ON A SMALLEST TOPOLOGICAL TRIANGLE FREE (n_4) POINT-LINE CONFIGURATION

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Dedicated to the memory of Branko Grünbaum

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Abstract

We study an abstract object, a finite generalized quadrangle $W(3)$, due to Jacques Tits, that can be seen as the Levi graph of a triangle free (40_4) point-line configuration. We provide for $W(3)$ representations as a topological (40_4) configuration, as a (40_4) circle representation, and a representation in the complex plane. These come close to a still questionable (real) geometric (40_4) point-line configuration realizing this finite generalized quadrangle. This abstract (40_4) configuration has interesting triangle free realizable geometric subconfigurations, which we also describe. A topological (n_4) configuration for $n < 40$ must contain a triangle, so our triangle free example is minimal.

Keywords: Finite generalized quadrangles, computational synthetic geometry, point-line configurations, oriented matroids, pseudoline arrangements

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1 Introduction

In Computational Synthetic Geometry, see [5], we search for an unknown geometric object when its abstract mathematical structure is given. Oriented matroids, see [2] or [4], play a central part within this field. Our article can be seen in this context. It provides a connection from the theory of finite generalized quadrangles, see [12], to the study of point-line configurations in the sense of recent books about these topics, see [9] and [14].

Definition 1.1. An (n_k) *configuration* is a set of n points and n *lines* such that every point lies on precisely k of these lines and every line contains precisely k of these points. We distinguish three concepts.

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Definition 1.2. When the lines are *straight lines* in the projective plane, we have a *geometric (n_k) configuration.*

Definition 1.3. When the lines are *pseudolines* forming a rank 3 oriented matroid, we have a *topological (n_k) configuration.*

Definition 1.4. When the lines are *abstract lines*, we have an *abstract (n_k) configuration.*

We assume the reader to know basic facts about rank 3 oriented matroids or pseudoline arrangements in the real projective plane.

This article provides, among other results, a triangle free topological (40_4) configuration. We remark that triangle free configurations have been studied so far only for smaller (n_3) configurations, see e.g. [3], [10], or [15].

Definition 1.5. The *generalized quadrangle* $W(3)$ is the point-line geometry where the points are the points of the projective 3-space $\mathbb{P}_3(3)$ over the field of 3 elements, and the lines are the lines of $\mathbb{P}_3(3)$ fixed under a symplectic polarity. A *symplectic polarity* is a permutation of the set of points, lines and planes of $\mathbb{P}_3(3)$ mapping the points to planes, lines to lines and planes to points, such that incidence and non-incidence are both preserved (that is, containment of points in lines and planes, and of lines in planes is transferred into reversed containment), and the permutation has order 2, that is, if a point p is mapped to the plane α, then the plane α is mapped to the point p. Such a polarity can be described, after suitable coordinatization, as mapping the point (a, b, c, d) to the plane with equation $bX - aY + dZ - cU = 0$, from which all other images follow.

As such, the full automorphism group of $W(3)$ is isomorphic to Aut($\mathsf{PSp}_4(3)$), a group of order 51840, containing $\mathsf{PSp}_4(3)$ as normal simple subgroup of index 2.

The geometry $W(3)$ is a member of the family of so-called *symplectic generalised quadrangles* $W(q)$, where q is any prime power. Each line of $W(q)$ contains $q+1$ points and each point is contained in $q+1$ lines. Moreover, $W(q)$ contains q^3+q^2+q+1 points. Hence it is an abstract $(q^3+q^2+q+1)_{q+1}$ configuration. For $q = 3$, we obtain an abstract $(40)_4$ configuration.

The name "generalised quadrangle" comes from the fact that the geometry does not contain any triangle, but every two elements are contained in a quadrangle. Hence every generalised quadrangle $W(q)$ defines a triangle free abstract configuration. We will not need the general definition of a generalised quadrangle, we content ourselves with mentioning that, conversely, when an abstract $(q^3 + q^2 + q + 1)_{q+1}$ configuration is triangle free, then it is a generalised quadrangle, i.e., every pair of elements is contained in a quadrangle. We also say that the generalised quadrangle has order q. When $q > 4$ is a power of 2, there are many non-isomorphic generalised quadrangles with order q known. For q a power of an odd prime, we know exactly two generalised quadrangles of order q. One of those is $W(q)$. The other one is obtained from the first one by interchanging the names "point" and "line". We say that the latter is *the dual* of the former. The dual of $W(q)$ is usually denoted by $Q(4, q)$; it arises as a non-singular parabolic quadric in the projective 4-space $\mathbb{P}_4(q)$ over the field of q elements, that is, a quadric with equation $X_1X_2 + X_3X_4 = X_0^2$, after suitable coordinatization. That $W(q)$ is really not isomorphic to $Q(4, q)$, q odd, can be seen by noting that $Q(4, q)$ admits substructures isomorphic to a $(q+1) \times (q+1)$ grid, while this is not the case for $W(q)$. If q is even, then $W(q)$ is isomorphic to $Q(4, q)$, and the isomorphism can be realised by projecting $Q(4, q)$ from its nucleus, that is, the intersection point of all tangent hyperplanes of $Q(4, q)$ (a hyperpane is *tangent* if it intersects $Q(4, q)$ in a cone).

For $q = 3$, it follows that there are at least two triangle free $(40)_4$ configurations. However, it is shown in 6.2.1 of [12] that these are the only examples.

The question whether a given generalised quadrangle of order q is a geometric $(q^3 + q^2 + q + 1)_{q+1}$ configuration seems to be extremely difficult. The only such quadrangle for which we know the answer is the one with $q = 2$: $W(2)$ is a geometric $(15)_3$ configuration. Already for the next cases $W(3)$ and $Q(4, 3)$ nothing is known. In the present paper, we focus on $W(3)$. We motivate this in Section 6.

For now we have only a conjecture concerning the main question:

Conjecture 1.6. *There is no geometric (40_4) configuration that represents the given finite generalized quadrangle $W(3)$.*

Here are some aspects about the missing methods for solving this problem. One way to prove that there is no such geometric configuration would be to show that there is even no corresponding topological configuration. Our theorem shows that this cannot be done. Another method would have been to start with a projective base and to apply the construction sequence method, see [6], that was very useful for the investigation of smaller (n_4) configurations. However, because of the missing triangles property, the number of variables for an algebraic investigation exceeds very soon the problem size that can be handled with computer algebra support. With a symmetry assumption we reduce the number of variables, however, by using these assumptions we very soon realized that the best results are those that we present in this article. Without any symmetry assumption, we never found a triangle free projective incidence theorem that should occur towards the end of a construction sequence; a property occurring in so many non-symmetric (n_4) configurations. For instance, if a configuration contains two triangles in perspective from a point, then we know by Desargues' theorem that an extra incidence occurs in the real plane, even if the "axis" of the corresponding Desargues' configuration is not a line of the configuration. But $W(3)$ does not contain triangles, and we are not aware of any incidence theorem in the real plane (like Desargues' theorem), which can be applied to $W(3)$. In particular, such an incidence theorem should be triangle free. What remains after this observation is a question.

