Ovoidal maximal subspaces of polar spaces

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Abstract

Let $\mathscr{S}$ be a polar space of rank $n \geq 2$. A set of mutually non-collinear points of $\mathscr{S}$ is trivially a subspace of $\mathscr{S}$. We call it an ovoidal subspace. It is well known that when $n = 2$ all ovoids are maximal subspaces. However, as we shall see in this paper, when $n > 2$ ovoids exist which are not maximal subspaces. Moreover, in the finite case, not all polar spaces admit ovoids. So, it is natural to ask whether ovoidal maximal subspaces exist in any polar space. In this paper we provide a basically affirmative answer to this question, proving that ovoidal maximal subspaces exist in all polar spaces but the following ones: $Q_{2n}(2)$ with n even and greater than 2, $Q^+_{2n-1}(2)$ with $n \equiv 2, 3 \pmod 4$ and greater than 2 and $Q^-_{2n+1}(2)$ with $n \equiv 0, 3 \pmod 4$.

1 Introduction

Let $\mathscr{S}$ be a non-degenerate thick-lined polar space of finite rank $n \geq 2$. According to Shult [17], a *subspace* of $\mathscr{S}$ is a set X of points of $\mathscr{S}$ such that if a line ℓ of $\mathscr{S}$ meets X in at least two points then $X \supseteq \ell$. If furthermore X is a *proper* subspace (that is, not the full point set of $\mathscr{S}$) and every line of $\mathscr{S}$ meets X non-trivially, then X is said to be a *hyperplane*. All hyperplanes are *maximal* subspaces (Shult [17, 7.5.1]), namely they are maximal in the family of proper subspaces of $\mathscr{S}$.

Trivially, a nonempty set of mutually non-collinear points of $\mathscr{S}$ is a subspace of $\mathscr{S}$. We call it an *ovoidal subspace*, also a *partial ovoid* of $\mathscr{S}$. In other words, an ovoidal subspace of $\mathscr{S}$ is a set O of points such that every generator (i.e. maximal singular subspace) of $\mathscr{S}$ meets O in at most one point. If every generator of $\mathscr{S}$ meets O in exactly one point then X is called an *ovoid*.

If O is an ovoid and $n = 2$ then O is a hyperplane, hence a maximal subspace of $\mathscr{S}$. On the other hand, when $n > 2$ all hyperplanes of $\mathscr{S}$ have rank at least $n - 1$, hence they cannot be ovoidal subspaces; moreover, if $\mathscr{S}$ is embeddable then the hyperplanes of $\mathscr{S}$ are precisely the maximal subspaces of $\mathscr{S}$ of rank at least 2, namely the non-ovoidal ones (see [4, Corollary 3]). Two questions arise quite naturally:

(A) Is it true that when $n = 2$ every ovoidal maximal subspace is an ovoid?

(B) Is it true that when $n > 2$ no ovoidal subspace is a maximal subspace?

The answer is NO for both questions. Question (A) is answered in the negative in [4, Note 1]. Further counterexamples will be offered in the present paper.

Question (B) is Problem 4 of [4]. In this paper we shall prove that all polar spaces admit ovoidal maximal subspaces, namely maximal subspaces which are ovoidal (not to be confused with maximal ovoidal subspaces, which trivially exist in any case). Explicitly, we shall prove the following:

Theorem 1 *Let $\mathscr{S}$ be a (non-degenerate thick-lined) polar space of finite rank $n \geq 2$. Then $\mathscr{S}$ admits ovoidal maximal subspaces except precisely when $\mathscr{S}$ is one of the following:*

(1) $Q_{2n}(2) \cong W_{2n-1}(2)$ *with* $n \equiv 0 \pmod 2$, $n > 2$;

(2) $Q^+_{2n-1}(2)$ *with* $n \equiv 2, 3 \pmod 4$, $n > 2$;

(3) $Q^-_{2n+1}(2)$ *with* $n \equiv 0, 3 \pmod 4$.

Section 2 of this paper is entirely devoted to the proof of this theorem. An interesting property of matrices is implicit in Theorem 1. In order to state it, we need the following definition: we say that a square matrix $A = (a_{i,j})_{i,j=1}^N$ is *anti-diagonal* when $a_{i,j} = 0$ if and only if $i = j$. We also recall some terminology. Let $M_N(\mathbb{F})$ be the ring of square matrices of order N with entries in a given field $\mathbb{F}$. We say that two matrices $A, B \in M_N(\mathbb{F})$ are *T-equivalent* if $B = C^T AC$ for a non-singular matrix $C \in M_N(\mathbb{F})$, where C^T stands for the transpose of C. Suppose that $\mathbb{F}$ is a separable quadratic extension of a field $\mathbb{F}_0$ and let σ be the unique non-trivial element of the Galois group of $\mathbb{F}$ over $\mathbb{F}_0$. Recall that the adjoint of a matrix $A = (a_{i,j})_{i,j=1}^N$ (with respect to σ) is the matrix $A^* := (a'_{i,j})_{i,j=1}^N$ where $a'_{i,j} = a^\sigma_{j,i}$ for every choice of $i, j = 1, 2, ..., N$. We say that two matrices $A, B \in M_N(\mathbb{F})$ are *$*$-equivalent* if $B = C^* AC$ for a non-singular matrix $C \in M_N(\mathbb{F})$. We are not going to recall the definitions of symmetric, anti-symmetric, hermitian or anti-hermitian matrices. We only remind the reader of the fact that when $\mathrm{char}(\mathbb{F}) = 2$ all entries on the main diagonal of an anti-symmetric matrix are null, by definition of anti-symmetry. The following will be proved at the end of Section 2.

Corollary 2 *In the following claims $\mathbb{F}$ is a field and we assume that $N > 1$.*

(1) *Let N be even. Then all non-singular anti-symmetric matrices of $M_N(\mathbb{F})$ are T-equivalent to anti-diagonal matrices.*

(2) *Let $N > 2$ and suppose that $\mathbb{F}$ is finite of odd order. When $N = 4$ suppose moreover that $\mathbb{F} \neq \mathbb{F}_3$. Then every non-singular symmetric matrix of $M_N(\mathbb{F})$ is T-equivalent to an anti-diagonal matrix.*

(3) *Let $\mathbb{F}$ be finite of square order. Then every non-singular hermitian or anti-hermitian matrix of $M_N(\mathbb{F})$ is $*$-equivalent to an anti-diagonal matrix.*

Turning back to ovoids, we know that in the rank 2 case ovoids are maximal subspaces. Is the same true for polar spaces of arbitrary rank, possibly modulo a few exceptions? The next theorem, to be proved in Section 4, provides an answer to this question in the finite case.

Theorem 3 *Let $\mathscr{S}$ be a (non-degenerate thick-lined) finite polar space, different from both $Q^+_5(q)$ and $Q^+_7(q)$, with q odd in the latter case. Suppose that $\mathscr{S}$ admits ovoids. Then all ovoids of $\mathscr{S}$ are maximal subspaces.*

All quadrics $Q_5^+(q)$ are actually counterexamples to the conclusion of Theorem 3 (see Section 4, Remark 14) while only a few of the quadrics $Q_7^+(q)$ are known to admit ovoids. We might regard Theorem 3 as an affirmative answer to the above question, albeit limited to the finite case. However ovoids seem to be rare in finite polar spaces of arbitrary rank; so, we are not sure if the set of exceptions considered in Theorem 3 can rightly be regarded as a small one.

We end the paper with a kind of counterpart to Theorem 3 for infinite polar spaces. It will imply examples of polar spaces containing ovoids that are not maximal subspaces, and examples of polar spaces all ovoids of which are automatically maximal subspaces.

Structure of the paper. Section 2 contains the proof of Theorem 1. The proof is divided in four parts. We prove first that every thick-lined generalized quadrangle admits ovoidal maximal subspaces. Next we consider embeddable polar spaces defined over division rings of order at least 3, proving that they also admit ovoidal maximal subspaces. After that, we turn to thick-lined non-embeddable polar spaces of rank 3, obtaining the conclusion with the help of a classification of their subspaces. Finally, we examine polar spaces defined over $\mathbb{F}_2$, thus completing the proof of Theorem 1.

The arguments exploited in Section 2 do not provide explicit descriptions of ovoidal maximal subspaces. However they show how to construct certain partial ovoids, called 'totally scattered' in Section 2, such that every maximal partial ovoid containing one of them is a maximal subspace.

In Section 3 we choose a more concrete approach. In the first part of Section 3 we offer an explicit construction of a family of ovoidal maximal subspaces in symplectic spaces. In the second part we consider embeddable polar spaces not of symplectic type, showing how totally scattered partial ovoids can be constructed for them. Section 4 is devoted to ovoids. It contains the proof of Theorem 3.

Notation. If X is a set of points of a polar space $\mathscr{S}$ we denote by $\langle X\rangle_{\mathscr{S}}$ the subspace of $\mathscr{S}$ spanned by X. Similarly, if X is a set of points of a projective space Σ then $\langle X\rangle_\Sigma$ is the subspace of Σ spanned by X. When no ambiguity will arise, we will freely omit the subscripts $\mathscr{S}$ or Σ from the symbols $\langle .\rangle_{\mathscr{S}}$ and $\langle .\rangle_\Sigma$, thus writing $\langle X\rangle$ instead of $\langle X\rangle_{\mathscr{S}}$, for instance, when $X \subseteq \mathscr{S}$.

For two points x and y of a polar space $\mathscr{S}$, if x and y are collinear we write $x \perp y$. Also, $x^\perp$ is the set of points of $\mathscr{S}$ collinear with x, with $x \in x^\perp$ by convention, and we put $X^\perp := \cap_{x\in X} x^\perp$, for X a set of points of $\mathscr{S}$.

Given a non-zero vector $\mathbf{v}$ of a vector space V, we denote by $[\mathbf{v}]$ the corresponding point of $\mathrm{PG}(V)$. We also use the symbol $\perp$ to denote orthogonality between vectors or projective points. Thus, given a $\mathbb{K}$-vector space V and a reflexive sesquilinear form $f : V \times V \to \mathbb{K}$, when writing $\mathbf{v} \perp \mathbf{w}$ for two vectors $\mathbf{v}, \mathbf{w} \in V$ (or $[\mathbf{v}] \perp [\mathbf{w}]$ for two points $[\mathbf{v}], [\mathbf{w}] \in \mathrm{PG}(V)$) we mean that $f(\mathbf{v}, \mathbf{w}) = 0$. Also, if x is a point of $\mathrm{PG}(V)$ we denote by $x^\perp$ the subspace of $\mathrm{PG}(V)$ formed by the points orthogonal to x and, for $X \subseteq \mathrm{PG}(V)$, we put $X^\perp := \cap_{x\in X} x^\perp$.

Thus, we use the same notation for collinearity in polar spaces and orthogonality in vector or projective spaces, including symbols as $x^\perp$ and $X^\perp$. However the context will always avoid any ambiguity.

The symbols $W_{2n-1}(q)$, $Q^+_{2n-1}(q)$, $Q_{2n}(q)$, $Q^-_{2n+1}(q)$, $H_{2n-1}(q)$ and $H_{2n}(q)$ have the usual meaning. Explicitly, $W_{2n-1}(q)$, $Q^+_{2n-1}(q)$ and $H_{2n-1}(q)$ are respectively the symplectic variety, the hyperbolic quadric and the hermitian variety of $PG(2n-1, q)$, $Q_{2n}(q)$ and $H_{2n}(q)$ are the quadric and the hermitian variety of $PG(2n, q)$ and $Q^-_{2n+1}(q)$ is the elliptic quadric of $PG(2n+1, q)$.

2 Proof of Theorem 1

2.1 The rank 2 case

Theorem 2.1 *All thick-lined generalized quadrangles admit ovoidal maximal subspaces.*

Proof. Let $\mathscr{S}$ be a thick-lined generalized quadrangle, let p, q be two non-collinear points of $\mathscr{S}$ and let $r \in \{p, q\}^\perp$ be arbitrary. Choose $r_p \in \langle p, r\rangle \setminus \{p, r\}$ and $r_q \in \langle q, r\rangle \setminus \{q, r\}$. Put

$$\{p, q\}^\perp_{r|r_p,r_q} \ := \ (\{p, q\}^\perp \setminus \{r\}) \cup \{r_p, r_q\}. \tag{1}$$

Clearly, $\{p, q\}^\perp_{r|r_p,r_q}$ is a partial ovoid. Let O be a maximal partial ovoid containing $\{p, q\}^\perp_{r|r_p,r_q}$ and, for a point $a \notin O$, let $\mathscr{S}_a := \langle O \cup \{a\}\rangle$. By the maximality of O, the subspace $\mathscr{S}_a$ is a (possibly degenerate) full subquadrangle. We shall prove that $\mathscr{S}_a = \mathscr{S}$.

We firstly prove that $p, q \in \mathscr{S}_a$. As $\mathscr{S}_a$ is a subquadrangle, it contains at least one line ℓ through r_q and, since $\mathscr{S}_a$ is full, all points of ℓ belong to $\mathscr{S}_a$. If $\ell = \langle r, q\rangle$ then $q, r \in \mathscr{S}_a$, hence $\mathscr{S}_a$ also contains the line $\langle r, r_p\rangle = \langle r, p\rangle$. In this case we are done: $p, q \in \mathscr{S}_a$. So, suppose that $\ell \neq \langle r, q\rangle$. Hence $p \notin \ell$. The unique point x on ℓ collinear to p belongs to $\mathscr{S}_a$, and the unique point y on $\langle p, x\rangle$ collinear to q is clearly distinct from r and hence belongs to $\{p, q\}^\perp_{r|r_p,r_q} \subseteq \mathscr{S}_a$. Note that $x \neq y$. Hence $p \in \langle x, y\rangle \subseteq \mathscr{S}_a$. Similarly, $q \in \mathscr{S}_a$.

As $p \in \mathscr{S}_a$ and every line of $\mathscr{S}_a$ through p meets $\{p, q\}^\perp_{r|r_p,r_q}$ non-trivially, all lines of $\mathscr{S}$ through p belong to $\mathscr{S}_a$. This implies that $\mathscr{S}_a$ is an ideal subquadrangle of $\mathscr{S}$ as defined in Section 1.8 of [20] (for every point $x \in \mathscr{S}_a$ all lines of $\mathscr{S}$ through x belong to $\mathscr{S}_a$). However $\mathscr{S}_a$ is a also a full subquadrangle of $\mathscr{S}$. Hence $\mathscr{S}_a = \mathscr{S}$ by 1.8.2 of [20]. □

Remark 1 When the generalized quadrangle $\mathscr{S}$ has order (s, t) with s infinite and $s \geq t$ the conclusion of Theorem 2.1 also follows from a result of Cameron [3], according to which every generalized quadrangle of order (s, t) as above admits ovoids (in fact, a partition in ovoids).

2.2 The embeddable case

2.2.1 Preliminaries on embeddings and subspaces

We recall that a *projective embedding* of a polar space $\mathscr{S}$ (an *embedding* of $\mathscr{S}$ for short) is an injective mapping $e : \mathscr{S} \to PG(V)$ from (the point set of) $\mathscr{S}$ to (the point set of) the projective geometry $PG(V)$ of a vector space V, such that the set $e(\mathscr{S})$ spans $PG(V)$ and e maps every line of $\mathscr{S}$ surjectively onto a projective line of $PG(V)$. If $\mathbb{K}$ is the underlying division ring of V, then e is said to be *defined over* $\mathbb{K}$.

A polar space is *embeddable* if it admits a projective embedding. If all embeddings of $\mathscr{S}$ are defined over the same division ring, say $\mathbb{K}$, then $\mathscr{S}$ is said to be *defined over* $\mathbb{K}$.

