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A CHARACTERIZATION OF $\tilde{C}_2$ -BUILDINGS BY FLOORS

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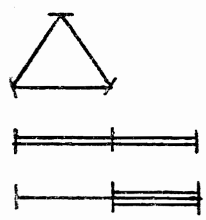
The n^{th} floor with basement b of an affine building Δ of type $\tilde{C}_2$ is the geometry arising from the vertices at distance n from the given vertex b. We show that these geometries are Hjelmslev quadrangles of level n forming an HQ-Artmann sequence (in the sense of [8]) and that Δ is completely determined by this sequence.

INTRODUCTION

When J.Tits classified all affine buildings of rank ≥ 4 showing they all arise from algebraic groups over a local field (see Tits [12]), there were counterexamples (so called *non-classical affine buildings*) for the rank 3 case (see e.g. Ronan [10] and Van Maldeghem [14]). There are actually three classes of rank 3 affine buildings, belonging to the respective diagrams



type $\tilde{A}_2$

type $\tilde{C}_2$

type $\tilde{G}_2$

A systematic approach to the first type was attempted in Van

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Maldeghem [14] and [15], resulting in the following characterization. All affine buildings of type $\tilde{A}_2$ arise from *planar ternary rings with valuation* in a sense explained in these papers. The most important intermediate step in the proof of this result is the characterization of the n^{th} floors of such affine buildings and the reconstruction of the building from these n^{th} floors. The analogue for buildings of type $\tilde{C}_2$ of this step is proved in the present paper. The complete algebraic characterization will be completed elsewhere (see Van Maldeghem [16] and [18]) after putting a valuation on *quadratic quaternary rings* (introduced by the authors in [5], it is the algebraic structure coordinatizing generalized quadrangles).

In section 1, we define affine buildings of type $\tilde{C}_2$ (as discrete systems of apartments of that type) and their building at infinity. All definitions are due to Tits [12]. In section 2, we list some known properties of $\tilde{C}_2$-buildings (proved in Tits [12]) and we define the n^{th} floor of a $\tilde{C}_2$-building with given basement. We show some preliminary results which we will need in section 4. Section 3 summarizes Hanssens and Van Maldeghem [8]. We define the notion of a Hjelmslev quadrangle of level n and of an HQ-Artmann sequence and show how such a sequence defines in an explicit way a $\tilde{C}_2$-building. Section 4 is devoted to the proof of our main result : any n^{th} floor of a $\tilde{C}_2$-building is a Hjelmslev quadrangle of level n. In section 5, we combine the results of sections 3 and 4 obtaining a characterization of all $\tilde{C}_2$-buildings by means of their n^{th} floors with a fixed but arbitrary basement.

MOTIVATION

This paper is the second one in a series of four which have as goal to obtain a characterization of $\tilde{C}_2$-buildings by means of *quadratic quaternary rings with valuation*, and it is the sequel to Hanssens and Van Maldeghem [8]. Later, we will use the results of this paper to construct explicitly defined (non classical) $\tilde{C}_2$-buildings (see Van Maldeghem [16]). The motivation for treating the geometric part seperately in this paper is basicly fourfold :

(1) The paper treats $\tilde{C}_2$-buildings in an axiomatic abstract way. We devellop a lot of geometric properties that can be easily generalized to affine buildings of other types and higher rank. A better geometrical understanding will improve our understanding of e.g. the automorphism groups.

(2) A natural generalization of the notion of a *projective plane* is the notion of a *projective Hjelmslev plane*, an object investigated by a number of people. A generalization in another direction yields the notion of *generalized polygon* (including the *generalized quadrangles*). Our definition of *Hjelmslev quadrangle of level n* offers a generalization in both directions. Questions beyond the scope of this paper can be asked and may be interesting (e.g. combinatorial properties, spectral sequences, etc...). But there are also possible applications within the theory of buildings where this local study has proved useful in distinguishing different isomorphism classes of certain affine buildings (see Tits [13]).

(3) Our geometrical construction has the advantage of knowing the building at infinity, which is the inverse limit of a sequence of Hjelmslev quadrangles. Hence, automorphisms may be described in terms of these Hjelmslev quadrangles. This analogue situation for $\tilde{A}_2$-buildings has led to the construction of vertex transitive $\tilde{A}_2$-buildings with non classical finite residues, showing there might be hope of finding finite $\tilde{A}_2$-GABs with non classical residues. So we may say that our construction is in a way complementary to Ronan's beautiful universal construction in [10]. Of course the latter applies to a lot of diagrams, but it is somewhat unclear how *free* this construction really is, because using the same set of residues, one can

obtain non isomorphic buildings. That question is settled for the $\tilde{A}_2$-case in Hanssens and Van Maldeghem [7] using our geometrical appraoch.

(4) The $\tilde{C}_2$-case is not just a rewritten copy of the $\tilde{A}_2$-case. It is much different in various ways, and therefore, it is interesting. Eventually, it must help us to construct and understand other types of buildings which are not understood yet such as other rank 3 buildings (of *hyperbolic* type). As a consequence, we consider the $\tilde{G}_2$-case as uninteresting since it is similar to the $\tilde{C}_2$-case.

1. AFFINE BUILDINGS OF TYPE $\tilde{C}_2$.

1.1. The standard apartment.

Let $\mathbf{A}$ be the real Euclidean plane endowed with the usual distance map $d_{\mathbf{A}}$. Denote by $\mathcal{T}$ a solid triangle with respective angles 45°, 45° and 90°. The length of the two shortest sides is 1. We denote the lines supporting the sides of $\mathcal{T}$ by $\mathcal{L}_i$, i=1, 2,3. Let $\mathcal{W}$ be the group of automorphisms of $\mathbf{A}$ generated by the reflections about the lines $\mathcal{L}_i$, i=1,2,3. The group $\mathcal{W}$ is called *the Weylgroup of type* $\tilde{C}_2$. The image of $\mathcal{T}$ under an arbitrary element of $\mathcal{W}$ is called a *chamber*. The set of all chambers determines a tessellation τ of $\mathbf{A}$ in congruent isosceles right triangles. We call $(\mathbf{A},\tau)$ the *standard apartment of type* $\tilde{C}_2$. The vertices of the triangles of τ are briefly called *vertices* and the sides of these triangles are called *panels*. Vertices, panels and chambers are also called *simplices*. Two vertices are called *adjacent* if they lie on a common panel. Panels of length 1 are called *short* panels, the other ones are called *long*. The lines supporting the panels are called *walls*. A line $\mathcal{L}$ is a wall if and only if the reflection about $\mathcal{L}$ is an element of $\mathcal{W}$. A vertex x is called *special* if for every wall $\mathcal{M}$, there exists a wall $\mathcal{M}^*$ parallel to $\mathcal{M}$ and incident with x. So x is special if and only if it lies on exactly four walls or eight panels. Now note that $\mathcal{W}$ defines two orbits on the set of walls. One orbit consists of all walls containing only special vertices and long panels. We call such walls *straight*. The walls of the other orbit, called *diagonal* walls, contain both special and non special vertices, but they contain only short panels. A *straight* (resp. *diagonal*) *interval* $[a,b]$ is a closed interval bounded by vertices a and b and contained in a straight (resp. diagonal) wall. Let x be a special vertex and denote by $\mathfrak{L}_x$ the set of all walls through x. The topological closure of any connected component of $\mathbf{A}-\cup\mathfrak{L}_x$ (where we consider $\mathbf{A}$ and the elements of $\mathfrak{L}_x$ as sets of points) is called a *sector (with source x)*. The closure of any connected component of $\cup\mathfrak{L}_x-\{x\}$ is called a *sectorpanel (with source x)*. Suppose Q_i ($i=1,2,\ldots,8$) is a sector with source x, $Q_i \neq Q_j$ for $i \neq j$ and Q_i meets Q_{i+1} in a sectorpanel (taking the subscripts modulo 8). Then $Q_1 \cup Q_2$ (resp. $Q_1 \cup \ldots \cup Q_j$, $0 < j < 8$) is called a *double* (resp. *j-fold*) *sector*. A 4-fold sector is also called a *half apartment*. Most of the above definitions are standard concepts and can be found in Bourbaki [2]. In fact, the standard apartment we just described is a geometric realization of the Coxeter complex of irreducible type $\tilde{C}_2$.

1.2. Discrete systems of apartments of type $\tilde{C}_2$.

An *affine building of type $\tilde{C}_2$*, also called a *discrete system of apartments of type $\tilde{C}_2$* (or briefly, a *$\tilde{C}_2$-building*), is by definition a pair $(\Delta,\mathcal{F})$, where Δ is a set and $\mathcal{F}$ is a family of injections from $\mathbf{A}$ into Δ satisfying the axioms (SA1),(SA2),(SA3) and (SA4) below. The building $(\Delta,\mathcal{F})$ is called *complete* if it also satisfies (SA5). Usually, we identify the building $(\Delta,\mathcal{F})$ with the set Δ and talk by abuse of language about the building Δ. The image of $\mathbf{A}$ under an arbitrary element of $\mathcal{F}$ is called an *apartment*. We suppose that $\mathbf{A}$ is provided with the tessellation τ

and we call the image of a chamber, panel, vertex, wall, etc... under the action of any element of $\mathfrak{F}$ also a *chamber*, resp. *a panel*, *a vertex*, etc... In particular, the elements of Δ are called *points* (just like the elements of $\mathbf{A}$). Here are the first four axioms.

(SA1) $\mathfrak{F} \cdot \mathfrak{W} = \mathfrak{F}$.

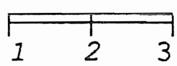
(SA2) *Let* $f, f' \in \mathfrak{F}$. *The set* $\mathfrak{B} = (f'^{-1} \cdot f)(\mathbf{A})$ *is a (not necessarily finite) union of simplices, it is closed and convex (with respect to* $d_{\mathbf{A}}$ *and the topology induced by* $d_{\mathbf{A}}$*) and there exists* $w \in \mathfrak{W}$ *such that* $f/\mathfrak{B} = f' \cdot w/\mathfrak{B}$.

(SA3) *Every two points of* Δ *lie in a commom apartment.*

(SA4) *If* $f \in \mathfrak{F}$ *and* x *is an arbitrary point of* $f(\mathbf{A})$*, then there exists a retraction (i.e. an idempotent surjection)* $\rho : \Delta \to f(\mathbf{A})$ *such that the restriction to every apartment diminishes distances (i.e.* $f^{-1} \cdot \rho \cdot f'$ *diminishes distances* $\mathbf{A}$*, for every* $f' \in \mathfrak{F}$*) and such that* $\rho^{-1}(x) = \{x\}$.

This set of axioms was introduced by J.Tits in [12].

J.Tits [12] shows that, in view of our slightly modified axiom (SA2), this definition is equivalent to the definition of an abstract building of type $\tilde{C}_2$. The way to go from a discrete system of apartments to an abstract building is symply by ignoring all points which are not vertices. Hence there exists a type map *typ* from the set of vertices of any affine building Δ of type $\tilde{C}_2$ to $\{1,2,3\}$ turning Δ into a *Buekenhout-Tits geometry of rank 3* with *Buekenhout-diagram* (see [4]) :



There exists a similar type map $typ_{\mathbf{A}}$ defined over the set of vertices of $(\mathbf{A},\tau)$ and typ can be considered as the image of $typ_{\mathbf{A}}$ under the action of the maps $\not{f} \in \mathcal{F}$. This is well defined by (SA2). The vertices of type 1 and 3 are special. The *residue* (cp. Buekenhout [4]) of such vertex is a generalized quadrangle (for definitions and properties, see the excellent monograph [9]) and the residue of a non special vertex (a vertex of type 2) is a *generalized digon*. We call two varieties of a generalized digon *opposite* if they have the same type but are distinct. We call two varieties of a generalized quadrangle *opposite* if they have the same type and no other variety is incident with both of them.

Now, the set of apartments of an affine building of type $\tilde{C}_2$ is not uniquely determined, but Tits proves that it always contains a *maximal set of apartments* (see Tits [12,théorème 1]). Hence the following axiom :

(SA5) *The set* $\{\not{f}(\mathbf{A}) \mid \not{f} \in \mathcal{F}\}$ *is a maximal set of apartments for* Δ.

Throughout, we will always assume that every affine building of type $\tilde{C}_2$ is *thick*, i.e. every panel is contained in at least three chambers. This conforms to the notion of buildings in Tits [11].

Let Δ be a thick complete building of type $\tilde{C}_2$, then we denote by d the *path metric*, i.e. a distance map defined on the set of vertices denoting the minimum number of panels needed to join two given vertices. Also, by (SA2) and (SA3), $d_{\mathbf{A}}$ induces a well defined metric d_{Δ} in Δ (in the obvious way, see also Tits [12]). Now by the choice of the unit length in $\mathbf{A}$, we have :

PROPOSITION 1.2.1. *If* x *and* y *are two arbitrary vertices of a* $\tilde{C}_2$ *- building* Δ, *then one has*

$$2.d(x,y) \geq d_\Delta(x,y)$$

and equality holds if and only if $[x,y]$ *is a straight interval.*

PROOF. This is obvious since by (SA2) we only need to check this in an apartment containing x and y. Q.E.D.

Note that by the definition of a combinatorial building we have

PROPOSITION 1.2.2. *Every two simplices of a* $\tilde{C}_2$*-building lie in an apartment.*

1.3. The geometry at infinity.

Let Δ be an affine building of type $\tilde{C}_2$. A *germ of sectors* is an equivalence class in the set of all sectors of Δ with respect to the equivalence relation "*Q_1 and Q_2 are equivalent if $Q_1 \cap Q_2$ contains a sector*". Two sectorpanels p and q are called *parallel* if they are at bounded distance from one another, i.e. the sets $\{d_\Delta(x,q) \mid x \in p\}$ and $\{d_\Delta(y,p) \mid y \in q\}$ are bounded (where $d_\Delta(x,q) = inf\{d_\Delta(x,y) \mid y \in q\}$ and similarly for $d_\Delta(y,p)$). This relation is apparently an equivalence relation and we denote the equivalence class of a sectorpanel p by $c(p)$. One can easily see that such a class contains either straight or diagonal sectorpanels (in view of Tits [12, proposition 17.3]). We can now define the following point-line incidence geometry $\Delta_\infty = (\mathcal{P}(\Delta_\infty), \mathcal{L}(\Delta_\infty), I)$. The points (elements of $\mathcal{P}(\Delta_\infty)$) are the parallel classes of straight sector panels ; the lines (elements of $\mathcal{L}(\Delta_\infty)$) are the parallel classes of diagonal sectorpanels and a point $c(p)$ is incident with a line $c(\ell)$ if there exists a sector containing at least one representative of both $c(p)$ and $c(\ell)$. By [12,proposition 5], we can identify the set of incident point-line pairs (the flags) with the set of germs of sectors. Proposition 5 of Tits [12] can now be rewritten as

PROPOSITION 1.3 (Tits [12]). *The geometry Δ_∞ as defined above is a generalized quadrangle. The eight germs of sectors in an arbitrary apartment Σ define eight flags in Δ_∞ which determine a customary non degenerate quadrangle Σ_∞ in Δ_∞. The map $\Sigma \to \Sigma_\infty$ is a bijection from the set of apartments of Δ to the set of customary non degenerate quadrangles in Δ_∞. The "trace at infinity" of a straight (resp. diagonal) wall of Δ is a pair of opposite points (resp. lines) in Δ_∞.*

We call Δ_∞ the *geometry at infinity of* Δ or also the *generalized quadrangle at infinity of* Δ. The spherical building naturally associated with Δ_∞ (for definitions, see Tits [11]) is called *the* (*spherical*) *building at infinity of* Δ (see Tits [12]).