Problem 1.7. Does there exist a triangle free incidence theorem in the real plane?

2 Description of the given abstract object

Our abstract object, an abstract (40_4) configuration, is known in the literature as $W(3)$. The authors attribute the discovery of classical finite quadrangles (including $W(3)$) to J. Tits and they are first described in 1968 in the book by P. Dembowski [8]

The second author mentioned the problem of realizing $W(3)$ long ago to the first author hoping for a solution with methods from computational synthetic geometry.

2.1 The Levi graph of a triangle free abstract $(40)_4$ point-line configuration

The *Levi graph* of a (point-line) configuration is the graph with vertices the points and the lines of the configuration, adjacent when incident. The Levi graph of the triangle free abstract $(40)_4$ configuration $W(3)$ is given by the following list of vertices with its following four neighbors. We have used the first 40 labels for the points.

(1, 41 42 43 44) (2, 45 46 47 48) (3, 49 50 51 52) (4, 53 54 55 56) (5, 41 45 49 53) (6, 41 57 58 59) (7, 41 60 61 62) (8, 45 63 64 65) (9, 49 66 67 68) (10, 53 69 70 71) (11, 45 72 73 74) (12, 53 75 76 77) (13, 49 78 79 80) (14, 42 46 50 54) (15, 42 63 66 69) (16, 42 72 75 78) (17, 46 60 70 79) (18, 50 61 64 76) (19, 54 62 67 73) (20, 46 57 68 77) (21, 54 58 65 80) (22, 50 59 71 74) (23, 43 47 51 55) (24, 43 65 68 71) (25, 43 73 76 79) (26, 47 61 69 80) (27, 51 62 63 77) (28, 55 60 66 74) (29, 47 59 67 75) (30, 55 57 64 78) (31, 51 58 70 72) (32, 44 48 52 56) (33, 44 64 67 70) (34, 44 74 77 80) (35, 48 62 71 78) (36, 52 60 65 75) (37, 56 61 68 72) (38, 48 58 66 76) (39, 56 59 63 79) (40, 52 57 69 73)

(41, 1 5 6 7) (42, 1 14 15 16) (43, 1 23 24 25) (44, 1 32 33 34) (45, 2 5 8 11) (46, 2 14 17 20) (47, 2 23 26 29) (48, 2 32 35 38) (49, 3 5 9 13) (50, 3 14 18 22) (51, 3 23 27 31) (52, 3 32 36 40) (53, 4 5 10 12) (54, 4 14 19 21) (55, 4 23 28 30) (56, 4 32 37 39) (57, 6 20 30 40) (58, 6 21 31 38) (59, 6 22 29 39) (60, 7 17 28 36) (61, 7 18 26 37) (62, 7 19 27 35) (63, 8 15 27 39) (64, 8 18 30 33) (65, 8 21 24 36) (66, 9 15 28 38) (67, 9 19 29 33) (68, 9 20 24 37) (69, 10 15 26 40) (70, 10 17 31 33) (71, 10 22 24 35) (72, 11 16 31 37) (73, 11 19 25 40) (74, 11 22 28 34) (75, 12 16 29 36) (76, 12 18 25 38) (77, 12 20 27 34) (78, 13 16 30 35) (79, 13 17 25 39) (80, 13 21 26 34)

2.2 A combinatorial construction

The generalised quadrangle $W(3)$ can be coordinatized and so a description using coordinates in the field of order 3 can be given, see [11]. However, we rather present a combinatorial description, which we will use later in Subsection 3.1 and in Section 6.

Let $N = \{1, 2, 3, 4\}$. Then the points of $W(3)$ are the elements $(i+)$, $(i-)$, $(ij+)$ and $(ij-)$, with $i, j \in N$. Sixteen of the forty lines can be described as the sets $L_{ij} := \{(i+), (j-), (ij+), (ij-)\}$, $i, j \in N$ (we emphasise that $i \geq j$ is allowed). Two of the remaining lines can be described as $L_\epsilon := \{(ii\epsilon) : i \in N\}$, $\epsilon \in \{+, -\}$. For each fixed point free involution σ of N we have the two lines $L_\sigma^{\epsilon_1} := \{(11^\sigma\epsilon_1), (22^\sigma\epsilon_2), (33^\sigma\epsilon_3), (44^\sigma\epsilon_4) : \epsilon_i = \epsilon_j \Leftrightarrow i^\sigma = j, \forall i, j \in \{1, 2, 3, 4\}\}$, which accounts for six more lines. Finally, let θ_0 be a fixed permutation of N with exactly one fixed point, say $i_0 \in N$. For each permutation θ of N with exactly one fixed point we define the two lines $L_\sigma^\epsilon := \{(ii^\theta\epsilon) : i \in N\}$, $\epsilon \in \{+, -\}$, if $\theta_0\theta$ has exactly one fixed point, and $L_\sigma^{\epsilon_j} := \{(ii^\theta\epsilon_i) : i \in N, \{\epsilon_i, \epsilon_j\} = \{+, -\}$ for $i \neq j = j^\theta\}$, otherwise (i.e., if $\theta_0\theta$ has no or four fixed points). Since there are exactly eight permutations with exactly one fixed point, this accounts for the remaining sixteen lines.

It is elementary to check that the abstract configuration defined in the previous paragraph is a triangle free $(40)_4$ configuration. The fact that it defines $W(3)$ can be deduced from the observation that it contains a dual 4×4 grid, namely, all points of the 4-set $\{(i+) : i \in N\}$ are collinear to all points of the 4-set $\{(i-) : i \in N\}$.

There are essentially two different choices for θ_0. We will choose θ_0 to be the permutation $(2\ 3\ 4)$, fixing 1.

Concretely, we see the correspondence with the construction in the previous section as follows (it is only one of the 51840 possible identifications).

