lambda - u) + (\lambda - u)^\sigma v + k_1(\lambda - u)^{\sigma+1} \\
&= x_2^\sigma(y_1 - x_1) + (y_1 - x_1)^\sigma x_2 + (y_1 - x_1)^{\sigma+1}(y_2 - x_2)(y_1 - x_1)^{-1}.
\end{aligned}$$

So, we see that the points $(x_1, x_2, x_3; k)$ and $(y_1, y_2, y_3; l)$ are distinct and J-collinear if and only if $y_1 \neq x_1$, $(y_2 - x_2)(y_1 - x_1)^{-1} \in \mathbb{K}$ and $l - k = x_2^\sigma(y_1 - x_1) + (y_1 - x_1)^\sigma x_2 + (y_1 - x_1)^{\sigma+1}(y_2 - x_2)(y_1 - x_1)^{-1}$.

Suppose that these three conditions hold. Then $(y_2 - x_2)(y_1 - x_1)^{-1} = (y_1 - x_1)^{-1}(y_2 - x_2)$ and hence $l - k = x_2^\sigma(y_1 - x_1) + (y_1 - x_1)^\sigma x_2 + (y_1 - x_1)^\sigma(y_2 - x_2) = y_1^\sigma y_2 + x_2^\sigma y_1 - x_1^\sigma y_2 - x_2^\sigma x_1$.

Conversely, suppose that $l - k = y_1^\sigma y_2 + x_2^\sigma y_1 - x_1^\sigma y_2 - x_2^\sigma x_1$ and $y_1 \neq x_1$. Then $(y_1 - x_1)^\sigma(y_2 - x_2) = y_1^\sigma y_2 + x_2^\sigma y_1 - x_1^\sigma y_2 - x_2^\sigma x_1 - (x_2^\sigma y_1 + y_1^\sigma x_2) + (x_2^\sigma x_1 + x_1^\sigma x_2) = (l - k) - (x_2^\sigma y_1 + y_1^\sigma x_2) + (x_2^\sigma x_1 + x_1^\sigma x_2) \in \mathbb{K}$ and hence $k_1 = (y_2 - x_2)(y_1 - x_1)^{-1} \in \mathbb{K}$.

We conclude that the points $(x_1, x_2, x_3; k)$ and $(y_1, y_2, y_3; l)$ are distinct and J-collinear if and only if $x_1 \neq y_1$ and $l - k = y_1^\sigma y_2 + x_2^\sigma y_1 - x_1^\sigma y_2 - x_2^\sigma x_1$.

(ii) Two points $(x_1,x_2,x_3,x_4;k)$ and $(y_1,y_2,y_3,y_4;l) \neq (x_1,x_2,x_3,x_4;k)$ are K-collinear if and only if there exist $x,y,z,u,v,w,\lambda \in \mathbb{O}$ and $k_1,k_2 \in \mathbb{K}$ such that $v = x^\sigma z + y^\sigma + k_1 u$, $(z,u,v,w;k_2) = (x_1,x_2,x_3,x_4;k)$ and $(z,\lambda,v+k_1(\lambda-u),w+x(\lambda-u);k_2+v^\sigma(\lambda-u)+(\lambda-u)^\sigma v+k_1(\lambda-u)^{\sigma+1}) = (y_1,y_2,y_3,y_4;l)$. If this is the case, then $x_1 = z = y_1$ and so the fact that $(y_1,y_2,y_3,y_4;l) \neq (x_1,x_2,x_3,x_4;k)$ implies that $x_2 \neq y_2$. The above conditions yield that $z = x_1 = y_1$, $u = x_2$, $v = x_3$, $w = x_4$, $k_2 = k$, $\lambda = y_2$,

$$\begin{aligned}
k_1 &= (y_3-x_3)(y_2-x_2)^{-1}, \\
x &= (y_4-x_4)(y_2-x_2)^{-1}, \\
y &= (v - x^\sigma z - k_1 u)^\sigma, \\
l-k &= v^\sigma(\lambda-u) + (\lambda-u)^\sigma v + k_1(\lambda-u)^{\sigma+1} \\
&= x_3^\sigma(y_2-x_2) + (y_2-x_2)^\sigma x_3 + (y_2-x_2)^{\sigma+1}(y_3-x_3)(y_2-x_2)^{-1}.
\end{aligned}$$

So, we see that the points $(x_1,x_2,x_3,x_4;k)$ and $(y_1,y_2,y_3,y_4;l)$ are distinct and K-collinear if and only if $y_1 = x_1$, $y_2 \neq x_2$, $(y_3-x_3)(y_2-x_2)^{-1} \in \mathbb{K}$ and $l-k = x_3^\sigma(y_2-x_2) + (y_2-x_2)^\sigma x_3 + (y_2-x_2)^{\sigma+1}(y_3-x_3)^\sigma(y_2-x_2)^{-1}$. With a reasoning completely similar to the one of Case (I), we see that this is the case precisely when $y_1 = x_1$, $y_2 \neq x_2$ and $l-k = y_2^\sigma y_3 + x_3^\sigma y_2 - x_2^\sigma y_3 - x_3^\sigma x_2$.

(iii) Two points $(x_1,x_2,x_3,x_4;k)$ and $(y_1,y_2,y_3,y_4;l) \neq (x_1,x_2,x_3,x_4;l)$ are L-collinear if and only if there exist $x,y,z,u,v,w,r,\lambda \in \mathbb{O}$ and $k_1,k_2 \in \mathbb{K}$ such that $r = z^\sigma - y^\sigma(xu - v) + k_1 u - x^\sigma w$, $(u,v,w,r,k_2) = (x_1,x_2,x_3,x_4;k)$ and $(\lambda, v+x(\lambda-u), w+y(\lambda-u), r+k_1(\lambda-u)-x^\sigma(y(\lambda-u)); k_2+(r^\sigma+w^\sigma x)(\lambda-u)+(\lambda-u)^\sigma(r+x^\sigma w)+k_1(\lambda-u)^{\sigma+1}) = (y_1,y_2,y_3,y_4;l)$. The condition $(y_1,y_2,y_3,y_4;l) \neq (x_1,x_2,x_3,x_4;k)$ is equivalent with $y_1 \neq x_1$. The above conditions yield $u = x_1$, $v = x_2$, $w = x_3$, $r = x_4$, $k_2 = k$, $\lambda = y_1$,

$$\begin{aligned}
x &= (y_2-x_2)(y_1-x_1)^{-1}, \\
y &= (y_3-x_3)(y_1-x_1)^{-1}, \\
k_1 &= \Big((y_4-x_4) + x^\sigma(y(y_1-x_1))\Big)(y_1-x_1)^{-1}, \\
z &= (r + y^\sigma(xu-v) - k_1 u + x^\sigma w)^\sigma, \\
l-k &= (r^\sigma + w^\sigma x)(\lambda-u) + (\lambda-u)^\sigma(r + x^\sigma w) + k_1(\lambda-u)^{\sigma+1}.
\end{aligned}$$

So, we see that $(x_1,x_2,x_3,x_4;k)$ and $(y_1,y_2,y_3,y_4;l)$ are distinct and L-collinear if and only if $y_1 \neq x_1$, $\Big((y_4-x_4) + x^\sigma(y(y_1-x_1))\Big)(y_1-x_1)^{-1} \in \mathbb{K}$ and $l-k = (r^\sigma + w^\sigma x)(\lambda-u) + (\lambda-u)^\sigma(r+x^\sigma w) + k_1(\lambda-u)^{\sigma+1}$.

Suppose that these three conditions hold. Since $k_1 = \Big((y_4-x_4)+x^\sigma(y(y_1-x_1))\Big)(y_1-x_1)^{-1} \in \mathbb{K}$, we also have $k_1 = (y_1-x_1)^{-1}\Big((y_4-x_4)+x^\sigma(y(y_1-x_1))\Big)$ and hence $k_1(\lambda-u)^{\sigma+1} = k_1(y_1-x_1)^{\sigma+1} = (y_1-x_1)^\sigma(y_4-x_4) + (y_1-x_1)^{-1}\Big(((y_1-x_1)(y_2-x_2)^\sigma)(y(y_1-x_1))\Big) = (y_1-x_1)^\sigma(y_4-x_4) + (y_1-x_1)^{-1}\Big((y_1-x_1)((y_2-x_2)^\sigma y)(y_1-x_1)\Big) = (y_1-x_1)^\sigma(y_4-x_4) +$