1.4. Notation.

From now on, Δ always denotes a complete thick affine building of type $\tilde{C}_2$. Its geometry at infinity is denoted by $\Delta_\infty = (\mathcal{P}(\Delta_\infty), \mathcal{L}(\Delta_\infty), I)$ as above. Also, $\mathfrak{b}$ denotes a special vertex of Δ chosen once and for all. Furthermore, we denote :

$$\begin{aligned}
\mathrm{Ap}(\Delta) &= \text{set of apartments of } \Delta,\\
\mathrm{Ch}(\Delta) &= \text{set of chambers of } \Delta,\\
\mathrm{Pa}(\Delta) &= \text{set of panels of } \Delta,\\
\mathrm{Ve}(\Delta) &= \text{set of vertices of } \Delta,\\
\mathrm{Se}(\Delta) &= \text{set of sectors of } \Delta,\\
\mathrm{Sp}(\Delta) &= \text{set of sectorpanels of } \Delta,\\
\mathrm{Se}(\Delta,\mathfrak{b}) &= \text{set of sectors of } \Delta \text{ with source } \mathfrak{b},\\
\mathrm{Sp}(\Delta,\mathfrak{b}) &= \text{set of sectorpanels of } \Delta \text{ with source } \mathfrak{b}.
\end{aligned}$$

REMARK. In contradistinction to [2] and [3], our simplices are *closed* subsets in any apartment Σ they lie (Σ viewed as Euclidean plane). This has only practical reasons and has no further significance.

2. SOME PROPERTIES OF Δ.

2.1. Definitions.

* A convex (in the usual sense) subset $\mathcal{E} \subseteq \Delta$ is called *chamber convex* if it is the union of simplices.

* Two chambers meeting in a panel are called *adjacent*.

* A *gallery* joining two points x and y is a chain $(\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_k)$ of chambers such that $x \in \mathcal{C}_0$, $y \in \mathcal{C}_k$ and two consecutive chambers of the sequence are adjacent. The positive integer k is called the *length* of the gallery. A gallery of minimal length joining x and y is called a gallery *stretched between x and y* (see Tits [11]).

* Suppose again $\mathcal{E} \subseteq \Delta$. The intersection of all chamber convex sets containing $\mathcal{E}$ is called the *chamber convex closure of $\mathcal{E}$* and denoted by $\mathrm{cl}(\mathcal{E})$. It is the smallest chamber convex subset of Δ containing $\mathcal{E}$. If $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \ldots \cup \mathcal{E}_n \cup \{v_1\} \cup \{v_2\} \cup \ldots \cup \{v_k\}$, then we denote $\mathrm{cl}(\mathcal{E})$ by $\mathrm{cl}(\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n, v_1, v_2, \ldots, v_k)$.

2.2. Known results and preliminary results.

In this section we collect a number of results either already known or with a short proof. We will need these results in the next paragraph and in section 4.

RESULT 2.2.1 (Tits [12]). (1) *In every germ of sectors $\mathcal{G}$, there exists a unique sector $Q \in \mathcal{G}$ with source b.*

(2) *In every parallel class $c(p)$ of sector panels, there exists a unique one $p' \in c(p)$ with source b.*

RESULT 2.2.2 (Tits [12]). *Every convex subset of Δ which can be isometrically embedded in the standard apartment* **A** *is contained in an apartment of Δ.*

RESULT 2.2.3 (Tits [11]). *Let $x, y \in \mathrm{Ve}(\Delta)$. If $\mathrm{cl}(x,y)$ containes at least one chamber, then $\mathrm{cl}(x,y)$ is the union of all chambers* *of all galleries stretched between x and y. In particular, $\mathrm{cl}(x,y)$ is contained in every apartment through x and y.*

RESULT 2.2.4 (Tits [12]). *Suppose $\Sigma,\Sigma' \in \mathcal{A}p(\Delta)$ and let $\Sigma_\infty,\Sigma'_\infty$ be the corresponding quadrangles in Δ_∞ as in proposition*(1.3). *Suppose*

$$\Sigma_\infty = \{X_j \mid j \in \mathbf{Z}(\mathrm{mod}8) \text{ and } X_j \; I \; X_{j+1}\},$$

$$\Sigma'_\infty = \{X'_j \mid j \in \mathbf{Z}(\mathrm{mod}8) \text{ and } X'_j \; I \; X'_{j+1}\}.$$

If $X_1 = X'_1$, $X_2 = X'_2$, $X_3 = X'_3$, $X_4 = X'_4$ and $X_5 = X'_5$, then $\Sigma \cap \Sigma'$ is a half apartment bounded by a wall $\mathcal{M}$ having $\{X_1, X_5\}$ as trace at infinity. The wall $\mathcal{M}$ is straight if $X_1 \in \mathcal{P}(\Delta_\infty)$ and $\mathcal{M}$ is diagonal if $X_1 \in \mathcal{L}(\Delta_\infty)$.

PROOF. Follows from the discussion in Tits [12,paragraph 8].

Q.E.D.

We now define a point-line geometry $\mathcal{W}_\infty = (\mathcal{P}(\mathcal{W}_\infty), \mathcal{L}(\mathcal{W}_\infty), I)$ as follows.

$\mathcal{P}(\mathcal{W}_\infty)$ = set of straight sectorpanels with source b,

$\mathcal{L}(\mathcal{W}_\infty)$ = set of diagonal sectorpanels with source b.

A straight sectorpanel p is incident with a diagonal sectorpanel ℓ if $p \cup \ell$ bounds some sector (necessarily with source b). Result 2.2.1 readily implies

RESULT 2.2.5. *The geometry $\mathcal{W}_\infty$ is a generalized quadrangle isomorphic with Δ_∞.*

NOTATION. Let $p \in \mathrm{Sp}(\Delta)$. Then we denote by $p^\infty \in c(p)$ the unique sectorpanel parallel to p having source b and we call p^∞ *the trace at infinity of p. Let* $Q \in \mathrm{Se}(\Delta)$ and suppose Q is bounded by the two sectorpanels p and ℓ. Then we denote $Q^\infty = \{p^\infty, \ell^\infty\}$ and call this *the trace at infinity of Q. The trace at infinity* *of a wall* $\mathcal{M}$ is the pair $\{p^\infty, q^\infty\}$ if $p^\infty, \ell^\infty \in \mathrm{Sp}(\Delta, b)$ and they are parallel to some respective sectorpanel contained in $\mathcal{M}$. Similarly for the *trace at infinity of an apartment*. Note that, if $\Sigma \in \mathrm{Ap}(\Delta)$ and if we identify $\mathcal{W}_\infty$ with Δ_∞ in the natural way induced by result 2.2.1, then we have identified Σ^∞ with Σ_∞ of proposition 1.3.

From now on, we assume that every sector and every sectorpanel we mention has source b, except when explicitly mentioned.

RESULT 2.2.6. *Suppose* $(\ldots, v_{-n}, \ldots, v_{-2}, v_{-1}, v_0, v_1, v_2, \ldots, v_n, \ldots)$ *is an infinite sequence of consecutive adjacent vertices of* Δ *such that* $\mathrm{cl}(v_{k-1}, v_{k+1}) = [v_{k-1}, v_{k+1}]$ *for every integer* k. *Then there exists a unique wall* $\mathcal{M}$ *for which the set of vertices on* $\mathcal{M}$ *is exactly* $\{v_k | k \in \mathbf{Z}\}$.

PROOF. If $\mathcal{M}$ exists, then $\mathcal{M}$ is apparently unique since all points of $\mathcal{M}$ are determined by the vertices on $\mathcal{M}$ via chamber convex closure. We now show that $\mathrm{cl}(v_{-j}, v_j) = [v_{-j}, v_j]$, $\forall j \in \mathbf{N}^*$ by induction on j. For j=1, this is part of the assumptions. Suppose now j>1. Consider an apartment Σ through the simplices $[v_{-j}, v_{-j+1}]$ and $[v_j, v_{j-1}]$ (Σ exists by proposition 1.2.2). By induction, $\mathrm{cl}(v_{j-1}, v_{-j+1}) = [v_{j-1}, v_{-j+1}] \subseteq \Sigma$. Hence $\mathrm{cl}(v_{j-1}, v_{-j+1})$ is contained in a wall $\mathcal{M}'$ of Σ. But since also $\mathrm{cl}(v_j, v_{j-2})$ and $\mathrm{cl}(v_{-j}, v_{-j+2})$ are intervals in Σ, apparently $\mathrm{cl}(v_{-j}, v_j)$ is an interval contained in $\mathcal{M}'$. The result now follows from result 2.2.2. Q.E.D.

For every vertex v of Δ, we denote by $\mathrm{R}(v)$ the residue of v in Δ. It is a rank 2 geometry corresponding to a generalized digon or a generalized quadrangle. In both cases we defined the notion of *opposite varieties*.

The next result follows readily from the previous.

RESULT 2.2.7. *Supppose* $(\ldots, v_{-n}, \ldots, v_{-2}, v_{-1}, v_0, v_1, v_2, \ldots, v_n, \ldots)$ *is an infinite sequence of consecutively adjacent vertices of* Δ *such that* v_{k-1} *and* v_{k+1} *are opposite in* $\mathcal{R}(v_k)$ *for every integer* k. *Then there is a unique wall having as set of vertices exactly* $\{v_k \mid k \in \mathbf{Z}\}$.

RESULT 2.2.8. *Suppose* $p, q \in \mathrm{Sp}(\Delta, b)$ *en let* v_p *resp.* v_q *be the vertex on* p *resp.* q *adjacent to* b. *If* $\mathrm{cl}(v_p, v_q) = [v_p, v_q]$ *or if* v_p *and* v_q *are opposite in* $\mathcal{R}(b)$, *then* $p \cup q$ *is a wall.*

PROOF. Let v_j resp. v_{-j} be the vertex on p resp. q for which $d(b, v_j) = j$ resp. $d(b, v_{-j}) = j$, $j \in \mathbf{N}$. Then $(v_k)_{k \in \mathbf{Z}}$ satisfies the assumptiond of result 2.2.6 or 2.2.7 and hence there is a wall M containing v_k for all k. By looking in an apartment containing M, one easily sees $M = p \cup q$. Q.E.D.

RESULT 2.2.9. *Suppose* $Q_1, Q_2 \in \mathrm{Se}(\Delta, b)$, $Q_1 \cap Q_2 = p \in \mathrm{Sp}(\Delta, b)$ *and let* $\Sigma \in \mathrm{Ap}(\Delta)$ *with* $Q_1^\infty \cup Q_2^\infty \subseteq \Sigma^\infty$. *Then there exist sectors* $Q_1' \subseteq Q_1$, $Q_2' \subseteq Q_2$, *not necessarily with source* b, *but with common source* $s \in p$ *and such that* $Q_1' \cup Q_2' \subseteq \Sigma$.

PROOF. Let $\mathcal{G}_i$ be the germ of sectors containing Q_i, $i=1,2$. By the assumptions, Σ contains sectors $Q_i^\star \in \mathcal{G}_i$. By definition of germ, there exist sectors $Q_i'' \subseteq Q_i \cap Q_i^\star$ and $Q_i'' \in \mathcal{G}_i$, $i=1,2$. Hence also $Q_i'' \subseteq \Sigma$. Let s_i be the source of Q_i'', $i=1,2$, then s_i lies in Q_i. Let p_i be the unique element of $c(p)$ with source s_i. It is clear that we can join p_1 and p_2 via $Q_1 \cup Q_2$ by a sequence of sectorpanels $(p_1 = q_1, q_2, \ldots, q_k = p, \ldots, q_n = p_2)$ where $q_j \subseteq Q_1$ for $1 \le j \le k$; $q_j \subseteq Q_2$ for $k \le j \le n$ and two consecutive sectorpanels are at minimal distance from one another. But a similar sequence with the same properties can be found in Σ since $s_1, s_2 \in \Sigma$. By Tits [12,proposition 1] both sequences must have same length and moreover the j^{th} element of one sequence must meet the j^{th} element of the other sequence in a sectorpanel. Hence p meets Σ.

Let s be a vertex on p in Σ and $Q_i' \in \mathcal{G}_i$ with source s, $i=1,2$, then the result follows. Q.E.D.

This result is crucial in the proof of the next one.

RESULT 2.2.10. *Suppose* $\{Q_i \mid i=1,\ldots,k\}$, $k \in \{2,3,4,5,6,7,8\}$ *but fixed, is a set of* k *distinct sectors with source* b *and such that* $Q_i \cap Q_{i+1} = p_i \in \mathrm{Sp}(\Delta,b)$, $i=1,2,\ldots,k-1$. *Suppose* $Q_1^\infty = \{p_0,p_1\}$ *(that defines* p_0*) and* $Q_k^\infty = \{p_{k-1},p_k\}$ *(that defines* p_k*). We assume* $p_i \cap p_j = \{b\}$ *for all* $i,j \in \{0,1,\ldots,k\}$, $i\neq j$. *If* $\{p_0,p_1,\ldots,p_k\}$ *as a set of varieties of* $\mathcal{W}_\infty$ *is contained in an ordinary quadrangle of* $\mathcal{W}_\infty$ *or if* $k \leq 5$, *then* $\cup\{Q_i \mid i=1,\ldots,k\}$ *is a* k*-fold sector.*

PROOF. In this proof, we take all indices modulo 8. We prove the result in two steps.

(1) If $k\neq 8$, then we show that there exist sectors $Q_{k+1},\ldots,$ Q_8 such that

(A) $Q_i \cap Q_{i+1} = p_i \in \mathrm{Sp}(\Delta,b)$, for all i,

(B) $p_i \cap p_j = \{b\}$ for all $i \neq j$,

(C) $\{p_i \mid i \in \mathbf{Z}(\mathrm{mod}\ 8)\}$ is an ordinary quadrangle in $\mathcal{W}_\infty$.

For $k = 8$, this is equivalent to the assumptions. If $k = 5, 6$ or 7, then we complete $\{p_0,p_1,\ldots,p_k\}$ to a quadrangle in $\mathcal{W}_\infty$ (possible by assumption). This quadrangle defines $8-k$ new sectors Q_i, $i = k,\ldots,8$ satisfying (A) and (C) obviously. Intersecting the sectorpanels with $\mathrm{R}(b)$, one sees that also (B) holds.

Now suppose $k = 2,3$ or 4. For every given sector $Q \in \mathrm{Se}(\Delta,b)$, $Q^\infty = \{p,\ell\}$, there exists at least one sector $Q' \in \mathrm{Se}(\Delta,b)$ such that $Q'^\infty = \{p,m\}$, where $\ell \cap m = \{b\}$ (indeed, consider an arbitrary apartment through Q). We call in this proof Q' a *p-neighbour of*

Q. Now define Q_{k+1} as an arbitrary p_k-neighbour of Q_k and let p_{k+1} be the sectorpanel with source b distinct from p_k and lying on the boundery of Q_{k+1}. For $k = 2$ or 3, we define similarly Q_{k+2} as a p_{k+1}-neighbour of Q_{k+1} and p_{k+2} is distinct from p_{k+1} and bounds Q_{k+2}. For $k = 2$ we define similarly again a sector Q_5 and a sectorpanel p_5. Similarly as above, we see that the five sectors $Q_1,\ldots,Q_5$ meet the assumptions of the present proposition for $k = 5$. Hence the assertion will follow if we show that $Q_1 \cup Q_2 \cup \ldots \cup Q_8$ is an apartment, $Q_6,\ldots Q_8$ defined above and satisfying (A),(B),(C).

(2) Let $\Sigma \in \mathrm{Ap}(\Delta)$ be such that $\Sigma^{\infty} = \{p_i \mid i=0,1,\ldots,7\}$. By result 2.2.9, Σ contains double sectors $\mathfrak{D}_{i,i+1} \subseteq Q_i \cup Q_{i+1}$. By the same result, Σ contains the sectorpanels $\mathfrak{D}_{1,2} \cap p_1$ and $\mathfrak{D}_{5,6} \cap p_5$. But by (B) and result 2.2.7, $p_1 \cup p_5$ is a wall (look at the residue of b). Since a wall is convex, $p_1 \cup p_5$ is contained in Σ. Hence b lies in Σ and so do all sectorpanels p_i, $i=0, 1,\ldots,7$, en hence also all sectors Q_i, $i=1,2,\ldots,8$. Hence the result. Q.E.D.

The next result is an immediate consequence of result 2.2.10.