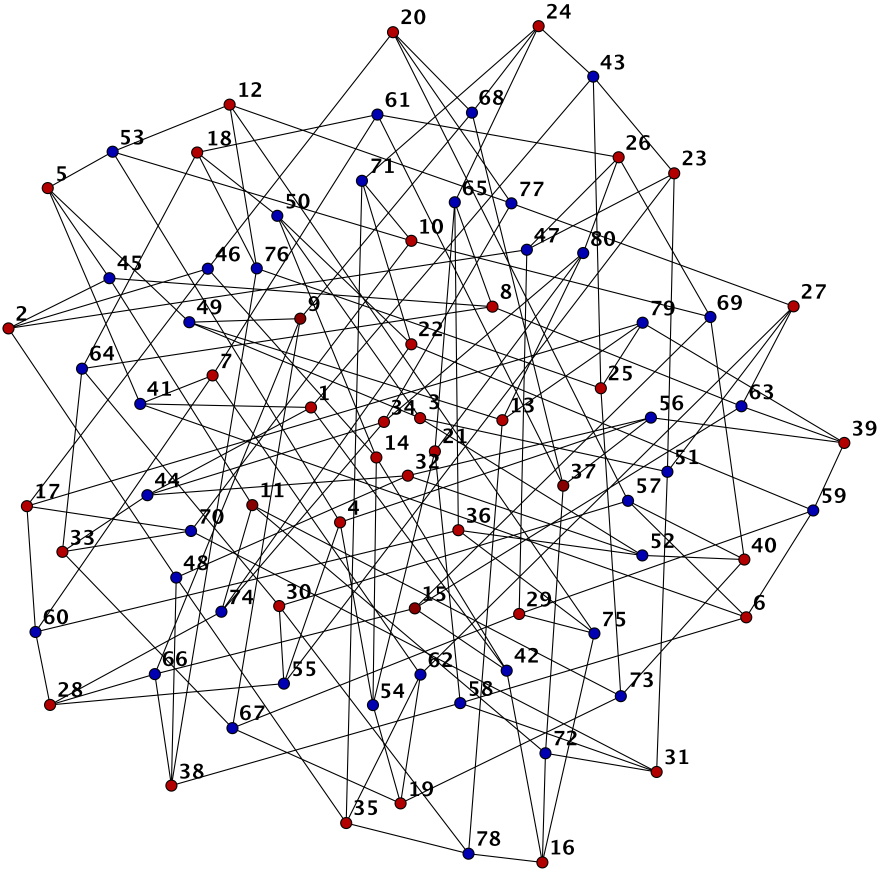

Figure 1: The Levi graph with a five-fold rotational symmetry of the triangle free (40_4) point-line configuration, we use red labels 1, 2, …, 40 as points and blue labels 41, 42, …80 as (abstract) lines.

$(1+) \mapsto 1$	$(2+) \mapsto 35$	$(3+) \mapsto 36$	$(4+) \mapsto 37$
$(1-) \mapsto 7$	$(2-) \mapsto 24$	$(3-) \mapsto 16$	$(4-) \mapsto 32$
$(11+) \mapsto 6$	$(22+) \mapsto 22$	$(33+) \mapsto 29$	$(44+) \mapsto 39$
$(11-) \mapsto 5$	$(22-) \mapsto 10$	$(33-) \mapsto 12$	$(44-) \mapsto 4$
$(12+) \mapsto 23$	$(21+) \mapsto 27$	$(12-) \mapsto 25$	$(21-) \mapsto 19$
$(13+) \mapsto 15$	$(31+) \mapsto 28$	$(13-) \mapsto 14$	$(31-) \mapsto 17$
$(14+) \mapsto 34$	$(41+) \mapsto 26$	$(14-) \mapsto 33$	$(41-) \mapsto 18$
$(23+) \mapsto 30$	$(32+) \mapsto 8$	$(23-) \mapsto 13$	$(32-) \mapsto 21$
$(24+) \mapsto 2$	$(42+) \mapsto 20$	$(24-) \mapsto 38$	$(42-) \mapsto 9$
$(34+) \mapsto 40$	$(43+) \mapsto 11$	$(34-) \mapsto 3$	$(43-) \mapsto 31$

This provides the following identification of the lines.

$L_{11} \mapsto 41$	$L_{22} \mapsto 71$	$L_{33} \mapsto 75$	$L_{44} \mapsto 56$
$L_{12} \mapsto 43$	$L_{23} \mapsto 78$	$L_{34} \mapsto 52$	$L_{41} \mapsto 61$
$L_{13} \mapsto 42$	$L_{24} \mapsto 48$	$L_{31} \mapsto 60$	$L_{42} \mapsto 68$
$L_{14} \mapsto 44$	$L_{21} \mapsto 62$	$L_{32} \mapsto 65$	$L_{43} \mapsto 72$
$L_{+} \mapsto 59$	$L_{-} \mapsto 53$	$L^{+}_{(12)(34)} \mapsto 51$	$L^{-}_{(12)(34)} \mapsto 73$
$L^{+}_{(13)(24)} \mapsto 66$	$L^{-}_{(13)(24)} \mapsto 46$	$L^{+}_{(14)(23)} \mapsto 80$	$L^{-}_{(14)(23)} \mapsto 64$
$L^{+}_{(123)} \mapsto 79$	$L^{-}_{(123)} \mapsto 55$	$L^{+}_{(321)} \mapsto 63$	$L^{-}_{(321)} \mapsto 54$
$L^{+}_{(124)} \mapsto 47$	$L^{-}_{(124)} \mapsto 76$	$L^{+}_{(421)} \mapsto 67$	$L^{-}_{(421)} \mapsto 77$
$L^{+}_{(134)} \mapsto 50$	$L^{-}_{(134)} \mapsto 69$	$L^{+}_{(431)} \mapsto 74$	$L^{-}_{(431)} \mapsto 70$
$L^{+}_{(234)} \mapsto 57$	$L^{-}_{(234)} \mapsto 49$	$L^{+}_{(432)} \mapsto 58$	$L_{(432)} \mapsto 45$

We now study some interesting subconfigurations.

3 Geometric subconfigurations

3.1 The geometric unique triangle free (20_3) configuration

There exist a lot of triangle free (v_3) configurations, for $v \geq 15$. The one with $v = 15$ is often called the *Cremona–Richmond configuration* and it is the unique generalized quadrangle $W(2)$ with three points per line and three lines per point. Its Levi graph is Tutte's 8-cage.

There is another remarkable triangle free (v_3) configuration with v relatively small, and that is the unique flag-transitive (20_3) configuration, denote it by $\mathcal{T}$. Note that there are 162 triangle-free (20_3) configurations altogether [1].

The Levi graph of $\mathcal{T}$ is the Kronecker cover (also sometimes called the bipartite double) of the dodecahedron graph. It can be described as follows. The point set $P_{\mathcal{T}}$ of $\mathcal{T}$ is the set of ordered non-identical pairs (a, b), with $a, b \in \{1, 2, 3, 4, 5\}$. The lines of $\mathcal{T}$ are the triples $\{(a, b), (b, c), (c, a)\}$, with a, b, c three distinct members of $\{1, 2, 3, 4, 5\}$. We denote the line set by $\mathcal{L}_{\mathcal{T}}$. The full collineation group $\mathsf{Sym}(5) \times \mathbb{Z}_2$ is now easy to see (the involution in the center corresponds to the "opposition" mapping $(a, b) \mapsto (b, a)$; we denote (b, a) by $\overline{(a, b)}$ and call these two points *opposite*).

The configuration $\mathcal{T}$ is realizable, see [3]. But it is also a subconfiguration of $W(3)$. This can be easily seen using the construction of Subsection 2.2. We present an embedding of $\mathcal{T}$ in $W(3)$, given explicitly as follows:

$$\begin{aligned} (5,i) &\mapsto (i+), && i \in \{1,2,3,4\}, \\ (i,5) &\mapsto (i-), && i \in \{1,2,3,4\}, \\ (i,j) &\mapsto (ij+), && i,j \in \{1,2,3,4\}, i \neq j. \end{aligned}$$

Using these explicit descriptions, the following properties can easily be checked. (A *hyperbolic line* in $W(3)$ is the set of points of an ordinary line of $\mathbb{P}_3(3)$ which is not a line of $W(3)$ in Definition 1.5 of $W(3)$.)