$((y_2-x_2)^\sigma y)(y_1-x_1) = (y_1-x_1)^\sigma(y_4-x_4)+\Big((y_2-x_2)^\sigma((y_3-x_3)(y_1-x_1)^{-1})\Big)(y_1-x_1) = (y_1-x_1)^\sigma(y_4-x_4)+\Big(((y_2-x_2)^\sigma(y_3-x_3))(y_1-x_1)^{-1}\Big)(y_1-x_1)-[(y_2-x_2)^\sigma, y_3-x_3, (y_1-x_1)^{-1}]\cdot(y_1-x_1) = (y_1-x_1)^\sigma(y_4-x_4)+(y_2-x_2)^\sigma(y_3-x_3)-[(y_2-x_2)^\sigma, y_3-x_3, (y_1-x_1)^\sigma]\cdot(y_1-x_1)^{-\sigma} = (y_1-x_1)^\sigma(y_4-x_4)+(y_2-x_2)^\sigma(y_3-x_3)-[y_1-x_1, y_2-x_2, y_3-x_3]\cdot(y_1-x_1)^{-\sigma}$.
We have

$$\begin{aligned}
(w^\sigma x)(\lambda-u) &= \Big(x_3^\sigma((y_2-x_2)(y_1-x_1)^{-1})\Big)(y_1-x_1) \\
&= \Big((x_3^\sigma(y_2-x_2))(y_1-x_1)^{-1}-[x_3^\sigma, y_2-x_2, (y_1-x_1)^{-1}]\Big)\cdot(y_1-x_1) \\
&= x_3^\sigma(y_2-x_2)-[x_3^\sigma, y_2-x_2, (y_1-x_1)^{-1}]\cdot(y_1-x_1) \\
&= x_3^\sigma(y_2-x_2)-[x_3^\sigma, y_2-x_2, (y_1-x_1)^\sigma]\cdot(y_1-x_1)^{-\sigma} \\
&= x_3^\sigma(y_2-x_2)+[y_1-x_1, y_2-x_2, x_3]\cdot(y_1-x_1)^{-\sigma}.
\end{aligned}$$

Hence,

$$\begin{aligned}
(\lambda-u)^\sigma(x^\sigma w) &= (y_2-x_2)^\sigma x_3-(y_1-x_1)^{-1}\cdot[y_1-x_1, y_2-x_2, x_3] \\
&= (y_2-x_2)^\sigma x_3-[y_1-x_1, (y_2-x_2)(y_1-x_1)^{-1}, x_3] \\
&= (y_2-x_2)^\sigma x_3+[y_1-x_1, (y_1-x_1)^{-\sigma}(y_2-x_2)^\sigma, x_3] \\
&= (y_2-x_2)^\sigma x_3+[y_1-x_1, (y_2-x_2)^\sigma, x_3]\cdot(y_1-x_1)^{-\sigma} \\
&= (y_2-x_2)^\sigma x_3-[y_1-x_1, y_2-x_2, x_3]\cdot(y_1-x_1)^{-\sigma}.
\end{aligned}$$

It follows that $(w^\sigma x)(\lambda-u)+(\lambda-u)^\sigma(x^\sigma w) = x_3^\sigma(y_2-x_2)+(y_2-x_2)^\sigma x_3$. We also have $l-k = r^\sigma(\lambda-u)+(\lambda-u)^\sigma r+(w^\sigma x)(\lambda-u)+(\lambda-u)^\sigma(x^\sigma w)+k_1(\lambda-u)^{\sigma+1} = x_4^\sigma(y_1-x_1)+(y_1-x_1)^\sigma x_4+(y_1-x_1)^\sigma(y_4-x_4)+x_3^\sigma(y_2-x_2)+(y_2-x_2)^\sigma x_3+(y_2-x_2)^\sigma(y_3-x_3)-[y_1-x_1, y_2-x_2, y_3-x_3]\cdot(y_1-x_1)^{-\sigma} = y_1^\sigma y_4+y_2^\sigma y_3+x_3^\sigma y_2+x_4^\sigma y_1-x_1^\sigma y_4-x_2^\sigma y_3-x_4^\sigma x_1-x_3^\sigma x_2-[y_1-x_1, y_2-x_2, y_3-x_3]\cdot(y_1-x_1)^{-\sigma}$.

Conversely, suppose that $y_1\neq x_1$ and $l-k = y_1^\sigma y_4+y_2^\sigma y_3+x_3^\sigma y_2+x_4^\sigma y_1-x_1^\sigma y_4-x_2^\sigma y_3-x_4^\sigma x_1-x_3^\sigma x_2-[y_1-x_1, y_2-x_2, y_3-x_3]\cdot(y_1-x_1)^{-\sigma}$. Then one has that $k_1'(\lambda-u)^{\sigma+1} = l-k-r^\sigma(\lambda-u)-(\lambda-u)^\sigma r-(w^\sigma x)(\lambda-u)-(\lambda-u)^\sigma(x^\sigma w)\in\mathbb{K}$, where $\lambda = y_1$, $u = x_1$, $r = x_4$, $w = x_3$, $x = (y_2-x_2)(y_1-x_1)^{-1}$, $y = (y_3-x_3)(y_1-x_1)^{-1}$ and $k_1' = (y_1-x_1)^{-1}\Big((y_4-x_4)+x^\sigma(y(y_1-x_1))\Big)$. It follows that $k_1'\in\mathbb{K}$ and hence also that $k_1 = \Big((y_4-x_4)+x^\sigma(y(y_1-x_1))\Big)(y_1-x_1)^{-1}\in\mathbb{K}$.

We conclude that the points $(x_1,x_2,x_3,x_4;k)$ and $(y_1,y_2,y_3,y_4;l)$ are distinct and L-collinear if and only if $y_1\neq x_1$ and $l-k = y_1^\sigma y_4+y_2^\sigma y_3+x_3^\sigma y_2+x_4^\sigma y_1-x_1^\sigma y_4-x_2^\sigma y_3-x_4^\sigma x_1-x_3^\sigma x_2-[y_1-x_1, y_2-x_2, y_3-x_3]\cdot(y_1-x_1)^{-\sigma}$. ∎

The following is a corollary of Proposition 4.4(2)+(3) and Proposition 4.5.

Corollary 4.6 *If p and p' are two distinct collinear points, then they are X-collinear for a unique $X\in\{\mathrm{A},\mathrm{B},\ldots,\mathrm{L}\}$. As a consequence, two distinct collinear points are contained in a unique line.*

The following corollary is also a consequence of Proposition 4.5. It gives necessary and sufficient conditions for two (not necessarily distinct) points to be collinear.

Corollary 4.7 *Let $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4 \in \mathbb{O}$ and $k, l \in \mathbb{K}$.*

• *The point (∞) is collinear with all points of Type 1, all points of Type 2, all points of Type 3 and all points of Type 4. The point (∞) is collinear with no point of Type 5.*

• *The point (x_1) is collinear with all points of Type 1, all points of Type 2 and all points of Type 3. The point (x_1) is collinear with no point of Type 4. The point (x_1) is collinear with the point $(y_1, y_2, y_3, y_4; l)$ of Type 5 if and only if $x_1 + y_1^\sigma = 0$.*

• *The point (x_1, x_2) is collinear with all points of Type 2 and no point of Type 3. The point (x_1, x_2) is collinear with the point $(y_1, y_2, y_3; l)$ if and only if $x_1 + y_1^\sigma = 0$. The point (x_1, x_2) is collinear with the point $(y_1, y_2, y_3, y_4; l)$ if and only if $y_2 + x_1^\sigma y_1 + x_2^\sigma = 0$.*

• *The point $(x_1, x_2; k)$ is collinear with the point $(y_1, y_2; l)$ if and only if $k = l$. The point $(x_1, x_2; k)$ is collinear with the point $(y_1, y_2, y_3; l)$ if and only if $y_2 - ky_1 - x_1^\sigma = 0$. The point $(x_1, x_2; k)$ is collinear with the point $(y_1, y_2, y_3, y_4; l)$ if and only if $y_3 - ky_2 - x_1^\sigma y_1 - x_2^\sigma = 0$.*

• *The point $(x_1, x_2, x_3; k)$ is collinear with the point $(y_1, y_2, y_3; l)$ if and only if $l - k = y_1^\sigma y_2 + x_2^\sigma y_1 - x_1^\sigma y_2 - x_2^\sigma x_1$. The point $(x_1, x_2, x_3; k)$ is collinear with the point $(y_1, y_2, y_3, y_4; l)$ if and only if $y_4 + x_1^\sigma y_3 - x_2^\sigma y_2 - ky_1 + x_2^\sigma(x_1 y_1) - x_3^\sigma = 0$.*

• *The points $(x_1, x_2, x_3, x_4; k)$ and $(y_1, y_2, y_3, y_4; l)$ are collinear if and only if either $x_1 = y_1$ and $l - k = y_2^\sigma y_3 + x_3^\sigma y_2 - x_2^\sigma y_3 - x_3^\sigma x_2$ or $x_1 \neq y_1$ and $l - k = y_1^\sigma y_4 + y_2^\sigma y_3 + x_3^\sigma y_2 + x_4^\sigma y_1 - x_1^\sigma y_4 - x_2^\sigma y_3 - x_4^\sigma x_1 - x_3^\sigma x_2 - [y_1 - x_1, y_2 - x_2, y_3 - x_3] \cdot (y_1 - x_1)^{-\sigma}$.*

The above-defined points and lines define a point-line geometry which we will denote by $\mathcal{P}_\mathcal{T}$. By Corollary 4.6, $\mathcal{P}_\mathcal{T}$ is a so-called partial linear space. Our next goal will be to show that $\mathcal{P}_\mathcal{T}$ is a polar space.

Proposition 4.8 *For every point p of $\mathcal{P}_\mathcal{T}$, there exists a point p' of $\mathcal{P}_\mathcal{T}$ which is not collinear with p.*

Proof. Let $(i, i') \in \{(0, 5), (1, 4), (2, 3)\}$. Then, by Corollary 4.7, no point of Type i is collinear with a point of Type i'. ■

In the following five propositions, we list a number of automorphisms of the point-line geometry $\mathcal{P}_\mathcal{T}$. The proof that the stated permutations of Ω actually define automorphisms is straightforward and involves no special difficulties. We will therefore omit the proofs. Notice that each of the listed automorphism preserves the types of the points and lines. All automorphisms we list are so-called *root-elations*, in particular unipotent elements in the corresponding algebraic group or group of mixed type. The set of all automorphisms defined in each proposition is a root group. Together, the five groups generate the unipotent radical of (∞).