RESULT 2.2.11. *Let* $\Sigma \in \mathrm{Ap}(\Delta)$, $\Sigma^{\infty}=\{p_i \mid i\,(\mathrm{mod}\ 8),\ p_i\ I\ p_{i+1}\ \textit{in}\ \mathcal{W}_{\infty}\}$ *and for every special vertex* s, *let* $p_i(s)$ *be the unique element of* $c(p_i)$ *with source* s. *Then the following conditions are equivalent.*

(1) $s \in \Sigma$,

(2) *the sectorpanels* $p_1(s)$, $p_3(s)$, $p_5(s)$ *and* $p_7(s)$ *have pairwise intersection* $\{s\}$,

(3) *the sectorpanels* $p_1(s)$, $p_4(s)$ *and* $p_7(s)$ *have pairwise intersection* $\{s\}$.

RESULT 2.2.12. (1) (Δ, d_Δ) *is a metric space.*

(2) *If* $x, y \in \mathrm{Ve}(\Delta)$, *then there exists a unique point* $z \in \Delta$ *such that for given* $n < d_\Delta(x,y)$ *we have* $d_\Delta(x,z) = n = d_\Delta(x,y) - d_\Delta(z,y)$.

PROOF. (1) follows from Tits [12].

(2). The proof in [3, §2.5.4] can be modified for axiomatic affine buildings. Q.E.D.

2.3. The n^{th} *floor of* Δ *with basement* b.

DEFINITION. We define a rank 2 geometry $\mathcal{W}_n = (\mathcal{P}(\mathcal{W}_n), \mathcal{L}(\mathcal{W}_n), I)$ as follows.

$$\mathcal{P}(\mathcal{W}_n) = \{\mathcal{P} \in \mathrm{Ve}(\Delta) \mid \mathcal{P} \in p \in \mathcal{P}(\Delta_\infty) \text{ and } d(\mathcal{P}, b) = n\},$$

$$\mathcal{L}(\mathcal{W}_n) = \{\mathcal{L} \in \mathrm{Ve}(\Delta) \mid \mathcal{L} \in \ell \in \mathcal{L}(\Delta_\infty) \text{ and } d(\mathcal{L}, b) = n\},$$

An element $\mathcal{P}$ of $\mathcal{P}(\mathcal{W}_n)$ is incident with an element $\mathcal{L}$ of $\mathcal{L}(\mathcal{W}_n)$ if $\mathcal{P}$ and $\mathcal{L}$ lie on some common sector with source b.

The geometry $\mathcal{W}_n$ is called the n^{th} floor of Δ with basement b, n a positive integer. If $n = 0$, $\mathcal{W}_0 = (\{b\}, \{b\}, I)$ with $b\, I\, b$. If $n = 1$, then $\mathcal{W}_1$ is exactly the residue $\mathrm{R}(b)$.

There exists a natural canonical epimorphism $\Pi^n_{n-1} : \mathcal{W}_n \to \mathcal{W}_{n-1}$ mapping a point or a line $\mathcal{X}$ of $\mathcal{W}_n$ to the vertex $\mathcal{X}'$ of $[b, \mathcal{X}]$ adjacent to $\mathcal{X}$.

If $\mathcal{X}$ is a point or a line of $\mathcal{W}_n$ and x is a sectorpanel through $\mathcal{X}$, then we say that x *represents* $\mathcal{X}$.

Now we will show some lemmas which will allow us to prove our main theorem in section 4.

LEMMA 2.3.1. *The inverse limit* $\varprojlim_{n \in \mathbf{N}} \mathcal{W}_n$ *with respect to* Π^n_{n-1} *is isomorphic to* $\mathcal{W}_\infty$.

PROOF. Let $(\mathcal{P}_n)_{n\in\mathbf{N}}$ be a point of the inverse limit with $\mathcal{P}_n \in \mathcal{P}(\mathcal{W}_n)$ and $\Pi^n_{n-1}(\mathcal{P}_n) = \mathcal{P}_{n-1}$. This defines a unique straight sectorpanel $p = \cup\{[b,\mathcal{P}_n] \mid n \in \mathbf{N}\}$ by result 2.2.2 (or result 2.2.6). Similarly for any line $(\mathcal{L}_n)_{n\in\mathbf{N}}$ (it defines a unique diagonal sectorpanel ℓ). Keeping this notation, we assume now that $\mathcal{P}_n$ I $\mathcal{L}_n$, for all $n \in \mathbf{N}$. Then we can embed $\cup\{\mathrm{cl}(b,\mathcal{P}_n,\mathcal{L}_n) \mid n \in \mathbf{N}\}$ in the standard apartment $(\mathbf{A},\tau)$ (since $\mathrm{cl}(b,\mathcal{P}_n,\mathcal{L}_n)$ is a subset of $\mathrm{cl}(b,\mathcal{P}_{n+1},\mathcal{L}_{n++}))$ and hence $\cup\{\mathrm{cl}(b,\mathcal{P}_n,\mathcal{L}_n) \mid n \in \mathbf{N}\}$ is a sector bounded by p and ℓ. Hence $p\ I\ \ell$ in $\mathcal{W}_\infty$. The inverse mapping is obvious and hence both geometries are indeed isomorphic. Q.E.D.

LEMMA 2.3.2. *Suppose $\mathcal{P} \in \mathcal{P}(\mathcal{W}_n)$ and $\mathcal{L} \in \mathcal{L}(\mathcal{W}_n)$ for $n \neq 0$ and let $\mathcal{P}$ I $\mathcal{L}$. For every sectorpanel p representing $\mathcal{P}$, there exists a sectorpanel ℓ representing $\mathcal{L}$ and such that $p\,I\,\mathcal{L}$ in $\mathcal{W}_\infty$.*

PROOF. Let Σ be an apartment containing b, $\mathcal{P}$ and $\mathcal{L}$ (existing since $\mathcal{P}\,I\,\mathcal{L}$). Let p' resp. ℓ' be the unique sectorpanel in Σ representing $\mathcal{P}$ resp. $\mathcal{L}$. Denote by q and m the unique sectorpanels in Σ such that $\ell'\ I\ q\ I\ m$ in $\mathcal{W}_\infty$ with $q \neq p'$ and $m \neq \ell'$. Since $\mathcal{W}_\infty$ is a generalized quadrangle, there exists a unique chain of sectorpanels $m\ I\ p''\ I\ \ell\ I\ p$. Looking in $\mathcal{W}_1$, one can see that the sectors bounded by these sectorpanels satisfy the assumptions of result 2.2.10. So $m \cup p$ bounds a 3-fold sector ${}^3Q$. The intersection $\Sigma \cap {}^3Q$ contains m and $\mathcal{P}$ and hence also $\mathrm{cl}(m,\mathcal{P})$, which clearly contains $\mathcal{L}$. Hence ℓ represents $\mathcal{L}$. Q.E.D.

LEMMA 2.3.3. *Suppose $\mathcal{P},\mathcal{P}^* \in \mathcal{P}(\mathcal{W}_n)$, $\mathcal{L} \in \mathcal{L}(\mathcal{W}_n)$, $\mathcal{P}\,I\,\mathcal{L}\,I\,\mathcal{P}^*$ and $[b,\mathcal{P}] \cap [b,\mathcal{P}^*] = \{b\}$. Then $b,\mathcal{P},\mathcal{L},\mathcal{P}^*$ lie in a common apartment and hence there exist sectorpanels p,ℓ,p^* representing resp. $\mathcal{P},\mathcal{L},\mathcal{P}^*$ such that $p\,I\,\ell\,I\,p^*$ in $\mathcal{W}_\infty$.*

PROOF. Suppose Σ is an apartment containing $b,\mathcal{P}$ and $\mathcal{L}$ and suppose Σ^* is an apartment containing $b,\mathcal{P}^*$ and $\mathcal{L}$. Let $\overline{p},\overline{\ell}$ resp. $\overline{p}^*,\overline{\ell}^*$ be the sectorpanels in Σ resp. Σ^* representing $\mathcal{P},\mathcal{L}$ resp.

$\mathcal{P}^*, \mathcal{L}$. Let m resp. m^* be the sectorpanel in Σ resp. Σ^* incident with $\bar{p}$ resp. $\bar{p}^*$ but distinct from $\bar{\ell}$ resp. $\bar{\ell}^*$. Since $[b, \mathcal{P}] \cap [b, \mathcal{P}^*] = \{b\}$, $m \cap \mathcal{W}_1$ and $m^* \cap \mathcal{W}_1$ are opposite in $\mathcal{W}_1$, hence by result 2.2.8, $m \cup m^*$ is a wall. Let $m_{\mathcal{L}}$ resp. $m^*_{\mathcal{L}}$ be the unique sectorpanel with source $\mathcal{L}$ and parallel to m resp. m^*. By looking in $R(\mathcal{L})$ and applying result 2.2.8. again (for $\mathcal{L}$ playing the role of b), we see that $m_{\mathcal{L}} \cup m^*_{\mathcal{L}}$ is a wall. But looking in Σ and Σ^*, we see that this wall contains $\mathcal{P}$, $\mathcal{L}$ and $\mathcal{P}^*$. By Tits [12, proposition 17.3], $m \cup m^*$ and $m_{\mathcal{L}} \cup m^*_{\mathcal{L}}$ lie in a common apartment $\Sigma_{\mathcal{L}}$ containing $b, \mathcal{P}, \mathcal{P}^*$ and $\mathcal{L}$. This shows the first part of the lemma. The second part now follows by considering the sectorpanels in $\Sigma_{\mathcal{L}}$ representing $\mathcal{P}$ resp. $\mathcal{P}^*, \mathcal{L}$. Q.E.D.

LEMMA 2.3.4. *Suppose $\mathcal{P}, \mathcal{P}^* \in \mathcal{P}(\mathcal{W}_r)$, $\mathcal{L} \in \mathcal{L}(\mathcal{W}_r)$, $\mathcal{P}\, I\, \mathcal{L}\, I\, \mathcal{P}^*$ and $[b, \mathcal{P}] \cap [b, \mathcal{P}^*] = \{b\}$. For every sectorpanel p representing $\mathcal{P}$, there exists an apartment containing $p, \mathcal{L}, \mathcal{P}^*$ and hence there exist sectorpanels ℓ, p^* representing resp. $\mathcal{L}, \mathcal{P}^*$ such that $p\, I\, \ell\, I\, p^*$ in $\mathcal{W}_\infty$.*

PROOF. Similarly to the proof of lemma 2.3.2, using lemma 2.3.3. Q.E.D.

LEMMA 2.3.5. *Suppose $\mathcal{P} \in \mathcal{P}(\mathcal{W}_r)$, $\mathcal{L}, \mathcal{L}^* \in \mathcal{L}(\mathcal{W}_r)$, $\mathcal{L}\, I\, \mathcal{P}\, I\, \mathcal{L}^*$ and $[b, \mathcal{L}] \cap [b, \mathcal{L}^*] = \{b\}$. For every sectorpanel p representing $\mathcal{P}$, there exists an apartment containing $p, \mathcal{L}, \mathcal{L}^*$ and hence there exist sectorpanels ℓ, ℓ^* representing resp. $\mathcal{L}$ and $\mathcal{L}^*$ such that $\ell\, I\, p\, I\, \ell^*$ in $\mathcal{W}_\infty$.*

PROOF. This is a consequence of result 2.2.10 and lemma 2.3.2. Q.E.D.

NOTATION. Two lines $\mathcal{L}$ and $\mathcal{L}'$ of $\mathcal{W}_r$ are called *concurrent* (notation $\mathcal{L} \perp \mathcal{L}'$) if they share a common point. Dually, one defines *collinear points* (denoted with the same symbol $\perp$).

LEMMA 2.3.6. *Suppose* $\mathcal{L},\mathcal{L}^* \in \mathcal{L}(\mathcal{W}_{\mathcal{b}})$, $\mathcal{L} \perp \mathcal{L}^*$ *and* $[\mathcal{b},\mathcal{L}] \cap [\mathcal{b},\mathcal{L}^*] = \{\mathcal{b}\}$. *For every sectorpanel* ℓ *representing* $\mathcal{L}$, *there exists an apartment containing* ℓ *and* $\mathcal{L}^*$ *and hence there exists a sectorpanel* ℓ^* *representing* $\mathcal{L}^*$ *such that* $\ell\ I\ \ell^*$ *in* $\mathcal{W}_\infty$.

PROOF. Similarly to the proof of lemma 2.3.2 using lemma 2.3.5.

Q.E.D.

LEMMA 2.3.7. *Suppose* $\mathcal{P},\mathcal{P}^* \in \mathcal{P}(\mathcal{W}_{\mathcal{b}})$ *and* $[\mathcal{b},\mathcal{P}] \cap [\mathcal{b},\mathcal{P}^*] = \{\mathcal{b}\}$. *The following conditions are equivalent.*

(1) $\mathcal{P} \perp \mathcal{P}^*$,

(2) $d_\Delta(\mathcal{P},\mathcal{P}^*) = 2n$,

(3) $d_\Delta(\mathcal{P},\mathcal{P}^*) \leq 2n$.

PROOF. This is trivial for $n=0$, so suppose $n\neq 0$.
(1) $\Rightarrow$ (2) by lemma 2.3.3.
(2) $\Rightarrow$ (3) is trivial.
(3) $\Rightarrow$ (1). Suppose that $\mathcal{P}$ is not collinear with $\mathcal{P}^*$. Denote by $\mathcal{P}_1$ resp. $\mathcal{P}_1^*$ the vertex on $[\mathcal{b},\mathcal{P}]$ resp. $[\mathcal{b},\mathcal{P}^*]$ adjacent to $\mathcal{b}$. Assume for a moment that $\mathcal{P}_1$ and $\mathcal{P}_1^*$ are opposite in $\mathcal{W}_1$. Then $\mathrm{cl}(\mathcal{P},\mathcal{P}^*) = [\mathcal{P},\mathcal{P}^*]$ by result 2.2.8. Since this is a straight interval, we have $d_\Delta(\mathcal{P},\mathcal{P}^*) = \sqrt{2}.n > n$ by proposition 1.2.1. Hence $\mathcal{P}_1$ and $\mathcal{P}_1^*$ are collinear in $\mathcal{W}_1$. But by assumption, they are distinct. Let Σ be an apartment through $\mathcal{P}$ and $\{\mathcal{b},\mathcal{P}_1^*\}$ and let p be the sectorpanel in Σ representing $\mathcal{P}$. Let ℓ be the sectorpanel in Σ incident with p in $\mathcal{W}_\infty$ but such that $\mathcal{L}_1 = \mathcal{W}_1 \cap \ell$ is not incident with $\mathcal{P}_1^*$ in $\mathcal{W}_1$. Let p^* be an arbitrary sector panel representing $\mathcal{P}^*$. By result 2.2.10, the unique chain (in $\mathcal{W}_\infty$) $\ell\ I\ q\ I\ m_q\ I\ p^*$ (defining q and m_q) determines a 3-fold sector (with boundary $\ell \cup p^*$) lying in some apartment $\Sigma_{q,\mathcal{r}}$; the sectorpanels of this apartment are given by (and this defines $\ell^*,\mathcal{r}^*,m_{\mathcal{r}}$ and $\mathcal{r}$) $\ell\ I\ q\ I\ m_q\ I\ p^*\ I\ \ell^*\ I\ \mathcal{r}^*\ I\ m_{\mathcal{r}}\ I\ \mathcal{r}\ I\ \ell$ (see figure 1).