- *For every point p of $\mathcal{T}$, the point $\overline{p}$ is the unique point of $\mathcal{T}$ at distance 6 from p in the Levi graph.*
- *For every line $L = \{p_1, p_2, p_3\}$ of $\mathcal{T}$, the line $\overline{L} := \{\overline{p}_1, \overline{p}_2, \overline{p}_3\}$ is the unique line at distance 6 from L in the Levi graph.*
- *Two distinct points p, q of $\mathcal{T}$ are collinear in $W(3)$ if and only if either they are collinear in $\mathcal{T}$, or $p = \overline{q}$. In the latter case, no other points of $\mathcal{T}$ are contained in the line of $W(3)$ determined by p and q. In the former case, only the points of the line in $\mathcal{T}$ determined by p and q are contained in the line of $W(3)$ determined by p and q.*
- *The lines of $W(3)$ corresponding to two distinct lines of $\mathcal{T}$ intersect in $W(3)$ if and only if they intersect in $\mathcal{T}$.*
- *For each $i \in \{1,2,3,4,5\}$, the point set $\{(i,j) : j \in \{1,2,3,4,5\} \setminus \{i\}\}$ forms a hyperbolic line in $W(3)$; the same thing holds for $\{(j,i) : j \in \{1,2,3,4,5\} \setminus \{i\}\}$.*
- *There are 20 lines of $W(3)$ containing exactly three (necessarily collinear) points of $\mathcal{T}$; there are 10 lines of $W(3)$ containing exactly two (necessarily opposite) points of $\mathcal{T}$; there are 10 lines of $W(3)$ disjoint from $\mathcal{T}$.*
- *The ten lines of $W(3)$ containing opposite points of $\mathcal{T}$ form a spread of $W(3)$, that is, a partition of the point set of $W(3)$ into lines.*

The geometry $\mathcal{T}$ is self-dual, even self-polar, see [3]. A polarity using our description is for instance given by the mapping

$$(i_1, i_2) \mapsto \{(i_3, i_4), (i_4, i_5), (i_5, i_3)\},$$

where $\{i_1, i_2, i_3, i_4, i_5\} = \{1,2,3,4,5\}$ and the permutation $(i_1\ i_2\ i_3\ i_4\ i_5)$ belongs to a preassigned conjugacy class of elements of order 5 in $\mathsf{Alt}(5)$. There are two such conjugacy classes of elements of order 5, and this gives rise to two distinct polarities, which differ by the opposition map.

This polarity cannot be induced by a duality of $W(3)$ as the latter is not self-dual. The question can be asked whether every collineation of $\mathcal{T}$ is induced by a collineation of $W(3)$. We now show that the answer is positive. To that aim we prove that $W(3)$ can be canonically recovered from $\mathcal{T}$. Given the abstract configuration $\mathcal{T}$, we define the following geometry $\Gamma = (P, \mathcal{L})$. The point set P consists of the union of the point set of $\mathcal{T}$ and the set

$$\{(\{p, \overline{p}\}, L) : p \in P_{\mathcal{T}}, L \in \mathcal{L}_{\mathcal{T}}, L \cap (p^\perp \cup \overline{p}^\perp) = \emptyset\},$$

where $p^\perp$ denotes the set of points of $\mathcal{T}$ collinear to p. It is easy to see that, for each $p \in P_{\mathcal{T}}$, there are exactly two lines of $\mathcal{T}$ not containing any point collinear to p or $\overline{p}$ (and those two lines are mutually opposite). Also, given a line $L \in \mathcal{L}_{\mathcal{T}}$, there is a unique pair of opposite points $p, \overline{p}$ with the property that neither of them is collinear to a point of L. Hence each line L defines a unique new point $(\{p, \overline{p}\}, L)$, which we denote by p_L. So in total, we have 40 points. We now define three types of lines of Γ.

Type 1 If L is a line of $\mathcal{T}$, then $L \cup \{p_L\}$ is a new line of Type 1.

Type 2 If $p \in P_{\mathcal{T}}$, then $\{p, \overline{p}\} \cup \{(\{p, \overline{p}\}, L) : L \in \mathcal{L}_{\mathcal{T}}, L \cap (p^{\perp} \cup \overline{p}^{\perp}) = \emptyset\}$ is a line of Type 2.

Type 3 Let L_1, L_2, L_3, L_4 be four pairwise disjoint lines of $\mathcal{T}$. Let $(\{p_i, \overline{p}_i\}, L_i)$, $i = 1, 2, 3, 4$, be the corresponding new points. If $P_{\mathcal{T}} = \{p_i, \overline{p}_i : i \in \{1, 2, 3, 4\}\} \cup L_1 \cup L_2 \cup L_3 \cup L_4$, then $\{p_{L_1}, p_{L_2}, p_{L_3}, p_{L_4}\}$ is a line of Type 3.

Type 3 lines require some explanation. First of all, it is clear that, if $p_1 = (a, b)$ and $p_2 = (c, d)$, then $\{a, b\} \cap \{c, d\} \neq \emptyset$, because otherwise L_2 contains one of (a, b) or (b, a), and we cannot obtain $P_{\mathcal{T}}$. Hence without loss, we can assume that $p_i = (5, i)$, for all $i \in \{1, 2, 3, 4\}$. Then there are only two possibilities for L_1, L_2, L_3, L_4 anymore. Indeed, if $L_1 = \{(2, 3), (3, 4), (4, 2)\}$, then L_j, $j = 2, 3, 4$ must be equal to $\{(4, 3), (3, 1), (1, 4)\}$, $\{(2, 4), (4, 1), (1, 2)\}$ and $\{(3, 2), (2, 1), (1, 3)\}$, respectively. The other possibility is by applying opposition all these points. Hence, for each member $j \in \{1, 2, 3, 4, 5\}$, we have exactly two lines of Type 3. So in total we have $20 + 10 + 10 = 40$ lines. Now it can be checked easily that Γ is a generalized quadrangle isomorphic to $W(3)$.

Hence every collineation (not duality) of $\mathcal{T}$ extends to a (unique) collineation of $W(3)$.

It can also be shown that the inclusion $\mathcal{T} \subseteq W(3)$ is unique, but we shall not insist on that.

3.2 Subconfigurations from (dual) geometric hyperplanes

A *geometric hyperplane* of a configuration is a subset H of the point set with the property that every line either has all its points in H, or intersects H in a unique point. A *dual geometric hyperplane* is the dual of that, hence a subset G of the line set with the property that for every point p either all the lines through p are in G, or a unique line through p is.