Proposition 4.9 *For every $\eta \in \mathbb{O}$, the permutation of Ω defined by*

$$\begin{array}{rcl} (x_1, x_2, x_3, x_4; k) & \mapsto & (x_1 + \eta, x_2, x_3, x_4; k), \\ (x_1, x_2, x_3; k) & \mapsto & (x_1, x_2, x_3 - \eta^\sigma k + (\eta^\sigma x_1^\sigma) x_2; k), \\ (x_1, x_2; k) & \mapsto & (x_1, x_2 - \eta^\sigma x_1; k), \\ (x_1, x_2) & \mapsto & (x_1, x_2 - \eta^\sigma x_1), \\ (x_1) & \mapsto & (x_1 - \eta^\sigma), \\ (\infty) & \mapsto & (\infty), \end{array}$$

is an automorphism of $\mathcal{P}_\mathcal{T}$.

Proposition 4.10 *For every $\eta \in \mathbb{O}$, the permutation of Ω defined by*

$$\begin{array}{rcl} (x_1, x_2, x_3, x_4; k) & \mapsto & (x_1, x_2 + \eta, x_3, x_4; k), \\ (x_1, x_2, x_3; k) & \mapsto & (x_1, x_2, x_3 - \eta^\sigma x_2; k), \\ (x_1, x_2; k) & \mapsto & (x_1, x_2 - k\eta^\sigma; k), \\ (x_1, x_2) & \mapsto & (x_1, x_2 - \eta^\sigma), \\ (x_1) & \mapsto & (x_1), \\ (\infty) & \mapsto & (\infty), \end{array}$$

is an automorphism of $\mathcal{P}_\mathcal{T}$.

Proposition 4.11 *For every $\eta \in \mathbb{O}$, the permutation of Ω defined by*

$$\begin{array}{rcl} (x_1, x_2, x_3, x_4; k) & \mapsto & (x_1, x_2, x_3 + \eta, x_4; k + \eta^\sigma x_2 + x_2^\sigma \eta), \\ (x_1, x_2, x_3; k) & \mapsto & (x_1, x_2, x_3 + \eta^\sigma x_1; k), \\ (x_1, x_2; k) & \mapsto & (x_1, x_2 + \eta^\sigma; k), \\ (x_1, x_2) & \mapsto & (x_1, x_2), \\ (x_1) & \mapsto & (x_1), \\ (\infty) & \mapsto & (\infty), \end{array}$$

is an automorphism of $\mathcal{P}_\mathcal{T}$.

Proposition 4.12 *For every $\eta \in \mathbb{O}$, the permutation of Ω defined by*

$$\begin{array}{rcl} (x_1, x_2, x_3, x_4; k) & \mapsto & (x_1, x_2, x_3, x_4 + \eta; k + \eta^\sigma x_1 + x_1^\sigma \eta), \\ (x_1, x_2, x_3; k) & \mapsto & (x_1, x_2, x_3 + \eta^\sigma; k), \\ (x_1, x_2; k) & \mapsto & (x_1, x_2; k), \\ (x_1, x_2) & \mapsto & (x_1, x_2), \\ (x_1) & \mapsto & (x_1), \\ (\infty) & \mapsto & (\infty), \end{array}$$

is an automorphism of $\mathcal{P}_\mathcal{T}$.

Proposition 4.13 *For every $k^* \in \mathbb{K}$, the permutation of Ω defined by*

$$\begin{array}{rcl}
(x_1, x_2, x_3, x_4; k) & \mapsto & (x_1, x_2, x_3, x_4; k + k^*), \\
(x_1, x_2, x_3; k) & \mapsto & (x_1, x_2, x_3; k), \\
(x_1, x_2; k) & \mapsto & (x_1, x_2; k), \\
(x_1, x_2) & \mapsto & (x_1, x_2), \\
(x_1) & \mapsto & (x_1), \\
(\infty) & \mapsto & (\infty),
\end{array}$$

is an automorphism of $\mathcal{P}_{\mathcal{T}}$.

We are now ready to prove that $\mathcal{P}_{\mathcal{T}}$ is a polar space.

Proposition 4.14 *For every point p and every line L, the point p is collinear with one or all points of L.*

Proof. There are 6 possible types for the point p and 12 possible types for the line L. This leads to 72 cases which we need to consider. Corollary 4.7 can be used to deal with each of these cases. Observe also that if p is a point of Type 5, then by Propositions 4.9, 4.10, 4.11, 4.12 and 4.13, we may assume that $p = (0, 0, 0, 0; 0)$. This observation can simplify the verification in some cases.

The verification of the proposition is straightforward (and often immediate) in many of the 72 cases. In fact, there are only four cases where some difficulty seems to occur. Before we discuss these four cases in detail, we treat a typical example among the 68 other cases.

Consider the point $(x_1, x_2, x_3; k_1)$ of Type 4 and the line $L = L_8(x, y, u, v, w, k)$ of Type H. Here, $x_1, x_2, x_3, x, y, u, v, w \in \mathbb{O}$ and $k, k_1 \in \mathbb{K}$ such that $u = -x^\sigma$. The line $L_8(x, y, u, v, w, k)$ contains the points (x, y) and $(u, \lambda, w + y(\lambda - v); k)$, $\lambda \in \mathbb{O}$. We have

$$\begin{array}{rcl}
(x_1, x_2, x_3; k_1) \sim (x, y) & \Leftrightarrow & x + x_1^\sigma = 0, \\
(x_1, x_2, x_3; k_1) \sim (u, \lambda, w + y(\lambda - v); k) & \Leftrightarrow & k - k_1 = u^\sigma \lambda + x_2^\sigma u - x_1^\sigma \lambda - x_2^\sigma x_1 \\
& \Leftrightarrow & (x + x_1^\sigma)\lambda = x_2^\sigma u - x_2^\sigma x_1 - k + k_1.
\end{array}$$

If $x + x_1^\sigma \neq 0$, then $(x_1, x_2, x_3; k_1)$ is collinear with a unique point of L, namely the point $(u, \lambda, w + y(\lambda - v); k)$ where $\lambda = (x + x_1^\sigma)^{-1}(x_2^\sigma u - x_2^\sigma x_1 - k + k_1)$. If $x + x_1^\sigma = 0 = x_2^\sigma u - x_2^\sigma x_1 - k + k_1$, then $(x_1, x_2, x_3; k_1)$ is collinear with all points of L. Finally, if $x + x_1^\sigma = 0 \neq x_2^\sigma u - x_2^\sigma x_1 - k + k_1$, then $(x_1, x_2, x_3; k_1)$ is collinear with a unique point of L, namely the point (x, y).

We now deal with the four cases where some difficulty is involved. With "some difficulty" we mean that after writing down the conditions for collinearity as given in Corollary 4.7, we still need to manipulate the obtained expressions before we can make the necessary conclusions.

(i) Consider the point $(x_1, x_2, x_3; k)$ of Type 4 and the line $L = L_{12}(x, y, z, u, v, w, r, k_1, k_2)$ of Type L. Here, $x_1, x_2, x_3, x, y, z, u, v, w, r \in \mathbb{O}$ and $k, k_1, k_2 \in \mathbb{K}$ such that $r = z^\sigma - y^\sigma(xu-v)+k_1u-x^\sigma w$. The line $L_{12}(x, y, z, u, v, w, r, k_1, k_2)$ contains the points $(x, y, z; k_1)$ and $p(\lambda) := (\lambda, v+x(\lambda-u), w+y(\lambda-u), r+k_1(\lambda-u)-x^\sigma(y(\lambda-u)); k_2+(r^\sigma+w^\sigma x)(\lambda-u)+(\lambda-u)^\sigma(r+x^\sigma w)+k_1(\lambda-u)^{\sigma+1})$, $\lambda \in \mathbb{O}$. The points $(x_1, x_2, x_3; k)$ and $(x, y, z; k_1)$ are collinear if and only if

$$k_1 - k = \eta := x^\sigma y + x_2^\sigma x - x_1^\sigma y - x_2^\sigma x_1.$$

The points $(x_1, x_2, x_3; k)$ and $p(\lambda)$ are collinear if and only if $r + k_1(\lambda - u) - x^\sigma(y(\lambda - u)) + x_1^\sigma(w + y(\lambda - u)) - x_2^\sigma(v + x(\lambda - u)) - k\lambda + x_2^\sigma(x_1\lambda) - x_3^\sigma = 0$, i.e. if and only if $\eta' := r - k_1u + x^\sigma(yu) + x_1^\sigma w - x_1^\sigma(yu) - x_2^\sigma v + x_2^\sigma(xu) - x_3^\sigma$ is equal to

$$x^\sigma(y\lambda) - x_1^\sigma(y\lambda) + x_2^\sigma(x\lambda) - x_2^\sigma(x_1\lambda) - k_1\lambda + k\lambda.$$

If $x = x_1$, then $\eta = 0$. In that case, the points $(x_1, x_2, x_3; k)$ and $(x, y, z; k_1)$ are collinear if and only if $k_1 = k$. Also, the points $(x_1, x_2, x_3; k)$ and $p(\lambda)$ are collinear if and only if $(k - k_1)\lambda = \eta'$. It is now easy to see that $(x_1, x_2, x_3; k)$ is collinear with one of all points of L.