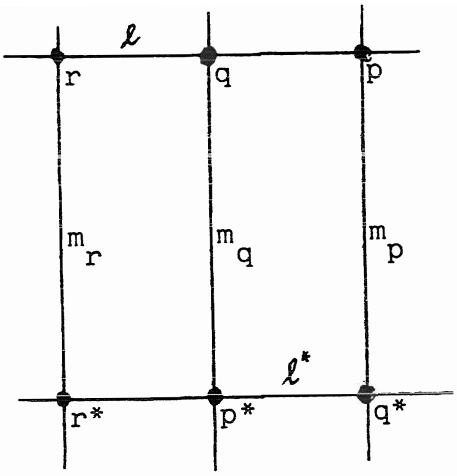


FIGURE 1

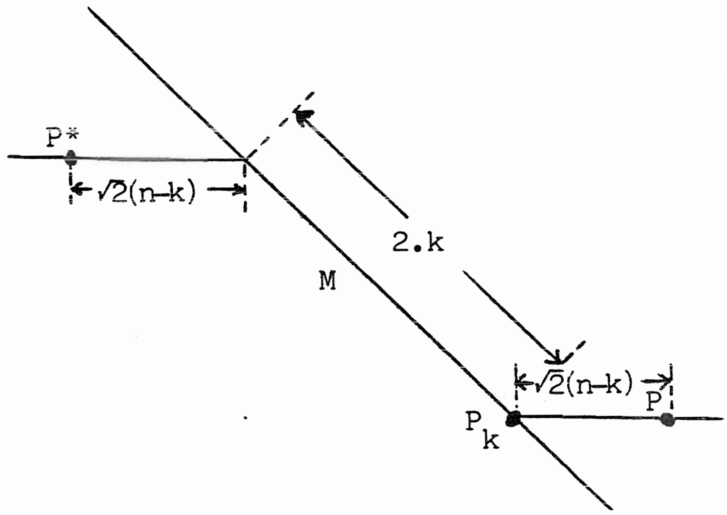


FIGURE 2

Note that $\Sigma \cap \Sigma_{q,r}$ contains $\mathcal{P}_1^*$ and ℓ and hence also $\mathcal{P}_1$ (in Σ obviously $\mathcal{P}_1 \in \mathrm{cl}(\ell,\mathcal{P}_1^*)$!). So $[s,\mathcal{P}] \cap q$ contains more than just s. So put $[s,\mathcal{P}] \cap q = [s,\mathcal{P}_k]$ with $\mathcal{P}_k \in \mathcal{P}(W_k)$, $0<k<n$ ($k=n$ implies $\mathcal{P}_k = \mathcal{P}$ and hence $\mathcal{P} \perp \mathcal{P}^*$). As above, the chain $p\ I\ m_p\ I\ q^*\ I\ \ell^*$ (and this defines uniquely m_p and q^*) determines a 3-fold sector. It lies in the apartment $\Sigma_{r,p}$ determined by $(\Sigma_{r,p})_\infty = \{c(p), c(m_p), c(q^*), c(\ell^*), c(r^*), c(m_r), c(r), c(\ell)\}$ since by result 2.2.11, $s \in \Sigma_{r,p}$. Hence $\Sigma_{r,p}$ contains the sectorpanels $p, m_p, q^*, \ell^*, r^*, m_r, r$ and ℓ. But by result 2.2.4, the intersection $\Sigma_{q,r} \cap \Sigma_{r,p}$ is a half apartment bounded by a wall M parallel to $\ell \cup \ell^*$ in both apartments. Since $p \cap q = [s,\mathcal{P}_k]$, we see that $\mathcal{P}_k \in M$. Considering the sectorpanels with source $\mathcal{P}_k$ belonging to resp. $c(p), c(m_p), c(q^*), c(\ell^*), c(p^*), c(m_q), c(q)$ and $c(\ell)$, then looking in $\Sigma_{q,r}$ and $\Sigma_{r,p}$, we see that the apartment $\Sigma_{p,q}$ defined by $(\Sigma_{p,q})_\infty = \{c(p), c(m_p), c(q^*), c(\ell^*), c(p^*), c(m_q), c(q), c(\ell)\}$ is exactly equal to $((\Sigma_{q,r} \cup \Sigma_{r,p}) - (\Sigma_{q,r} \cap \Sigma_{r,p})) \cup M$. Now figure 2 shows us the situation in that apartment. We easily deduce that $d_\Delta^2(\mathcal{P},\mathcal{P}^*) = 2(2n-k)^2 + 2.k^2$ and so $d_\Delta(\mathcal{P},\mathcal{P}^*) > 2.n$ since $k<n$.

Q.E.D.

LEMMA 2.3.8. *Suppose* $\mathcal{P} \in \mathcal{P}(\mathcal{W}_n)$ *and* $\mathcal{L} \in \mathcal{L}(\mathcal{W}_n)$. *Then the following conditions are equivalent.*

(1) $\mathcal{P}\ I\ \mathcal{L}$,

(2) $d_\Delta(\mathcal{P},\mathcal{L}) = n$,

(3) $d_\Delta(\mathcal{P},\mathcal{L}) \leq n$.

PROOF. This is trivial for $n = 0$, so suppose $n > 0$.
(1) $\Rightarrow$ (2) follows by looking in any apartment containing $b, \mathcal{P}$ and $\mathcal{L}$.
(2) $\Rightarrow$ (3) is trivial.
(3) $\Rightarrow$ (1). Suppose $d_\Delta(\mathcal{P},\mathcal{L}) \leq n$ and let $\mathcal{P}_1$ resp. $\mathcal{L}_1$ be the vertex on $[b,\mathcal{P}]$ resp. $[b,\mathcal{L}]$ adjacent with b. If $\mathcal{P}_1$ and $\mathcal{L}_1$ are not incident in $\mathcal{W}_1$, then (as before) any representative p of $\mathcal{P}$ and any representative ℓ of $\mathcal{L}$ determine a 3-fold sector bounded by $p \cup \ell$. Hence $d_\Delta(\mathcal{P},\mathcal{L}) = \sqrt{5}.n > n$. So $\mathcal{P}_1\ I\ \mathcal{L}_1$ in $\mathcal{W}_1$. Put $\mathcal{C}$ = cl$(b,\mathcal{P}_1,\mathcal{L}_1)$; $\mathcal{C}$ is a chamber. Let Σ be an apartment through $\mathcal{L}$ and $\mathcal{C}$ and let $\mathcal{P}^* \in \mathcal{P}(\mathcal{W}_n)$ be the vertex in Σ satisfying $\mathcal{P}^*\ I\ \mathcal{L}$ in $\mathcal{W}_n$ and the unique vertex of $[b,\mathcal{P}^*]$ adjacent to b is not $\mathcal{P}_1$. Then $[b,\mathcal{P}] \cap [b,\mathcal{P}^*] = \{b\}$ and $d_\Delta(\mathcal{P},\mathcal{P}^*) \leq d_\Delta(\mathcal{P},\mathcal{L}) + d_\Delta(\mathcal{L},\mathcal{P}^*) \leq 2.n$. By lemma 2.3.7, $\mathcal{P} \perp \mathcal{P}^*$ and so $d_\Delta(\mathcal{P},\mathcal{P}^*) = 2.n$, hence $d_\Delta(\mathcal{P},\mathcal{L}) = n$. Let Σ^* be the apartment through $b,\mathcal{P}$ and $\mathcal{P}^*$ (existing by lemma 2.3.3) and let $\mathcal{L}^* \in \mathcal{L}(\mathcal{W}_n)$ be such that $\mathcal{L}^* \in \Sigma^*$ and $\mathcal{P}\ I\ \mathcal{L}^*\ I\ \mathcal{P}^*$ in $\mathcal{W}_n$. Clearly $n = d_\Delta(\mathcal{P},\mathcal{L}^*) = d_\Delta(\mathcal{L}^*,\mathcal{P}^*) = \frac{1}{2}.d_\Delta(\mathcal{P},\mathcal{P}^*) = d_\Delta(\mathcal{P},\mathcal{L}) = d_\Delta(\mathcal{L},\mathcal{P}^*)$ and by result 2.2.12(2), $\mathcal{L} = \mathcal{L}^*$ and hence $\mathcal{P}\ I\ \mathcal{L}$. Q.E.D.

3. HJELMSLEV QUADRANGLES OF LEVEL n.

3.1. Notation.

Suppose $X = (\mathcal{P}(X), \mathcal{L}(X), I)$ is a point-line incidence geometry with point set $\mathcal{P}(X)$, line set $\mathcal{L}(X)$ and symmetric incidence relation I. We denote the set of points incident with a given line $\mathcal{L}$ by

$\sigma(\mathcal{L})$ and call it the *shadow (of $\mathcal{L}$)* (see Buekenhout [4]). We also keep the notation $\perp$ previously defined. A *flag* in X is an incident point-line pair of X. The set of flags of X is denoted by $\mathcal{F}(X)$. A *morphism* from X to some other point-line incidence geometry $X' = (\mathcal{P}(X'),\mathcal{L}(X'),I)$ maps $\mathcal{P}(X)$ to $\mathcal{P}(X')$, $\mathcal{L}(X)$ to $\mathcal{L}(X')$ and the map induced on $\mathcal{F}(X)$ maps $\mathcal{F}(X)$ to $\mathcal{F}(X')$. An *epimorphism* is a morphism which is surjective on the set of flags. We call X *thick* if every line is incident with at least three points and every point is incident with at least three lines.

Suppose $\mathcal{A}$ is an arbitrary set and $\mathbf{P}_1(\mathcal{A})$ and $\mathbf{P}_2(\mathcal{A})$ are two arbitrary partitions of $\mathcal{A}$. Then we say that $\mathbf{P}_1(\mathcal{A})$ is *properly finer than* $\mathbf{P}_2(\mathcal{A})$ if *every* class of $\mathbf{P}_2(\mathcal{A})$ is the union of at least *two* classes of $\mathbf{P}_1(\mathcal{A})$. In that case, we denote

$$\mathbf{P}_2(\mathcal{A})/\mathbf{P}_1(\mathcal{A}) = \{\{\mathcal{C} \in \mathbf{P}_1(\mathcal{A}) \mid \mathcal{C} \subseteq \mathcal{D}\} \mid \mathcal{D} \in \mathbf{P}_2(\mathcal{A})\},$$

which is a partition of $\mathbf{P}_1(\mathcal{A})$. If $\mathcal{D} \in \mathbf{P}_2(\mathcal{A})$, then we call the set $\{\mathcal{C} \in \mathbf{P}_1(\mathcal{A}) \mid \mathcal{C} \subseteq \mathcal{D}\}$ the *canonical image of $\mathcal{D}$ in $\mathbf{P}_2(\mathcal{A})/\mathbf{P}_1(\mathcal{A})$*.

1.1.3. *Definition of a Hjelmslev quadrangle of level n.* Throughout, n, i, j and k denote positive integers. We define a Hjelmslev quadrangle of level n by induction on n. The induction will start with $n = 1$. We give a separate definition for the level 0. We abbreviate "Hjelmslev quadrangle of level n" by "level n HQ".

A *level* 0 *HQ* is any trivial geometry $\mathcal{V}_0 = (\{*\},\{*\},=)$, where * is any arbitrary (but twice the same) symbol.

A *level* 1 *HQ* is any 6-tuple $\mathcal{V}_1 =$

$$(\mathcal{P}(\mathcal{V}_1),\mathcal{L}(\mathcal{V}_1),I,(\mathbf{P}_i(\mathcal{V}_1))_{i\le 1},(\mathbf{L}_i(\mathcal{V}_1))_{i\le 1},(\mathcal{W}_0(\mathcal{V}_1,\{\mathcal{P}\}),\{\mathcal{P}\})_{\mathcal{P}\in\mathcal{P}(\mathcal{V}_1)}),$$

where $(\mathcal{P}(\mathcal{V}_1), \mathcal{L}(\mathcal{V}_1), I)$ is an arbitrary thick generalized quadrangle ; $\mathbf{P}_0(\mathcal{V}_1)$ is the partition of $\mathcal{V}_1$ determined by: every class is a singleton ; $\mathbf{P}_1(\mathcal{V}_1)$ is the partition of $\mathcal{P}(\mathcal{V}_1)$ consisting of one class ; similar for $(\mathbf{L}_i(\mathcal{V}_n))_{i\leq 1}$, and for every $\mathcal{P} \in \mathcal{P}(\mathcal{V}_1)$, $\mathcal{W}_0(\mathcal{V}_1, \{\mathcal{P}\}) = (\{\mathcal{P}\}, \{\mathcal{P}\}, =)$. The last three elements of $\mathcal{V}_1$ do not add more structure to the generalized quadrangle, but they are necessary to start the induction. So in fact, a level 1 HQ "is" a generalized quadrangle.

Now suppose $n \geq 2$. Let $(\mathcal{P}(\mathcal{V}_n), \mathcal{L}(\mathcal{V}_n), I)$ be a thick incidence geometry with at least one non incident point-line pair. Suppose $(\mathbf{P}_i(\mathcal{V}_n))_{i\leq n}$, resp. $(\mathbf{L}_i(\mathcal{V}_n))_{i\leq n}$ is a family of partitions of $\mathcal{P}(\mathcal{V}_n)$, resp. $\mathcal{L}(\mathcal{V}_n)$ satisfying :

(PS1) $\mathbf{P}_0(\mathcal{V}_n) = \{\{\mathcal{P}\} \mid \mathcal{P} \in \mathcal{P}(\mathcal{V}_n)\}$; $\mathbf{P}_n(\mathcal{V}_n) = \{\mathcal{P}(\mathcal{V}_n)\}$,

(PS2) $\mathbf{L}_0(\mathcal{V}_n) = \{\{\mathcal{L}\} \mid \mathcal{L} \in \mathcal{L}(\mathcal{V}_n)\}$; $\mathbf{L}_n(\mathcal{V}_n) = \{\mathcal{L}(\mathcal{V}_n)\}$,

(PS3) $\mathbf{P}_i(\mathcal{V}_n)$ is properly finer than $\mathbf{P}_{i+1}(\mathcal{V}_n)$, for all $i < n$,

(PS4) $\mathbf{L}_i(\mathcal{V}_n)$ is properly finer than $\mathbf{L}_{i+1}(\mathcal{V}_n)$, for all $i < n$,

The elements of $\mathbf{P}_i(\mathcal{V}_n)$, resp $\mathbf{L}_i(\mathcal{V}_n)$ are called *i-point-neighbourhoods*, resp. *i-line-neighbourhoods (of their elements)*. An i-point-neighbourhood is also called a *point-neighbourhood*, an *i-neighbourhood* or briefly a *neighbourhood*. Similar definitions for i-line-neighbourhoods. If $\mathcal{P} \in \mathcal{P}(\mathcal{V}_n)$ and $\mathcal{L} \in \mathcal{L}(\mathcal{V}_n)$, then we denote by $\mathcal{O}^i(\mathcal{P})$, resp. $\mathcal{O}^i(\mathcal{L})$ the unique i-point-neighbourhood of $\mathcal{P}$, resp. i-line- neighbourhood of $\mathcal{L}$.

Suppose for every $\mathcal{C} \in \mathbf{P}_{n-1}(\mathcal{V}_n)$, we have a level $(n-1)$ HQ, denoted by $\mathcal{W}_{n-1}(\mathcal{V}_n, \mathcal{C})$ (this is an element of a well-defined class of objects by induction) and select in every $\mathcal{W}_{n-1}(\mathcal{V}_n, \mathcal{C})$ an $(n-2)$-point-neighbourhood $\mathcal{N}_{\mathcal{C}}$. Then we call the 6-tuple $\mathcal{V}_n$ =

$$(\mathcal{P}(\mathcal{V}_n), \mathcal{L}(\mathcal{V}_n), I, (\mathbf{P}_i(\mathcal{V}_n))_{i\leq n}, (\mathbf{L}_i(\mathcal{V}_n))_{i\leq n}, (\mathcal{W}_{n-1}(\mathcal{V}_n, \mathcal{C}), \mathcal{N}_{\mathcal{C}})_{\mathcal{C}\in\mathbf{P}_{n-1}(\mathcal{V}_n)})$$

a *level* n *HQ* if $\mathcal{V}_n$ satisfies the axioms (IS), (GQ) and (NP) below. The geometry $(\mathcal{P}(\mathcal{V}_n), \mathcal{L}(\mathcal{V}_n), I)$ is called the *base geometry* of $\mathcal{V}_n$. Before stating the actual axioms, we need some preliminaries.