The interest in (dual) geometric hyperplanes for us lies in the fact that, removing a (dual) geometric hyperplane of size s, together with all lines (points) completely contained in it, from a configuration (v_k), always gives a $((v-s)_k, v'_{k-1})$-configuration, or (in the dual case) a $(v'_{k-1}, (v-s)_k)$-configuration, where $v' = k\frac{v-s}{k-1}$. To find geometric realizations of a given configuration, it can help to first find those of such subconfigurations. We give two examples, one with a dual geometric hyperpane and one with a geometric hyperplane. First a dual geometric hyperplane.

In the description of $W(3)$ given in Subsection 2.2, the lines $\{(i+), (j-), (ij+), (ij-)\}$, $i, j \in \{1, 2, 3, 4\}$, $i \neq j$, form a dual geometric hyperplane G. Removing all lines of G and all points $(i+)$ and $(i-)$, $i \in \{1, 2, 3, 4\}$, from $W(3)$ gives rise to a geometric $(32_3, 24_4)$ configuration. A realization is provided in Figure 2. It has a rotational symmetry of order 4.

An example of a geometric hyperplane is given by the set H_p of all points collinear to a given point p. Removing such a set of points, together with all lines through p, gives rise to a $(27_4, 36_3)$-configuration, which is a subconfiguration of the unique triangle free $(27_5, 45_3)$-configuration, which is the unique generalized quadrangle with 3 points per line and 5 lines per point. It is realizable by Theorem 1.4 of [16].

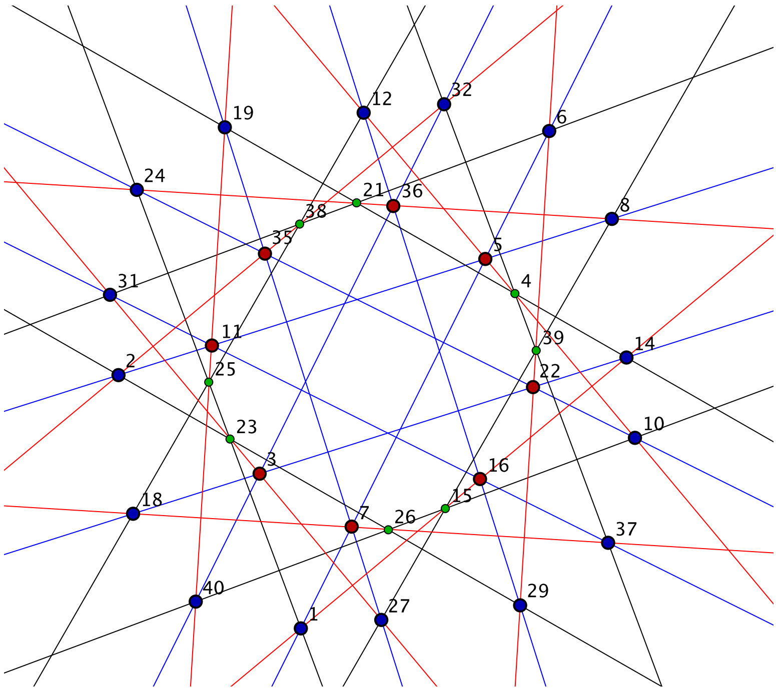

Figure 2: A realized part of the questionable triangle free (40_4) configuration, an incidence structure with 32 3-valent points and 24 4-valent lines.

Remark 3.1. Another example of a dual geometric hyperplane G_L is given by a line L and the set of lines intersecting L nontrivially. The intersection of the configurations arising as complements of H_p and G_L, for a point p incident with L is the dual of the so-called *Gray configuration*, that is, dual to the triple Cartesian product $K \times K \times K$ of a line K of size 3 with itself. More information about the Gray configuration can be found in [13].

3.3 An incidence structure with 40 points and 35 lines

The group $\mathsf{Aut}(\mathsf{PSp}_4(3))$ contains a single conjugacy class of elements of order 5. Each such element acts fixed-point freely on $W(3)$, and hence semi-regularly (this can immediately be deduced from the character table in [7]). Therefore, $W(3)$ is a *polycyclic* configuration. Remarkably, if we remove one line orbit, then we can realize the rest of $W(3)$. This is shown in Figure 3.

Starting with this geometric point-line incidence structure of 40 points and 35 lines, we will be able to construct a circle configuration in Section 5.

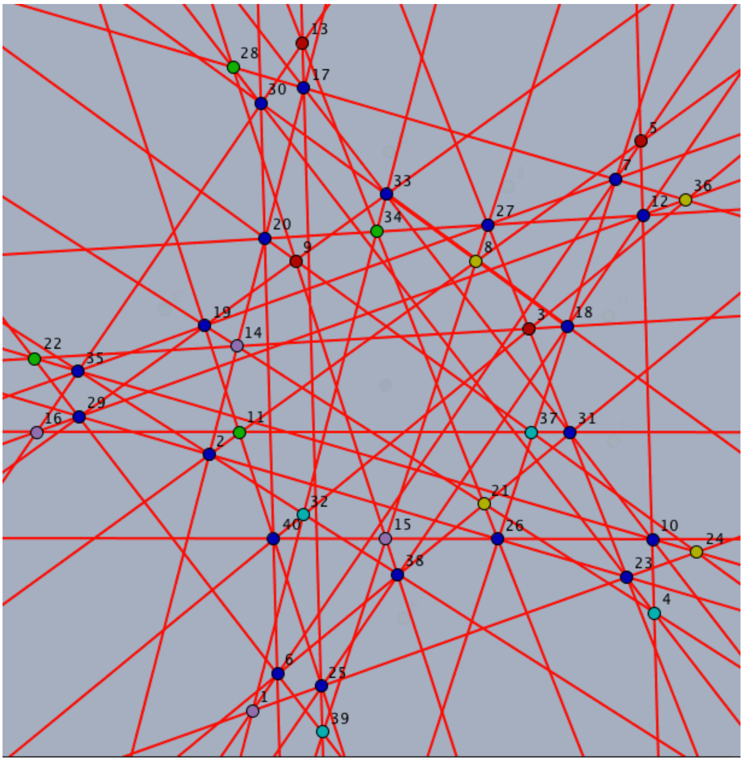

Figure 3: A realized part of the questionable triangle free (40_4) configuration, an incidence structure with 40 points and 35 lines

4 Topological solution

In this section we provide our first main result of this article.