Two embeddings $e : \mathscr{S} \to PG(V)$ and $e' : \mathscr{S} \to PG(V')$ are *isomorphic* if $e' = \gamma \cdot e$ for an isomorphism γ from $PG(V)$ to $PG(V')$. An embedding $\tilde{e} : \mathscr{S} \to PG(\widetilde{V})$ is *universal* if every embedding of $\mathscr{S}$ is isomorphic to $e_X := \pi_X \cdot \tilde{e}$ for a suitable (possibly trivial) subspace X of $\widetilde{V}$, where π_X stands for the projection of $\mathrm{PG}(\widetilde{V})$ onto $\mathrm{PG}(\widetilde{V}/X)$. The universal embedding, if it exists, is unique (modulo isomorphisms). Clearly, if $\mathscr{S}$ admits the universal embedding then all of its embeddings are defined over the same dvision ring, namely the underlying division ring of its universal embedding.

We recall that all (non-degenerate thick-lined) polar spaces of rank $n > 3$ are embeddable and all those of rank 3 are embeddable but for the following two families: line-grassmannians of 3-dimensional projective spaces defined over non-commutative division rings and a family of thick polar spaces of rank 3 with non-desarguesian Moufang planes, which live inside buildings of type E_7 (see Tits [19, Chapters 7-9]; also Buekenhout and Cohen [2, Chapters 7-11]). We call the polar spaces of the latter family *Freudenthal-Tits polar spaces*; they are implicit in Freudenthal [9] and explicitly defined in Tits [19, Chapter 9].

Remark 2 The approach chosen by Freudenthal [9] and Tits's definition in [19, Chapter 9] are rather algebraic. The reader is referred to Mühlherr [13] for a more geometric approach and De Bruyn and Van Maldeghem [8] for an explicit concrete description of Freudenthal-Tits polar spaces.

As proved by Tits [19, 8.6], all embeddable polar spaces admit the universal embedding but for the following two families of rank 2: grids of order at least 4 (at least five points on each line) and certain generalized quadrangles defined over quaternion division rings (Tits [19, 8.6(II)(a)]). We also recall that, if $\mathscr{S}$ admits the universal embedding, say $\tilde{e}$, and is defined over a division ring of characteristic different from 2, than $\tilde{e}$ is the unique embedding of $\mathscr{S}$ (Tits [19, Chapter 8]). In this case, the universal property of $\tilde{e}$ is ultimately vacuous.

Suppose that $\mathscr{S}$ is embeddable and let $e : \mathscr{S} \to \Sigma = \mathrm{PG}(V)$ be an embedding of $\mathscr{S}$. A subspace X of $\mathscr{S}$ *arises from* e if $e^{-1}(\langle e(X)\rangle_\Sigma) = X$. Clearly, if $\mathscr{S}$ admits the universal embedding and a subspace X of $\mathscr{S}$ arises from an embedding of $\mathscr{S}$, then X also arises from the universal embedding.

Every subspace of $\mathscr{S}$ is a possibly degenerate polar space. The *rank* of a subspace X of $\mathscr{S}$ is its rank as a polar space. If m is the rank of X and r the rank of the radical $X \cap X^\perp$ of X, then $m - r$ is the *reduced rank* of X. Clearly, the rank and the reduced rank of X are equal if and only if X is non-degenerate. The subspaces of rank 1 are precisely the ovoidal subspaces.

Let $n \geq 2$ be the rank of $\mathscr{S}$. Then all hyperplanes of $\mathscr{S}$ have reduced rank at least $n-1$. For instance, let $H = x^\perp$ for a point x of $\mathscr{S}$. Then H is a hyperplane of $\mathscr{S}$, called a *singular* hyperplane; its rank is n and its reduced rank is $n-1$.

Suppose that $\mathscr{S}$ is embeddable and let e be an embedding of $\mathscr{S}$. Then all singular hyperplanes of $\mathscr{S}$ arise from e but, if e is not universal, then hyperplanes of $\mathscr{S}$ exist which do not arise from e. However, let $n > 2$. Then $\mathscr{S}$ admits the universal embedding and all hyperplanes of $\mathscr{S}$ arise from it (as it follows from a theorem of Ronan [15]). In fact this is a special case of a more general result, proved in [4]:

Proposition 2.2 *Let $n \geq 2$ and suppose that $\mathscr{S}$ admits the universal embedding, say $\tilde{e}$. Then all subspaces of $\mathscr{S}$ of reduced rank at least 2 arise from $\tilde{e}$.*

2.2.2 Preliminaries on ovoidal maximal subspaces

As above, let $\mathscr{S}$ be an embeddable non-degenerate polar space of rank $n \geq 2$. As noticed in Section 2.2.1, if $n > 2$ then $\mathscr{S}$ admits the universal embedding. When $n = 2$, assume that the universal embedding of $\mathscr{S}$ exists. Let $\tilde{e} : \mathscr{S} \to \mathrm{PG}(\widetilde{V})$ be the universal embedding of $\mathscr{S}$.

Proposition 2.3 *Let $n > 2$. Then a partial ovoid O of $\mathscr{S}$ is a maximal subspace of $\mathscr{S}$ if and only if it is maximal as a partial ovoid and $\tilde{e}(O)$ spans $\mathrm{PG}(\widetilde{V})$.*

Proof. Let O be an ovoidal maximal subspace of $\mathscr{S}$. Then O is a maximal ovoidal subspace. By way of contradiction, suppose that $\langle\tilde{e}(O)\rangle \subset \mathrm{PG}(\widetilde{V})$ and let X be a hyperplane of $\mathrm{PG}(\widetilde{V})$ containing $\tilde{e}(O)$. Then $H := \tilde{e}^{-1}(X)$ is a hyperplane of $\mathscr{S}$. However H contains at least a line of $\mathscr{S}$, since all hyperplanes of $\mathscr{S}$ have rank at least $n-1$ and $n \geq 3$ by assumption. Hence $H \supset O$. This contradicts the hypothesis that O is a maximal subspace. Therefore $\tilde{e}(O)$ spans $\mathrm{PG}(\widetilde{V})$.

Conversely, suppose that $\tilde{e}(O)$ spans $\mathrm{PG}(V)$ and O is a maximal partial ovoid. Let X be a (possibly improper) subspace of $\mathscr{S}$ properly containing O and choose a point $x \in X \setminus O$. As O is maximal as a partial ovoid, if x is a point of $\mathscr{S}$ exterior to O then $x \perp y$ for some point $y \in O$. Accordingly, X contains a line of $\mathscr{S}$, namely the line $\langle x, y\rangle$. Hence X has rank at least 2. If X is degenerate, then X is contained in a singular hyperplane $x^\perp$ of $\mathscr{S}$. Hence $\tilde{e}(O) \subseteq \tilde{e}(x)^\perp \subset \mathrm{PG}(\widetilde{V})$. However $\langle\tilde{e}(O)\rangle = \mathrm{PG}(\widetilde{V})$ by assumption. We get a contradiction, which forces us to conclude that X is non-degenerate. Accordingly, X has reduced rank at least 2. Therefore $X = \tilde{e}^{-1}(\langle\tilde{e}(X)\rangle)$, by Proposition 2.2. However $\langle\tilde{e}(X)\rangle \supseteq \langle\tilde{e}(O)\rangle$ and $\langle\tilde{e}(O)\rangle = \mathrm{PG}(\widetilde{V})$ by assumption. Hence $\langle\tilde{e}(X)\rangle = \mathrm{PG}(\widetilde{V})$. It follows that $X = \mathscr{S}$. We have proved that O is a maximal subspace of $\mathscr{S}$. $\square$

The next corollary trivially follows from Proposition 2.3.

Corollary 2.4 *If $n > 2$ then no ovoidal maximal subspace of $\mathscr{S}$ arises from $\tilde{e}$.*

Proposition 2.5 *Let $n = 2$. Then a partial ovoid O of $\mathscr{S}$ is a maximal subspace of $\mathscr{S}$ if and only if one of the following holds:*

(1) *O is an ovoid;*

(2) *O is a maximal partial ovoid and $\tilde{e}(O)$ spans $\mathrm{PG}(\widetilde{V})$.*

Proof. This statement can be proved by essentially the same arguments used to prove Proposition 2.3. We leave the details for the reader. $\square$

Remark 3 Conditions (1) and (2) of Proposition 2.5 are not mutually exclusive. Indeed ovoids are maximal partial ovoids and if an ovoid O does not arise from $\tilde{e}$ (in short, it is *non-classical*) then $\tilde{e}(O)$ spans $\mathrm{PG}(\widetilde{V})$.

The following is an obvious consequence of Propositions 2.3 and 2.5.

Corollary 2.6 *Suppose that $\mathscr{S}$ admits a partial ovoid O such that $\tilde{e}(O)$ spans $\mathrm{PG}(\widetilde{V})$. Then every maximal partial ovoid of $\mathscr{S}$ containing O is a maximal subspace of $\mathscr{S}$.*

2.2.3 Setting and more notation and terminology

Henceforth $\mathscr{S}$ is a (non-degenerate) embeddable polar space of rank $n \geq 2$ and, if $\mathscr{S}$ admits the universal embedding, then $e : \mathscr{S} \to \Sigma = \mathrm{PG}(V)$ is its universal embedding; otherwise, if $\mathscr{S}$ is a grid or a generalized quadrangle as in [19, 8.6(II)(a)], then e is any of the embeddings of $\mathscr{S}$. We denote by $\mathbb{K}$ the underlying division ring of V (which is also the underlying division ring of $\mathscr{S}$, except when $\mathscr{S}$ is a grid and $\mathbb{K}$ is infinite).

We shall keep the distinction between $\mathscr{S}$ and its e-image $e(\mathscr{S}) \subseteq \Sigma$. This distinction might look futile (especially when e is the unique embedding of $\mathscr{S}$), but it helps to avoid misunderstandings.

In the sequel we shall often deal with the subspace $\langle e(\{p,q\}^\perp)\rangle_\Sigma$ of Σ, for two non-collinear points p, q if $\mathscr{S}$. This subspace is the same as $\{e(p), e(q)\}^\perp$, where $\perp$ stands for the orthogonality relation of Σ associated to $e(\mathscr{S})$ rather than the collinearity relation in $\mathscr{S}$. Clearly, the subspace $\langle e(\{p,q\}^\perp)\rangle_\Sigma = \{e(p), e(q)\}^\perp$ has codimension 2 in Σ. In order to have a symbol not so clumsy as $\langle e(\{p,q\}^\perp)\rangle_\Sigma$, we put

$$\Sigma_{p,q} \;:=\; \langle e(\{p,q\}^\perp)\rangle_\Sigma \;(= \{e(p), e(q)\}^\perp). \tag{2}$$

Finally, we say that a partial ovoid O of $\mathscr{S}$ is *totally scattered* (in Σ) if $e(O)$ spans Σ. In view of Proposition 2.3, when $n > 2$, proving that $\mathscr{S}$ admits an ovoidal maximal subspace is the same as proving that $\mathscr{S}$ admits a totally scattered partial ovoid.

2.2.4 Back to the rank 2 case

With $\mathscr{S}$ and Σ as above, suppose that $n = 2$. Let p, q be two non-collinear points of $\mathscr{S}$.

Lemma 2.7 *Suppose that $\mathscr{S}$ is not a grid and $\mathbb{K} \neq \mathbb{F}_2$. Then the set $e(\{p,q\}^\perp)$ properly contains a basis of $\Sigma_{p,q}$.*

Proof. By way of contradiction, suppose that $e(\{p,q\}^\perp)$ is a basis of $\Sigma_{p,q}$. As this basis contains all singular points of $\Sigma_{p,q}$, the quadrangle $e(\mathscr{S})$ cannot be symplectic. Accordingly, $e(\mathscr{S})$ is defined by a $(\sigma, 1)$-quadratic form, say ϕ. Choose any three points p_1, p_2, p_3 of $\{p,q\}^\perp$ and let $\pi = \langle e(p_1), e(p_2), e(p_3)\rangle_\Sigma$ (recall that $\mathscr{S}$ is not a grid, by assumption). So, $\pi \cap e(\mathscr{S}) = \{e(p_1), e(p_2), e(p_3)\}$, since $e(\{p,q\}^\perp)$ is independent by assumption and contains all singular points of $\Sigma_{p,q}$. Accordingly, $e(p_1), e(p_2)$ and $e(p_3)$ are the unique points of π which are singular for ϕ and they span π. It follows that $\pi \cap e(\mathscr{S})$ is a conic defined over $\mathbb{F}_2$, namely σ is the identity and $\mathbb{K} = \mathbb{F}_2$. However $\mathbb{K} \neq \mathbb{F}_2$ by assumption. A contradiction has been reached. □

The set $\{p,q\}^\perp_{r|r_p,r_q}$ defined in (1) (see the proof of Theorem 2.1) is a partial ovoid. Moreover, by Lemma 2.7, when neither $\mathscr{S}$ is a grid nor $\mathbb{K} = \mathbb{F}_2$, then $e(\{p,q\}^\perp \setminus \{r\})$ spans $\Sigma_{p,q}$ for some $r \in \{p,q\}^\perp$, whence for any $r \in \{p,q\}^\perp$, since the stabilizer of p and q in $\mathrm{Aut}(\mathscr{S})$ acts transitively on $\{p,q\}^\perp$. However the span of $\Sigma_{p,q} \cup \{e(r_p), e(r_q)\}$ contains the line $\langle e(p), e(q)\rangle_\Sigma$ and $\langle e(p), e(q)\rangle_\Sigma \cup \Sigma_{p,q}$ spans Σ. Therefore $e(\{p,q\}^\perp_{r|r_p,r_q})$ spans Σ, consequently $\{p,q\}^\perp_{r|r_p,r_q}$ is totally scattered.

Remark 4 Actually, when $\mathscr{S}$ is as in [19, 8.6(II)(a)] the previous argument is incorrect, since the full automorphism group of $\mathscr{S}$ does not stabilize the chosen embedding e. However, a suitable subgroup of $\mathrm{Aut}(\mathscr{S})$ of index 2 does the job.

Lemma 2.8 *Suppose that $\mathbb{K}$ is infinite but $\mathscr{S}$ is not a grid. Then $\{p,q\}^{\perp}_{r|r_p,r_q}$ contains an infinite subset C such that the partial ovoid $\{p,q\}^{\perp}_{r|r_p,r_q} \setminus C$ is still totally scattered.*

Proof. As $\mathscr{S}$ is embeddable but not a grid, $\mathscr{S}$ cannot be semi-finite. Hence $\{p,q\}^{\perp}$ is infinite, since $\mathbb{K}$ is infinite by assumption. If $\dim(\Sigma_{p,q}) > 1$, then we can choose three points p_1, p_2, p_3 in $\{p,q\}^{\perp} \setminus \{r\}$, different from r and such that $e(p_1), e(p_2)$ and $e(p_3)$ span a plane π of Σ. This plane contains infinitely many points of $e(\mathscr{S})$, which obviously belong to $e(\{p,q\}^{\perp})$. The point $e(r)$ might be one of them. Put $C := \pi \setminus \{p_1, p_2, p_4, r\}$ Then C is an infinite subset of $\{p,q\}^{\perp}_{r|r_p,r_q}$ and $\{p,q\}^{\perp}_{r|r_p,r_q} \setminus C$ still spans Σ. When $\dim(\Sigma_{p,q}) = 1$ we argue in the same way, but considering two points $p_1, p_2 \in \{p,q\}^{\perp}$ instead of three. $\square$

The case where $\mathscr{S}$ is a grid has been put aside in the above. We deal with it in the following lemma:

Lemma 2.9 *Suppose that $\mathscr{S}$ is a grid and $|\mathbb{K}| > 3$. Then $\mathscr{S}$ admits an ovoid O containing a subset C of size $|\mathbb{K}| - 3$ ($= |\mathbb{K}|$ if $\mathbb{K}$ is infinite) such that $O \setminus C$ is totally scattered.*

Proof. As $|\mathbb{K}| \geq 4$ by assumption, $\mathscr{S}$ admits several non-equivalent embeddings and the embedding e which we have chosen for $\mathscr{S}$ is just one of them. The full automorphisms group of $\mathscr{S}$ acts transitively on the set of embeddings of $\mathscr{S}$ as well as on the set of ovoids of $\mathscr{S}$. None of the ovoids of O arises as a plane section from all embeddings of $\mathscr{S}$. So, there exists an ovoid O of $\mathscr{S}$ such that $e(O)$ is not a conic of Σ, namely $e(O)$ spans Σ. Hence $e(O)$ contains a basis $\{e(p_1), e(p_2), e(p_3), e(p_4)\}$ of Σ. The ovoid O and the set $C = O \setminus \{p_1, p_2, p_3, p_4\}$ have the required properties. $\square$

2.2.5 The general embeddable case

Let now $n \geq 2$. Note that, if e is universal then, for any non-degenerate subspace $\mathscr{S}'$ of $\mathscr{S}$ of rank at least 2, the embedding $e' : \mathscr{S}' \to \Sigma' := \langle e(\mathscr{S}') \rangle_{\Sigma}$ induced by e on $\mathscr{S}'$ is still universal except when $|\mathbb{K}| > 3$ and $\mathscr{S}'$ is a grid or $\mathscr{S}'$ is a generalized quadrangle as in [19, 8.6(II)(a)]. However, in the latter two cases we can always assume that e' is the embedding chosen for $\mathscr{S}'$ as in Section 2.2.3.