In the sequel, we will suppose that $x \neq x_1$. We then have that $x^\sigma(y\lambda)-x_1^\sigma(y\lambda)+x_2^\sigma(x\lambda)-x_2^\sigma(x_1\lambda)$ is equal to

$$\begin{aligned}
& (x-x_1)^\sigma(y\lambda) + x_2^\sigma((x-x_1)\lambda) \\
= \;& ((x-x_1)^\sigma y + x_2^\sigma(x-x_1)) \cdot \lambda - [(x-x_1)^\sigma, y, \lambda] - [x_2^\sigma, x-x_1, \lambda] \\
= \;& \eta\lambda + [x-x_1, y, \lambda] - [x-x_1, x_2, \lambda] \\
= \;& \eta\lambda + [x-x_1, y-x_2, \lambda] \\
= \;& \eta\lambda + [x-x_1, (x-x_1)^\sigma(y-x_2), \lambda] \cdot (x-x_1)^{-\sigma} \\
= \;& \eta\lambda + \Big([x-x_1, \eta, \lambda] - [x-x_1, x_2^\sigma(x-x_1) + (x-x_1)^\sigma x_2, \lambda]\Big) \cdot (x-x_1)^{-\sigma} \\
= \;& \eta\lambda + [x-x_1, \eta, \lambda] \cdot (x-x_1)^{-\sigma} \\
= \;& \eta\lambda - [(x-x_1)^\sigma, \eta, \lambda] \cdot (x-x_1)^{-\sigma} \\
= \;& \eta\lambda - [(x-x_1)^\sigma, (x-x_1)^{-\sigma}\eta, \lambda] \\
= \;& (x-x_1)^\sigma(((x-x_1)^{-\sigma}\eta)\lambda).
\end{aligned}$$

So, the points $(x_1, x_2, x_3; k)$ and $p(\lambda)$ are collinear if and only if

$$(x-x_1)^\sigma(((x-x_1)^{-\sigma}(\eta - k_1 + k))\lambda) = \eta'.$$

If $\eta \neq k_1 - k$, then there is a unique solution for λ and hence the point $(x_1, x_2, x_3; k)$ is collinear with a unique point of L. If $\eta = k_1 - k$ and $\eta' = 0$, then $(x_1, x_2, x_3; k)$ is collinear with all points of L. Finally, if $\eta = k_1 - k$ and $\eta' \neq 0$, then $(x, y, z; k_1)$ is the unique point of L collinear with $(x_1, x_2, x_3; k)$.

(ii) Consider the point $(0,0,0,0;0)$ of Type 5 and the line $L = L_9(x,y,z,u,v,w,k)$ of Type I. Here, $x,y,z,u,v,w \in \mathbb{O}$ and $k \in \mathbb{K}$ such that $y = -u^\sigma - z^\sigma x$. The line $L_9(x,y,z,u,v,w,k)$ contains the points (x,y) and $(z,u,\lambda,w+x(\lambda-v);k)$, $\lambda \in \mathbb{O}$.

The points $(0,0,0,0;0)$ and (x,y) are collinear if and only if $y=0$. The points $(0,0,0,0;0)$ and $(z,u,\lambda,w+x(\lambda-v);k)$ are collinear if and only if either $z=0$ and $k=u^\sigma\lambda$ or $z\neq 0$ and $k = z^\sigma(w+x(\lambda-v)) + u^\sigma\lambda - [z,u,\lambda]\cdot z^{-\sigma}$.

If $z=0$, then $y=-u^\sigma$ and in that case it is easily seen that $(0,0,0,0;0)$ is collinear with one or all points of L. So, we may suppose that $z\neq 0$. The condition $k = z^\sigma(w+x(\lambda-v)) + u^\sigma\lambda - [z,u,\lambda]\cdot z^{-\sigma}$ can then be rewritten as

$$\begin{aligned}
k &= z^\sigma w - z^\sigma(xv) + z^\sigma(x\lambda) + u^\sigma\lambda - [z^\sigma,u^\sigma,\lambda]\cdot z^{-\sigma}\\
k &= z^\sigma w - z^\sigma(xv) + z^\sigma(x\lambda) + u^\sigma\lambda - [z^\sigma,z^{-\sigma}u^\sigma,\lambda]\\
k &= z^\sigma w - z^\sigma(xv) + z^\sigma((x+z^{-\sigma}u^\sigma)\lambda)\\
k &= z^\sigma w - z^\sigma(xv) - z^\sigma((z^{-\sigma}y)\lambda).
\end{aligned}$$

So, depending on which of the values y and $k - z^\sigma w + z^\sigma(xv)$ are equal to 0, the point $(0,0,0,0;0)$ is collinear with one or all points of L.

(iii) Consider the point $(0,0,0,0;0)$ of Type 5 and the line $L = L_{11}(x,y,z,u,v,w,k_1,k_2)$ of Type K. Here, $x,y,z,u,v,w \in \mathbb{O}$ and $k_1,k_2 \in \mathbb{K}$ such that $v = x^\sigma z + y^\sigma + k_1 u$. The line $L_{11}(x,y,z,u,v,w,k_1,k_2)$ contains the points $(x,y;k_1)$ and $p(\lambda) := (z,\lambda,v+k_1(\lambda-u),w+x(\lambda-u);k_2+v^\sigma(\lambda-u)+(\lambda-u)^\sigma v+k_1(\lambda-u)^{\sigma+1})$. The point $(z,u,v,w;k_2)$ on L can be chosen in such a way that $u=0$. Then $v = x^\sigma z + y^\sigma$ and $p(\lambda) = (z,\lambda,v+k_1\lambda,w+x\lambda;k_2+v^\sigma\lambda+\lambda^\sigma v+k_1\lambda^{\sigma+1})$.

The points $(0,0,0,0;0)$ and $(x,y;k_1)$ are collinear if and only if $y=0$. The points $(0,0,0,0;0)$ and $p(\lambda)$ are collinear if and only if either $z=0$ and $k_2+v^\sigma\lambda+\lambda^\sigma v+k_1\lambda^{\sigma+1} = \lambda^\sigma(v+k_1\lambda)$ or $z\neq 0$ and $k_2+v^\sigma\lambda+\lambda^\sigma v+k_1\lambda^{\sigma+1} = z^\sigma(w+x\lambda)+\lambda^\sigma(v+k_1\lambda)-[z,\lambda,v+k_1\lambda]\cdot z^{-\sigma}$.

Suppose $z=0$. The condition $k_2+v^\sigma\lambda+\lambda^\sigma v+k_1\lambda^{\sigma+1} = \lambda^\sigma(v+k_1\lambda)$ then becomes $k_2+v^\sigma\lambda = k_2+y\lambda = 0$. So, depending on which of the values y and k_2 are equal to 0, the point $(0,0,0,0;0)$ is collinear with one or all points of L.

Suppose $z\neq 0$. The condition $k_2+v^\sigma\lambda+\lambda^\sigma v+k_1\lambda^{\sigma+1} = z^\sigma(w+x\lambda)+\lambda^\sigma(v+k_1\lambda) - [z,\lambda,v+k_1\lambda]\cdot z^{-\sigma}$ then becomes

$$\begin{aligned}
k_2+v^\sigma\lambda &= z^\sigma w + z^\sigma(x\lambda) - [z,\lambda,v]\cdot z^{-\sigma}\\
k_2+v^\sigma\lambda &= z^\sigma w + z^\sigma(x\lambda) + [z^\sigma,v^\sigma,\lambda]\cdot z^{-\sigma}\\
k_2+v^\sigma\lambda &= z^\sigma w + z^\sigma(x\lambda) + [z^\sigma,z^{-\sigma}v^\sigma,\lambda]\\
k_2+v^\sigma\lambda &= z^\sigma w + z^\sigma(x\lambda) + v^\sigma\lambda - z^\sigma((z^{-\sigma}v^\sigma)\lambda)\\
k_2 - z^\sigma w &= z^\sigma((x-z^{-\sigma}v^\sigma)\lambda)\\
k_2 - z^\sigma w &= -z^\sigma((z^{-\sigma}y)\lambda).
\end{aligned}$$

So, depending on which of the values $k_2 - z^\sigma w$ and y are equal to 0, the point $(0,0,0,0;0)$ is collinear with one or all points of L.

(iv) Suppose p is a point of Type 5 and $L = L_{12}(x,y,z,u,v,w,r,k_1,k_2)$ is a line of Type L. Here, $x,y,z,u,v,w,r \in \mathbb{O}$ and $k_1,k_2 \in \mathbb{K}$ such that $r = z^\sigma - y^\sigma(xu-v) + k_1 u - x^\sigma w$. We can choose the point $(u,v,w,r;k_2)$ of L in such a way that u is equal to the first coordinate of p. By Propositions 4.9, 4.10, 4.11, 4.12 and 4.13, we may suppose that $p = (0,0,x_1,x_2;0)$ and $(u,v,w,r;k_2) = (0,v,0,0;k_2)$. We then have $z^\sigma + y^\sigma v = 0$. The line L contains the points $(x,y,z;k_1)$ and $p(\lambda) := (\lambda, v+x\lambda, y\lambda, k_1\lambda - x^\sigma(y\lambda); k_2 + k_1\lambda^{\sigma+1})$, $\lambda \in \mathbb{O}$.