Theorem 4.1. *We have a topological* (40_4) *configuration that represents the given finite generalized quadrangle* $W(3)$

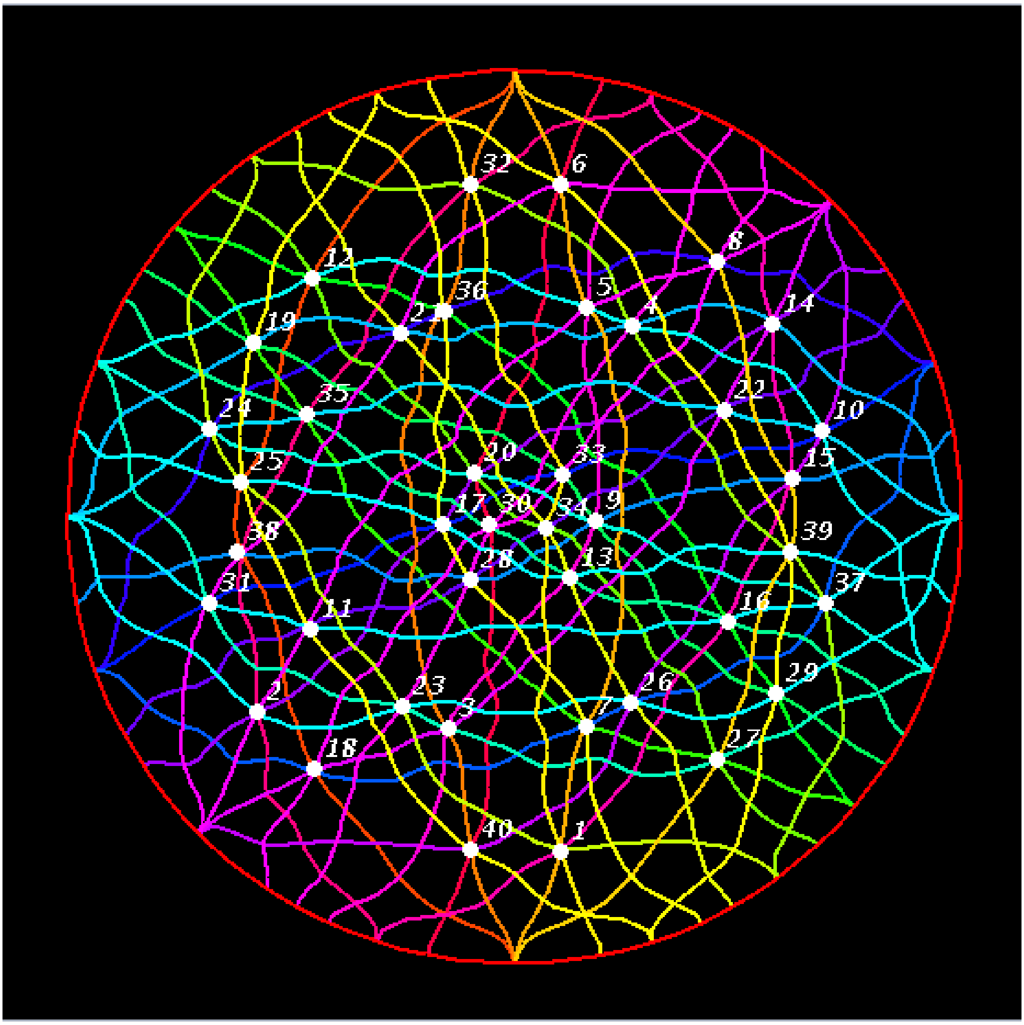

Figure 4: A topological triangle free (40_4) point-line configuration, i.e., a pseudoline arrangement, with a symmetry of order 2.

Proof. We describe our result according to the picture in Figure 4. It shows the pseudoline arrangement with a two-fold symmetry about a vertical axis (which is not drawn). The circular disc provides a model of the projective plane, the outer circle is not an element of the configuration. It is easy to confirm the properties of this configuration. •

A triangle free (n_4)-configuration must have at least 40 lines, which implies that our configuration is minimal.

This can be seen as follows: Consider a first point P of a triangle free (n_4)-configuration with its four lines L_1, L_2, L_3, and L_4 that are incident with P. On each of these four lines L_i, $i \in \{1,2,3,4\}$ we have three additional points $P_{(i,j)}$, $j \in \{1,2,3\}$. There are three additional lines incident with each of these twelve points $P_{(i,j)}$, $j \in \{1,2,3\}$. These 36 lines have to be all different. Otherwise such a line forms a triangle together with P. This was our claim, the four lines we started with, together with these 36 lines have to be part of any triangle free (n_4)-configuration.

5 A circle configuration representing W(3)

We also have a "realization" of W(3) as a "circle configuration" in which 35 "circles" are degenerated, they are lines, see Figure 5. It has a rotational symmetriy of order 5, the same symmetry as the Levi graph of Figure 1. A realization with 40 proper circles can be obtained by applying inversion.

6 Realization in higher dimensions and over other fields

In this section, we further motivate the study of the $(40)_4$ configuration $W(3)$.

In [16], the geometric realisations of all so-called *classical* generalised quadrangles in finite projective spaces of dimension at least 3 are studied, except for the class of symplectic generalised quadrangles. The reason is that all methods break down for these examples. Now, for the other classes, the generic result is that, up to a very few exceptions, if a quadrangle defined over a finite field $\mathbb{F}_q$ of order q admits a representation spanning a projective space of dimension at least 3 defined over the field $\mathbb{F}_{q'}$ of q' elements, then $\mathbb{F}_q$ is a subfield of $\mathbb{F}_{q'}$ and the representation in obtained by a field extension and a (possibly trivial) projection of the standard representation. Although the results in [16] are stated and proved for finite projective spaces, most results also hold for the infinite case, in particular over the reals and the complex numbers. We summarize the results for $Q(4,q)$ below in Theorem 6.1, but first we'd like to point out that, as a consequence of the results in [16], in the generic case, the characteristic of the field over which the quadrangle is defined coincides with the characteristic of the field over which the projective space is defined. If this is not the case, the representation has been called *grumbling* in [17]. Hence, any representation of a finite (classical) quadrangle in a real or complex projective space is necessarily grumbling.

The following theorem can be proved similar to the results in [16].

Theorem 6.1. *Let $Q(4,q)$ be the dual of $W(q)$ and let $\mathbb{P}_n(k)$ be the n-dimensional projective space over the field k, with $n \geq 3$. Then $Q(4,q)$ admits a grumbling representation spanning $\mathbb{P}_n(k)$, for some $n \geq 3$, if and only if either $q = 2$ (and k is any field), or $q = 3$ and k admits a nontrivial cubic root of -1, say ζ. Let k' be the prime field of k. Then, if $q = 2$, the embedding is a (possibly trivial) projection of a projectively unique embedding in $\mathbb{P}_4(k')$. If $q = 3$, then the embedding is a (possibly trivial) projection of a projectively unique embedding in $\mathbb{P}_4(k')$ (if $\zeta \in k'$) or $\mathbb{P}_4(k'(\zeta))$ (if $\zeta \notin k'$).*