Lemma 2.10 *Let $\mathbb{K}$ be infinite. Then $\mathscr{S}$ admits a partial ovoid O containing an infinite subset C such that $O \setminus C$ is totally scattered.*

Proof. We argue by induction on n. We firstly fix the inductive step. Suppose that the claim holds true for a given n, let $\mathrm{rank}(\mathscr{S}) = n + 1$ and let p, q be two non-collinear points of $\mathscr{S}$. By the induction hypothesis, $\mathscr{S}' := \{p,q\}^{\perp}$ admits a partial ovoid O' containing an infinite subset C' such that $e(O' \setminus C')$ spans $\Sigma_{p,q}$ ($= \langle e(\{p,q\}^{\perp}) \rangle_{\Sigma}$ according to definition (2) of Section 2.2.3).

Choose $r \in C'$ and points $r_p \in \langle p, r\rangle \setminus \{p, r\}$ and $r_q \in \langle q, r\rangle \setminus \{q, r\}$. Then $O := (O' \setminus \{r\}) \cup \{r_p, r_q\}$ is a partial ovoid. Moreover, the set $C := C' \setminus \{r\} \subseteq O$ is infinite and $e(O \setminus C)$ spans Σ, since $e(O' \setminus C') \subset e(O \setminus C)$ spans $\Sigma_{p,q}$ and $\{e(r_p), e(r_q)\} \cup \Sigma_{p,q}$ spans Σ. So, $O \setminus C$ is totally scattered.

The initial step remains to be fixed. Suppose $n = 2$. When $\mathscr{S}$ is thick Lemma 2.8 does the job. When $\mathscr{S}$ is a grid we can use Lemma 2.9. Note that, as $\mathbb{K}$ is infinite, the set C of lemma 2.9 is infinite as well. □

We turn now to the finite case.

Lemma 2.11 *Let $\mathbb{K}$ be finite but different from $\mathbb{F}_2$ and let $U \subset e(\mathscr{S})$ be a basis of Σ formed by mutually non-orthogonal singular points. Then there exists a point $v \in \Sigma$ such that $v^\perp \cap U = \emptyset$.*

Proof. Since $\mathbb{K}$ is finite, $\dim(\Sigma)$ is finite as well. Hence $H^\perp \neq \emptyset$ for every hyperplane H of Σ. If either $\mathrm{char}(\mathbb{K}) \neq 2$ or $e(\mathscr{S})$ is not a quadric then $H^\perp$ is a point, say v, and $v^\perp = H$. Clearly Σ admits hyperplanes disjoint from U. If H is such a hyperplane then $v^\perp \cap U = \emptyset$. In this case we are done. When $\mathrm{char}(\mathbb{K}) = 2$, $e(\mathscr{S})$ is a quadric and $\dim(\Sigma)$ is odd (hence $\dim(\Sigma) = 2n \pm 1$ since $\mathbb{K}$ is finite), then $H^\perp$ is a point and, if v is that point, then $v^\perp = H$. As before, we can choose H disjoint from U thus obtaining that $v^\perp \cap U = \emptyset$.

Suppose now that $\mathrm{char}(\mathbb{K}) = 2$, $e(\mathscr{S})$ is a quadric and $\dim(\Sigma)$ is even. Then $\dim(\Sigma) = 2n$ and $e(\mathscr{S}) = Q_{2n+2}(\mathbb{K})$. Accordingly, Σ contains a unique point v_0 (the nucleus of the quadric $e(\mathscr{S})$) such that $v_0^\perp = \Sigma$. We shall prove that Σ admits a hyperplane H disjoint from U and such that $H^\perp \neq \{v_0\}$. With H chosen in that way, if $v \in H^\perp \setminus \{v_0\}$ then $v^\perp = H$, and we are done.

By assumption, $\Sigma = \mathrm{PG}(V)$ with $V = V(2n+1, \mathbb{K})$, $e(\mathscr{S})$ is the quadric defined by a non-singular quadratic form $\phi : V \to \mathbb{K}$ and V admits a basis $(\mathbf{u}_i)_{i=1}^{2n+1}$ such that $\phi(\mathbf{u}_i) = 0$ for any $i = 1, 2, ..., 2n+1$ and $f(\mathbf{u}_i, \mathbf{u}_j) \neq 0$ for $1 \leq i < j \leq 2n+1$, where $f : V \times V \to \mathbb{K}$ is the bilinearization of ϕ. We shall prove that Σ admits a hyperplane H containing the nucleus v_0 of $e(\mathscr{S})$ and disjoint from $U = ([\mathbf{u}_i])_{i=1}^{2n+1}$.

Put $a_{i,j} = f(\mathbf{u}_i, \mathbf{u}_j)$. Then $A = (a_{i,j})_{i,j=1}^{2n+1}$ is the representative matrix of f with respect to $([\mathbf{u}_i])_{i=1}^{2n+1}$. By assumption, $a_{i,j} = 0$ if and only if $i = j$, namely A is anti-diagonal. Moreover $\mathrm{rank}(A) = 2n$ and, if $\mathbf{v}_0 = \sum_{i=1}^{2n+1} \mathbf{u}_i \lambda_i$ represents v_0, then $A\mathbf{v}_0 = \mathbf{0}$. With no loss, we can assume that the first $2n$ rows of A are independent. Then necessarily $\lambda_{2n+1} \neq 0$. With no loss, $\lambda_{2n+1} = 1$, namely $\mathbf{v}_0 = \mathbf{u}_{2n+1} + \sum_{i=1}^{2n} \mathbf{u}_i \lambda_i$. Moreover, up to rescaling the vectors $\mathbf{u}_1, ..., \mathbf{u}_{2n}$, we can assume that $a_{i,2n+1} = 1$ for $i = 2, 3, ..., 2n$ and $a_{1,2n+1} = \mu \neq 1$ (recall that $|\mathbb{K}| > 2$ by assumption). Hence $\mu\lambda_1 + \sum_{i=2}^{2n} \lambda_i = a_{2n+1,2n+1} = 0$. Consequently $\sum_{i=1}^{2n} \lambda_i \neq 0$, because $\mu \neq 1$. Let H be the hyperplane of Σ defined by the following equation: $\sum_{i=1}^{2n} x_i + \sum_{i=1}^{2n} \lambda_i \cdot x_{2n+1} = 0$, where unknowns are taken with respect to the basis $(\mathbf{u}_i)_{i=1}^{2n+1}$ of V.

Clearly $v_0 \in H$ and $[\mathbf{u}_i] \notin H$ for every $i = 1, 2, ..., 2n$. Moreover $[\mathbf{u}_{2n+1}] \notin H$ because $\sum_{i=1}^{2n} \lambda_i \neq 0$. So, H contains the nucleus v_0 of $\mathscr{S}$ and is disjoint from U. As H contains v_0, $H^\perp$ is a line through v_0. If v is a point of that line different from v_0 then $v^\perp = H$. □

Lemma 2.12 *Let $\mathbb{K}$ be finite but different from $\mathbb{F}_2$. If $\mathscr{S} = Q^+_{2n-1}(3)$ assume moreover that $n > 2$. Then $\mathscr{S}$ admits a completely scattered partial ovoid.*

Proof. As in the proof of Lemma 2.10, we argue by induction on n. We firstly fix the inductive step. Suppose that the claim holds true for n and let $\mathrm{rank}(\mathscr{S}) = n+1$. Let p, q be non-collinear points of $\mathscr{S}$. By the inductive hypothesis, $\{p,q\}^\perp$ admits a partial ovoid O' such that $e(O')$ spans $\Sigma_{p,q}$. Clearly, $e(O')$ contains a basis U of $\Sigma_{p,q}$ formed by singular mutually non-orthogonal points. By Lemma 2.11 applied to $\Sigma_{p,q}$ the subspace $\Sigma_{p,q}$ contains a point v such that $v^\perp \cap U = \emptyset$. The triple $\{e(p), e(q), v\}$ spans a plane π of Σ and $\Sigma_{p,q} \cup \pi$ spans Σ. As $v^\perp \cap U = \emptyset$, we have $u^\perp \cap \pi = \langle e(p), e(q)\rangle$ for every point $u \in U$. It is clear that if $\mathbb{K} \neq \mathbb{F}_2$ then the plane π contains at least two singular points $e(a)$ and $e(b)$ exterior to the line $\langle e(p), e(q)\rangle_\Sigma$ and such that $v, e(a)$ and $e(b)$ are non-collinear in Σ. Accordingly, $O := \{a, b\} \cup e^{-1}(U)$ is a partial ovoid and $e(O) = \{e(a), e(b)\} \cup U$ spans Σ, namely O is totally scattered. The inductive step is performed.

The initial step remains to be done. When $\mathscr{S}$ is thick or $|\mathbb{K}| > 3$ then we can start the induction at $n = 2$. Indeed when $n = 2$ and $\mathscr{S}$ is not a grid then $\{p,q\}^\perp_{r|r_p,r_q}$ is a totally scattered partial ovoid. When $|\mathbb{K}| > 3$ and $\mathscr{S}$ is a grid, then we can apply Lemma 2.9.

Suppose that $\mathbb{K} = \mathbb{F}_3$ and $\mathscr{S}$ is non-thick, namely $\mathscr{S} = Q^+_{2n-1}(3)$. In this case the claim of the lemma is false for $n = 2$. Thus, we are forced to start the induction at $n = 3$. Recall that, given a non-singular 6×6 symmetric matrix A with entries in $\mathbb{F}_3$, the matrix A represents a bilinear form f_A of $V(6,3)$ and the Witt index of f_A is 3 or 2 according to whether $\det(A)$ is equal to -1 or 1 respectively. Choose A anti-diagonal as follows:

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 1 & 2 & 0 \end{pmatrix}.$$

It is easily seen that $\det(A) = -1$ (in $\mathbb{F}_3$). Hence the form f_A has Witt index 3, namely it defines $Q^+_5(3)$. The canonical basis of $V(6,3)$ consists of mutually non-orthogonal singular vectors (with respect to f_A). Hence the corresponding points of $\mathrm{PG}(5,3)$ form a partial ovoid of $Q^+_5(3)$ with the required properties. $\square$

Remark 5 The following is the main obstacle we face when trying to generalize Lemma 2.11 to the infinite case: if $\mathbb{K}$ is infinite then $\dim(\Sigma)$ might be infinite; when $\dim(\Sigma)$ is infinite, it can happen that $H^\perp = \emptyset$. On the other hand, it is likely that we can safely replace the hypothesis that $|\mathbb{K}| < \infty$ with the weaker hypothesis that $\dim(\Sigma) < \infty$.

By combining Lemmas 2.10 and 2.12 with Proposition 2.3 we immediately obtain the following:

Theorem 2.13 *Let $\mathscr{S}$ be embeddable of rank $n > 2$ and defined over a division ring different from $\mathbb{F}_2$. Then $\mathscr{S}$ admits ovoidal maximal subspaces.*

2.3 The non-embeddable case

Throughout this subsection $\mathscr{S}$ is a non-embeddable thick-lined polar space of rank $n \geq 3$. As recalled in Section 2.2.1, we have $n = 3$ and the following are the only possibilities for $\mathscr{S}$:

(1) $\mathscr{S}$ is the line-grassmannian of $\Sigma = \mathrm{PG}(3, \mathbb{K})$ with $\mathbb{K}$ a non-commutative division ring; explicitly, the points and the lines of $\mathscr{S}$ are the lines and the full planar line pencils of Σ respectively. In this case $\mathscr{S}$ is not thick; explicitly, every line of $\mathscr{S}$ belongs to just two planes (generators) of $\mathscr{S}$.

(2) $\mathscr{S}$ a Freudenthal-Tits polar space, namely a polar space as defined in Tits [19, Chapter 9] (see also Mühlherr [13] and De Bruyn and Van Maldeghem [8]). In this case $\mathscr{S}$ is thick. Its planes (generators) are Moufang but non-desarguesian.

We shall prove that $\mathscr{S}$ admits ovoidal maximal subspaces. We will obtain this result as a by-product of a classification of all subspaces of $\mathscr{S}$.

2.3.1 The non-thick case

Let $\mathscr{S}$ be the line-grassmannian of $\Sigma = \mathrm{PG}(3, \mathbb{K})$, with $\mathbb{K}$ non-commutative.

Lemma 2.14 *Let $\mathscr{S}'$ be a proper subspace of $\mathscr{S}$. Then one of the following occurs.*

(i) *$\mathscr{S}'$ is a partial ovoid;*

(ii) *$\mathscr{S}'$ consists of a set of lines through some point x which form a partial ovoid in the residue at x;*

(iii) *$\mathscr{S}' = \{p, q\}^{\perp}$ for two non-collinear points;*

(iv) *$\mathscr{S}'$ is either a plane or the union of two planes through a given line;*

(v) *$\mathscr{S}'$ is a singular hyperplane, namely $\mathscr{S}' = p^{\perp}$ for some point p.*

Proof. Viewing $\mathscr{S}'$ as a set of lines of Σ, the property of being a subspace of $\mathscr{S}$ corresponds to the following: if $\mathscr{S}'$ contains two intersecting lines then it contains all lines of the planar pencil containing those two lines. The above cases (i)-(v) can be rephrased as follows:

(i) $\mathscr{S}'$ is a partial spread of Σ.

(ii) All members of $\mathscr{S}'$ meet a fixed line L of Σ, if a point $x \in L$ is contained in at least two members of $\mathscr{S}'$ then the members of $\mathscr{S}'$ through x form a full line pencil in a plane $\pi_x \supset L$ and $\pi_x \neq \pi_y$ for distinct points $x, y \in L$ for which π_x and π_y are defined; dually, if a plane $\pi \supset L$ contains at least two members of $\mathscr{S}'$ then the lines of $\mathscr{S}'$ in π form a pencil with center on L and no two such pencils have the same center.

(iii) There are two skew lines L_1, L_2 of Σ such that $\mathscr{S}'$ is the set of lines of Σ intersecting both L_1 and L_2.

(iv) The members of $\mathscr{S}'$ are all lines of a plane of Σ, all lines through a point of Σ or all lines of some plane π of Σ together with all lines through some point $x \in \pi$.

(v) The members of $\mathscr{S}'$ are the lines meeting a fixed line L (including L).

If $\mathscr{S}'$ does not contain intersecting members then clearly (i) holds. Suppose that $\mathscr{S}'$ does contain intersecting members, but not all lines of a plane of Σ and not all lines through a point of Σ. If all line pencils contained in $\mathscr{S}'$ contain the same line L, then $\mathscr{S}'$ is as in (ii).