The point $p = (0,0,x_1,x_2;0)$ is collinear with $(x,y,z;k_1)$ if and only if

$$x_2 = z^\sigma - x^\sigma x_1.$$

The point $(0,0,x_1,x_2;0)$ is collinear with the point $p(0) = (0,v,0,0;k_2)$ if and only if

$$x_1^\sigma v = k_2.$$

Suppose $\lambda \neq 0$. Then the point $(0,0,x_1,x_2;0)$ is collinear with the point $p(\lambda)$ if and only if

$$(v+x\lambda)^\sigma(y\lambda) + \lambda^\sigma(k_1\lambda - x^\sigma(y\lambda)) + x_1^\sigma(v+x\lambda) + x_2^\sigma\lambda - [\lambda, v+x\lambda, y\lambda - x_1]\cdot\lambda^{-\sigma} = k_2 + k_1\lambda^{\sigma+1}.$$

This simplifies to

$$v^\sigma(y\lambda) + (\lambda^\sigma x^\sigma)(y\lambda) - \lambda^\sigma(x^\sigma(y\lambda)) + x_1^\sigma v + x_1^\sigma(x\lambda) + x_2^\sigma\lambda - [\lambda, v+x\lambda, y\lambda - x_1]\cdot\lambda^{-\sigma} = k_2.$$

This can be rewritten as

$$(v^\sigma y)\lambda + (x_1^\sigma x)\lambda + x_2^\sigma\lambda - [v^\sigma, y, \lambda] + [\lambda^\sigma, x^\sigma, y\lambda] - [x_1^\sigma, x, \lambda] - [\lambda, v+x\lambda, y\lambda - x_1]\cdot\lambda^{-\sigma} = k_2 - x_1^\sigma v.$$

Now,

$$[\lambda, v+x\lambda, y\lambda - x_1] = [\lambda, v, y\lambda] + [\lambda, x\lambda, y\lambda] - [\lambda, v, x_1] - [\lambda, x\lambda, x_1]$$

with

$$\begin{aligned}
[\lambda, v, y\lambda]\cdot\lambda^{-\sigma} &= -[\lambda, v, \lambda^\sigma y^\sigma]\cdot\lambda^{-\sigma} = -[\lambda, v, y^\sigma] = -[v^\sigma, y, \lambda],\\
[\lambda, x\lambda, y\lambda]\cdot\lambda^{-\sigma} &= -[\lambda, \lambda^\sigma x^\sigma, y\lambda]\cdot\lambda^{-\sigma} = -[\lambda, x^\sigma, y\lambda] = [\lambda^\sigma, x^\sigma, y\lambda],\\
[\lambda, x\lambda, x_1]\cdot\lambda^{-\sigma} &= -[\lambda, \lambda^\sigma x^\sigma, x_1]\cdot\lambda^{-\sigma} = -[\lambda, x^\sigma, x_1] = [x_1^\sigma, x, \lambda].
\end{aligned}$$

So, the point $(0,0,x_1,x_2;0)$ is collinear with the point $p(\lambda)$, $\lambda \neq 0$, if and only if

$$(x_2^\sigma - z + x_1^\sigma x)\lambda = k_2 - x_1^\sigma v - [\lambda, v, x_1]\cdot\lambda^{-\sigma}.$$

• Suppose $x_1 = 0$. Then the latter condition becomes $(x_2^\sigma - z)\lambda = k_2$. So, depending on which of the values $x_2^\sigma - z$ and y are equal to 0, the point $(0,0,x_1,x_2;0) = (0,0,0,x_2;0)$ is collinear with one or all points of L.

- Suppose $x_1 \neq 0$ and put $\alpha := k_2 - x_1^\sigma v$. Then the above condition becomes

$$\begin{aligned}
(x_2^\sigma - z + x_1^\sigma x)\lambda &= \alpha - [\lambda, x_1^{-\sigma}(k_2 - \alpha), x_1] \cdot \lambda^{-\sigma} \\
(x_2^\sigma - z + x_1^\sigma x)\lambda &= \alpha - \Big([\lambda, k_2 - \alpha, x_1] \cdot x_1^{-\sigma}\Big) \cdot \lambda^{-\sigma} \\
(x_2^\sigma - z + x_1^\sigma x)\lambda &= \alpha + \Big([\lambda, \alpha, x_1] \cdot x_1^{-\sigma}\Big) \cdot \lambda^{-\sigma} \\
(x_2^\sigma - z + x_1^\sigma x)\lambda &= \alpha - \Big([\alpha, \lambda^\sigma, x_1^\sigma] \cdot x_1^{-\sigma}\Big) \cdot \lambda^{-\sigma} \\
(x_2^\sigma - z + x_1^\sigma x)\lambda &= \alpha - \alpha + ((\alpha(\lambda^\sigma x_1^\sigma))x_1^{-\sigma})\lambda^{-\sigma} \\
(x_2^\sigma - z + x_1^\sigma x)\lambda &= ((\alpha(\lambda^{-1} x_1^\sigma))x_1^{-\sigma})\lambda \\
x_2^\sigma - z + x_1^\sigma x &= (\alpha(\lambda^{-1} x_1^\sigma))x_1^{-\sigma}.
\end{aligned}$$

If $x_2^\sigma - z + x_1^\sigma x = 0 = \alpha$, then $(0, 0, x_1 x_2; 0)$ is collinear with all points of L. If $x_2^\sigma - z + x_1^\sigma x = 0 \neq \alpha$, then $(x, y, z; k_1)$ is the unique point of L which is collinear with $(0, 0, x_1, x_2; 0)$. If $x_2^\sigma - z + x_1^\sigma x \neq 0 = \alpha$, then $p(0)$ is the unique point of L collinear with $(0, 0, x_1, x_2; 0)$. If $x_2^\sigma - z + x_1^\sigma x \neq 0 \neq \alpha$, then $p(\lambda)$ is the unique point of L collinear with $(0, 0, x_1, x_2; 0)$, where $\lambda \in \mathbb{O} \setminus \{0\}$ is the unique solution of the equation $x_2^\sigma - z + x_1^\sigma x = (\alpha(\lambda^{-1} x_1^\sigma))x_1^{-\sigma}$.
■

4.3 The explicit description of the planes

We define eight families of subsets of Ω which we call *planes*.

(I) We denote by $[\infty]$ the set consisting of the points

$$\begin{aligned}
p_1(a, b) &:= (a, b), \\
p_2(s) &:= (s), \\
p_3^* &:= (\infty),
\end{aligned}$$

where $a, b, s \in \mathbb{O}$. We call $[\infty]$ the *plane of Type I.*

(II) For every $k \in \mathbb{K}$, we denote by $[k]$ the set consisting of the points

$$\begin{aligned}
p_1(a, b) &:= (a, b; k), \\
p_2(s) &:= (s), \\
p_3^* &:= (\infty),
\end{aligned}$$

where $a, b, s \in \mathbb{O}$. We call $[k]$ a *plane of Type II.*

(III) For every $x \in \mathbb{O}$ and every $k \in \mathbb{K}$, we denote by $[x; k]$ the set consisting of the points

$$\begin{aligned}
p_1(a, b) &:= (x, a, b; k), \\
p_2(s) &:= (-x^\sigma, s), \\
p_3^* &:= (\infty),
\end{aligned}$$

where $a, b, s \in \mathbb{O}$. We call $[x; k]$ a *plane of Type III.*

(IV) For every $x \in \mathbb{O}$ and all $k, l \in \mathbb{K}$, we denote by $[x; k, l]$ the set consisting of the points

$$\begin{aligned} p_1(a,b) &:= (a, x + la, b; k + x^\sigma a + a^\sigma x + la^{\sigma+1}), \\ p_2(s) &:= (x^\sigma, s; l), \\ p_3^* &:= (\infty), \end{aligned}$$

where $a, b, s \in \mathbb{O}$. We call $[x; k, l]$ a *plane of Type IV.*

(V) For all $x_1, x_2 \in \mathbb{O}$ and every $k \in \mathbb{K}$, we denote by $[x_1, x_2; k]$ the set consisting of the points

$$\begin{aligned} p_1(a,b) &:= (-x_2^\sigma, -x_1^\sigma, a, b; k), \\ p_2(s) &:= (s, x_1 + x_2 s), \\ p_3^* &:= (x_2), \end{aligned}$$

where $a, b, s \in \mathbb{O}$. We call $[x_1, x_2; k]$ a *plane of Type V.*

(VI) For all $x_1, x_2 \in \mathbb{O}$ and all $k, l \in \mathbb{K}$, we denote by $[x_1, x_2; k, l]$ the set consisting of the points

$$\begin{aligned} p_1(a,b) &:= (-x_2^\sigma, a, x_1^\sigma + ka, b; l + x_1 a + a^\sigma x_1^\sigma + ka^{\sigma+1}), \\ p_2(s) &:= (s, x_1 + x_2 s; k), \\ p_3^* &:= (x_2), \end{aligned}$$

where $a, b, s \in \mathbb{O}$. We call $[x_1, x_2; k, l]$ a *plane of Type VI.*

(VII) For all $x_1, x_2, x_3 \in \mathbb{O}$ and all $k, l \in \mathbb{K}$, we denote by $[x_1, x_2, x_3; k, l]$ the set consisting of the points

$$\begin{aligned} p_1(a,b) &:= (a, -x_3^\sigma + x_1 a, b, x_2^\sigma + ka - x_1^\sigma b; l + x_2 a + a^\sigma x_2^\sigma + ka^{\sigma+1}), \\ p_2(s) &:= (x_1, s, x_2 + x_3 s; k), \\ p_3^* &:= (-x_1^\sigma, x_3), \end{aligned}$$

where $a, b, s \in \mathbb{O}$. We call $[x_1, x_2, x_3; k, l]$ a *plane of Type VII.*