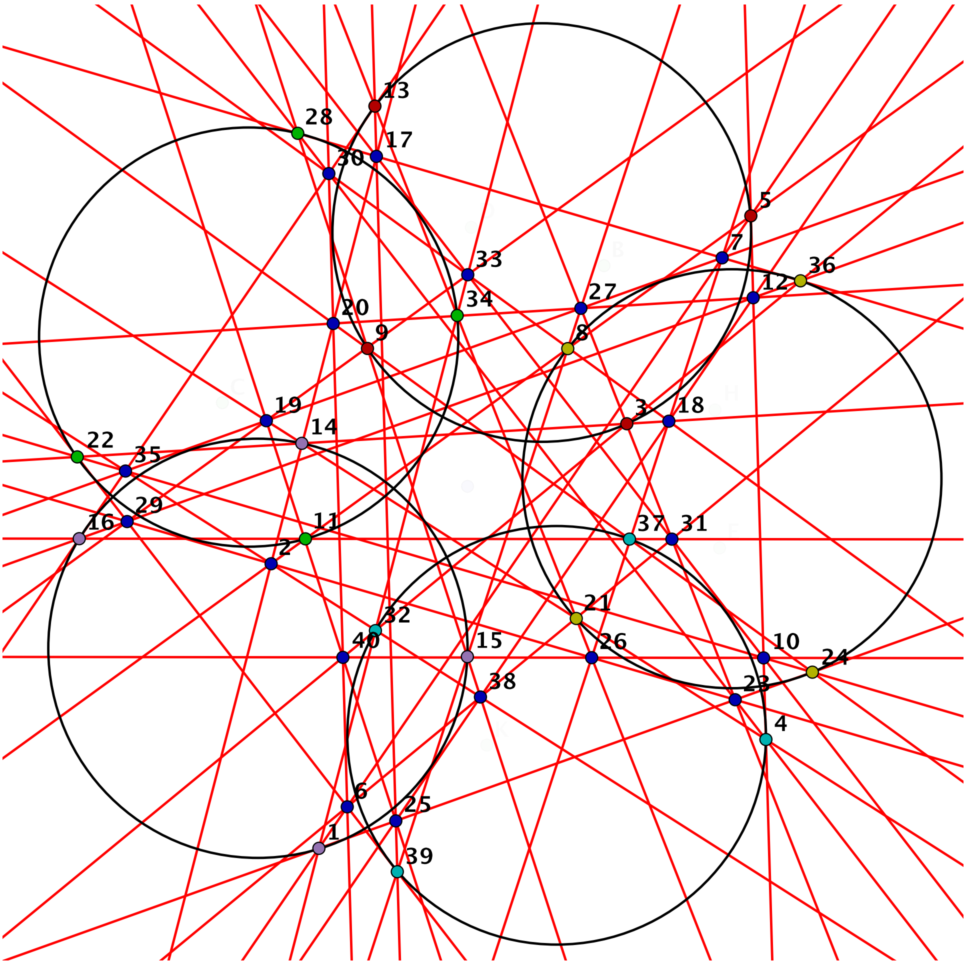

Figure 5: (40_4) circle configuration.

A similar result for $W(q)$, q odd, is now known, and probably out of reach for the moment (although we do know that $W(q)$ does not admit a representation spanning $\mathbb{P}_n(k)$, for $n \geq 4$ and any skew field k). That is why the geometry $W(3)$ is interesting to us. It is the smallest case for which we do not know a result like the previous theorem, and it is small enough to possibly behave exceptionally. In general, it is the belief that $W(q)$ does not admit a grumbling embedding, but the case $q = 3$ could be exceptional. In fact, we will now show that it does admit a grumbling embedding, but unfortunately, not over the reals, though it does over the complex numbers. To that aim, we classify its embeddings spanning a projective 3-space and such that, with the notation of Section 2.2, the points $(i+)$, $i \in N$, are contained in a single line, and the same thing holds for the points $(i-)$, $i \in N$.

Theorem 6.2. *The abstract* $(40)_4$ *configuration* $W(3)$ *admits no representation spanning* $\mathbb{P}_n(k)$ *for* $n \geq 4$. *It admits a unique grumbling representation spanning* $\mathbb{P}_3(k)$*, for* k *a field, with the property that, with the notation of Section* 2.2*, the points* $(i+)$*,* $i \in N$*, are contained in a single line, and the same thing holds for the points* $(i-)$*,* $i \in N$*, if and only the characteristic of* k *is not equal to* 2 *and* k *admits a nontrivial cubic root of* -1.

Proof. Suppose $W(3)$ admits a representation spanning $\mathbb{P}_n(k)$, $n \geq 3$. We show that $n = 3$. The lines $\{(11+), (22+), (33+), (44+)\}$ and $\{(11-), (22-), (33-), (44-)\}$ span a subspace of dimension at most 3. But now all points must be contained in that subspace, since $(i\epsilon)$ is contained in the line defined by $(ii+)$ and $(ii-)$, for all $i \in N$, and the arbitrary point $(ij\epsilon)$, with $i, j \in N$ and $\epsilon \in \{+, -\}$ is contained in the line defined by $(i+)$ and $(j-)$. Hence $W(3)$ spans a subspace of dimension at most 3 and so $n \leq 3$.

Now suppose $n = 3$ and the points $(i+)$, $i \in N$, are contained in a single line, and the same thing holds for the points $(i-)$, $i \in N$. We can introduce coordinates in $\mathbb{P}_3(k)$ in the following way (where "$\longrightarrow$" means "gets the coordinates").

$$\begin{array}{rcl}
(1+) & \longrightarrow & (1,0,0,0), \\
(1-) & \longrightarrow & (0,1,0,0), \\
(2+) & \longrightarrow & (0,0,1,0), \\
(2-) & \longrightarrow & (0,0,0,1), \\
(3+) & \longrightarrow & (1,0,1,0), \\
(11+) & \longrightarrow & (1,1,0,0), \\
(22+) & \longrightarrow & (0,0,1,1).
\end{array}$$

We denote the line of $\mathbb{P}_3(k)$ joining the points P and Q by $\langle P, Q\rangle$. Expressing that $\langle (3+), (3-)\rangle$ and $\langle (11+), (22+)\rangle$ meet in $(33+)$, and that $(3-)$ belongs to $\langle (1-), (2-)\rangle$ by assumption, we obtain

$$\begin{array}{rcl}
(3-) & \longrightarrow & (0,1,0,1), \\
(33+) & \longrightarrow & (1,1,1,1).
\end{array}$$

Since the point $(4+)$ belongs to $\langle (1+), (2+)\rangle$, there exists $x \in k$ so that $(4+)$ has coordinates $(x, 0, 1, 0)$. Expressing that $(4+), (4-)$ and $(44+)$ are collinear, that $(4-) \in \langle (1-), (2-)\rangle$ and $(44+) \in \langle (11+), (22+)\rangle$, we easily see that $(4-)$ has coordinates $(0, x, 0, 1)$ and $(44+)$ has coordinates $(x, x, 1, 1)$.