So we may assume that $\mathscr{S}'$ contains two disjoint line pencils, say with vertex x_i and plane π_i, $i = 1, 2$. Our assumption implies that the line $L_1 := \langle x_1, x_2\rangle$ is not contained in $\pi_1 \cup \pi_2$. Set $L_2 = \pi_1 \cap \pi_2$. Clearly, L_1 and L_2 are skew, for each point $y \in L_2$ the planar line pencil with vertex y and plane $\langle y, L_1\rangle$ is contained in $\mathscr{S}'$ and for each point $x \in L_1$ the planar line pencil with vertex x and plane $\langle x, L_2\rangle$ is contained in $\mathscr{S}'$. So $\mathscr{S}'$ contains all lines intersecting both L_1 and L_2. If $\mathscr{S}'$ does not contain any additional line, then we have case (iii).

Suppose now that $\mathscr{S}'$ contains a line L not intersecting both L_1 and L_2. If L intersects L_1 in a point, then by considering the plane $\langle L, L_1\rangle$, we readily deduce that also L_1 belongs to $\mathscr{S}'$; if $L = L_1$, then similarly we readily deduce that every line intersecting L_1 belongs to $\mathscr{S}'$. But this contradicts our assumption that $\mathscr{S}'$ does not contain all lines of any plane of Σ. Hence L is disjoint from Ł$_1 \cup L_2$. We claim that every point x of Σ is the vertex of a unique planar line pencil contained in $\mathscr{S}'$. It is certainly contained in at most one such line pencil by assumption, so we only need to show it is contained in at least one.

Suppose first that $x \in L$. Hence $x \notin L_1 \cup L_2$ and there is a unique line L' through x meeting both L_1 and L_2. This line belongs to $\mathscr{S}'$. Consequently, the pencil containing L and L' is contained in $\mathscr{S}'$.

Assuming that $x \notin L$, set $\pi = \langle x, L\rangle$ and $x_i = \pi \cap L_i$, $i = 1, 2$. Suppose first that x, x_1, x_2 are not collinear. Then $\mathscr{S}'$ contains the planar line pencil with vertex $y := \langle x_1, x_2\rangle \cap L$ and plane π, so $\langle x, y\rangle$ is a member of $\mathscr{S}'$. But also the unique line through x intersecting both L_1 and L_2 belongs to $\mathscr{S}'$ and the claim follows in this case. If x, x_1, x_2 are collinear, then we can find a line $M \neq \langle x_1, x_2\rangle$ in π through x containing two points u_1, u_2 which are vertices of planar line pencils not containing M, with respective planes α_1 and α_2; a previous argument then shows that $\mathscr{S}'$ contains all lines intersecting both M and $\alpha_1 \cap \alpha_2$ and the claim follows also in this case.

Hence the members of $\mathscr{S}'$ form the line set of a generalized quadrangle $\mathscr{Q}$ fully embedded in Σ such that each point of Σ is also a point of $\mathscr{Q}$. This property forces $\mathscr{Q}$ to be a symplectic generalized quadrangle. Accordingly, the underlying division ring $\mathbb{K}$ of Σ is commutative, a contradiction.

Hence we may assume that $\mathscr{S}'$ contains all lines of a certain plane π or all lines through a certain point x. If $\mathscr{S}'$ is not as in (iv), then, up to duality of Σ, we may assume that $\mathscr{S}'$ contains all lines of π and additionally two line L_1, L_2 intersecting π in distinct respective points x_1, x_2. Then clearly all lines of Σ through x_1 and all those through x_2 are contained in $\mathscr{S}'$. For an arbitrary line M intersecting $L := \langle x_1, x_2\rangle$, we choose an arbitrary point $y \in M \setminus (L \cap M)$. Then $\langle y, x_i\rangle \in \mathscr{S}'$ for $i = 1, 2$ implies $M \in \mathscr{S}'$. Hence all lines meeting L belong to $\mathscr{S}'$. If no further lines belong to $\mathscr{S}'$, then (v) holds.

If $N \in \mathscr{S}'$ does not meet L, then we quickly deduce that all lines meeting N are contained in $\mathscr{S}'$. Now, since for an arbitrary point $z \notin L \cup N$, the line pencils with vertex z and respective planes $\langle z, L\rangle$ and $\langle z, N\rangle$ are contained in $\mathscr{S}'$, all lines through z are contained in $\mathscr{S}'$ and so $\mathscr{S}' = \mathscr{S}$. □

We now define a partial ovoid O' of $\mathscr{S}$, which amounts to defining a partial spread of Σ, as follows. Let L_1, L_2 be two disjoint lines of Σ and $\beta : L_1 \to L_2$ an arbitrary bijection. Choose $r \in L_1$ arbitrarily. Choose a line M_1 through r in the plane $\langle L_1, \beta(r)\rangle$ distinct from both L_1 and $\langle r, \beta(r)\rangle$. Similarly, choose a line M_2 through $\beta(r)$ in the plane $\langle L_2, r\rangle$ distinct from both L_2 and $\langle r, \beta(r)\rangle$. Then O' consists of M_1, M_2 and every line $K_x := \langle x, \beta(x)\rangle$, for $x \in L_1 \setminus \{r\}$.

Let O be a maximal partial ovoid containing O'.

Theorem 2.15 *The maximal partial ovoid O is a maximal subspace of $\mathscr{S}$.*

Proof. We show that O cannot be properly contained in any of the subspaces listed in Lemma 2.14.

(1) By definition of maximal partial ovoid, O is not contained in a strictly larger partial ovoid. This rules out (i).

(2) If a line L intersects M_1, M_2 and every line K_x, $x \in L_1 \setminus \{r\}$ then $L \notin \{L_1, L_2\}$. It is then readily deduced that L is disjoint from $L_1 \cup L_2$ and the line through r intersecting L and L_2 intersects L_2 in $\beta(r)$. Hence none of M_1 and M_2 intersects L, a contradiction. This implies that no line of Σ intersects every member of O. This rules out (ii), (iii) and (v). In case (iv), picking a line in π through x also shows that every member of O should intersect a fixed line. □

2.3.2 The thick case

Let $\mathscr{S}$ be a Freudenthal-Tits polar space. Then there is an octonion division algebra $\mathbb{O}$ over some field $\mathbb{K}$ coordinatizing the planes of $\mathscr{S}$ (see [19, Chapter 9]). From the description given in [8] it easily follows that, for two opposite points p_1, p_2, the set of points $\{p_1, p_2\}^{\perp\perp}$ is an infinite set containing p_1, p_2 and determined by any pair of its points. We call such a set a *hyperbolic line.*

We could again give a complete classification of the subspaces of $\mathscr{S}$ (see Remark 6), but the following 'quasi-classification' is sufficient for our purposes.

Lemma 2.16 *Let $\mathscr{S}'$ be a proper subspace of $\mathscr{S}$. Then one of the following holds*

(i) *$\mathscr{S}'$ is a partial ovoid of $\mathscr{S}$;*

(ii) *$\mathscr{S}'$ consists of a set of lines through some point x which form a partial ovoid in the residue at x;*

(iii) *$\mathscr{S}'$ is a (non-degenerate) generalized quadrangle closed under taking hyperbolic lines;*

(iv) *$\mathscr{S}'$ is a set of planes through some fixed line;*

(v) *$\mathscr{S}' = p^\perp$ for some point p.*

Proof. If $\mathscr{S}'$ does not contain lines then we clearly have (i). Suppose now $\mathscr{S}'$ contains lines but no planes. If all such lines contain a common point then we have (ii). So suppose that $\mathscr{S}'$ contains two non-intersecting lines L_1, L_2. If these are not opposite, then $\mathscr{S}'$ contains a plane, a contradiction. Hence L_1 and L_2 are opposite. Let p_1 and p_2 be two opposite points with

$(L_1 \cup L_2) \subseteq \{p_1, p_2\}^\perp =: U$ (we obtain p_i by choosing two planes through L_1 and projecting L_2 onto these planes). Now U is a generalized quadrangle which, according to Proposition 5.9.4 of [20], does not contain proper full subquadrangles. Hence $U = \langle L_1, L_2 \rangle$ and therefore $U \subseteq \mathscr{S}'$. Also, since by definition $\{x_1, x_2\}^{\perp\perp}$ is contained in $\{p_1, p_2\}^\perp$ for each pair of opposite points $x_1, x_2 \in U$, the subspace U and hence $\mathscr{S}'$ is closed under taking hyperbolic lines. This is (iii).

So we may assume that $\mathscr{S}'$ contains planes. If all planes contain a common point x, and not a common line, then we have two opposite lines in the residue of x, which implies that $\mathscr{S}'$ is the full residue at x since the residual quadrangle does not have proper full subquadrangles. Hence (iv) or (v) arises.

Suppose that $\mathscr{S}'$ contains two opposite planes. The construction in [8] reveals the following property of $\mathscr{S}$. Restricting $\mathbb{O}$ in the construction to a quaternion subalgebra over $\mathbb{K}$, we obtain a sub polar space $\mathscr{S}_0$ of $\mathscr{S}$, which is not a subspace, but with the property that every plane of $\mathscr{S}$ through any line of $\mathscr{S}_0$ is also a plane of $\mathscr{S}_0$. We refer to the latter property as *idealness*. Now two opposite planes can always be included in such a sub polar space $\mathscr{S}_0$. Moreover, $\mathscr{S}_0$ lives in a projective 5-space. Therefore, by Proposition 2.2, every pair of opposite planes of $\mathscr{S}_0$ spans $\mathscr{S}_0$. Consequently $\mathscr{S}_0$ is a sub polar space of $\langle X_0 \cup Y_0 \rangle_{\mathscr{S}}$ for any two mutually disjoint planes X_0, Y_0 of $\mathscr{S}_0$. On the other hand, every pair of opposite planes of $\mathscr{S}$ can be regarded as a pair of opposite planes of a suitable copy of $\mathscr{S}_0$. Since we have assumed that $\mathscr{S}'$ contains two opposite planes, we can also assume that $\mathscr{S}_0$ is a sub polar space of $\mathscr{S}'$. Since $\mathscr{S}_0$ is ideal as a sub polar space, $\mathscr{S}'$ is ideal as a polar subspace. This forces $\mathscr{S}' = \mathscr{S}$. Indeed, for a point p of $\mathscr{S} \setminus \mathscr{S}'$, if any such point exists, select a plane π in $\mathscr{S}'$ and let $\pi' = \langle p, p^\perp \cap \pi \rangle$. The line $p^\perp \cap \pi$ belongs to $\mathscr{S}'$. Hence π' is a plane of $\mathscr{S}'$, by idealness of $\mathscr{S}'$. Accordingly, $p \in \mathscr{S}'$. □

Now we choose two opposite points x_1, x_2 in $\mathscr{S}$ and an infinite ovoidal maximal subspace O' of $U := \{x_1, x_2\}^\perp$ as constructed in Theorem 2.1. Hence

$$O' \supseteq ((y_1^\perp \cap y_2^\perp \cap U) \setminus \{r\}) \cup \{u_1, u_2\}$$

where y_1, y_2 are opposite points of U, $r \in y_1^\perp \cap y_2^\perp \cap U$ and $u_i \in \langle y_i, r \rangle \setminus \{y_i, r\}$ for $i = 1, 2$. Notice that $y_1^\perp \cap y_2^\perp \cap U$ is a hyperbolic line. Indeed let z_1 and z_2 be any two distinct points of $\{y_1, y_2\}^\perp \cap U$, necessarily opposite since $\{y_1, y_2\}^\perp \cap U$ consists of mutually opposite points. Put $U' := \{z_1, z_2\}^\perp$. Then $U'^\perp = \{z_1, z_2\}^{\perp\perp}$ is a hyperbolic line. Moreover, any two opposite lines of U' span U'. In particular, U' is spanned by the lines $\langle x_1, y_1 \rangle$ and $\langle x_2, y_2 \rangle$. Therefore $U'^\perp = \{x_1, x_2, y_1, y_2\}^\perp = U \cap \{y_1, y_2\}^\perp$, namely $U \cap \{y_1, y_2\}^\perp = \{z_1, z_2\}^{\perp\perp}$.

Choose $r' \in (y_1^\perp \cap y_2^\perp \cap U) \setminus \{r\}$ and select points $w_i \in \langle x_i, r' \rangle \setminus \{x_i, r'\}$, $i = 1, 2$. Then define $O'' = (O' \setminus \{r'\}) \cup \{w_1, w_2\}$. Let O be a maximal partial ovoid of $\mathscr{S}$ containing O''.

Theorem 2.17 *The maximal partial ovoid O is a maximal subspace of $\mathscr{S}$.*

Proof. We first show that no point of $\mathscr{S}$ is collinear to all points of O''. Indeed, all points collinear to $(y_1^\perp \cap y_2^\perp \cap U) \setminus \{r, r'\}$ constitute a generalized quadrangle $\mathscr{Q}$ contained in $\{v_1, v_2\}^\perp$, for two distinct points $v_1, v_2 \in (y_1^\perp \cap y_2^\perp \cap U) \setminus \{r, r'\}$. However $\{v_1, v_2\}^\perp$ admits no proper full subquadrangle. Hence $\mathscr{Q} = \{v_1, v_2\}^\perp$ and any two opposite lines of $\mathscr{Q}$ span $\mathscr{Q}$. In particular, $\mathscr{Q}$ is spanned by the opposite lines $\langle x_1, y_1 \rangle$ and $\langle x_2, y_2 \rangle$. Moreover, $\mathscr{Q} = \{r, r'\}^\perp$, since $r, r' \in \{x_1, x_2, y_1, y_2\}^\perp$ and $\mathscr{Q} = \langle x_1, y_1, x_2, y_2 \rangle$.

If a point z is collinear to all points of O'', then it belongs to $\mathscr{Q}$. However $\mathscr{Q} = \{r, r'\}^\perp$. Hence $z \perp r, r'$. On the other hand, z is also collinear to u_1, u_2, w_1, w_2, hence to x_1, x_2, y_1, y_2. This is impossible within $\mathscr{Q}$.

Hence a subspace containing lines and containing O cannot be of any of the types (i), (ii), (iv) or (v) of Lemma 2.16. So suppose O is contained in a subspace $\mathscr{S}'$ isomorphic to a generalized quadrangle and closed under taking hyperbolic lines, as in (iii) of Lemma 2.16. Then, for $z_1, z_2 \in O$, we deduce $y_1^\perp \cap y_2^\perp \cap U \subseteq \mathscr{S}'$. Hence $r \in \mathscr{S}'$, implying $y_i \in \langle r, u_i \rangle \subseteq \mathscr{S}'$, $i = 1, 2$. Also, $r' \in \mathscr{S}'$ implying $x_i \in \langle r', w_i \rangle \subseteq \mathscr{S}'$, $i = 1, 2$. Hence the two opposite planes $\langle x_1, y_1, z_1 \rangle$ and $\langle x_2, y_2, z_2 \rangle$ belong to $\mathscr{S}'$, and so $\mathscr{S}'$ coincides with $\mathscr{S}$. We have proved that O is indeed a maximal subspace. □

Remark 6 Actually, in case (iii) of Lemma 2.16 we have $\mathscr{S}' = \{p, q\}^\perp$ for two opposite points p and q of $\mathscr{S}$. This can be proved by dimension arguments and a little more work, but we are not going into the details of that proof here.

With (iii) stated in this sharper way, Lemma 2.16 yields a complete classification of all non-ovoidal subspaces of $\mathscr{S}$, which can be summarized as follows: $\mathscr{S}$ admits only those subspaces that exists in any polar space, namely those of reduced rank at most 1 and intersections of singular hyperplanes. As a consequence, the maximal non-ovoidal subspaces of $\mathscr{S}$ are precisely its singular hyperplanes. Exactly the same holds in the non-embeddable non-thick case (Lemma 2.14). This also improves a result of Cohen and Shult [5], according to wich all hyperplanes of a non-embeddable thick-lined polar space of rank 3 are singular.