(VIII) For all $x_1, x_2, x_3 \in \mathbb{O}$ and all $k, l, m \in \mathbb{K}$, we denote by $[x_1, x_2, x_3; k, l, m]$ the set consisting of the points

$$\begin{aligned} p_1(a,b) &:= (a, b, {x_3}^\sigma + lb + x_1 a, {x_2}^\sigma + ka + x_1^\sigma b; \\ &\qquad m + x_2 a + a^\sigma x_2^\sigma + x_3 b + b^\sigma {x_3}^\sigma + ka^{\sigma+1} + lb^{\sigma+1} + (a^\sigma x_1^\sigma) b + b^\sigma (x_1 a)), \\ p_2(s) &:= (s, x_1 + ls, x_2 + x_3 s; k + x_1^\sigma s + s^\sigma x_1 + ls^{\sigma+1}), \\ p_3^* &:= (x_1^\sigma, x_3; l), \end{aligned}$$

where $a, b, s \in \mathbb{O}$. We call $[x_1, x_2, x_3; k, l, m]$ a *plane of Type VIII.*

Recall that Figure 1 pictures the incidences between the different types of points, lines ad planes.

Proposition 4.15 *The following holds for a plane α.*

(1) *If p and p' are two distinct points contained in α, then p and p' are collinear and the unique line through them is contained in α.*

(2) *The points and lines contained in α define a point-line geometry $\widetilde{\alpha}$ which is isomorphic to the Moufang projective plane $\mathsf{PG}(2, \mathbb{O})$.*

Proof. We give a scheme that can be used to prove the proposition. After that we will apply it to the most difficult case, namely the case where α is a plane of Type VIII. The verification of the other cases is straightforward. In fact, in the case that α is a plane of Type VIII, we need to rely on some properties (of associators) mentioned in Propositions 4.2 and 4.3. This is not the case for the other seven cases.

(1) Consider a plane (depending on some parameters) corresponding to one of the eight types considered above.

(2) Consider the points p_3^* and $p_2(s)$ of that plane, where s is some arbitrary element of $\mathbb{O}$. Corollary 4.7 can be used to prove that these points are collinear and Proposition 4.5 provides the unique value of $X \in \{\mathrm{A}, \mathrm{B}, \ldots, \mathrm{L}\}$ such that p_3^* and $p_2(s)$ are X-collinear. An explicit description of the unique line of Type X containing p_3^* and $p_2(s)$ easily follows from the information provided when we defined the twelve types of lines. This unique line of Type X is equal to $\{p_3^*\} \cup \{p_2(\lambda) : \lambda \in \mathbb{O}\}$ and hence is contained in α.

(3) Consider the points p_3^* and $p_1(a, b)$ where a and b are arbitrary elements of $\mathbb{O}$. With a similar method as in (2), one can verify that these points are X-collinear for a unique $X \in \{\mathrm{A}, \mathrm{B}, \ldots, \mathrm{L}\}$. Again, the unique line of Type X through p_3^* and $p_1(a, b)$ can easily be determined. This line is equal to $\{p_3^*\} \cup \{p_1(a, \lambda) : \lambda \in \mathbb{O}\}$ and hence is completely contained in α.

(4) Consider the points $p_2(s)$ and $p_1(a, b)$, where a, b, s are arbitrary elements of $\mathbb{O}$. With a similar method as explained in (2), one can verify that these points are X-collinear for a unique $X \in \{\mathrm{A}, \mathrm{B}, \ldots, \mathrm{L}\}$. Again, the unique line of Type X containing $p_2(s)$ and $p_1(a, b)$ can easily be determined. This line is equal to $\{p_2(s)\} \cup \{p_1(\lambda, b + s(\lambda - a)) : \lambda \in \mathbb{O}\}$ and hence is completely contained in α.

(5) Consider the points $p_2(s)$ and $p_2(s')$ where s and s' are two distinct elements of $\mathbb{O}$. By (2), these points are contained in the line $\{p_3^*\} \cup \{p_2(\lambda) : \lambda \in \mathbb{O}\}$ which is completely contained in α.

(6) Consider the points $p_1(a, b)$ and $p_1(a, b')$ where $a, b, b' \in \mathbb{O}$ with $b \neq b'$. By (3), these points are contained in the unique line through the points p_3^* and $p_1(a, b)$. Hence, the

points $p_1(a,b)$ and $p_1(a,b')$ are collinear and the unique line through them is contained in α.

(7) Consider the points $p_1(a,b)$ and $p_1(a',b')$ where $a,b,a',b' \in \mathbb{O}$ with $a \neq a'$. By (4), these points are contained in the unique line through the points $p_2((b'-b)(a'-a)^{-1})$ and $p_1(a,b)$. Hence, the points $p_1(a,b)$ and $p_1(a',b')$ are collinear and the unique line through them is contained in α.

(8) By (2), (3), ..., (7) above, we know that the lines contained in α are precisely the lines described in (2), (3) and (4) above. From these descriptions, it immediately follows that the map $p_1(a,b) \mapsto (a,b)$, $p_2(s) \mapsto (s)$, $p_3^* \mapsto (\infty)$ defines an isomorphism between $\widetilde{\alpha}$ and the projective plane $\mathsf{PG}(2,\mathbb{O})$ which is coordinatized by the alternative division ring $\mathbb{O}$ as explained in Section 2.

We will now apply the above scheme to the case where α is the plane $[x_1,x_2,x_3;k,l,m]$ of Type VIII ($x_1,x_2,x_3 \in \mathbb{O}$ and $k,l,m \in \mathbb{K}$). We need to verify the claims mentioned in the paragraphs (2), (3) and (4).

(i) Consider the points $p_3^* = (x_1^\sigma, x_3; l)$ and $p_2(s) = (s, x_1 + ls, x_2 + x_3 s; k + x_1^\sigma s + s^\sigma x_1 + ls^{\sigma+1})$. These points are J-collinear by Proposition 4.5. The unique line through them contains the points $p_3^* = (x_1^\sigma, x_3; l)$ and

$$\begin{aligned} p(\lambda) &:= (\lambda, x_1 + ls + l(\lambda - s), x_2 + x_3 s + x_3(\lambda - s); \\ &\quad\;\; k + x_1^\sigma s + s^\sigma x_1 + ls^{\sigma+1} + x_1^\sigma(\lambda - s) + (\lambda - s)^\sigma x_1 + l(\lambda^{\sigma+1} - s^{\sigma+1})) \\ &= (\lambda, x_1 + l\lambda, x_2 + x_3\lambda, k + x_1^\sigma \lambda + \lambda^\sigma x_1 + l\lambda^{\sigma+1}) \\ &= p_2(\lambda) \end{aligned}$$

for every $\lambda \in \mathbb{O}$.

(ii) Consider the points $p_3^* = (x_1^\sigma, x_3; l)$ and $p_1(a,b) = (a, b, x_3^\sigma + lb + x_1 a, x_2^\sigma + ka + x_1^\sigma b; m + x_2 a + a^\sigma x_2^\sigma + x_3 b + b^\sigma x_3^\sigma + ka^{\sigma+1} + lb^{\sigma+1} + (a^\sigma x_1^\sigma)b + b^\sigma(x_1 a))$. These points are K-collinear by Proposition 4.5. The unique line through them contains the points $p_3^* = (x_1^\sigma, x_3; l)$ and $p'(\lambda)$, $\lambda \in \mathbb{O}$, where $p'(\lambda)$ is the following point:

$$(a, \lambda, x_3^\sigma + lb + x_1 a + l(\lambda - b), x_2^\sigma + ka + x_1^\sigma b + x_1^\sigma(\lambda - b); m + x_2 a + a^\sigma x_2^\sigma + x_3 b + b^\sigma x_3^\sigma$$

$$+ka^{\sigma+1} + lb^{\sigma+1} + (a^\sigma x_1^\sigma)b + b^\sigma(x_1 a) + (x_3 + a^\sigma x_1^\sigma)(\lambda - b) + (\lambda - b)^\sigma(x_3^\sigma + x_1 a) + l(\lambda^{\sigma+1} - b^{\sigma+1})).$$

We have

$$\begin{aligned} p'(\lambda) &= (a, \lambda, x_3^\sigma + l\lambda + x_1 a, x_2^\sigma + ka + x_1^\sigma \lambda; \\ &\quad\;\; m + x_2 a + a^\sigma x_2^\sigma + x_3\lambda + \lambda^\sigma x_3^\sigma + ka^{\sigma+1} + l\lambda^{\sigma+1} + (a^\sigma x_1^\sigma)\lambda + \lambda^\sigma(x_1 a)) \\ &= p_1(a, \lambda). \end{aligned}$$