We first define the canonical $(n-1)$-image of $\mathcal{V}_n$ by induction on n. The *canonical* 0-*image* of a level 1 HQ $\mathcal{V}_1$ is by definition the trivial geometry $(\{\mathcal{P}(\mathcal{V}_1)\}, \{\mathcal{P}(\mathcal{V}_1)\}, =)$. Now let $n \geq 2$. Define the geometry $(\mathbf{P}_1(\mathcal{V}_n), (\mathbf{L}_1(\mathcal{V}_n), I)$ as follows. If $\mathscr{C} \in \mathbf{P}_1(\mathcal{V}_n)$ and $\mathfrak{D} \in (\mathbf{L}_1(\mathcal{V}_n)$, then $\mathscr{C}\ I\ \mathfrak{D}$ if and only if there exist $\mathcal{P} \in \mathscr{C}$ and $\mathcal{L} \in \mathfrak{D}$ which are incident in $(\mathcal{P}(\mathcal{V}_n), \mathcal{L}(\mathcal{V}_n), I)$. Furthermore, denote by $\mathcal{W}_{n-2}(\mathcal{V}_n, \mathscr{C})$ the canonical $(n-2)$-image of $\mathcal{W}_{n-1}(\mathcal{V}_n, \mathscr{C})$ (well-defined by the induction hypothesis). Denote by $\mathcal{N}'_{\mathscr{C}}$ the canonical image of $\mathcal{N}_{\mathscr{C}}$ in $\mathbf{P}_{n-2}(\mathcal{W}_{n-1}(\mathcal{V}_n, \mathscr{C}))/\mathbf{P}_1(\mathcal{W}_{n-1}(\mathcal{V}_n, \mathscr{C}))$ if $n > 2$ and $\mathcal{N}'_{\mathscr{C}} = \{\mathcal{P}(\mathcal{W}_1(\mathcal{V}_2, \mathscr{C}))\}$ if $n = 2$. Obviously, there is a bijective correspondence between $\mathbf{P}_{n-1}(\mathcal{V}_n)$ and $\mathbf{P}_{n-1}(\mathcal{V}_n)/\mathbf{P}_1(\mathcal{V}_n)$ and the unique element of $\mathbf{P}_{n-1}(\mathcal{V}_n)/\mathbf{P}_1(\mathcal{V}_n)$ corresponding to the element $\mathscr{C}$ of $\mathbf{P}_{n-1}(\mathcal{V}_n)$ is denoted by $\mathscr{C}^*$. In particular, all elements of $\mathbf{P}_{n-1}(\mathcal{V}_n)/\mathbf{P}_1(\mathcal{V}_n)$ are denoted with a *. We define the *canonical* $(n-1)$-*image* of $\mathcal{V}_n$ as the 6-tuple $\mathcal{V}_{n-1} =$

$$(\mathbf{P}_1(\mathcal{V}_n), (\mathbf{L}_1(\mathcal{V}_n), I, (\mathbf{P}_{i+1}(\mathcal{V}_n)/\mathbf{P}_1(\mathcal{V}_n))_{i \leq n-1}, (\mathbf{L}_{i+1}(\mathcal{V}_n)/\mathbf{L}_1(\mathcal{V}_n))_{i \leq n-1}, (\mathcal{W}_{n-2}(\mathcal{V}_n, \mathscr{C}), \mathcal{N}'_{\mathscr{C}})_{\mathscr{C}^* \in \mathbf{P}_{n-1}(\mathcal{V}_n)/\mathbf{P}_1(\mathcal{V}_n)}).$$

We can now state the very natural axiom (IS).

(IS) *The canonical* $(n-1)$-*image* $\mathcal{V}_{n-1}$ *of* $\mathcal{V}_n$ *is a level* $n-1$ *HQ.*

Using a similar notation for $\mathcal{V}_{n-1}$ as for $\mathcal{V}_n$, (IS) implies e.g. $\mathbf{P}_i(\mathcal{V}_{n-1}) = \mathbf{P}_{i+1}(\mathcal{V}_n)/\mathbf{P}_1(\mathcal{V}_n)$ and similarly for the line-partitions.

Define inductively the *canonical* $(n-j)$-*image* of $\mathcal{V}_n$ $(0 < j \leq n)$ as the canonical $(n-j)$-image $\mathcal{V}_{n-j}$ of the canonical $(n-j+1)$-image $\mathcal{V}_{n-j+1}$ of $\mathcal{V}_n$, or as $\mathcal{V}_n$ (for $j = 0$). Note that $\mathcal{O}^1$ defines a mapping from $\mathcal{P}(\mathcal{V}_n)$ to $\mathcal{P}(\mathcal{V}_{n-1})$ and from $\mathcal{L}(\mathcal{V}_n)$ to $\mathcal{L}(\mathcal{V}_{n-1})$. By the definition of the incidence relation in $\mathcal{V}_{n-1}$, we can see that this mapping is an epimorphism from $(\mathcal{P}(\mathcal{V}_n), \mathcal{L}(\mathcal{V}_n), I)$ onto $(\mathbf{P}_1(\mathcal{V}_n), (\mathbf{L}_1(\mathcal{V}_n), I)$. We denote this epimorphism by Π^n_{n-1}. By the induction hypothesis, a similar epimorphism exists from the base geometry of $\mathcal{V}_{n-j+1}$ onto the base geometry of $\mathcal{V}_{n-j}$ and we denote it by Π^{n-j+1}_{n-j}. By induction, we can put

$$\Pi^n_{n-j} = \Pi^{n-j+1}_{n-j} \circ \Pi^n_{n-j+1}.$$

From now on, we denote the canonical j-image $\mathcal{V}_j$ of $\mathcal{V}_n$ by

$$(\mathcal{P}_i(\mathcal{V}_j), \mathcal{L}_i(\mathcal{V}_j), I, (\mathbf{P}_i(\mathcal{V}_j))_{i \le j}, (\mathbf{L}_i(\mathcal{V}_j))_{i \le j}, (\mathcal{W}_{j-1}(\mathcal{V}_j, \mathcal{C}), \mathcal{N}_{\mathcal{C}})_{\mathcal{C} \in \mathbf{P}_{j-1}(\mathcal{V}_j)}),$$

for all j, $0 < j \le n$. The epimorphism Π^n_j is called a *projection* or a *partition map*. We define the *valuation map*

$$u : (\mathcal{P}(\mathcal{V}_n) \cup \mathcal{L}(\mathcal{V}_n)) \times (\mathcal{P}(\mathcal{V}_n) \cup \mathcal{L}(\mathcal{V}_n)) \to \mathbf{N}$$

as follows. Let x, y be either both points or both lines of $\mathcal{V}_n$, then

$$u(x, y) = \sup\{ j \le n \mid \Pi^n_j(x) = \Pi^n_j(y) \}$$

If $\mathcal{P} \in \mathcal{P}(\mathcal{V}_n)$ and $\mathcal{L} \in \mathcal{L}(\mathcal{V}_n)$, then

$$u(\mathcal{P}, \mathcal{L}) = u(\mathcal{L}, \mathcal{P}) = (u_1(\mathcal{P}, \mathcal{L}), u_2(\mathcal{P}, \mathcal{L}))$$

with

$$u_1(\mathcal{P}, \mathcal{L}) = u_1(\mathcal{L}, \mathcal{P}) = \sup\{ j \le n \mid \exists Q \, I \, \mathcal{L} \text{ such that } \Pi^n_j(Q) = \Pi^n_j(\mathcal{P}),\ Q \in \mathcal{P}(\mathcal{V}_n) \}$$

$$u_2(\mathcal{P}, \mathcal{L}) = u_2(\mathcal{L}, \mathcal{P}) = \sup\{ j \le n \mid \exists \mathcal{M} \, I \, \mathcal{P} \text{ such that } \Pi^n_j(\mathcal{M}) = \Pi^n_j(\mathcal{L}),\ \mathcal{M} \in \mathcal{L}(\mathcal{V}_n) \}$$

We now write down the axiom (GQ), consisting of two statements (GQ1) and (GQ2).

(GQ1) *If* $\mathcal{P},Q \in \mathcal{P}(\mathcal{V}_n)$, $\mathcal{L},\mathcal{M} \in \mathcal{L}(\mathcal{V}_n)$, $Q\ I\ \mathcal{L}\ I\ \mathcal{P}\ I\ \mathcal{M}$, $u(\mathcal{P},Q)=0$ *and* $\mathcal{L} \neq \mathcal{M}$, *then*

$$\sigma^{n-j}(Q) \cap \sigma(\mathcal{M}) \neq \emptyset \iff 2.j \leq u(\mathcal{L},\mathcal{M})$$

(GQ2) *If* $\mathcal{P} \in \mathcal{P}(\mathcal{V}_n)$, $\mathcal{L} \in \mathcal{L}(\mathcal{V}_n)$ *and* $u(\mathcal{P},\mathcal{L}) = (k,2k)$ *for some* $k \leq \frac{n}{2}$, *then there exists a unique* $\mathcal{M} \in \mathcal{L}(\mathcal{V}_n)$ *such that* $\mathcal{P}\ I\ \mathcal{M} \perp \mathcal{L}$. *Moreover,* $u(\mathcal{L},\mathcal{M}) = 2k$ *and* $u(\mathcal{P},Q) = 0$, *for all* $Q \in \sigma(\mathcal{L}) \cap \sigma(\mathcal{M})$. *If* $k=0$, *then* $u(Q_1,Q_2) \geq \frac{n}{2}$, *for all* Q_1, $Q_2 \in \sigma(\mathcal{L}) \cap \sigma(\mathcal{M})$.

We now define an affine structure on level j HQs. Suppose X_j is a level j HQ, $0<j<n$, with X_j =

$$(\mathcal{P}_i(X_j),\mathcal{L}_i(X_j),I,(\mathbf{P}_i(X_j))_{i\leq j},(\mathbf{L}_i(X_j))_{i\leq j},(\mathcal{W}_{j-1}(X_j,\mathcal{C}),\mathcal{N}_{\mathcal{C}})_{\mathcal{C}\in\mathbf{P}_{j-1}(X_j)}).$$

Let $X_1 = (\mathcal{P}(X_1),\ldots)$ be its canonical 1-image. Let $\mathcal{D} \in \mathbf{P}_{j-1}(X_j)$ be arbitrary. We denote:

* $\mathcal{L}_{\mathcal{D}}^{\infty} = \{\mathcal{L} \in \mathcal{L}(X_j) \mid \sigma(\mathcal{L}) \cap \mathcal{D} \neq \emptyset\}$,
* $\mathcal{P}_{\mathcal{D}}^{\infty} = \{\mathcal{P} \in \mathcal{P}(X_j) \mid \exists \mathcal{L} \in \mathcal{L}_{\mathcal{D}}^{\infty}$ such that $\mathcal{P}\ I\ \mathcal{L}\}$,
* $\mathcal{AP}(X_j,\mathcal{D}) = \mathcal{P}(X_j) - \mathcal{P}_{\mathcal{D}}^{\infty}$,
* $\mathcal{AL}(X_j,\mathcal{D}) = \mathcal{L}(X_j) - \mathcal{L}_{\mathcal{D}}^{\infty}$.

We call the elements of $\mathcal{AP}(X_j,\mathcal{D})$ the *affine points (of* $(X_j,\mathcal{D})$*, or of* X_j *if there is no confusion possible)* and the elements of $\mathcal{AL}(X_j,\mathcal{D})$ the *affine lines (of* $(X_j,\mathcal{D})$*)*. The elements of $\mathcal{P}_{\mathcal{D}}^{\infty}-\mathcal{D}$, resp. of $\mathcal{L}_{\mathcal{D}}^{\infty}$ are called the *points*, resp. the *lines at infinity (of* $(X_j,\mathcal{D})$*)*. The elements of $\mathcal{D}$ are the *hyperpoints (of* $(X_j,\mathcal{D})$*)*. The pair $(X_j,\mathcal{D})$ is called an *affine HQ (of level* n*)*. In Hanssens and Van Maldeghem [8], it is shown that every element of the $(j-1)$-point-neighbourhood of any affine point is again an affine point. Hence every element of the $(j-1)$-point-neighbour hood of any point at infinity, resp. hyperpoint, is again a point at infinity, resp. hyperpoint. This will give sense to axiom (NP) below.

We now introduce the notion of a "strip of width i" in an affine HQ $(X_j, \mathfrak{D})$. Suppose $\mathcal{P} \in \mathcal{P}(X_j)$ is a point at infinity of $(X_j, \mathfrak{D})$ and $\mathcal{L} \in \mathcal{L}(X_j)$ is an affine line incident with $\mathcal{P}$. If $i<j$, then we call the set

$$\{Q \in \mathcal{AP}(X_j, \mathfrak{D}) \mid Q \; I \; \mathcal{M} \; I \; \mathcal{P} \text{ for some } \mathcal{M} \in \mathcal{O}^i(\mathcal{L})\}$$

a *strip of width* i *(in* $(X_j, \mathfrak{D})$*)*. If $i \geq j$, then the set

$$\{Q \in \mathcal{AP}(X_j, \mathfrak{D}) \mid Q \perp \mathcal{P}\}$$

is called a *strip of width* i *(in* $(X_j, \mathfrak{D})$*)*. In every case, we call $\mathcal{P}$ a *base point (of the strip)*. It is not necessarily unique, even if the strip has width > 0 (cp.[8,property(2.26)]).

We can now state the first part of (NP).

(NP1) *If* $\mathcal{C} \in \mathbf{P}_{n-1}(\mathcal{V}_n)$, *then* $\mathcal{AP}(\mathcal{W}_{n-1}(\mathcal{V}_n, \mathcal{C}), \mathcal{N}_{\mathcal{C}}) = \mathcal{C}$. *Moreover, the* i*-point-neighbourhood of any point* $\mathcal{P} \in \mathcal{C}$ *in* $\mathcal{W}_{n-1}(\mathcal{V}_n, \mathcal{C})$ *coincides with the* i*-point-neighbourhood of* $\mathcal{P}$ *in* $\mathcal{V}_n$, *for all* $i \leq n-2$.

Suppose $\mathcal{C}_{n-j} \in \mathbf{P}_{n-j}(\mathcal{V}_n)$ and let $\mathcal{C}_{n-k}$ be the unique element of $\mathbf{P}_{n-k}(\mathcal{V}_n)$ containing $\mathcal{C}_{n-j}$ as a subset, $0 \leq k \leq j < n$. By (NP1),

$$\mathcal{C}_{n-2} \in \mathbf{P}_{n-2}(\mathcal{W}_{n-1}(\mathcal{V}_n, \mathcal{C}_{n-1})),$$
$$\mathcal{C}_{n-3} \in \mathbf{P}_{n-3}(\mathcal{W}_{n-2}(\mathcal{W}_{n-1}(\mathcal{V}_n, \mathcal{C}_{n-1}), \mathcal{C}_{n-2})), \text{ etc....}$$

This way, we justify the following notation.

$$\mathcal{W}_{n-j}(\mathcal{V}_n, \mathcal{C}_{n-j}) = \mathcal{W}_{n-j}(\mathcal{W}_{n-j+1}(\ldots(\mathcal{W}_{n-1}(\mathcal{V}_n, \mathcal{C}_{n-1}), \ldots), \mathcal{C}_{n-j+1}), \mathcal{C}_{n-j}).$$

Moreover, $\mathcal{C}_{n-j} = \mathcal{AP}(\mathcal{W}_{n-j}(\mathcal{V}_n, \mathcal{C}_{n-j}), \mathcal{N}_{\mathcal{C}_{n-j}})$.

The axiom (NP1) was about points of the point-neighbourhoods. The last axiom, (NP2), which we call the *strip axiom,* says something about the lines in the affine HQs corresponding to these neighbourhoods.

(NP2) *If* $\mathcal{P} \in \mathcal{P}(\mathcal{V}_n)$, $\mathcal{L} \in \mathcal{L}(\mathcal{V}_n)$, $0<j<n$ *and* $\sigma(\mathcal{L}) \cap \mathcal{O}^{n-j}(\mathcal{P}) \neq \emptyset$, *then the set*

$$S_j^n(\mathcal{P},\mathcal{L}) = \sigma(\mathcal{L}) \cap \mathcal{O}^{n-j}(\mathcal{P})$$

is a strip of width j *in* $(\mathcal{W}_{n-j}(\mathcal{V}_n, \mathcal{O}^{n-j}(\mathcal{P})), \mathcal{N}_{\mathcal{O}^{n-j}(\mathcal{P})})$. *Every strip of width* 1 *in* $(\mathcal{W}_{n-1}(\mathcal{V}_n, \mathcal{O}^{n-1}(\mathcal{P})), \mathcal{N}_{\mathcal{O}^{n-1}(\mathcal{P})})$ *can be obtained in this way (putting* j=1).