2.4 The case $\mathbb{K} = \mathbb{F}_2$

In order to finish the proof of Theorem 1, the case where $\mathscr{S}$ is a quadric defined over $\mathbb{F}_2$ remains to be examined.

Theorem 2.18 *Let $\mathscr{S}$ be a non-degenerate quadric of rank $n \geq 2$, defined over $\mathbb{F}_2$. Then $\mathscr{S}$ admits ovoidal maximal subspaces precisely in the following cases:*

(1) $\mathscr{S} = Q_{2n}(2)$ *with either $n = 2$ or n odd;*
(2) $\mathscr{S} = Q^+_{2n-1}(2)$ *with either $n = 2$ or $n \equiv 0, 1 \pmod 4$;*
(3) $\mathscr{S} = Q^-_{2n+1}(2)$ *with $n \equiv 1, 2 \pmod 4$.*

Proof. The quadric $\mathscr{S}$ lives in $\mathrm{PG}(N-1, 2)$, with $N \in \{2n, 2n+1, 2n+2\}$. Since it is well known that $Q_4(2)$ and $Q^+_3(2)$ admit ovoids while $Q^-_5(2)$ admits no ovoids, we can assume that $N > 5$ and, in view of Propositions 2.3 and 2.5, the existence of an ovoidal maximal subspace of $\mathscr{S}$ is equivalent to the existence of a totally scattered partial ovoid of $\mathscr{S}$, which in turn is equivalent to the existence of a basis $E = (\mathbf{e}_1, ..., \mathbf{e}_N)$ of V such that, if ϕ is the quadratic form giving rise to $\mathscr{S}$ and f its bilinearization, we have $\phi(\mathbf{e}_i) = 0$ for $i = 1, 2, ..., N$ and $f(\mathbf{e}_i, \mathbf{e}_j) = 1$ for $1 \leq i < j \leq N$. Equivalently, ϕ admits the following expression with respect to E:

$$\phi(x_1 x_2, ..., x_N) \; = \; \sum_{i<j} x_i x_j. \qquad (3)$$

So, the representative matrix of f with respect to E is $A = J + I$, where J is the $N \times N$ matrix with all entries equal to 1 and I is the identity matrix of order N.

Assume firstly $N = 2n+1$. Then A has rank $2n = N-1$ with kernel $\{\mathbf{0}, \mathbf{r}\}$, $\mathbf{r} = \sum_i \mathbf{e}_i$. Clearly, $\phi(\mathbf{r}) = \binom{N}{2} = n(2n+1)$ (computed modulo 2). However, ϕ is non degenerate if and only if $\phi(\mathbf{r}) = 1$; equivalently, n is odd. So, as ϕ is non-degenerate by assumption, ϕ can be expressed as in (3) if and only if n is odd. This proves the claim of Theorem 2.18 for N odd.

Suppose that N is even, say $N = 2m$. Hence A is non-singular. Define

$$\begin{array}{lcl} \phi_m(x_1, ..., x_{2m}) & := & \sum_{1\leq i<j\leq 2m} x_i x_j \quad \text{(see (3))}; \\ \psi_m(x_1, ..., x_{2m}) & := & \phi_m(x_1, ..., x_{2m}) + \sum_{i=1}^{2m} x_i^2. \end{array}$$

In these definitions $m = 1$ is allowed. Both ϕ_m and ψ_m are non-degenerate quadratic forms. Let $\perp$ be the orthogonality relation associated with ϕ_m and suppose $m > 1$. Then ϕ_m induces ψ_{m-1} on $\{\mathbf{e}_{2m-1}, \mathbf{e}_{2m}\}^\perp$. As the form induced by ϕ_m on $\langle \mathbf{e}_{2n-1}, \mathbf{e}_{2n}\rangle$ is hyperbolic, ϕ_m and ψ_{m-1} have the same type, namely they are either both hyperbolic or both elliptic.

Suppose $m > 2$. Then ϕ_m induces $\phi_{\lambda,m-2}$ on $\{\mathbf{e}_{2m-3}, \mathbf{e}_{2m-2}, \mathbf{e}_{2m-1}, \mathbf{e}_{2m}\}^\perp$. As ψ_{m-1} induces $\psi_{1,2}(x_{2m-3}, x_{2m-2}) = x_{2m-3}^2 + x_{2m-3}x_{2m-2} + x_{2m-2}^2$ on $\langle \mathbf{e}_{2m-3}, \mathbf{e}_{2m-2}\rangle$, which is elliptic, the types of ϕ_{m-2} and ψ_{m-1} are opposite. Hence ϕ_{m-2} and ϕ_m have opposite types. Clearly, ϕ_1 is hyperbolic. Therefore ϕ_{1+2k} is hyperbolic if k is even and elliptic if k is odd.

Consider now ϕ_2. We know that ϕ_2 and ψ_1 have the same type. However $\psi_1(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2$ is elliptic. Hence ϕ_2 is elliptic. By the above, ϕ_{2k} is elliptic if k is odd and hyperbolic if k is even. The theorem follows from the fact that the existence of an ovoidal maximal subspace of $\mathscr{S}$ is equivalent to ϕ admitting the expression (3), which is precisely the expression called ϕ_m in the above. □

Remark 7 Suppose that the form ϕ defined in (3) is non-degenerate (hence either N is even or $N-1 \equiv 0 \pmod 4$) and let $\mathscr{S}$ be the quadric associated to ϕ. With $\mathbf{e}_1, ..., \mathbf{e}_N$ as in the proof of Theorem 2.18, the set $O := \{[\mathbf{e}_1], ..., [\mathbf{e}_N]\}$ is a partial ovoid of $\mathscr{S}$ and, if it is maximal as a partial ovoid, then it is a maximal subspace of $\mathscr{S}$.

As we shall see in a few lines, O is non-maximal if and only if $N \equiv 0 \pmod 4$. Indeed, for a vector $\mathbf{u} \in V$, we have $\mathbf{e}_k \not\perp \mathbf{u}$ for every $k = 1, 2, ..., N$ only if $\mathbf{u} = \sum_{i=1}^N \mathbf{e}_i$ and N is even. Consequently, when N is odd O is maximal. Let N be even, with $\mathbf{u}$ as above, and suppose that $\phi(\mathbf{u}) = 0$. Then $\binom{N}{2} = 0$ (in $\mathbb{F}_2$). Hence $N \equiv 0 \pmod 4$. So, if $N \equiv 0 \pmod 4$ then O is not maximal and $O \cup \{[\sum_{i=1}^N \mathbf{e}_i]\}$ is the unique maximal partial ovoid containing O (hence it is the unique ovoidal maximal subspace containing O); otherwise, O is maximal.

Note that neither O nor $O \cup \{\mathbf{u}\}$ (when $N \equiv 0 \pmod 4$) are ovoids. Indeed, apart from the fact $Q_4(2), Q_3^+(2), Q_5^+(2)$ and $Q_7^+(2)$ are the only quadrics defined over $\mathbb{F}_2$ which admit ovoids (see Section 4.1), when $N > 5$ both O and $O \cup \{\mathbf{u}\}$ are far too small to be ovoids.

2.5 Conclusions

End of the proof of Theorem 1. By combining Theorems 2.1, 2.13, 2.15, 2.17 and 2.18 we obtain Theorem 1.

Proof of Corollary 2. We could obtain part (1) of Corollary 2 from Lemmas 2.10 and 2.12, but the following elementary argument is enough. It is well known that all non-singular anti-symmetric matrices of $M_N(\mathbb{F})$ (N even) are mutually T-equivalent. So, all we have to do is finding a non-singular, anti-diagonal and anti-symmetric matrix $A = (a_{i,j})_{i,j=1}^N$. Here is one: $a_{i,j} = 1$ if $i < j$, $a_{i,i} = 0$ and $a_{i,j} = -1$ if $i > j$. It is readily seen that, with A defined in this way, $\det(A) = 1$. Claim (1) of Corollary 2 is proved.

Turning to claims (2) and (3), let $\mathscr{S}_M$ be the polar space associated with the appropriate bilinear or hermitian form represented by M with respect to the canonical basis of $V = V(N, q)$. Claims (2) and (3) amount to the following: $\mathscr{S}_M$ contains a basis of $\mathrm{PG}(N-1, q)$ formed by mutually non-orthogonal points. So, assuming that $N > 2$ and $(N, q) \neq (4, 3)$ when M is symmetric, we must prove the following: if $\mathrm{rank}(\mathscr{S}_M) > 1$ then $\mathscr{S}_M$ admits a totally scattered partial ovoid; otherwise $\mathscr{S}_M$ has rank 1 and spans $\mathrm{PG}(N-1, q)$.

When $\mathrm{rank}(\mathscr{S}_M) > 1$, the claim follows from Lemma 2.12. Note that $q = 2$ is forbidden here. Indeed q is odd in claim (2) while $q \geq 4$ in (3). Suppose that $\mathrm{rank}(\mathscr{S}_M) \leq 1$. Note that $\mathscr{S}_M \neq \emptyset$. Indeed $N \geq 3$ in (2), which forces $|\mathscr{S}_M| \geq q+1$, while if M is hermitian then $|\mathscr{S}_M| \geq 1 + \sqrt{q}$. Therefore $\mathscr{S}_M$ has rank 1 and spans $\mathrm{PG}(N-1, q)$. Explicitly, $\mathscr{S}_M$ is either a conic of $\mathrm{PG}(2, q)$, an elliptic quadric of $\mathrm{PG}(3, q)$, a set of $1+\sqrt{q}$ points of $\mathrm{PG}(1, q)$ or a hermitian unital of $\mathrm{PG}(2, q)$.

Remark 8 The hypothesis $(N, \mathbb{F}) \neq (4, \mathbb{F}_3)$ cannot be dropped from claim (2) of Corollary 2. Indeed it is easily checked that every anti-diagonal symmetric matrix M of $M_4(\mathbb{F}_3)$ has determinant equal to -1. Therefore, if $M \in M_4(\mathbb{F}_3)$ is symmetric and anti-diagonal then $\mathscr{S}_M$ is an elliptic quadric of $\mathrm{PG}(3, 3)$. We miss hyperbolic quadrics. This is in conformity with the fact that each ovoid of $Q_3^+(3)$ arises from each embedding.

Remark 9 In claims (2) and (3) of Corollary 2 the hypothesis that $\mathbb{F}$ is finite is not strictly necessary, provided that those claims are rephrased as follows: every isotropic non-singular symmetric (hermitian) matrix of $M_N(\mathbb{F})$ is T-equivalent ($*$-equivalent) to an anti-diagonal matrix, a symmetric or hermitian matrix $M \in M_N(\mathbb{F})$ being called *isotropic* if $\mathscr{S}_M \neq \emptyset$.

3 Constructions

3.1 A construction in symplectic varieties

Let $\mathscr{S}$ be a symplectic polar space of rank $n \geq 2$, namely $\mathscr{S}$ admits an embedding $e : \mathscr{S} \to \mathrm{PG}(V)$ such that $\dim(V) = 2n$ and $e(\mathscr{S})$ is the polar space associated to a non-degenerate alternating form $f : V \times V \to \mathbb{K}$, with $\mathbb{K}$ a commutative division ring.

Let A be a generator of $\mathscr{S}$ and $\{a_1, a_2, ..., a_n\}$ a basis of A. For every $k = 1, 2..., n$, let L_k be a hyperbolic line of $\mathscr{S}$ containing a_k and contained in $\{a_1, ..., a_{k-1}, a_{k+1}, ..., a_n\}^\perp$. Let $\{\mathbf{e}_1, \mathbf{f}_2, ..., \mathbf{e}_n, \mathbf{f}_n\}$ be a basis of V such that $f(\mathbf{e}_i, \mathbf{f}_j) = \delta_{i,j}$ (Kronecker symbol) and $f(\mathbf{e}_i, \mathbf{e}_j) = f(\mathbf{f}_i, \mathbf{f}_j) = 0$ for any choice of $i, j = 1, ..., n$. We can assume that $[\mathbf{e}_i] = e(a_i)$ for $i = 1, 2, ..., n$. Thus, there exist scalars $\lambda_{i,j} \in \mathbb{K}$ such that $e(L_k) = \langle [\mathbf{e}_k], [\mathbf{f}_k + \sum_{i \neq k} \mathbf{e}_i \lambda_{i,k}] \rangle$. With the lines L_k defined as above, for $k \neq h$ we have $L_k \not\perp L_h$ (namely $L_k^\perp \cap L_h = \{a_h\}$) if and only if

$$\lambda_{i,j} \neq \lambda_{j,i}, \quad \text{for any choice of } \; i \neq j. \tag{4}$$

Condition (4) is very easy to satisfy. For instance, we can choose $\lambda_{i,j}$ arbitrarily for $i \leq j$ and put $\lambda_{j,i} = \lambda_{i,j} + 1$ for $i < j$. Suppose to have chosen $L_1, L_2, ..., L_n$ in such a way that (4) holds and let a be a point in $A \setminus \cup_{k=1}^{n} \langle a_i \rangle_{i \neq k}$, namely $e(a) = [\sum_{i=1}^{n} \mathbf{e}_i \mu_i]$ for $\mu_1, \mu_2, ..., \mu_n \in \mathbb{K}^*$. Put $O' := \bigcup_{k=1}^{n} (L_k \setminus \{a_k\})$ and $O := O' \cup \{a\}$.

Lemma 3.1 *The set O is a maximal partial ovoid of $\mathscr{S}$.*

Proof. We firstly prove that O is a partial ovoid. No two points of the set $\bigcup_{k=1}^{n}(L_k \setminus \{a_k\})$ are orthogonal, since $L_1, ..., L_n$ are hyperbolic lines and if $k \neq h$ then $L_k^{\perp} \cap L_h = \{a_h\}$. It remains to prove that $a^{\perp} \cap L_k = \{a_k\}$. Suppose the contrary. Then $L_k \subseteq a^{\perp}$. However $L_k \perp a_i$ for every $i \neq k$. Moreover, the set $\{a_1, ..., a_{k-1}, a_{k+1}, ..., a_n, a\}$ spans A. It follows that $L_k \subseteq A^{\perp} = A$, a contradiction.

Maximality remains to be proved. Let b be a point exterior to O. We shall prove that $b^{\perp} \cap O \neq \emptyset$. We have $b^{\perp} \cap L_k \neq \emptyset$ for every k. If $b^{\perp}$ contains a point of $L_k \setminus \{a_k\}$ then we are done. So, suppose that $b^{\perp} \cap L_k = \{a_k\}$ for every k. Then $b \in A$. Accordingly, $b \perp a \in O$. $\square$

As $O' \subset O$, the set O' is also a partial ovoid, but not a maximal one. However,

Lemma 3.2 *Let $X \subseteq O'$ be such that $|X \cap L_k| \geq 2$ for every $k = 1, 2, ..., n$. Then $e(X)$ spans $\mathrm{PG}(V)$. In particular, $e(O')$ spans $\mathrm{PG}(V)$.*

Proof. With X as in the hypotheses of the lemma, $\langle e(X) \rangle$ contains the line $e(L_k)$ for every $k = 1, 2, ..., n$. Accordingly, $\langle e(X) \rangle$ contains $[\mathbf{e}_k]$ and $[\mathbf{f}_k]$ for every k. However V is spanned by $\mathbf{e}_1, ..., \mathbf{e}_n, \mathbf{f}_1, ..., \mathbf{f}_n$. Therefore $\langle e(X) \rangle = \mathrm{PG}(V)$. $\square$.