(iii) Consider the points $p_2(s) = (s, x_1 + ls, x_2 + x_3 s; k + x_1^\sigma s + s^\sigma x_1 + ls^{\sigma+1})$ and $p_1(a, b) = (a, b, x_3^\sigma + lb + x_1 a, x_2^\sigma + ka + x_1^\sigma b; m + x_2 a + a^\sigma x_2^\sigma + x_3 b + b^\sigma x_3^\sigma + ka^{\sigma+1} + lb^{\sigma+1} + (a^\sigma x_1^\sigma)b + b^\sigma(x_1 a))$. Since $x_2^\sigma + ka + x_1^\sigma b - x_2^\sigma - s^\sigma x_3^\sigma + (x_1^\sigma + ls^\sigma)(sa - b) - (k + x_1^\sigma s + s^\sigma x_1 + ls^{\sigma+1})a + s^\sigma(x_3^\sigma + lb + x_1 a) = x_1^\sigma(sa) - (x_1^\sigma s)a - (s^\sigma x_1)a + s^\sigma(x_1 a) = -[x_1^\sigma, s, a] - [s^\sigma, x_1, a] = [x_1, s, a] + [s, x_1, a] = 0$, these points are L-collinear by Proposition 4.5. The unique line through them contains the points $p_2(s) = (s, x_1 + ls, x_2 + x_3 s; k + x_1^\sigma s + s^\sigma x_1 + ls^{\sigma+1})$ and $p''(\lambda) = (f_1(\lambda), f_2(\lambda), f_3(\lambda), f_4(\lambda); f_5(\lambda))$, $\lambda \in \mathbb{O}$, where

$$\begin{aligned}
f_1(\lambda) &= \lambda, \\
f_2(\lambda) &= b + s(\lambda - a), \\
f_3(\lambda) &= x_3^\sigma + lb + x_1 a + (x_1 + ls)(\lambda - a) \\
&= x_3^\sigma + l(b + s(\lambda - a)) + x_1\lambda, \\
f_4(\lambda) &= x_2^\sigma + ka + x_1^\sigma b + (k + x_1^\sigma s + s^\sigma x_1 + ls^{\sigma+1})(\lambda - a) - s^\sigma((x_1 + ls)(\lambda - a)) \\
&= x_2^\sigma + k\lambda + x_1^\sigma b + (x_1^\sigma s + s^\sigma x_1)(\lambda - a) - s^\sigma(x_1(\lambda - a)) \\
&= x_2^\sigma + k\lambda + x_1^\sigma b + x_1^\sigma(s(\lambda - a)) + [x_1^\sigma, s, \lambda - a] + [s^\sigma, x_1, \lambda - a] \\
&= x_2^\sigma + k\lambda + x_1^\sigma(b + s(\lambda - a)) - [x_1, s, \lambda - a] - [s, x_1, \lambda - a] \\
&= x_2^\sigma + k\lambda + x_1^\sigma(b + s(\lambda - a)), \\
f_5(\lambda) &= m + x_2 a + a^\sigma x_2^\sigma + x_3 b + b^\sigma x_3^\sigma + ka^{\sigma+1} + lb^{\sigma+1} + (a^\sigma x_1^\sigma)b + b^\sigma(x_1 a) \\
&\quad + (x_2 + ka^\sigma + b^\sigma x_1 + x_3 s + lb^\sigma s + (a^\sigma x_1^\sigma)s)(\lambda - a) \\
&\quad + (\lambda - a)^\sigma(x_2^\sigma + ka + x_1^\sigma b + s^\sigma x_3^\sigma + ls^\sigma b + s^\sigma(x_1 a)) \\
&\quad + (k + x_1^\sigma s + s^\sigma x_1 + ls^{\sigma+1})(\lambda - a)^{\sigma+1}.
\end{aligned}$$

In order to prove that $p''(\lambda) = p_1(\lambda, b + s(\lambda - a))$, it suffices to prove that $f_5(\lambda)$ is equal to

$$m + x_2\lambda + \lambda^\sigma x_2^\sigma + x_3(b + s(\lambda - a)) + (b^\sigma + (\lambda - a)^\sigma s^\sigma)x_3^\sigma + k\lambda^{\sigma+1} + l(b + s(\lambda - a))^{\sigma+1}$$
$$+(\lambda^\sigma x_1^\sigma)(b + s(\lambda - a)) + (b^\sigma + (\lambda - a)^\sigma s^\sigma)(x_1\lambda),$$

i.e., equal to

$$m + x_2 a + x_2(\lambda - a) + (\lambda - a)^\sigma x_2^\sigma + a^\sigma x_2^\sigma + x_3(b + s(\lambda - a)) + (b^\sigma + (\lambda - a)^\sigma s^\sigma)x_3^\sigma + k(\lambda - a)^{\sigma+1}$$
$$+ka^{\sigma+1} + ka^\sigma(\lambda - a) + k(\lambda - a)^\sigma a + lb^{\sigma+1} + lb^\sigma(s(\lambda - a)) + l((s(\lambda - a))^\sigma b) + l(\lambda - a)^{\sigma+1}s^{\sigma+1}$$
$$+((\lambda - a)^\sigma x_1^\sigma)(b + s(\lambda - a)) + (a^\sigma x_1^\sigma)(b + s(\lambda - a)) + (b^\sigma + (\lambda - a)^\sigma s^\sigma)(x_1(\lambda - a))$$
$$+(b^\sigma + (\lambda - a)^\sigma s^\sigma)(x_1 a).$$

If we subtract the latter expression from $f_5(\lambda)$ and cancel the equal terms, we see that we need to prove that the following expression is equal to 0:

$$[b^\sigma, x_1, \lambda - a] + [x_3, s, \lambda - a] + l \cdot [b^\sigma, s, \lambda - a] + [a^\sigma x_1^\sigma, s, \lambda - a] - [(\lambda - a)^\sigma, x_1^\sigma, b] - [(\lambda - a)^\sigma, s^\sigma, x_3^\sigma]$$
$$-l \cdot [(\lambda - a)^\sigma, s^\sigma, b] - [(\lambda - a)^\sigma, s^\sigma, x_1 a] + (x_1^\sigma s + s^\sigma x_1)(\lambda - a)^{\sigma+1} - ((\lambda - a)^\sigma x_1^\sigma)(s(\lambda - a))$$

$$-((\lambda-a)^\sigma s^\sigma)(x_1(\lambda-a)).$$

Since

$$[b^\sigma, x_1, \lambda-a] = -[b, x_1^\sigma, (\lambda-a)^\sigma] = [(\lambda-a)^\sigma, x_1^\sigma, b],$$
$$[x_3, s, \lambda-a] = -[x_3^\sigma, s^\sigma, (\lambda-a)^\sigma] = [(\lambda-a)^\sigma, s^\sigma, x_3^\sigma],$$
$$[b^\sigma, s, \lambda-a] = -[b, s^\sigma, (\lambda-a)^\sigma] = [(\lambda-a)^\sigma, s^\sigma, b],$$
$$[a^\sigma x_1^\sigma, s, \lambda-a] = -[x_1 a, s^\sigma, (\lambda-a)^\sigma] = [(\lambda-a)^\sigma, s^\sigma, x_1 a],$$

we need to prove that

$$((\lambda-a)^\sigma x_1^\sigma)(s(\lambda-a)) + ((\lambda-a)^\sigma s^\sigma)(x_1(\lambda-a)) = (x_1^\sigma s + s^\sigma x_1)(\lambda-a)^{\sigma+1}.$$

Now,

$$((\lambda-a)^\sigma x_1^\sigma)(s(\lambda-a)) = (((\lambda-a)^\sigma x_1^\sigma)s)(\lambda-a) - [(\lambda-a)^\sigma x_1^\sigma, s, \lambda-a]$$
$$= ((\lambda-a)^\sigma (x_1^\sigma s))(\lambda-a) + [(\lambda-a)^\sigma, x_1^\sigma, s]\cdot(\lambda-a) - [x_1^\sigma, s, \lambda-a]\cdot(\lambda-a)^\sigma.$$

In a similar way, one proves that $((\lambda-a)^\sigma s^\sigma)(x_1(\lambda-a))$ is equal to

$$((\lambda-a)^\sigma (s^\sigma x_1))(\lambda-a) + [(\lambda-a)^\sigma, s^\sigma, x_1]\cdot(\lambda-a) - [s^\sigma, x_1, \lambda-a]\cdot(\lambda-a)^\sigma.$$

Since

$$[(\lambda-a)^\sigma, x_1^\sigma, s] = -[(\lambda-a)^\sigma, x_1, s] = [(\lambda-a)^\sigma, s, x_1] = -[(\lambda-a)^\sigma, s^\sigma, x_1],$$
$$[x_1^\sigma, s, \lambda-a] = -[x_1, s, \lambda-a] = [s, x_1, \lambda-a] = -[s^\sigma, x_1, \lambda-a],$$
$$x_1^\sigma s + s^\sigma x_1 \in \mathbb{K},$$

we have $((\lambda-a)^\sigma x_1^\sigma)(s(\lambda-a)) + ((\lambda-a)^\sigma s^\sigma)(x_1(\lambda-a)) = ((\lambda-a)^\sigma(x_1^\sigma s))(\lambda-a) + ((\lambda-a)^\sigma(s^\sigma x_1))(\lambda-a) = ((\lambda-a)^\sigma(x_1^\sigma s + s^\sigma x_1))(\lambda-a) = (x_1^\sigma s + s^\sigma x_1)(\lambda-a)^{\sigma+1}$, and this is precisely what we needed to prove. ■

In case $\mathbb{O}$ is non-associative, this proposition implies that $\mathcal{P}_\mathcal{T}$ contains non-Desarguesian planes and hence is a non-embeddable polar space. At this point, we have already collected sufficient information to identify the constructed polar spaces. However, we still want to show that we have all planes. This fact is implied by the next proposition.