This completes our list of axioms for a level n HQ.

We keep the same notation as above. Suppose $\mathcal{M}$ is an affine line of $(\mathcal{W}_{n-j}(\mathcal{V}_n, \mathcal{O}^{n-j}(\mathcal{P})), \mathcal{N}_{\mathcal{O}^{n-j}(\mathcal{P})})$ such that the set of affine points of $\mathcal{M}$ is a subset of $\sigma(\mathcal{L}) \cap \mathcal{O}^{n-j}(\mathcal{P})$ (with the notation of (NP2) above), then we call $\mathcal{M}$ a *component of* $\mathcal{L}$, or a *component of the strip* $S_j^n(\mathcal{P},\mathcal{L})$ and we denote $\mathcal{M} < \mathcal{L}$. The set of affine points of $\mathcal{M}$ is called the *affine shadow of* $\mathcal{M}$. As an extension, we call every point of $\mathcal{V}_n$ incident with $\mathcal{L}$ a *component of* $\mathcal{L}$.

Now let $\mathcal{V}_n' = (\mathcal{P}(\mathcal{V}_n'), \mathcal{L}(\mathcal{V}_n'), \ldots)$ be a second level n HQ and suppose

$$\Psi : (\mathcal{P}(\mathcal{V}_n), \mathcal{L}(\mathcal{V}_n), I) \to (\mathcal{P}(\mathcal{V}_n'), \mathcal{L}(\mathcal{V}_n'), I)$$

is an isomorphism of incidence geometries mapping the affine shadow of every component of any line $\mathcal{L}$ onto the affine shadow of a component of $\Psi(\mathcal{L})$ and mapping i-neighbourhoods onto i-neighbourhoods, for all i, $0 < i \leq n$, then we call $\mathcal{V}_n$ and $\mathcal{V}_n'$ *equivalent.* This way, we can extend Ψ to the set of all components of all lines of $\mathcal{V}_n$ and this extended map, which we also denote by Ψ, preserves "being component of". We call Ψ an *equivalence*.

We now define by induction the notion of an isomorphism between $\mathcal{V}_n$ and $\mathcal{V}'_n = (\mathcal{P}(\mathcal{V}'_n), \mathcal{L}(\mathcal{V}'_n), \ldots)$. If $n=1$, then $\mathcal{V}_1$ and $\mathcal{V}'_1$ are called *isomorphic* if their base geometries are isomorphic generalized quadrangles. Now let $n \geq 2$, then we call $\mathcal{V}_n$ and $\mathcal{V}'_n$ *isomorphic* if they are equivalent (denote in that case the corresponding equivalence by Ψ) and if for all $\mathscr{C} \in \mathbf{P}_{n-1}(\mathcal{V}_n)$, $\mathcal{W}_{n-1}(\mathcal{V}_n, \mathscr{C})$ is isomorphic to $\mathcal{W}_{n-1}(\mathcal{V}'_n, \Psi(\mathscr{C}))$ and this isomorphism $\Psi_{\mathscr{C}}$ coincides with $\Psi/\mathscr{C}$ over $\mathscr{C}$. We can now extend Ψ with every $\Psi_{\mathscr{C}}$ and if we denote this extension still by Ψ, then we call Ψ an *isomorphism*. Obviously, isomorphic level n HQs are also equivalent.

Recall that Π^n_{n-1} is the projection mapping the base geometry of $\mathcal{V}_n$ onto the base geometry of the canonical $(n-1)$-image $\mathcal{V}_{n-1} = (\mathcal{P}(\mathcal{V}_{n-1}), \ldots)$. We can extend Π^n_{n-1} to all $\mathcal{W}_{n-j}(\mathcal{V}_n, \mathscr{C})$, $\mathscr{C} \in \mathbf{P}_{n-j}(\mathcal{V}_n)$ and $0<j<n$, with the projection of $\mathcal{W}_{n-j}(\mathcal{V}_n, \mathscr{C})$ onto $\mathcal{W}_{n-j-1}(\mathcal{V}_{n-1}, \Pi^n_{n-1}(\mathscr{C}))$. We denote that extension still by Π^n_{n-1}. Suppose now that $\mathcal{V}_{n-1}$ is isomorphic with some level $n-1$ HQ X_{n-1} and call the corresponding isomorphism Ψ. Then we call $\Psi \circ \Pi^n_{n-1}$ a *HQ-epimorphism*. Suppose now that $(X_n, \nabla^{n+1}_n)_{n \in \mathbf{N}}$ is an infinite sequence with X_n a level n HQ and ∇^{n+1}_n an HQ-epimorphism from X_{n+1} onto X_n, then we call $(X_n, \nabla^{n+1}_n)_{n \in \mathbf{N}}$ an *HQ-Artmann-sequence*. This name is inspired by the work of Artmann [1], who studied similar sequences of level n Hjelmslev planes, giving rise to affine buildings of type $\tilde{A}_2$ (by Hanssens and Van Maldeghem [6] and Van Maldeghem [14] and [15]).

If Z_n is the base geometry of X_n, for all $n \in \mathbf{N}$, then we call the sequence $(Z_n, \nabla^{n+1}_n / Z_{n+1})_{n \in \mathbf{N}}$ the *base sequence of* $(X_n, \nabla^{n+1}_n)_{n \in \mathbf{N}}$.

Hanssens and Van Maldeghem [8] have shown :

LEMMA 3.2. *Let, with the same notation as above,* $\mathcal{V}_n$ *be a level* n *HQ and suppose* $\mathcal{P} \in \mathcal{P}(\mathcal{V}_n)$ *and* $\mathcal{L} \in \mathcal{L}(\mathcal{V}_n)$. *Then* $u(\mathcal{P},\mathcal{L}) = 0$ *if and only if* $\Pi_1^n(\mathcal{P})$ *is not incident with* $\Pi_1^n(\mathcal{L})$.

3.3. The building corresponding to an HQ-Artmann-sequence. In this paragraph, we denote by $\mathcal{V} = (\mathcal{V}_n, \Pi_n^{n+1})_{n \in \mathbf{N}}$ an HQ-Artmann-sequence with $\mathcal{V}_n$ as above. Suppose $\mathcal{P} \in \mathcal{P}(\mathcal{V}_n)$. Denote $\mathcal{O}^j(\mathcal{P})$ briefly by $\mathcal{C}$. By Hanssens and Van Maldeghem [8], there is a natural projection

$$\nabla_\ell^k : \mathcal{W}_k(\mathcal{V}_n, \mathcal{C}) \to \mathcal{W}_\ell(\mathcal{V}_n, \mathcal{C}), \quad 0 \le \ell \le k \le j \le n-1.$$

We define a simplicial complex $\Delta(\mathcal{V}) = (X, \mathcal{S})$ with X de set of vertices and $\mathcal{S}$ the set of simplices. The dimension of $(X,\mathcal{S})$ will be 2, i.e. the cardinallity of the maximal simplices is 3. We denote

$$\mathbf{B}_n^j = \{\mathcal{L} \in \mathcal{AL}(\mathcal{W}_j(\mathcal{V}_n, \mathcal{C}), N_{\mathcal{C}}) \mid \mathcal{C} \in \mathbf{P}_j(\mathcal{V}_n)\}, \quad 0 < j < n,$$

$$\mathbf{B}_n^0 = \mathcal{P}(\mathcal{V}_n),$$

$$\mathbf{B}_n^n = \mathcal{L}(\mathcal{V}_n).$$

In all other cases $\mathbf{B}_n^j$ is the empty set (j and n integers). We define :

$$X = \cup \{\mathbf{B}_n^j \mid 0 \le j \le n \in \mathbf{N}\}$$

We now define $\mathcal{S}$. let $x \in \mathbf{B}_n^j$ and $y \in X$, then $\{x,y\} \in \mathcal{S}$ if one of the following conditions are satisfied :

(A1) $y \in \mathbf{B}_n^{j+1}$ and $x < y$,

(A1') $y \in \mathbf{B}_n^{j-1}$ and $y < x$,

(A2) $y \in \mathbf{B}_{n-1}^{j-1}$ and $\nabla_{j-1}^j(x) = y$, (A2') $y \in \mathbf{B}^{j+1}_{n+1}$ and $\nabla^{j+1}_{j}(y) = x$,

(A3) $y \in \mathbf{B}^{j}_{n-1}$ and $\nabla^{j}_{j-1}(x) < y$ and j is even,

(A3') $y \in \mathbf{B}^{j}_{n+1}$ and $\nabla^{j}_{j-1}(y) < x$ and j is even,

(A4) $y \in \mathbf{B}^{j-2}_{n-1}$ and $y < \nabla^{j}_{j-1}(x)$ and j is even,

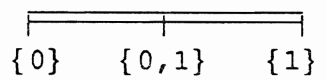
(A4') $y \in \mathbf{B}^{j+2}_{n+1}$ and $x < \nabla^{j+2}_{j+1}(y)$ and j is even.

where we identified for shortness'sake "$\Pi^{\bullet}_{\bullet}$" with "$\nabla^{\bullet}_{\bullet}$". The 2-dimensional simplices are by definition the 3-sets $\{x, y, z\}$ where every 2-subset is a 1-dimensional simplex. We define a type-map typ on the set X of vertices :

$$typ : X \to \{\{0\},\{1\},\{0,1\}\} : x \in \mathbf{B}^{j}_{n} \to \{n (\mathrm{mod} 2), n-j (\mathrm{mod} 2)\}.$$

The two following results were proved in Hanssens and Van Maldeghem [8].

RESULT 3.3.1. *The rank 3 incidence structure $\Delta(\mathcal{V})$, as defined above, is a thick geometry of type $\tilde{C}_2$ with diagram*



RESULT 3.3.2. *Equivalent HQ-Artmann-sequences give rise to isomorphic buildings by the above construction.*

4. PROOF OF THE MAIN RESULT.

In this section we show, with the notations of paragraph 3 and section 3

MAIN THEOREM. *The n^{th} floor of Δ with basement b is the base geometry of a level n HQ $\mathcal{V}^{b}_{n}$ =* .

$$(\mathcal{P}(\mathcal{V}^{b}_{n}), \mathcal{L}(\mathcal{V}^{b}_{n}), I, (\mathbf{P}_i(\mathcal{V}^{b}_{n}))_{i \le n}, (\mathbf{L}_i(\mathcal{V}^{b}_{n}))_{i \le n}, (\mathcal{W}_{n-1}(\mathcal{V}^{b}_{n}, \mathcal{C}), \mathcal{N}_{\mathcal{C}})_{\mathcal{C} \in \mathbf{P}_{n-1}(\mathcal{V}^{b}_{n})}).$$

PROOF. We show this result in a number of paragraphs.

4.1. Definition of $\mathcal{V}_n^b$.

We define $\mathcal{V}_n^b$ by induction on the non-negative integer n for every special vertex b (so here, b can vary) in Δ. For n=0, $\mathcal{V}_0^b = (\{b\},\{b\},=)$. For n=1, $\mathcal{V}_1^b$ is the unique level 1 HQ having R(b) as base geometry. Suppose now $n \geq 2$. The base geometry of $\mathcal{V}_n^b$ is by definition the n^{th} floor of Δ with basement b. We denote the natural canonical epimorphism onto the $(n-1)^{\text{th}}$ floor by Π_{n-1}^n (see paragraph 2.3). It will turn out below that Π_{n-1}^n is the partition map of $\mathcal{V}_n^b$, justifying this notation. Define

$$\Pi_j^i = \Pi_j^{j+1} \circ \Pi_{j+1}^{j+2} \circ \ldots \circ \Pi_{i-1}^i$$

$$\mathbf{P}_i(\mathcal{V}_n^b) = \{(\Pi_{n-i}^n)^{-1}(\mathcal{P}) \mid \mathcal{P} \in \mathcal{P}(\mathcal{V}_{n-i}^b)\}, \quad 0 < i \leq n,$$

and $\mathbf{P}_0(\mathcal{W}_n)$ is as in (PS1). Similarly for the line-neighbourhoods. By abuse of language, we can identify the point $\mathcal{P} \in \mathcal{P}(\mathcal{V}_{n-i}^b)$ with the i-point-neighbourhood $(\Pi_{n-i}^n)^{-1}(\mathcal{P})$, $0 < i \leq n$. Now let $\mathcal{C}$ be an $(n-1)$-point-neighbourhood, then $\mathcal{C}$ can be regarded as a special vertex adjacent to b. We define

$$\mathcal{W}_{n-1}(\mathcal{V}_n^b, \mathcal{C}) = \mathcal{V}_{n-1}^{\mathcal{C}},$$

well defined by induction. Reciprocally, b can be regarded as an $(n-2)$-point-neighbourhood of $\mathcal{W}_{n-1}(\mathcal{V}_n^b,\mathcal{C})$. Well, we define $N_{\mathcal{C}} = b$. The conditions (PS1),(PS1'),(PS2) and (PS2') are readily verified (since Δ is a thick building, cp. Van Maldeghem [15,§4.3.3]). It is also clear that Π_j^i is a partition map. The axiom (IS) follows immediately by an easy inductive argument. The other axioms are less trivial and we show them in detail.

Throughout $n \geq 2$.

4.2. $\mathcal{V}_n^b$ satisfies axiom (NP).

4.2.1. $\mathcal{V}_n^b$ satisfies axiom (NP1).

Let ∇_1^{n-1} be the partition map $\nabla_1^{n-1} : \mathcal{W}_{n-1}(\mathcal{V}_n^b, \mathcal{C}) \to \mathcal{W}_1(\mathcal{V}_n^b, \mathcal{C})$, $\mathcal{C} \in \mathbf{P}_{n-1}(\mathcal{V}_n^b)$. The set of affine points of $(\mathcal{W}_{n-1}(\mathcal{V}_n^b, \mathcal{C}), b)$ is the union of the sets $(\nabla_1^{n-1})^{-1}(\mathcal{P})$, $\mathcal{P} \in \mathcal{P}(\mathcal{W}_1(\mathcal{V}_n^b, \mathcal{C}))$ and $\mathcal{P}$ not collinear with b in $R(\mathcal{C}) = \mathcal{W}_1(\mathcal{V}_n^b, \mathcal{C})$ (this is a general property of level $(n-1)$ HQ following directly from Hanssens and Maldeghem [8, properties(2.3) and (2.4])). So $\mathcal{P}$ and b are opposite in $R(\mathcal{C})$. By result 2.2.7, $(\Pi_1^n)^{-1}(\mathcal{C}) = \mathcal{AP}(\mathcal{W}_{n-1}(\mathcal{V}_n^b, \mathcal{C}), b)$. The second assertion of (NP1) follows immediately from the definition of point-neighbourhoods.

4.2.2. $\mathcal{V}_n^b$ satisfies axiom (NP2).