Now we consider the line defined by θ_0 and $(11+)$. Since $\theta_0\theta_0$ has exactly one fixed point in N, the points $(23+), (34+)$ and $(41+)$ are on a line with $(11+)$ and belong to $\langle (2+), (3-)\rangle$, $\langle (3+), (4-)\rangle$ and $\langle (4+), (1-)\rangle$, respectively. Hence, we can give $(23+)$ the coordinates $(0,1,a,1)$, for some $a \in k$, so that $(34+)$ gets assigned $(b, b+1, a, 1)$, for some $b \in k$. Since $(34+) \in \langle (3+), (4-) \rangle = \langle (1,0,1,0), (0,x,0,1) \rangle$, we see that $a = b = x-1$. Finally, the point $(41+)$ is the intersection of $\langle (4+), (1-) \rangle = \langle (x,0,1,0), (0,0,0,1) \rangle$ and $\langle (11+), (23+) \rangle = \langle (1,1,0,0), (0,1,x-1,1) \rangle$, which easily implies $(41+) = (x^2 - x, 0, x-1, 1) = (-1, 0, x-1, 1)$. This is only possible if $x^2 - x + 1 = 0$, hence if x is a nontrivial third root of -1, since our assumption "grumbling" implies that the characteristic of k is unequal to 3. So we can put $x = \zeta$, with ζ one of the two nontrivial cubic roots of -1. We now calculate:

$$\begin{array}{rcl}
(4+) & \longrightarrow & (\zeta, 0, 1, 0), \\
(4-) & \longrightarrow & (0, \zeta, 0, 1), \\
(44+) & \longrightarrow & (\zeta, \zeta, 1, 1), \\
(23+) & \longrightarrow & (0, -\zeta, 1, -\zeta), \\
(34+) & \longrightarrow & (1, -\zeta^2, 1, -\zeta), \\
(42+) & \longrightarrow & (\zeta, 0, 1, -\zeta).
\end{array}$$

In a similar way, we calculate the points on the line defined by θ_0^{-1} and $(11+)$.

$$\begin{array}{rcl}
(24-) & \longrightarrow & (0, \zeta, -\zeta, 1), \\
(32-) & \longrightarrow & (-\zeta, 0, -\zeta, 1), \\
(43-) & \longrightarrow & (-\zeta^2, 1, -\zeta, 1).
\end{array}$$

We continue similarly with calculating the coordinates of the points of the lines $\{(22+), (13-), (34-), (41-$ and $\{(22+), (14+), (31+), (43+)\}$:

$$\begin{array}{rcl}
(13-) & \longrightarrow & (-\zeta, 1, 0, 1), \\
(34-) & \longrightarrow & (1, \zeta^2, 1, \zeta), \\
(41-) & \longrightarrow & (\zeta, -1, 1, 0), \\
(14+) & \longrightarrow & (1, -\zeta, 0, -1), \\
(31+) & \longrightarrow & (1, -\zeta, 1, 0), \\
(43+) & \longrightarrow & (\zeta^2, 1, \zeta, 1).
\end{array}$$

Comparing $(43+)$ and $(43-)$, or equivalently, $(34+)$ and $(34-)$, we see that $\zeta \neq -\zeta$, implying that the characteristic of k cannot be equal to 2.

Continuing like this, we obtain the coordinates of all remaining points.

$$\begin{array}{rcl}
(11-) & \longrightarrow & (1, -1, 0, 0), \\
(12+) & \longrightarrow & (1, 0, 0, \zeta), \\
(12-) & \longrightarrow & (1, 0, 0, -\zeta), \\
(13+) & \longrightarrow & (\zeta, 1, 0, 1), \\
(14-) & \longrightarrow & (1, \zeta, 0, 1), \\
(21+) & \longrightarrow & (0, 1, -\zeta, 0), \\
(21-) & \longrightarrow & (0, 1, \zeta, 0), \\
(22-) & \longrightarrow & (0, 0, 1, -1), \\
(23-) & \longrightarrow & (0, \zeta, 1, \zeta), \\
(24+) & \longrightarrow & (0, \zeta, \zeta, 1), \\
(31-) & \longrightarrow & (1, \zeta, 1, 0), \\
(32+) & \longrightarrow & (\zeta, 0, \zeta, 1), \\
(33-) & \longrightarrow & (1, -1, 1, -1), \\
(41+) & \longrightarrow & (\zeta, 1, 1, 0), \\
(42-) & \longrightarrow & (\zeta, 0, 1, \zeta), \\
(44-) & \longrightarrow & (\zeta, -\zeta, 1, -1).
\end{array}$$

It is not difficult to check now that the four points on any line of $W(3)$ are collinear in $\mathbb{P}_3(k)$. This concludes the proof of Theorem 6.2. •

One also checks that the group induced on $W(3)$ by the linear transformation of $\mathbb{P}_3(k)$ is isomorphic to $2 \times \mathsf{Alt}(4) \times \mathsf{Alt}(4)$. The first 2 is realized by the involution sending (x_0, x_1, x_2, x_3) to $(x_0, -x_1, x_2, -x_3)$ and fixes all points of shape $(i\epsilon)$, $i \in N$ and $\epsilon \in \{+, -\}$. The $\mathsf{Alt}(4)$ part can be derived from the mapping

$$(x_0, x_1, x_2, x_3) \mapsto (x_0, x_1 - \zeta x_3, x_2, -\zeta x_3)$$

and the uniqueness of the representation.

Discussion. Since $\mathbb{R}$ does not admit nontrivial cubic roots of -1, and $\mathbb{C}$ does, we deduce that $W(3)$ is not embeddable in $\mathbb{P}_3(\mathbb{R})$ with the restrictions of Theorem 3, but it is embeddable in the complex plane, and also as a spanning set of points in complex 3-space (this was not known before). Hence it feeds our conjecture stated before.

It is perhaps remarkable that the condition of k having a nontrivial cubic root of -1 turns up in both theorems of this section. The explanation could be that every planar grumbling representation of $W(3)$ and of $Q(4, q)$ over a field k arises from a projection of a 3-dimensional spacial grumbling representation over the field k. In that case, a planar grumbling embedding of $Q(4, q)$ in $\mathbb{P}_2(k)$ exists if and only if k admits a nontrivial cubic root of -1. In the dual plane, this gives rise to a grumbling embedding of $W(3)$. Hence the existence conditions for grumbling embeddings of $W(3)$ and $Q(4, 3)$ are exactly the same! This, however, leaves us wondering about the additional condition of Theorem 6.2, namely that the characteristic of k is not 2. This could be explained by the fact that the condition, for each $\epsilon \in \{+, -\}$, of the four points $(i\epsilon)$, $i \in N$, being collinear, is too strong in the characteristic 2 case.

If our claim that every planar grumbling embedding of $W(3)$ is obtained from a 3-dimensional one is right, then there certainly exist embeddings spanning $\mathbb{P}_3(k)$, with k not of characteristic 2 or 3, and k admitting nontrivial roots of unity, such that the points of no dual grid are contained in two lines of $\mathbb{P}_3(k)$. Indeed, there are projections of $Q(4, 3)$ that do not satisfy the dual of this condition (as the dual of that condition is never satisfied in any 4-dimensional representation of $Q(4, 3)$ (meaning to span the 4-space), and we can choose the projection line appropriately).

However, as already mentioned, it is not clear whether proving the claims in this discussion is feasible. For the moment we either have to make assumptions that make the calculations feasible, or use ad hoc methods and trial and error to find a representation.

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