Proposition 4.16 *Let p_1, p_2 and p_3 be three mutually collinear points which are not contained in a line. Then p_1, p_2 and p_3 are contained in a unique plane.*

Proof. Let i_1, i_2 and i_3 be the types of the respective points p_1, p_2 and p_3. Without loss of generality, we may suppose that $i_1 \geq i_2 \geq i_3$.

(I) We prove that there exist three points p_1', p_2' and p_3' such that: • p_1', p_2' and p_3' are mutually collinear; • the subspaces of $\mathcal{P}_\mathcal{T}$ containing $\{p_1', p_2', p_3'\}$ are precisely the subspaces of $\mathcal{P}_\mathcal{T}$ containing $\{p_1, p_2, p_3\}$; • the types of the points p_1', p_2' and p_3' are mutually distinct.

(a) Suppose that $i_1 > i_2 > i_3$. Then there is nothing to prove. Just take $p_1' := p_1$, $p_2' := p_2$ and $p_3' := p_3$.

(b) Suppose that $i_1 > i_2 = i_3$. Let p_3' be the unique point of smallest type on the line p_2p_3 and put $p_1' := p_1$, $p_2' := p_2$. Then $\{p_1', p_2', p_3'\}$ satisfies the required conditions. Observe that p_1', p_2' and p_3' are mutually collinear by Proposition 4.14.

(c) Suppose $i_1 = i_2 > i_3$. Let p_2'' be the unique point of smallest type contained in the line p_1p_2. Then p_1, p_2'' and p_3 are mutually collinear and the subspaces of $\mathcal{P}_\mathcal{T}$ containing $\{p_1, p_2, p_3\}$ are precisely the subspaces of $\mathcal{P}_\mathcal{T}$ containing $\{p_1, p_2'', p_3\}$. Now, by either (a) or (b), there exist 3 mutually collinear points p_1', p_2' and p_3' whose types are mutually distinct such that the subspaces of $\mathcal{P}_\mathcal{T}$ containing $\{p_1', p_2', p_3'\}$ are precisely the subspaces of $\mathcal{P}_\mathcal{T}$ containing $\{p_1, p_2'', p_3\}$, i.e. the subspaces of $\mathcal{P}_\mathcal{T}$ containing $\{p_1, p_2, p_3\}$.

(d) Suppose $i_1 = i_2 = i_3$. Let p_3'' be the unique point of smallest type contained in the line p_2p_3. Then p_1, p_2 and p_3'' are mutually collinear and the subspaces of $\mathcal{P}_\mathcal{T}$ containing $\{p_1, p_2, p_3''\}$ are precisely the subspaces of $\mathcal{P}_\mathcal{T}$ containing $\{p_1, p_2, p_3\}$. Now, by (c), there exist three mutually collinear points p_1', p_2' and p_3' whose types are mutually distinct such that the subspaces of $\mathcal{P}_\mathcal{T}$ containing $\{p_1', p_2', p_3'\}$ are precisely the subspaces of $\mathcal{P}_\mathcal{T}$ containing $\{p_1, p_2, p_3''\}$, i.e. the subspaces of $\mathcal{P}_\mathcal{T}$ containing $\{p_1, p_2, p_3\}$.

(II) By (I), we may suppose that $i_1 > i_2 > i_3$. Since no point of Type $i \in \{0, 1, 2\}$ is collinear with a point of Type $5-i$, we have $(i_1, i_2, i_3) \in \{(2, 1, 0), (3, 1, 0), (4, 2, 0), (4, 3, 0), (5, 2, 1), (5, 3, 1), (5, 4, 2), (5, 4, 3)\}$, If $(i_1, i_2, i_3) = (3, 1, 0)$, then we put $Y := II$, $\bar{x} = k$ and $\mathcal{A} := \mathbb{K}$. If $(i_1, i_2, i_3) = (4, 2, 0)$, then we put $Y := III$, $\bar{x} = (x, k)$ and $\mathcal{A} := \mathbb{O} \times \mathbb{K}$. If $(i_1, i_2, i_3) = (4, 3, 0)$, then we put $Y := IV$, $\bar{x} = (x, k, l)$ and $\mathcal{A} := \mathbb{O} \times \mathbb{K} \times \mathbb{K}$. If $(i_1, i_2, i_3) = (5, 2, 1)$, then we put $Y := V$, $\bar{x} = (x_1, x_2, k)$ and $\mathcal{A} := \mathbb{O} \times \mathbb{O} \times \mathbb{K}$. If $(i_1, i_2, i_3) = (5, 3, 1)$, then we put $Y := VI$, $\bar{x} = (x_1, x_2, k, l)$ and $\mathcal{A} := \mathbb{O} \times \mathbb{O} \times \mathbb{K} \times \mathbb{K}$. If $(i_1, i_2, i_3) = (5, 4, 2)$, then we put $Y := VII$, $\bar{x} = (x_1, x_2, x_3, k, l)$ and $\mathcal{A} := \mathbb{O} \times \mathbb{O} \times \mathbb{O} \times \mathbb{K} \times \mathbb{K}$. If $(i_1, i_2, i_3) = (5, 4, 3)$, then we put $Y := VIII$, $\bar{x} = (x_1, x_2, x_3, k, l, m)$ and $\mathcal{A} := \mathbb{O} \times \mathbb{O} \times \mathbb{O} \times \mathbb{K} \times \mathbb{K} \times \mathbb{K}$.

If $(i_1, i_2, i_3) = (2, 1, 0)$, then there exists a unique plane containing the points p_1, p_2 and p_3, namely the plane $[\infty]$.

If $(i_1, i_2, i_3) \neq (2, 1, 0)$, then every plane containing p_1, p_2 and p_3 necessarily has Type Y. Using Corollary 4.7, it is straightforward to verify that there exists a unique $\bar{x} \in \mathcal{A}$ and a unique $(a, b, s) \in \mathbb{O}^3$ such that $p_1 = p_1(a, b)$, $p_2 = p_2(s)$ and $p_3 = p_3^*$, where $p_1(a, b)$, $p_2(s)$ and p_3^* are the points which we defined when we gave an explicit description of the plane of Type Y with parameters $\bar{x}$. This shows that there exists a unique plane containing p_1, p_2 and p_3, namely the plane of Type Y with parameters $\bar{x}$. ∎

The following theorem identifies the constructed polar spaces with known ones.

Theorem 4.17 (1) *If $\mathbb{O} = \mathbb{K}$ is a field, then $\mathcal{P}_\mathcal{T}$ is isomorphic to the symplectic polar space of rank 3 defined over the field $\mathbb{K}$.*

(2) *If* $\mathbb{O}$ *and* $\mathbb{K}$ *are fields such that* $\mathbb{O}$ *is a separable quadratic extension of* $\mathbb{K}$*, then* $\mathcal{P}_{\mathcal{T}}$ *is isomorphic to the Hermitian polar space of rank* 3 *associated with* $(\mathbb{O}, \mathbb{K})$*.*

(3) *If* $\mathbb{O}$ *is a field of characteristic* 2 *and* $\mathbb{O}^2 \subseteq \mathbb{K} \neq \mathbb{O}$*, then* $\mathcal{P}_{\mathcal{T}}$ *is isomorphic to the polar space of rank* 3 *of mixed type associated with* $(\mathbb{O}, \mathbb{K})$*.*

(4) *If* $\mathbb{O}$ *is a quaternion division algebra and* $\mathbb{K} = Z(\mathbb{O})$*, then* $\mathcal{P}_{\mathcal{T}}$ *is isomorphic to the quaternionic polar space of rank* 3 *associated with* $\mathbb{O}$*.*

(5) *If* $\mathbb{O}$ *is a Cayley-Dickson division algebra and* $\mathbb{K} = Z(\mathbb{O})$*, then* $\mathcal{P}_{\mathcal{T}}$ *is isomorphic to the nonclassical polar space of rank* 3 *associated with the Cayley-Dickson division algebra* $\mathbb{O}$*.*

Proof. As $\mathcal{P}_{\mathcal{T}}$ contains planes, its rank r is at least 3. If $r \geq 4$, then Proposition 4.4(1) would imply that each maximal singular subspace contains points of (at least) four different types. But this is impossible. Indeed, we already know that if $(i, i') \in \{(0,5),(1,4),(2,3)\}$, then no point of Type i is collinear with a point of Type i'. So, $r = 3$.

Now, a polar space of rank 3 is completely determined by its point set and the collinearity relation defined on this point set. So, Claims (1), (2), (3) and (4) of the theorem are immediate consequences of Proposition 3.1 and Corollary 4.7. As for Claim (5), this follows from Tits' classification of polar spaces [13] and the fact that $\mathcal{P}_{\mathcal{T}}$ is a polar space of rank 3 all whose planes are isomorphic to the Moufang plane $\mathsf{PG}(2, \mathbb{O})$, where $\mathbb{O}$ is a Cayley-Dickson division algebra. ■

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