We show (NP2) only for the case $j \le \frac{n}{2}$. The case $j \ge \frac{n}{2}$ is proved in the same way. Suppose $\mathcal{P} \in \mathcal{P}(\mathcal{V}_n^b)$. Consider $\sigma^{n-j}(\mathcal{P})$ and denote $\mathcal{P}_j = \Pi_j^n(\mathcal{P})$. Note that we identified $\sigma^{n-j}(\mathcal{P})$ and $\mathcal{P}_j$ above, but for clearness' sake, we will have to treat them here as distinct objects. Let $\mathcal{L} \in \mathcal{L}(\mathcal{V}_n^b)$ with $\mathcal{P}\, I\, \mathcal{L}$. Let Σ be an apartment

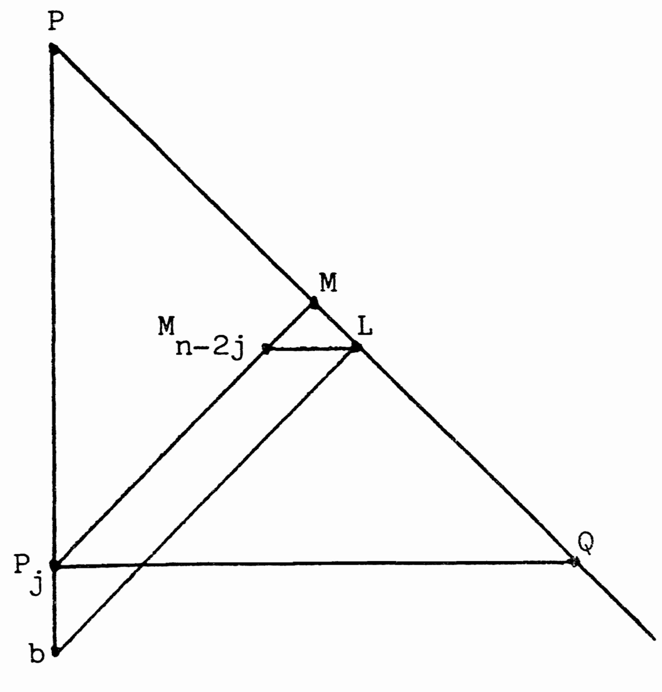


FIGURE 3

containing $b, \mathcal{P}$ and $\mathcal{L}$. Let p, resp. ℓ be the sectorpanel in Σ representing $\mathcal{P}$ resp. $\mathcal{L}$. Let $Q \in \mathcal{P}(\mathcal{W}_{n-j}(\mathcal{V}_n^b, \mathcal{O}^{n-j}(\mathcal{P})))$ and $\mathcal{M} \in \mathcal{L}(\mathcal{W}_{n-j}(\mathcal{V}_n^b, \mathcal{O}^{n-j}(\mathcal{P})))$ be defined as on figure 3. Also let $\mathcal{M}_{n-2j} \in$ Ve(Δ) be as on figure 3. Then $\mathcal{M}_{n-2j}$ is the projection of $\mathcal{M}$ on $\mathcal{W}_{n-2j}(\mathcal{V}_n^b, \mathcal{O}^{n-j}(\mathcal{P}))$, i.e. $\mathcal{M}_{n-2j} = \nabla_{n-2j}^{n-j}(\mathcal{M})$, where ∇_{n-2j}^{n-j} is the appropriate partition map. We show (NP2) in four steps.

(1). We first show that, given $\mathcal{P}^* \in \mathcal{O}^{n-j}(\mathcal{P}) \cap \sigma(\mathcal{L})$, we can find a unique $\mathcal{M}^* \in \mathcal{L}(\mathcal{W}_{n-j}(\mathcal{V}_n^b, \mathcal{O}^{n-j}(\mathcal{P})))$ such that $\mathcal{P}^*\ I\ \mathcal{M}^*\ I\ Q$ in $\mathcal{W}_{n-j}(\mathcal{V}_n^b, \mathcal{O}^{n-j}(\mathcal{P}))$. Moreover $\nabla_{n-2j}^{n-j}(\mathcal{M}^*) = \mathcal{M}_{n-2j}$. So let $\mathcal{P}^* \in \mathcal{O}^{n-j}(\mathcal{P}) \cap \sigma(\mathcal{L})$. By result 2.3.8, $d_\Delta(\mathcal{P}^*, \mathcal{L}) = n$ and hence since $d_\Delta(\mathcal{L}, Q) = n-2j$, $d_\Delta(\mathcal{P}^*, Q) \le 2n-2j$. By lemma 2.3.7, $\mathcal{P}^* \perp Q$ in $\mathcal{W}_{n-j}(\mathcal{V}_n^b, \mathcal{O}^{n-j}(\mathcal{P}))$ and $d_\Delta(\mathcal{P}^*, Q) = 2n-2j$. By result 2.2.12(2) we also have $\mathcal{L} \in [\mathcal{P}^*, Q]$. So let Σ^* be an apartment containing $\mathcal{P}_j$, $\mathcal{P}^*$ and Q (Σ^* exists by lemma 2.3.3), then Σ^* also contains $\mathcal{L}$ and hence also $\mathcal{M}_{n-2j} \in \mathrm{cl}(\mathcal{P}_j, \mathcal{L})$. One deduces $\mathcal{M}_{n-2j} = \nabla_{n-2j}^{n-j}(\mathcal{M}^*)$ where $\mathcal{M}^*$ is the unique vertex (in Σ^*) on distance $n-j$ from $\mathcal{P}^*$ and Q.

(2). We now show that, given $\mathcal{M}^* \in \mathcal{L}(\mathcal{W}_{n-j}(\mathcal{V}_n^b, \mathcal{O}^{n-j}(\mathcal{P})))$ satisfying $Q\ I\ \mathcal{M}^*$ in $\mathcal{W}_{n-j}(\mathcal{V}_n^b, \mathcal{O}^{n-j}(\mathcal{P}))$ and $\nabla_{n-2j}^{n-j}(\mathcal{M}^*) = \mathcal{M}_{n-2j}$, and given any point $Q^* \in \mathcal{P}(\mathcal{W}_{n-j}(\mathcal{V}_n^b, \mathcal{O}^{n-j}(\mathcal{P})))$ incident with $\mathcal{M}^*$ in $\mathcal{W}_{n-j}(\mathcal{V}_n^b, \mathcal{O}^{n-j}(\mathcal{P}))$ and such that $u'(Q, Q^*) = 0$ (u' is the valuation map in $\mathcal{W}_{n-j}(\mathcal{V}_n^b, \mathcal{O}^{n-j}(\mathcal{P}))$), then this implies that $Q^* \in \mathcal{O}^{n-j}(\mathcal{P})$ and $Q^*\ I\ \mathcal{L}$ in $\mathcal{V}_n^b$. Well, any apartment through $\mathcal{P}_j$, Q and $\mathcal{M}^*$ contains $\mathcal{M}_{n-2j}$ and hence also $\mathcal{L} \in \mathrm{cl}(Q, \mathcal{M}_{n-2j})$. So one can see that $d_\Delta(\mathcal{M}^*, \mathcal{L}) = j$. Denote $Q_1^* = R(\mathcal{P}_j) \cap [\mathcal{P}_j, Q^*]$, $Q_1 = R(\mathcal{P}_j') \cap [\mathcal{P}_j, Q]$, $b_1 = R(\mathcal{P}_j) \cap [\mathcal{P}_j, b]$ and $\mathcal{M}_1^* = R(\mathcal{P}_j) \cap [\mathcal{P}_j, b]$. Then $Q_1^* \perp Q_1 \perp b_1$ in $R(\mathcal{P}_j)$ while clearly b_1 is not incident with $\mathcal{M}_1^*$ and Q_1^* is. Hence Q_1^* and b_1 are opposite in $R(\mathcal{P}_j)$ and $Q^* \in \mathcal{O}^{n-j}(\mathcal{P})$. But now, since $Q^*\ I\ \mathcal{M}^*$ in $\mathcal{W}_{n-j}(\mathcal{V}_n^b, \mathcal{O}^{n-j}(\mathcal{P}))$, $d_\Delta(Q^*, \mathcal{M}^*) = n-j$. Hence $d_\Delta(Q^*, \mathcal{L}) \le n$ and by result 2.3.8, $Q^*\ I\ \mathcal{L}$ in $\mathcal{V}_n^b$.

(3). Let b_1 be as above. We show that every affine line $\mathcal{K} \in \mathcal{AL}(\mathcal{W}_{n-j}(\mathcal{V}_n^b, \mathcal{O}^{n-j}(\mathcal{P})), b_1)$ whose affine shadow is a subset of $\sigma(\mathcal{L})$

is incident with Q in $\mathcal{W}_{n-j}(\mathcal{V}_n^b, \sigma^{n-j}(\mathcal{P}))$. Suppose therefore $\mathcal{K}$ is not incident with Q. Consider through an arbitrary point $\mathcal{P}^*$ of $\mathcal{K}$ the line $\mathcal{M}^*$ as in **(1)**. By **(2)**, the affine shadow of $\mathcal{M}^*$ is a subset of $\sigma(\mathcal{L})$, but $\mathcal{M}^* \neq \mathcal{K}$ since $\mathcal{M}^* \, I \, Q$. Hence there is an affine point $\mathcal{P}^{**}$ of $\mathcal{K}$ not incident with $\mathcal{M}^*$. Let $\mathcal{M}^{**}$ be the unique line of $\mathcal{W}_{n-j}(\mathcal{V}_n^b, \sigma^{n-j}(\mathcal{P}))$ corresponding to $\mathcal{P}^{**}$ as in **(1)**. then again $\mathcal{M}^* \neq \mathcal{M}^{**}$, but both are incident with Q. By induction (using (GQ2)) and properties(2.9) and (2.14) of [8], $\mathcal{M}^* = \mathcal{M}^{**}$, a contradiction. Hence $\mathcal{K}$ is incident with Q. Combining **(1)**, **(2)** and **(3)**, the first part of (NP2) follows.

(4). There remains to show that for $j = 1$, every strip of width 1 arises in this way. So let $Q \in \mathcal{P}(\mathcal{W}_{n-1}(\mathcal{V}_n^b, \sigma^{n-1}(\mathcal{P}))$ be a point at infinity and let $\mathcal{M}^*$ be an affine line of $(\mathcal{W}_{n-1}(\mathcal{V}_n^b, \sigma^{n-1}(\mathcal{P})), b)$ incident with Q. Let Σ be an apartment through $\{\mathcal{P}_1, b\}$ and Q and let q be the sectorpanel in Σ with source $\mathcal{P}_1$ containing Q. By lemma 3.3.2, q and $\mathcal{M}^*$ lie in a common sector with source $\mathcal{P}_1$. Let $\mathcal{D}$ be the double sector in Σ bounded by the sectorpanels in Σ

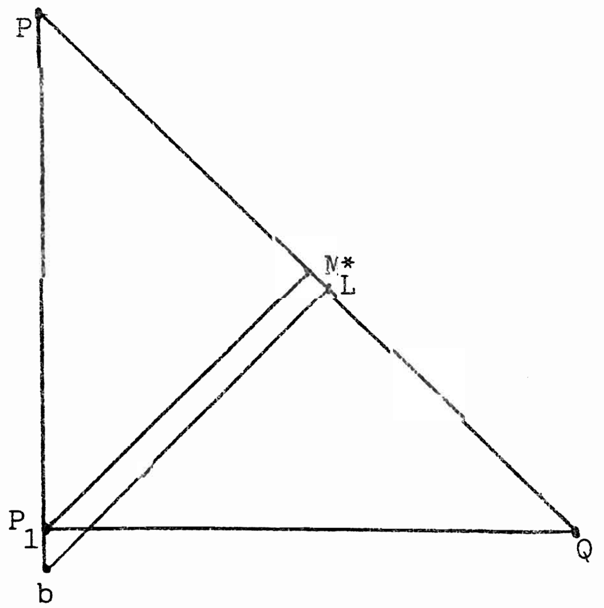


FIGURE 4

with source $\mathcal{P}_1$ containing resp. q and b. Since $R(\mathcal{P}_1) \cap [\mathcal{P}_1, \mathcal{M}^*]$ is not incident with b in $R(b_1)$, by result 2.2.10, there is an apartment Σ^* containing $\mathcal{D}$ and $\mathcal{M}^*$. Let $\mathcal{L}$ and $\mathcal{P}$ be as on figure 4 (figure 4 pictures Σ^*). By the proof of the first part of (NP2), $\mathcal{C}_1^n(\mathcal{P},\mathcal{L})$ is exactly the strip of width 1 with base point Q and containing $\mathcal{M}^*$ as an element.

4.3. V_n^b satisfies the axiom (GQ).

4.3.1. V_n^b satisfies the axiom (GQ1).

Let $\mathcal{P}, Q \in \mathcal{P}(V_n^b)$ and $\mathcal{L}, \mathcal{M} \in \mathcal{L}(V_n^b)$ be such that $Q\, I\, \mathcal{L}\, I\, \mathcal{P}\, I\, \mathcal{M}$, $u(\mathcal{P}, Q) = 0$, $u(\mathcal{L},\mathcal{M}) = k' < n$ and $\mathcal{L} \neq \mathcal{M}$ (u is the valuation map in V_n^b). Assume k' is odd, then, since $\mathrm{cl}(\Pi_{k'}^n(\mathcal{L}),\mathcal{P})$ contains $\Pi_{k'+1}^n(\mathcal{L})$, we have $\Pi_{k'+1}^n(\mathcal{L}) = \Pi_{k'+1}^n(\mathcal{M})$, whence $u(\mathcal{L},\mathcal{M}) \geq k'+1$, a contradiction. Hence k' is even en we put $k' = 2k$. Since $u(\mathcal{P},Q) = 0$, we can find representatives p, q, ℓ of resp. $\mathcal{P}, Q, \mathcal{L}$ lying in a common apartment Σ (by lemma 2.3.3). We show (GQ1) in two steps.

(1). Suppose $\mathcal{P}^* \in \mathcal{P}(V_n^b)$ incident with $\mathcal{M}$ and put $u(\mathcal{P}^*, Q) = j$. We show $j \leq k$. So assume $j > k$. Since $j > 0$ there exists by lemma

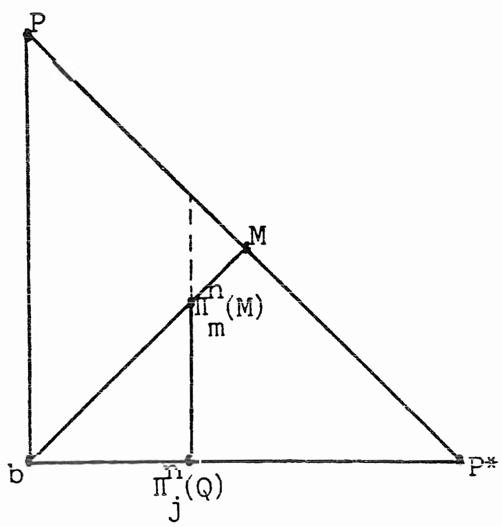


FIGURE 5

3.2.2 an apartment Σ^* containing $\mathcal{b},\mathcal{P},\mathcal{M}$ and $\mathcal{P}^*$ (note $u(\mathcal{P},\mathcal{P}^*) = 0$). So $\Sigma \cap \Sigma^*$ contains $cl(\mathcal{P},\Pi_j^{\cdot}(Q))$ which itself contains $\Pi_i^{\cdot}(\mathcal{L})$, $i = inf\{n, 2j\}$ (see figure 5). Hence $\Pi_i^{\cdot}(\mathcal{L}) = \Pi_i^{\cdot}(\mathcal{M})$ contradicting $u(\mathcal{L},\mathcal{M}) = 2k$.

(2). It suffices to show that there is a point $\mathcal{P}^*$ in $\mathcal{V}_n^{\mathcal{b}}$ incident with $\mathcal{M}$ and such that $u(\mathcal{P}^*,Q) \geq k$. If $k = 0$ this is trivial, so suppose $k > 0$. Let q^* be the sectorpanel in Σ (with source $\mathcal{b}$) opposite to p, i.e. $q^* \cup p$ is a wall. Let m be any sectorpanel representing $\mathcal{M}$ and let Σ^{**} be an apartment through q^* and m (Σ^{**} exists by result 2.2.10 and the fact that $m \cap R(\mathcal{b})$ ($= \ell \cap R(\mathcal{b})$) is not incident with $q^* \cap R(\mathcal{b})$ in $R(\mathcal{b})$). Let $\ell^*, p^* \in Sp(\Delta,\mathcal{b})$ be such that $q^*\, I\, \ell^*\, I\, p^*\, I\, m$ in $\mathcal{W}_\infty$, then $\ell^*, p^* \subseteq \Sigma^{**}$. So Σ^{**} contains q^* and $\Pi_{2k}^{\cdot}(\mathcal{L}) = \Pi_{2k}^{\cdot}(\mathcal{M})$, hence also $cl(q^*,\Pi_{2k}^{\cdot}(\mathcal{L}))$. The latter contains $\Pi_k^{\cdot}(Q)$. Hence the unique point $\mathcal{P}^*$ in $\mathcal{V}_n^{\mathcal{b}}$ on p^* is incident with $\mathcal{M}$ and $u(\mathcal{P}^*,Q) \geq k$.

4.3.2. $\mathcal{V}_n^{\mathcal{b}}$ satisfies axiom (GQ2).

Suppose $\mathcal{P} \in \mathcal{P}(\mathcal{W}_n)$ and $\mathcal{L} \in \mathcal{L}(\mathcal{W}_n)$ with $u(\mathcal{P},\mathcal{L}) = (k,2k)$ for some $k < \frac{n}{2}$. We show the assertion in six steps.