Let $\tilde{e} : \mathscr{S} \to \mathrm{PG}(\widetilde{V})$ be the universal embedding of $\mathscr{S}$. If $\mathrm{char}(\mathbb{K}) \neq 2$ then $\tilde{e} = e$ (hence $\widetilde{V} = V$). In this case $\tilde{e}(O')$ spans $\mathrm{PG}(\widetilde{V})$, by Lemma 3.2 (with the terminology in Section 2, the partial ovod O' is totally scattered). On the other hand, when $\mathrm{char}(\mathbb{K}) = 2$ the embedding e is a proper quotient of $\tilde{e}$. In this case we cannot use Lemma 3.2 to obtain that $\langle \tilde{e}(O') \rangle = \mathrm{PG}(\widetilde{V})$. In fact $\langle \tilde{e}(O') \rangle \subset \mathrm{PG}(\widetilde{V})$ when $\mathbb{K} = \mathbb{F}_2$. Nevertheless:

Lemma 3.3 *Suppose that* $\mathrm{char}(\mathbb{K}) = 2$ *but* $\mathbb{K} \neq \mathbb{F}_2$. *Then* $\tilde{e}(O')$ *spans* $\mathrm{PG}(\widetilde{V})$.

Proof. Under the hypotheses of the lemma, $\tilde{e}(\mathscr{S})$ is the quadric associated to a non-degenerate quadratic form $\phi : \widetilde{V} \to \mathbb{K}$ and $\langle \tilde{e}(O') \rangle = \mathrm{PG}(V')$ for a subspace V' of $\widetilde{V}$. We shall prove that $V' = \widetilde{V}$.

Let ϕ' be the form induced by ϕ on V' and $\mathscr{S}'$ the quadric defined by ϕ' in $\mathrm{PG}(V')$. As $|\mathbb{K}| > 2$, for every $k = 1, 2, ..., n$ the sets $\tilde{e}(L_k)$ and $\tilde{e}(L_k \setminus \{a_k\})$ span the same subspace X_k of $\mathrm{PG}(\widetilde{V})$. Hence $\mathrm{PG}(V') \supseteq \cup_{i=1}^{n} X_i$. Moreover $\tilde{e}(a_k) \in \mathrm{PG}(V')$ for every $k = 1, 2, ..., n$, as $a_k \in L_k$. It follows that $\mathscr{S}'$ contains $\tilde{e}(A)$, which is a generator of $\tilde{e}(\mathscr{S})$. However no point of A is collinear with all points of O, by construction. Therefore $\mathscr{S}'$ is non-degenerate of rank n. Turning back to X_k, its codimension in $\mathrm{PG}(\widetilde{V})$ is equal to $2n - 2$. Indeed $X_k = \langle \tilde{e}(L_k) \rangle$ and $\tilde{e}(L_k) = A_k^{\perp} \cap B_k^{\perp}$ (in $\tilde{e}(\mathscr{S})$) for any two submaximal singular subspaces A_k and B_k of $\tilde{e}(\mathscr{S})$ such that $A_k^{\perp} \cap B_k^{\perp} = \emptyset$ and $|A_k^{\perp} \cap B_k^{\perp} \cap \tilde{e}(L_k)| > 1$. Since $\mathscr{S}'$ is non-degenerate and has the same rank as $\tilde{e}(\mathscr{S})$, two singular subspaces A_k and B_k as required can always be chosen in $\mathscr{S}'$. Hence the equality $\tilde{e}(L_k) = A_k^{\perp} \cap B_k^{\perp}$ holds true in $\mathscr{S}'$, too. Consequently, X_k has codimension $2n - 2$ in $\mathrm{PG}(V')$ as well as in $\mathrm{PG}(\widetilde{V})$. This forces $V' = \widetilde{V}$. $\square$

Lemma 3.4 *If $\mathbb{K} = \mathbb{F}_2$ and n is odd then $\tilde{e}(O)$ spans $\mathrm{PG}(\widetilde{V})$.*

Proof. Suppose that $\mathbb{K} = \mathbb{F}_2$. Let ϕ be the quadratic form of $\widetilde{V}$ associated to $\tilde{e}(\mathscr{S})$. An ordered basis $(\mathbf{u}_1, \mathbf{v}_1, ..., \mathbf{u}_n, \mathbf{v}_n, \mathbf{w})$ can be chosen in $\widetilde{V}$ such that ϕ admits the following expression with respect to it:

$$\phi(x_1, y_1, ..., x_n, y_n, z) \;=\; \sum_{i=1}^{n} x_i y_i + z^2.$$

With no loss, we can assume to have chosen the vectors $\mathbf{u}_k$ in such a way that $[\mathbf{u}_k] = \tilde{e}(a_k)$ for every $k = 1, 2, ..., n$ and

$$\tilde{e}(L_k) \;=\; \{[\mathbf{u}_k],\ [\mathbf{v}_k + \sum_{i \neq k} \mathbf{u}_i \lambda_{i,k}],\ [\mathbf{u}_k + \mathbf{v}_k + \sum_{i \neq k} \mathbf{u}_i \lambda_{i,k} + \mathbf{w}]\}$$

for suitable scalars $\lambda_{i,j}$ such that $\lambda_{k,h} \neq \lambda_{k,h}$ when $k \neq h$ (see (4)). Since $[\mathbf{u}_k] = \tilde{e}(a_k)$ for every k, we also have $\tilde{e}(a) = [\sum_{k=1}^{n} \mathbf{u}_k]$.

Let X be the span of $\tilde{e}(O)$ in $\mathrm{PG}(\widetilde{V})$. Then X contains both $[\mathbf{v}_k + \sum_{i \neq k} \mathbf{u}_i \lambda_{i,k}]$ and $[\mathbf{u}_k + \mathbf{v}_k + \sum_{i \neq k} \mathbf{u}_i \lambda_{i,k} + \mathbf{w}]$, for every $k = 1, 2, ..., n$. Hence X also contains $[\mathbf{u}_k + \mathbf{w}]$. Consequently $[\mathbf{u}_k + \mathbf{u}_h] \in X$ for any choice of $k \neq h$. However X also contains $\tilde{e}(a) = [\sum_{k=1}^{n} \mathbf{u}_k]$ and n is odd, by assumption. Therefore $[\mathbf{u}_k] \in X$ for any k. Hence $\mathbf{w} \in X$ and $\mathbf{v}_k \in X$ for any k. So, X contains all of $[\mathbf{u}_1], ..., [\mathbf{u}_n], [\mathbf{v}_1], ..., [\mathbf{v}_n]$ and $[\mathbf{w}]$. In short, $X = \mathrm{PG}(\widetilde{V})$. $\square$

Theorem 3.5 *The set O is an ovoidal maximal subspace of $\mathscr{S}$, except precisely when $\mathbb{K} = \mathbb{F}_2$ and n is even and different from 2.*

Proof. In view of Lemmas 3.3 and 3.4 and Propositions 2.3 and 2.5, if either $\mathbb{K} \neq \mathbb{F}_2$ or n is odd then O is a maximal subspace of $\mathscr{S}$. If $\mathbb{K} = \mathbb{F}_2$ and $n = 2$ then $|O| = 5$, which is just the size of an ovoid of $\mathscr{S}$. Hence O is an ovoid. Finally, when $\mathbb{K} = \mathbb{F}_2$ and $n > 2$ is even then $\mathscr{S}$ admits no ovoidal maximal subspace, by the isomorphism $W_{2n-1}(2) \cong O_{2n}(2)$ and Theorem 1. $\square$

Remark 10 If $\mathbb{K} \neq \mathbb{F}_2$ then O is not an ovoid. Indeed, let $|\mathbb{K}| > 2$. Suppose firstly that for every $k = 1, ..., n$ there exists $i \neq k$ such that $\lambda_{i,k} \neq 0$. Then $B := \langle [\mathbf{f}_1], ..., [\mathbf{f}_n] \rangle$ is a generator of $\mathscr{S}$ but $O \cap B = \emptyset$. On the other hand, let $\lambda_{i,k} = 0$ for some k and every $i \neq k$, say $\lambda_{i,1} = 0$ for every $i > 1$. Then $\lambda_{1,2} \neq 0$ by (4). Given $t \in \mathbb{K} \setminus \{0, \lambda_{1,2}\}$ ($\neq \emptyset$ since $|\mathbb{K}| > 2$), put $\mathbf{f}'_1 := \mathbf{e}_2 t + \mathbf{f}_1$ and $\mathbf{f}'_2 := \mathbf{e}_1 t + \mathbf{f}_2$. Then $B' := \langle [\mathbf{f}'_1], [\mathbf{f}'_2], [\mathbf{f}_3], ..., [\mathbf{f}_n] \rangle$ is a generator of $\mathscr{S}$ and $B' \cap O = \emptyset$. In any case, O is not an ovoid.

Finally, let $\mathbb{K} = \mathbb{F}_2$. Then $|O| = 2n + 1$ while an ovoid of $W_{2n-1}(2)$, if it exists, has size $2^n + 1$. As $2n + 1 < 2^n + 1$ if $n > 2$, O is an ovoid if and only if $n = 2$. (Anyway, it is well known that no ovoids exist in $W_{2n-1}(2)$ when $n > 2$; see also Section 4.1.)

3.2 More constructions

Throughout this subsection $\mathscr{S}$ is the polar space associated to a non-degenerate $(\sigma, 1)$-pseudoquadratic form $\phi : V \to \mathbb{K}/\mathbb{K}_{\sigma,1}$ of finite Witt index $n \geq 2$, where $\mathbb{K}$ is a (possibly non-commutative) division ring, V is a $\mathbb{K}$-vector space, σ is an involutory anti-automorphism of $\mathbb{K}$ and $\mathbb{K}_{\sigma,1} :=$

$\{t - t^\sigma\}_{t\in\mathbb{K}}$. So, $\mathscr{S}$ is regarded as a subgeometry of $\Sigma := \mathrm{PG}(V)$ and the inclusion mapping of $\mathscr{S}$ into Σ provides the universal embedding of $\mathscr{S}$ except when $\dim(V) = 4$, $|\mathbb{K}| > 3$ and $\mathscr{S}$ is a hyperbolic quadric or a quadrangle as in [19, 8.6(II)(a)].

We shall show how to construct a totally scattered partial ovoid of $\mathscr{S}$, namely a partial ovoid O such that $\langle O\rangle_\Sigma = \Sigma$. We will only describe the constructions. The verifications that they indeed hit the target will be left to the reader.

We put $N := \dim(V) \geq 2n$. For ease of exposition, we assume that $N < \infty$, but everything we are going to say also holds when N is infinite.

3.2.1 The orthogonal case with $N = 2n$

Assume that $\sigma = \mathrm{id}_\mathbb{K}$ (hence $\mathbb{K}$ is commutative) and $N = 2n$. Then ϕ can be expressed as follows with respect to a suitable basis $(\mathbf{u}_1, \mathbf{v}_1, ..., \mathbf{u}_n, \mathbf{v}_n)$ of V:

$$\phi(x_1, y_1, ..., x_n, y_n) \;=\; \sum_{i=1}^{n} x_i y_i.$$

Suppose that $\mathrm{char}(\mathbb{K}) \neq 2$. Let $L = (\lambda_{i,j})_{i,j=2}^n$ be an $(n-1)\times(n-1)$ anti-symmetric matrix. For every $k = 2, 3, ..., n$, put $I_k = \{2, 3, ..., n\} \setminus \{k\}$ and

$$\mathbf{x}_k(t) \;=\; -\mathbf{u}_k t^2 + \sum_{i\in I_k} \mathbf{u}_i\lambda_{i,k} + \mathbf{v}_k + (\mathbf{u}_1 + \mathbf{v}_1)t.$$

Put $X_k := \{\mathbf{x}_k(t)\}_{t\in\mathbb{K}^*}$. The set $O := \{\mathbf{u}_1\} \cup (\cup_{k=2}^n X_k)$ is a partial ovoid. Moreover, if $|\mathbb{K}| > 3$ then $\langle O\rangle_\Sigma$ contains $[\mathbf{u}_1], ..., [\mathbf{u}_n], [\mathbf{v}_1], ..., [\mathbf{v}_n]$. Hence $\langle O\rangle_\Sigma = \Sigma$.

The same construction applies when $\mathrm{char}(\mathbb{K}) = 2$ but now choosing $L = (\lambda_{i,j})_{i,j=2}^n$ in such a way that $\lambda_{i,j} + \lambda_{j,i} \neq 0$ for any choice of $i \neq j$ instead of $\lambda_i + \lambda_j = 0$.

Proposition 3.6 *If $|\mathbb{K}| > 3$ then the partial ovoid O is totally scattered.*

Remark 11 The partial ovoid O is not maximal. Indeed $O \cup \{\mathbf{v}_1\}$ is still a partial ovoid. However, if $\mathrm{char}(\mathbb{K}) \neq 2$ and $\mathbb{K}$ is quadratically closed then $O \cup \{\mathbf{v}_1\}$ is maximal, whence it is a maximal subspace.

3.2.2 The orthogonal case with $N > 2n$ and $\mathrm{char}(\mathbb{K}) \neq 2$

Still with $\sigma = \mathrm{id}_\mathbb{K}$ suppose that $N > 2n$ and $\mathrm{char}(\mathbb{K}) \neq 2$. Put $m = N - 2n$. A basis $(\mathbf{u}_1, \mathbf{v}_1, ..., \mathbf{u}_n, \mathbf{v}_n, \mathbf{w}_1, ..., \mathbf{w}_m)$ of V exists such that ϕ admits the following expression with respect to it:

$$\phi(x_1, y_1, ..., x_n, y_n, z_1, ..., z_m) \;=\; \sum_{i=1}^{n} x_i y_i + \sum_{j=1}^{m} \mu_j z_j^2$$

and ϕ is anisotropic on $\langle \mathbf{w}_1, ..., \mathbf{w}_m\rangle$, namely $\sum_{j=1}^m \mu_j z_j^2 = 0$ only if $z_1 = z_2 = ... = z_m = 0$. We choose an anti-symmetric matrix $L = (\lambda_{i,j})_{i,j=1}^n$ and a non-singular matrix $A = (\alpha_{i,j})_{i,j=1}^m$ such that

$$\sum_{j=1}^{m} \mu_j \alpha_{j,r}\alpha_{j,s} \;\neq\; 0 \quad \text{for} \quad 1 \leq r < s \leq m. \tag{5}$$

Example 3.7 Many non-singular matrices exist which satisfy condition (5). For instance, put $\alpha_{1,j} = 1$ for every j and $\alpha_{i,j} = \delta_{i,j}$ (Kronecker symbol) when $i > 1$. This is perhaps the easiest choice, but here is another one: choose m pairwise different square elements $\alpha_1, ..., \alpha_m \in \mathbb{K}^*$ and put $\alpha_{i,j} = \alpha_i^{j-1}$.

For $k = 1, 2, ..., n$ and $r = 1, 2, ..., m$ we put

$$\mathbf{x}_{k,r}(t) := -\mathbf{u}_k \cdot (\sum_{j=1}^{m} \mu_j \alpha_{j,r}^2) t^2 + \sum_{i \neq k} \mathbf{u}_i \lambda_{i,k} + \mathbf{v}_k + \sum_{j=1}^{m} \mathbf{w}_j \alpha_{j,r} t, \quad (t \in \mathbb{K}),$$

$X_{k,r} := \{[\mathbf{x}_{k,r}(t)]\}_{t \in \mathbb{K}^*}$ and $O := \cup_{k=1}^{n} \cup_{r=1}^{m} X_{k,r}$. The set O is a partial ovoid (but not a maximal one).

Proposition 3.8 *If $|\mathbb{K}| > 3$ then the partial ovoid O is totally scattered. If $\mathbb{K} = \mathbb{F}_3$ but $m > 1$ then the matrix A can be chosen in such a way that O is totally scattered.*

Remark 12 The hypothesis that $\mathbb{K} \neq \mathbb{F}_3$ when $m = 1$ cannot be removed from Lemma 3.8. Indeed, when $m = 1$ and $\mathbb{K} = \mathbb{F}_3$ our construction yields a partial ovoid of size $n + 1$, too small to span Σ.

3.2.3 The orthogonal case with $N > 2n$ and $\mathrm{char}(\mathbb{K}) = 2$

Still assuming that $\sigma = \mathrm{id}_{\mathbb{K}}$ and $N > 2n$, let now $\mathrm{char}(\mathbb{K}) = 2$. Then V admits a basis $(\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_{2n-1}, \mathbf{u}_{2n}, \mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_{2\ell-1}, \mathbf{v}_{2\ell}, \mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_m)$, with $2\ell + m = N - 2n$, such that ϕ is expressed as follows with respect to it:

$$\phi(x_1, ..., x_{2n}, y_1, ..., y_{2\ell}, z_1, ..., z_m) = \sum_{i=1}^{n} x_{2i-1} x_{2i} + \psi(y_1, ..., y_{2\ell}, z_1, ..., z_m)$$

where $\psi(y_1, ..., y_{2\ell}, z_1, ..., z_m) = \sum_{i=1}^{\ell} (\kappa_i y_{2i-1}^2 + y_{2i-1} y_{2i} + y_{2i}^2 \chi_i) + \sum_{j=1}^{m} z_j^2 \mu_j$ for suitable scalars $\kappa_1, \chi_1, ..., \kappa_\ell, \chi_\ell, \mu_1, ..., \mu_m \in \mathbb{K}^*$ such that

$$\psi(y_1, ..., y_{2\ell}, z_1, ..., z_m) = 0 \ \text{ only if } \ y_1 = ... = y_{2\ell} = z_1 = ... = z_m = 0.$$

By assumptions, at least one of ℓ or m is positive. To fix ideas, assume that both ℓ and m are positive. Let $L = (\lambda_{i,j})_{i,j=1}^{n}$ and $N = (\nu_{i,j})_{i,j=1}^{n}$ be $n \times n$ matrices such that $\lambda_{i,j} + \lambda_{j,i} \neq 0 \neq \nu_{i,j} + \nu_{j,i}$ and $\nu_{i,j} \neq \lambda_{i,j}, \lambda_{j,i}$ for any choice of $i \neq j$ and let $A = (\alpha_{i,j})_{i,j=1}^{\ell}$, $B = (\beta_{i,j})_{i,j=1}^{\ell}$ and $C = (\gamma_{i,j})_{i,j=1}^{m}$ be invertible matrices with A and B satisfying the following conditions:

$$\left.\begin{array}{rl} \sum_{j=1}^{\ell} \kappa_j \alpha_{j,r} \alpha_{j,s} \neq 0 & \text{for } 1 \leq r < s \leq \ell, \\ \sum_{j=1}^{\ell} \chi_j \beta_{j,r} \beta_{j,s} \neq 0 & \text{for } 1 \leq r < s \leq \ell, \\ \sum_{j=1}^{\ell} \alpha_{j,r} \beta_{j,s} \neq 0 & \text{for any choice of } r, s \in \{1, 2, ..., \ell\}. \end{array}\right\} \quad (6)$$

We know from Example 3.7 that several ways exist to choose matrices A and B in such a way that the first two conditions of (6) are satisfied. Satisfying the third condition is not so difficult. For instance, the ‘easiest’ choice of Example 3.7 for A and a slight modification of that choice for B do the job.

With A, B and C as above, for every $k = 1, 2, ..., n$, every $r = 1, 2, ..., \ell$ and every $s = 1, 2, ..., m$ we put:

$$\begin{aligned}
\mathbf{x}_{k,r}(t) &:= \mathbf{u}_{2k-1} \cdot \sum_{j=1}^{\ell} \kappa_j \alpha_{j,r}^2 t^2 + \sum_{i \neq k} \mathbf{u}_{2i-1} \lambda_{i,k} + \mathbf{u}_{2k} + \sum_{j=1}^{\ell} \mathbf{v}_{2j-1} \alpha_{j,r} t, \\
\mathbf{y}_{k,r}(t) &:= \mathbf{u}_{2k-1} \cdot \sum_{j=1}^{\ell} \chi_j \beta_{j,r}^2 t^2 + \sum_{i \neq k} \mathbf{u}_{2i-1} \lambda_{k,i} + \mathbf{u}_{2k} + \sum_{j=1}^{\ell} \mathbf{v}_{2j} \beta_{j,r} t, \\
\mathbf{z}_{k,s}(t) &:= \mathbf{u}_{2k-1} \cdot \sum_{j=1}^{m} \mu_j \gamma_{j,s}^2 t^2 + \sum_{i \neq k} \mathbf{u}_{2i-1} \nu_{i,k} + \mathbf{u}_{2k} + \sum_{j=1}^{m} \mathbf{w}_j \gamma_{j,s} t.
\end{aligned}$$

Put $X_{k,r} := \{\mathbf{x}_{k,r}(t)\}_{t \neq 0}$, $Y_{k,r} := \{\mathbf{y}_{k,r}(t)\}_{t \neq 0}$, $Z_{k,s} := \{\mathbf{z}_{k,s}(t)\}_{t \neq 0}$ and

$$O := \bigcup_{k,r,s=1}^{n,\ell,m} (X_{k,r} \cup Y_{k,r} \cup Z_{k,s}).$$

Then O is a partial ovoid and spans Σ. So far we have assumed that $\ell, m > 0$ (hence $\mathbb{K}$ is infinite). If $\ell = 0$ then we define only $\mathbf{z}_{k,s}(t)$ and put $O := \cup_{k,s} Z_{k,s}$. Similarly, when $m = 0$ then $O := \cup_{k,r}(X_{k,r} \cup Y_{k,r})$. Again, O is a partial ovoid; moreover, it spans Σ except when $(\ell, m) \in \{(1, 0), (0, 1)\}$ and $\mathbb{K} = \mathbb{F}_2$. In the end, the following holds:

Proposition 3.9 *If $\mathbb{K} \neq \mathbb{F}_2$ then the partial ovoid O is totally scattered.*

Remark 13 With $R := \langle \mathbf{w}_1 ..., \mathbf{w}_m \rangle$, let π_R be the canonical projection of $\Sigma = \mathrm{PG}(V)$ onto $\mathrm{PG}(V/R)$. Then π_R provides an embedding of $\mathscr{S}$ in $\mathrm{PG}(V/R)$. Suppose that $\ell = 0$ and $\phi(R) = \mathbb{K}$. Then $\pi_R(\mathscr{S})$ is symplectic and we are driven back to Section 3.1. With O as above, let O_R be the partial ovoid of $\pi_R(\mathscr{S})$ as constructed in Section 3.1 and let $\widetilde{O}_R = \pi_R^{-1}(O_R) \cap \mathscr{S}$ be its lifting to Σ. Then $O \neq \widetilde{O}_R$. Indeed $\widetilde{O}_R$ contains exactly one pointed conic for every $k = 1, 2, ..., n$ while O contains m bi-pointed conics for every k.

3.2.4 The hermitian case with $N = 2n$

Let $\sigma \neq \mathrm{id}_{\mathbb{K}}$ and $N = 2n$. A basis $(\mathbf{u}_1, \mathbf{v}_1, ..., \mathbf{u}_n, \mathbf{v}_n)$ of V exists such that ϕ admits the following expression with respect to it:

$$\phi(x_1, y_1, ..., x_n, y_n) = \sum_{i=1}^{n} x_i^\sigma y_i + \mathbb{K}/\mathbb{K}_{\sigma,1}.$$

Given a matrix $L = (\lambda_{i,j})_{i,j=1}^n$ such that $\lambda_{i,j} + \lambda_{j,i}^\sigma \neq 0$ for any $i \neq j$, we put $\mathbf{x}_k(t) := \mathbf{u}_k t + \sum_{i \neq k} \mathbf{u}_i \lambda_{i,k} + \mathbf{v}_k$ for $k = 1, 2, ..., n$ and $t \in \mathbb{K}_{\sigma,1}$, $X_k := \{[\mathbf{x}_k(t)]\}_{t \in \mathbb{K}_{\sigma,1} \setminus \{0\}}$ and $O := \cup_{k=1}^n X_k$.

Proposition 3.10 *The set O is a totally scattered partial ovoid.*

3.2.5 The hermitian case with $N > 2n$

Still with $\sigma \neq \mathrm{id}_{\mathbb{K}}$, let now $N > 2n$. Put $\mathbb{K}^{\sigma,1} := \{t \in \mathbb{K} \mid t + t^\sigma = 0\}$. Note that $\mathbb{K}_{\sigma,1} \subseteq \mathbb{K}^{\sigma,1}$. The vector space V admits a basis

$$(\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_{2n-1}, \mathbf{u}_{2n}, \mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_{2\ell-1}, \mathbf{v}_{2\ell}, \mathbf{e}_1, ..., \mathbf{e}_{m_0}, \mathbf{f}_1, ..., \mathbf{f}_{m_1})$$

(possibly $\ell = 0$, $m_0 = 0$ or $m_1 = 0$) such that ϕ is expressed as follows with respect to it, with values taken modulo $\mathbb{K}_{\sigma,1}$:

$$\begin{array}{l} \phi(x_1, ..., x_{2n}, y_1, ..., y_{2\ell}, t_1, ..., t_{m_0}, s_1, ..., s_{m_1}) \;= \\ = \;\sum_{i=1}^n x_{2i-1}^\sigma x_{2i} + \psi(y_1, ..., y_{2\ell}, t_1, ..., t_{m_0}, s_1, ..., s_{m_1}) \end{array}$$

where $\psi(y_1, ..., y_{2\ell}, t_1, ..., t_{m_0}, s_1, ..., s_{m_1})$ stands for the following

$$\sum_{i=1}^{\ell}(y_{2i-1}^\sigma \kappa_i y_{2i-1} + y_{2i-1}^\sigma y_{2i} + y_{2i}^\sigma \chi_i y_{2i}) + \sum_{j=1}^{m_0} t_j^\sigma \mu_j t_j + \sum_{j=1}^{m_1} s^\sigma \nu_j s_j$$

for suitable scalars $\kappa_1, \chi_1, ..., \kappa_\ell, \chi_\ell, \mu_1, ..., \mu_{m_0} \in \mathbb{K}^{\sigma,1} \setminus \mathbb{K}^{\sigma,1}$ and $\nu_1, ..., \nu_{m_1} \in \mathbb{K} \setminus \mathbb{K}^{\sigma,1}$ such that

$$\left.\begin{array}{l} \phi(y_1, ..., y_{2\ell}, t_1, ..., t_{m_0}, s_1, ..., s_{m_1}) \in \mathbb{K}_{\sigma,1} \;\text{ only if} \\ y_1 = y_2 = ... = y_{2\ell} = t_1 = ... = t_{m_0} = s_1 = ... = s_{m_1} = 0. \end{array}\right\}$$

Clearly, if $\mathbb{K}_{\sigma,1} = \mathbb{K}^{\sigma,1}$ (as it is the case when either $\mathrm{char}(\mathbb{K}) \neq 2$ or σ acts non-trivially on the center of $\mathbb{K}$) then $\ell = m_0 = 0$. In this case necessarily $m_1 > 0$, since $2\ell + m_0 + m_1 = N - 2n$ and $N > 2n$ by assumption.

To fix ideas, suppose that each of ℓ, m_0 and m_1 is positive. Choose matrices $L = (\lambda_{i,j})_{i,j=1}^n$, $M = (\mu_{i,j})_{i,j=1}^n$ and $N = (\nu_{i,j})_{i,j=1}^n$ such that $\lambda_{i,j} + \lambda_{j,i}^\sigma \neq 0$, $\mu_{i,j} + \mu_{j,i}^\sigma \neq 0$, $\nu_{i,j} + \nu_{j,i}^\sigma = 0$, $\mu_{i,j} \neq \lambda_{i,j}, \lambda_{j,i}^\sigma$, $\nu_{i,j} \neq \lambda_{i,j}, \lambda_{j.i}^\sigma$ and $\nu_{i,j} \neq \mu_{j,i}^\sigma$ for any choice of $i \neq j$. Moreover $A = (\alpha_{i,j})_{i,j=1}^\ell$, $B = (\beta_{i,j})_{i,j=1}^\ell$, $C = (\gamma_{i,j})_{i,i=1}^{m_0}$ and $D = (\delta_{i,j})_{i,j=1}^{m_1}$ are invertible matrices with A, B and D satisfying the following:

$$\begin{array}{rl} \sum_{j=1}^\ell \alpha_{j,r}^\sigma \kappa_j \alpha_{j,s} \neq 0 & \text{for } 1 \leq r < s \leq \ell, \\ \sum_{j=1}^\ell \beta_{j,r}^\sigma \chi_j \beta_{j,s} \neq 0 & \text{for } 1 \leq r < s \leq \ell, \\ \sum_{j=1}^\ell \alpha_{j,r}^\sigma \beta_{j,s} \neq 0 & \text{for } r, s = 1, ..., \ell. \\ \sum_{j=1}^{m_1} \delta_{j,r}^\sigma (\nu_j + \nu_j^\sigma) \delta_{j,s} \neq 0 & \text{for } 1 \leq r < s \leq m_1. \end{array}$$

For every $k = 1, 2, ..., n$, every $r = 1, 2, ..., \ell$ every $r_0 = 1, 2, ..., m_0$, every $r_1 = 1, 2, ..., m_1$ and any $s, t \in \mathbb{K}$ put:

$$\begin{array}{rcl} \mathbf{a}_{k,r}(s,t) & := & \mathbf{u}_{2k-1}s + \sum_{i \neq k} \mathbf{u}_{2i-1}\lambda_{i,k} + \mathbf{u}_{2k} + \sum_{j=1}^\ell \mathbf{v}_{2j-1}\alpha_{j,r}t, \\ \mathbf{b}_{k,r}(s,t) & := & \mathbf{u}_{2k-1}s + \sum_{i \neq k} \mathbf{u}_{2i-1}\lambda_{k,i} + \mathbf{u}_{2k} + \sum_{j=1}^\ell \mathbf{v}_{2j}\beta_{j,r}t, \\ \mathbf{c}_{k,r_0}(s,t) & := & \mathbf{u}_{2k-1}s + \sum_{i \neq k} \mathbf{u}_{2i-1}\mu_{i,k} + \mathbf{u}_{2k} + \sum_{j=1}^{m_0} \mathbf{e}_j\gamma_{j,r_0}t, \\ \mathbf{d}_{k,r_1}(s,t) & := & \mathbf{u}_{2k-1}s + \sum_{i \neq k} \mathbf{u}_{2i-1}\nu_{i,k} + \mathbf{u}_{2k} + \sum_{j=1}^{m_1} \mathbf{f}_j\delta_{j,r_1}t. \end{array}$$

Next put

$$\begin{array}{rcl} A_{k,r} & := & \{\mathbf{a}_{k,r}(s,t) \mid s + \sum_{j=1}^{\ell} t^\sigma \alpha_{j,r}^\sigma \kappa_j \alpha_{j,r} t \ \in \ \mathbb{K}_{\sigma,1}, \ t \neq 0\}, \\ B_{k,r} & := & \{\mathbf{b}_{k,r}(s,t) \mid s + \sum_{j=1}^{\ell} t^\sigma \beta_{j,r}^\sigma \chi_j \beta_{j,r} t \ \in \ \mathbb{K}_{\sigma,1}, \ t \neq 0\}, \\ C_{k,r_0} & := & \{\mathbf{c}_{k,r_0}(s,t) \mid s + \sum_{j=1}^{m_0} t^\sigma \gamma_{j,r_0}^\sigma \mu_j \gamma_{j,r_0} t \ \in \ \mathbb{K}_{\sigma,1}, \ t \neq 0\}, \\ D_{k,r_1} & := & \{\mathbf{d}_{k,r_1}(s,t) \mid s + \sum_{j=1}^{m_1} t^\sigma \delta_{j,r_1}^\sigma \nu_j \delta_{j,r_1} t \ \in \ \mathbb{K}_{\sigma,1}, \ t \neq 0\}. \end{array}$$

Finally, $O := \bigcup_{k,r,r_0,r_1=1}^{n,\ell,m_0,m_1} (A_{k,r} \cup B_{k,r} \cup C_{k,r_0} \cup D_{k,r_1})$.

We have assumed that neither ℓ nor m_0 is 0. Of course, when one of them is 0 we must accordingly modify the previous definition. For instance, when $\ell = 0 < m_0, m_1$ we form O as the union of the sets C_{k,r_0} and D_{k,r_1}. In this case we omit to introduce the matrices L, A and B; we only need M, N, C and D. Similarly, if $m_0 = 0 < \ell, m_1$ then O is formed only by the sets $A_{k,r}$, $B_{k,r}$ and D_{k,r_1}. In this case M and C are omitted. If $\ell = m_1 = 0$ then $O = \cup_{k,r_0} C_{k,r_1}$ and if $m_0 = m_1 = 0$ then $O = \cup_{k,r}(A_{k,r} \cup B_{k,r})$. Finally, if $\ell = m_0 = 0$ then only N and D are needed and $O = \cup_{k,r_1} D_{k,r_1}$. With O defined in this way,

Proposition 3.11 *The set O is a totally scattered partial ovoid.*

4 Ovoids and maximal subspaces

Throughout this section, but for the very last remark, $\mathscr{S}$ if a finite (non-degenerate, thick-lined) polar space of rank $n > 2$. Let q be the order of the underlying field of $\mathscr{S}$ and $t+1$ the number of generators of $\mathscr{S}$ which contain a given singular subspace of rank $n-1$. Recall that $t \in \{1, q, q^2, q^{1/2}, q^{3/2}\}$. The parameters q and t are the *orders* of $\mathscr{S}$.

We recall that $W_{2n-1}(q) \cong Q_{2n}(q)$ when q is even. Accordingly, the properties of $W_{2n-1}(q)$ with q even are not the same as when q is odd. This fact causes slight complications in the exposition, which we prefer to avoid. So, henceforth, when referring to $W_{2n-1}(q)$ we implicitly assume that q is odd.

We have defined ovoids in the Introduction of this paper. We are not going to repeat that definition here. Instead we recall that, for a singular subspace X of $\mathscr{S}$ of rank $m < n$, the *star* $\mathscr{S}_X$ of X is the polar space of rank $n-m$ formed by the singular subspaces of $\mathscr{S}$ which properly contain X, those of rank $m+1$ being taken as points of $\mathscr{S}_X$ and those of rank $m+2$ as lines (if $m \leq n-2$). Obviously, the generators of $\mathscr{S}_X$ are the generators of $\mathscr{S}$ which contain X.

4.1 Basics on ovoids and non-existence results

Let q and t be the orders of $\mathscr{S}$. Since a partial ovoid meets every generator in at most one point, every partial ovoid O of $\mathscr{S}$ has size at most $q^{n-1}t+1$ and it is an ovoid if and only if $|O| = q^{n-1}t+1$.

Let O be an ovoid of $\mathscr{S}$ and let X be a non-maximal singular subspace of $\mathscr{S}$, disjoint from O. Let $O_X := \{\langle X, x\rangle \mid x \in O \cap X^\perp\}$. Then O_X is an ovoid of $\mathscr{S}_X$. In particular, if x is a point of $\mathscr{S}$ exterior to O then O_x is the set of lines which join x to points of O and it is an ovoid of the star $\mathscr{S}_x$ of x. Accordingly, $|O_x| = |x^\perp \cap O| = q^{n-2}t+1$.

The fact that O_X is an ovoid of $\mathscr{S}_X$ can be exploited to prove the non-existence of ovoids in certain polar spaces. Indeed, if we already know that $\mathscr{S}_X$ admits no ovoid, then we can conclude that no ovoids exist in $\mathscr{S}$. In this way, by reduction to the rank 2 case, we immediately see that $W_{2n-1}(q)$, $Q^-_{2n+1}(q)$ and $H_{2n}(q)$ admit no ovoid. Indeed it is well known that $W_3(q)$, $Q^-_5(q)$ and $H_4(q)$ admit no ovoids (see e.g. Payne and Thas [16]). Moreover, no ovoid exists in $Q_8(q)$, for any prime power q (Gunarwardena and Moorhouse [10]). Therefore $Q_{2n}(q)$ has no ovoid, for any $n \geq 4$.

As for $Q_6(q)$, the following is known. If q is even then $Q_6(q)$ has no ovoids (Thas [18]) and no ovoids exist in $Q_6(p)$, for p prime, $p > 3$ (O'Keefe and Thas [14]). On the other hand, $Q_6(3^h)$ admits ovoids, for any positive integer h (Kantor [11]).

Not so much is known about $H_{2n-1}(q)$ for $n > 2$. It is known that $H_5(4)$ has no ovoid (De Beule and Metsch [6]), hence no ovoids exist in $H_{2n-1}(4)$ for any $n \geq 3$. It is also known that no ovoid exists in $H_{2n-1}(q)$ when the prime basis p of q satisfies the following inequality (Moorhouse [12]):

$$p^{2n-1} \; > \; \binom{2n+p-3}{p-1}^2 + 2\cdot\binom{2n+p-3}{p-1}\cdot\binom{2n+p-3}{p-2}. \tag{7}$$

Note that, for a given p, the second term of (7), say $f_p(n)$, is a polynomial of degree $2(p-1)$ in the unknown n. Therefore the ratio $p^{2n-1}/f_p(n)$ diverges as n diverges. So, (7) says that $H_{2n-1}(q)$ admits ovoids only if n is not too large compared to p. In other words, for every given prime p, if $n > 2$ then $H_{2n-1}(p^{2h})$ admits no ovoids except possibly for a finite number of choices of n. Inequality (7) embodies an upper bound for the number of those lucky choices, which only depends on p. We are not aware of any further existence or non-existence result for ovoids of $H_{2n-1}(q)$ when $n > 2$.

A few existence results are known for ovoids of $Q^+_{2n-1}(q)$, $n > 2$. For instance, $Q^+_5(q)$ admits ovoids for any q and $Q^+_7(q)$ admits ovoids for q a power of 2 or 3, for q an odd power of a prime $p \equiv 2 \pmod 3$ and for q prime (we refer to Table 1 of [7] for this information). On the other hand, no ovoids exist in $Q^+_{2n-1}(q)$ if the prime basis p of q satisfies the following inequality (Blokhuis and Moorhouse [1])

$$p^{n-1} \; > \; \binom{2n+p-4}{p-1} + 2\cdot\binom{2n+p-4}{p-2}. \tag{8}$$

Likewise (7), inequality (8) says that $Q^+_{2n-1}(p^h)$ admits ovoids only if n is not too large compared to p.

4.2 Proof of Theorem 3

Let $\mathscr{S}$ be finite with orders $q > 1$ and t and rank $n > 2$. Suppose that $\mathscr{S}$ is neither $Q^+_5(q)$ (for any q) nor $Q^+_7(q)$ (with q odd). Let O be an ovoid of $\mathscr{S}$. Given a point $x \notin O$, consider the subspace $\mathscr{S}(O,x) = \langle O \cup \{x\}\rangle$ of $\mathscr{S}$ generated by $O \cup \{x\}$. If $\mathscr{S}(O,x) = \mathscr{S}$ for every $x \notin O$ then O is a maximal subspace of $\mathscr{S}$.

By way of contradiction, suppose there exists a point $a \notin O$ such that $\mathscr{S}(O,a) \subset \mathscr{S}$. The subspace $\mathscr{S}(O,a)$ has rank at least 2, since it contains at least the $q^{n-2}t+1$ lines which join a to points of O. The polar space $\mathscr{S}$ is embeddable and admits the universal embedding (indeed it is finite of rank $n > 2$). Hence the subspace $\mathscr{S}(O,a)$ arises from the universal embedding $\tilde{e} : \mathscr{S} \to \mathrm{PG}(\widetilde{V})$ of $\mathscr{S}$ (Theorem 2.2), namely $\mathscr{S}(O,a) = \tilde{e}^{-1}(\langle \tilde{e}(O \cup \{a\})\rangle)$. However

$\mathscr{S}(O,a) \subset \mathscr{S}$ by assumption. Hence $\langle \tilde{e}(O \cup \{a\}) \rangle$ is contained in a hyperplane X of $\mathrm{PG}(\widetilde{V})$. Accordingly, $O \cup \{a\}$ is contained in the hyperplane $H := \tilde{e}^{-1}(X)$ of $\mathscr{S}$.

If $H = b^{\perp}$ for some point b of $\mathscr{S}$, then b is joined with all points of $|O|$. However $|O| = q^{n-1}t + 1$ while $|b^{\perp} \cap O|$ is either 1 or $q^{n-2}t + 1$ according to whether $b \in O$ or $b \notin O$. In any case, we get a contradiction. Therefore H is non-singular. So, H is a non degenerate polar space of rank $m \in \{n-1, n\}$ and order (q, t'), where t' depends on m and the type of $\mathscr{S}$. However O, being contained in H, is also an ovoid of H. Consequently $q^{n-1}t + 1 = q^{m-1}t' + 1$. If $m = n$ then $t' = t$. This is impossible, as H cannot have the same rank and the same orders as $\mathscr{S}$. Therefore $m = n - 1$ and $t' = qt$.

On the other hand, if $t' > q$ then H is necessarily isomorphic to either $Q^-_{2m+1}(q)$ or $H_{2m}(q)$. However, as remarked in Section 4.1, neither $Q^-_{2m+1}(q)$ nor $H_{2m}(q)$ admit ovoids, while O is an ovoid of H. Therefore $t' \leq q$ and the equality $t' = qt$ now forces $t = q$ and $t = 1$, namely $\mathscr{S} \cong Q^+_{2n-1}(q)$ and $H \cong Q_{2m}(q) = Q_{2n-2}(q)$.

As noticed in Section 4.1, the quadric $Q_{2m}(q)$ admits no ovoids when $m > 3$ and $Q_6(q)$ admits no ovoids if q is even. However O is an ovoid of H. Therefore $m \leq 3$ and $m = 2$ if q is even, namely $n \leq 4$ and $n = 3$ if q is even. However, according to the hypotheses of the theorem, $t > 1$ when either $n = 3$ or $n = 4$ and q is even. We have reached a final contradiction. Consequently, $\mathscr{S}(O, x) = \mathscr{S}$ for every point $x \notin O$. $\square$

Remark 14 The quadrics $Q^+_5(q)$ and $Q^+_7(3^h)$ are counterexamples to the conclusion of Theorem 3. Indeed $Q^+_5(q)$ contains $Q^-_3(q)$, which is indeed an ovoid of $O^+_5(q)$. However $Q^-_3(q)$ is not a maximal subspace of $Q^+_5(q)$, as it is contained in a hyperplane $Q_4(q)$ of $Q^+_5(q)$. Similarly, $Q_6(3^h)$ is a hyperplane of $Q^+_7(3^h)$ and contains ovoids (Section 4.1). The ovoids of $Q_6(3^h)$ are still ovoids in $Q^+_7(3^h)$, but they cannot be maximal subspaces of $Q^+_7(3^h)$.

Remark 15 The statement of Theorem 3 can be made slightly sharper by allowing $\mathscr{S} = Q^+_7(p)$ with p prime and different from 3. Indeed $Q_6(p)$ admits no ovoids for a prime $p \neq 3$.

Remark 16 Remark 14 suggest the following counterexample in the infinite case. Let $\mathscr{S} = Q^+_{2n+1}(\mathbb{K})$, with $\mathbb{K}$ an infinite field and $n \geq 2$. Then $\mathscr{S}$ contains hyperplanes isomorphic to $Q_{2n}(\mathbb{K})$. If $\mathscr{S}'$ is one of them, every generator of $\mathscr{S}'$ is contained in exactly two generators of $\mathscr{S}$ and every generator of $\mathscr{S}$ contains exactly one generator of $\mathscr{S}'$. The polar space $\mathscr{S}'$ admits ovoids, by Cameron [3] (in fact, it admits even a partition in ovoids). Let O be an ovoid of $\mathscr{S}'$. Since every generator of $\mathscr{S}$ contains exactly one generator of $\mathscr{S}'$, every generator of $\mathscr{S}$ meets O in exactly one point. Accordingly, O is also an ovoid of $\mathscr{S}$. However O is not a maximal subspace of $\mathscr{S}$, since it is contained in the hyperplane $\mathscr{S}'$ of $\mathscr{S}$. This is a special case of the following more general observation, which is the best we can do for the infinite case:

Proposition 4.1 *Let $\mathscr{S}$ be an infinite polar space of rank $r \geq 3$. If $\mathscr{S}$ is the line-grassmannian of $\mathrm{PG}(3, \mathbb{K})$, with $\mathbb{K}$ non-commutative, then every ovoid of $\mathscr{S}$ is a maximal subspace. If $\mathscr{S}$ is embeddable, and its universal embedding is finite-dimensional, then $\mathscr{S}$ possesses ovoids that are not maximal subspaces if and only if $\mathscr{S}$ admits non-singular hyperplanes of rank $r - 1$.*

Proof. We first note that no ovoid is contained in a singular hyperplane. Indeed, suppose for a contradiction that the ovoid O is contained in $p^{\perp}$ for some point p of $\mathscr{S}$. Then we can find a submaximal singular subspace W in $p^\perp$ not containing p and disjoint from O (since every projective space has a hyperplane avoiding two given points). Every generator containing W and distinct from $\langle W, p\rangle$ is disjoint from O, a contradiction.

Suppose first that $\mathscr{S}$ is the line-grassmannian of $\mathrm{PG}(3, \mathbb{K})$, with $\mathbb{K}$ non-commutative, and suppose, for a contradiction, that an ovoid O is contained in a proper subspace X of $\mathscr{S}$. By Lemma 2.14, X is contained in the perp of a point, contradicting the previous paragraph.

Now suppose that $\mathscr{S}$ is embeddable, and its universal embedding is finite-dimensional, say in the projective space Σ. First suppose that $\mathscr{S}$ admits a non-singular hyperplane H of rank $r-1$. By [3], H admits an ovoid O. Since H is a hyperplane, every generator of $\mathscr{S}$ intersects H in a submaximal singular subspace, that is, a generator of H, which by definition contains a member of O. Hence O is an ovoid of $\mathscr{S}$, which is not a maximal subspace as $O \subset H \subset \mathscr{S}$.

Now suppose that $\mathscr{S}$ possesses an ovoid O that is not a maximal subspace, and let $O \subset \mathscr{S}' \subset \mathscr{S}$, with $\mathscr{S}'$ a subspace. Then $\mathscr{S}'$ is a non-degenerate polar space by the first paragraph of this proof. Let S be the subspace of Σ corresponding to $\mathscr{S}'$. Suppose $\mathscr{S}'$ has rank r'. We claim that $r' < r$. Indeed, suppose for a contradiction that $r' = r$. There exists a submaximal singular subspace M' of $\mathscr{S}'$ disjoint from O. Since $\mathscr{S}'$ is not an ideal subspace (as otherwise $\mathscr{S}'$ would coincide with $\mathscr{S}$), there is a generator M not contained in $\mathscr{S}'$ but containing M'. Since also O is contained in $\mathscr{S}'$, $M \cap O = \emptyset$, a contradiction. The claim is proved. Now we can extend $\mathscr{S}'$ to a hyperplane of Σ and the proposition follows. □

Note that the sharpened version of Lemma 2.16 referred to in Remark 6 implies that also each ovoid of the Freudenthal-Tits polar space is a maximal subspace, providing alternative evidence of the existence of ovoidal maximal subspaces in these polar spaces, using [3] once again.

Proposition 4.1 has many applications. We content ourselves by mentioning that each ovoid of every non-degenerate quadric in an even-dimensional projective space over an algebraically closed field is a maximal subspace, and that every quadric over $\mathbb{R}$ (and likewise every hermitian polar space over $\mathbb{C}$) admits ovoids which are not maximal as subspaces.

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