(1). We first show (GQ2) for $k = 0$. Let p, resp. ℓ be representatives of $\mathcal{P}$, resp. $\mathcal{L}$. Define $m, q \in Sp(\Delta,\mathcal{b})$ by $p\, I\, m\, I\, q\, I\, \ell$ in $\mathcal{W}_\infty$ then the conditions of result 2.2.10 are satisfied for the sectors bounded by resp. $p \cup m$, $m \cup q$, $q \cup \ell$. So $p \cup \ell$ bounds some 3-fols sector. Let $\mathcal{M} = m \cap \mathcal{V}_n^{\mathcal{b}}$ (obvious notation), then $\mathcal{P}\, I\, \mathcal{M} \perp \mathcal{L}$. Since $\mathcal{M} \in cl(\mathcal{P},\mathcal{L})$, $\mathcal{M}$ is unique. Clearly every point $Q \in \sigma(\mathcal{L}) \cap \sigma(\mathcal{M})$ has the property $u(\mathcal{P},Q) = 0$ (look in $R(\mathcal{b})$) and if Q_1, $Q_2 \in \sigma(\mathcal{L}) \cap \sigma(\mathcal{M})$, then $\Pi_j^{\cdot}(Q_i) \in cl\mathcal{b},\mathcal{L},\mathcal{M})$, $i=1,2$, for all $j \leq \frac{n+1}{2}$. Hence $u(Q_1,Q_2) \geq \frac{n}{2}$.

From now on we assume $k > 0$.

(2). We now show that for every $\mathcal{L}^* \in \mathcal{L}(\mathcal{V}_{\tau}^{b})$ with the properties $u(\mathcal{L},\mathcal{L}^*) = 2k$ and $\mathcal{P}\, I\, \mathcal{L}^*$, there exists $\mathcal{P}^* \in \mathcal{P}(\mathcal{V}_{\tau}^{b})$ such that $u(\mathcal{P},\mathcal{P}^*) = k$, $\mathcal{P}^*\, I\, \mathcal{L}$ and $\mathrm{cl}(b,\mathcal{P},\mathcal{L}^*) \cap \mathrm{cl}(b,\mathcal{P}^*,\mathcal{L}) = \mathrm{cl}(b,\Pi_k'(\mathcal{P}),\Pi_{2k}'(\mathcal{L}))$. Set $\Pi_k'(\mathcal{P}) = \mathcal{P}_k$ and $\Pi_{2k}'(\mathcal{L}) = \mathcal{L}_{2k}$. Let $\mathcal{L}^*$ be as just described and let Σ be an apartment through $b,\mathcal{P}$ and $\mathcal{L}^*$. Let p,ℓ^* and ℓ' be the sectorpanels in Σ defined as on figure 6. Let ℓ be any sectorpanel (with source b) representing $\mathcal{L}$. Define $p^* \in sp(\Delta,b)$ by $p' \perp p^*\, I\, \ell$ in $\mathcal{W}_\infty$. Since $k > 0$, $p' \cup \ell$ bounds a 3-fold sector ${}^3Q$ containing p^*. Let $\mathcal{P}^* = p^* \cap \mathcal{V}_{\tau}^{b}$, then we see in ${}^3Q$ that $\mathcal{P}^*\, I\, \mathcal{L}$.

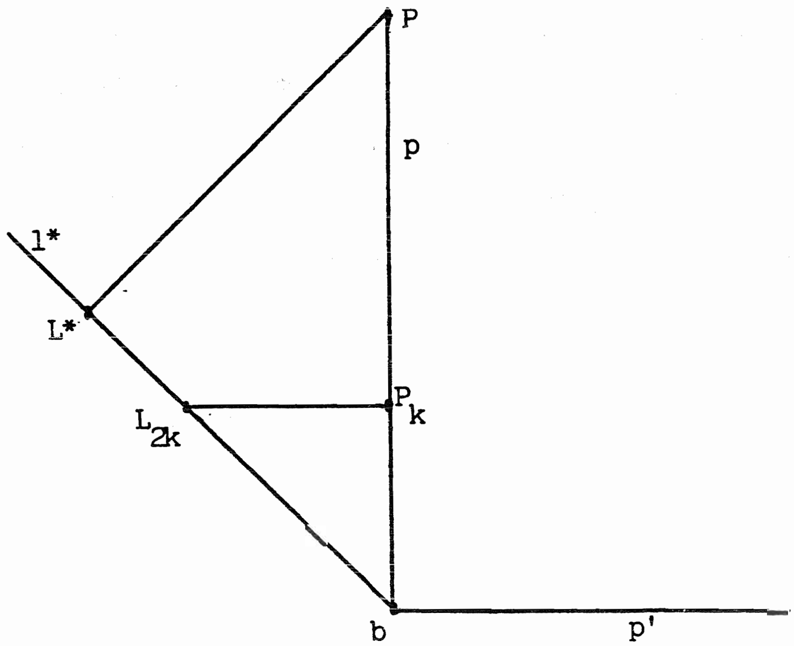


FIGURE 6

Since ${}^3Q$ contains $\mathcal{L}_{2k}$ and p', it will also contain $\mathcal{P}_k \in \mathrm{cl}(p',\mathcal{L}_{2k})$. Hence $u(\mathcal{P},\mathcal{P}^*) \geq k$. Suppose that $\mathrm{cl}(b,\mathcal{P},\mathcal{L}^*) \cap \mathrm{cl}(b,\mathcal{P}^*,\mathcal{L})$ contains more than only $\mathrm{cl}(b,\mathcal{P}_k,\mathcal{L}_{2k})$, say a vertex t, then $t \in {}^3Q \cap \Sigma$ and since $\mathrm{cl}(p',t) \subseteq {}^3Q$, $\mathrm{cl}(p',t)$ would contain a vertex $\Pi_j'(\mathcal{P})$ with $j>k$ and so $u(\mathcal{P},\mathcal{P}^*) > k$, contradicting $u_1(\mathcal{P},\mathcal{L}) = k$. Hence $\mathrm{cl}(b,\mathcal{P},\mathcal{L}^*) \cap \mathrm{cl}(b,\mathcal{P}^*,\mathcal{L}) \subseteq \mathrm{cl}(b,\mathcal{P}_k,\mathcal{L}_{2k})$. The inverse inclusion is obvious.

(3). We now show that there exists a line $\mathcal{M} \in \mathcal{L}(\mathcal{V}_n^b)$ such that $\mathcal{P}\ I\ \mathcal{M} \perp \mathcal{L}$ and $u(\mathcal{L},\mathcal{M}) = 2k$. Let $\mathcal{L}^*, \mathcal{P}^*, p, p^*, \ell, \ell^*$ be as in **(2)** (the existence of $\mathcal{L}^*$ follows from $u_2(\mathcal{P},\mathcal{L}) = 2k$). Let $b_0 = \mathcal{L}_{2k}$ and consider the level $n-2k$ HQ $\mathcal{V}^{b_0}_{n-2k}$. Let p_0 resp. p_0^*, ℓ_0, ℓ_0^* be the unique sectorpanel with source b_0 and parallel to p, resp. p^*, ℓ, ℓ^*. Let $\mathcal{P}_0^1$ resp. $\mathcal{L}_0^1$, $\mathcal{P}_0^{*1}$ be the intersection of $R(b_0)$ with p_0 resp. ℓ_0, p_0^*. Define the vertices $\mathcal{K}_0^1$, $\mathcal{K}_0^{*1}$ and $\mathcal{J}_0^1 = \Pi'_{2k-1}(\mathcal{L})$ as on figure 7. They are all points or lines in $R(b_0)$. If $\mathcal{P}_0^1\ I\ \mathcal{L}_0^1$ in $R(b_0)$, then the chain $\mathcal{L}_0^1 \perp \mathcal{K}_0^{*1} \perp \mathcal{K}_0^1 \perp \mathcal{L}_0^1$ constitutes a

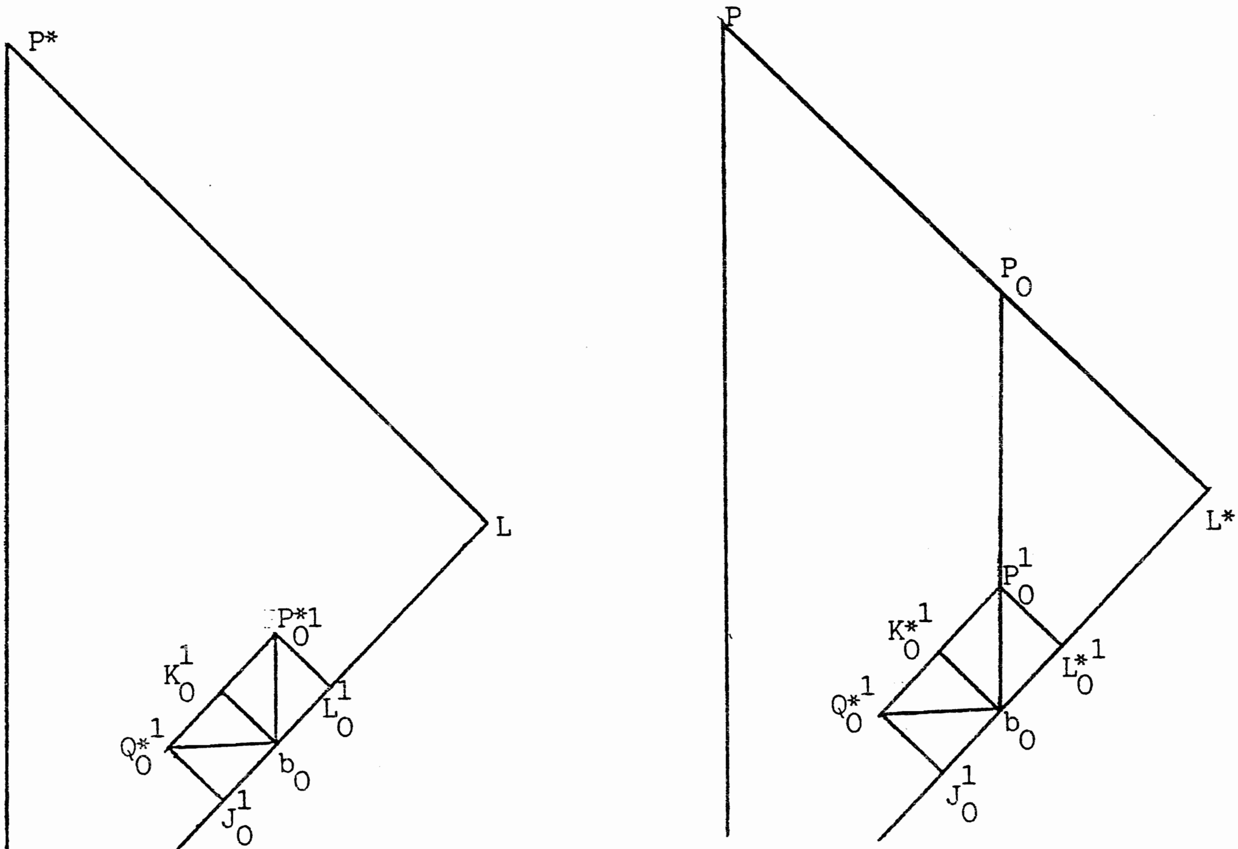


FIGURE 7

triangle in $R(b_0)$ (by **(2)**, it is non-degenerate), hence $\mathcal{P}_0^1$ and $\mathcal{L}_0^1$ are not incident in $R(b_0)$. Now p_0 represents in $\mathcal{V}^{b_0}_{n-2k}$ some point $\mathcal{P}_0$, and by lemma 3.2 and the induction hypothesis, $u'(\mathcal{L},\mathcal{P}_0) = (0,0)$, where u' is the valuation map in $\mathcal{V}^{b_0}_{n-2k}$. So we can apply induction (or **(1)**) and obtain a line $\mathcal{M}$ in $\mathcal{V}^{b_0}_{n-2k}$ incident with $\mathcal{P}_0$ and concurrent with $\mathcal{L}$ in $\mathcal{V}^{b_0}_{n-2k}$. Moreover, there exists a sectorpanel $m_0 \in \mathrm{Sp}(\Delta, b_0)$ containing $\mathcal{M}$ and contained in a 3-fold sector bounded by p_0 and ℓ_0. Let m be the sectorpanel with source b parallel to m_0, then $p\, I\, m \perp \ell$ in $\mathcal{W}_\infty$. We show that $\mathcal{J}_0^1$ and the vertex $\mathcal{M}_0^1$ (which is the line in $R(b_0)$ represented by m_0) are opposite in $R(b_0)$. Assume therefore $\mathcal{J}_0^1 \perp \mathcal{M}_0^1$. Well, $\mathcal{J}_0^1 \neq \mathcal{M}_0^1$ since $\mathcal{P}_0^1$ is incident with $\mathcal{M}_0^1$ but not with $\mathcal{J}_0^1$; $\mathcal{J}_0^1 \neq \mathcal{K}_0^{*1}$ (by definition, see figure 7); $\mathcal{K}_0^{*1} \neq \mathcal{M}_0^1$ since $\mathcal{L}_0^1$ is concurrent with $\mathcal{M}_0^1$ (because $\mathcal{M} \perp \mathcal{L}$ in $\mathcal{V}^{b_0}_{n-2k}$), but not with $\mathcal{K}_0^{*1}$ (otherwise $\mathcal{K}_0^{*1} \perp \mathcal{L}_0^1 \perp \mathcal{K}_0^1 \perp \mathcal{K}_0^{*1}$ is a triangle in $R(b_0)$); $\mathcal{P}_0^1$ is incident with $\mathcal{K}_0^{*1}$ and with $\mathcal{M}_0^1$, but not with $\mathcal{J}_0^1$. Hence $\mathcal{M}_0^1 \perp \mathcal{K}_0^{*1} \perp \mathcal{J}_0^1 \perp \mathcal{M}_0^1$ forms a triangle in $R(b_0)$. So $\mathcal{J}_0^1$ and $\mathcal{M}_0^1$ are opposite in $R(b_0)$. By result 2.2.7, $[b, b_0] \cup m_0$ is a sectorpanel parallel to m_0 and m. But it has the same source as m, hence $m = [b, b_0] \cup m_0$. But then m represents $\mathcal{M} \in \mathcal{L}(\mathcal{V}^{b}_{n})$ and $\mathcal{P}\, I\, \mathcal{M} \perp \mathcal{L}$. Since $m \cap \ell \supseteq [b, b_0]$, $u(\mathcal{L}, \mathcal{M}) \geq 2k$. Equality follows from $u_2(\mathcal{P}, \mathcal{L}) = 2k$.

(4). Let $\mathcal{M}$ be as in **(3)** and let $\mathcal{P}^* \in \mathcal{P}(\mathcal{V}^{b}_{n})$ be such that $u(\mathcal{P}, \mathcal{P}^*) = k$, $\mathcal{P}^*\, I\, \mathcal{L}$ and $\mathrm{cl}(b, \mathcal{P}, \mathcal{M}) \cap \mathrm{cl}(b, \mathcal{P}^*, \mathcal{L}) = \mathrm{cl}(b, \mathcal{P}_k, \mathcal{L}_{2k})$ with $\mathcal{P}_k$ and $\mathcal{L}_{2k}$ as in **(2)**. We show that there exists an apartment Σ_1 containing all the vertices $\mathcal{P}$, $\mathcal{P}^*$, $\mathcal{P}_k$, $\mathcal{L}_{2k}$, $\mathcal{M}$ and $\mathcal{L}$. Set $b_1 = \mathcal{P}_k$ and denote $\Pi^{n}_{k+1}(\mathcal{P}) = \mathcal{P}_1^1$, $\Pi^{n}_{k+1}(\mathcal{P}^*) = \mathcal{P}_1^{*1}$. Let $\mathcal{T}_1^1$ denote the vertex of Δ adjacent to b_1 and lying on $[b_1, b_0]$ (with the above notation) and let $\mathcal{R}_1^1$, resp. $\mathcal{R}_1^{*1}$ be the middle of $[\mathcal{P}_1^1, \mathcal{T}_1^1]$, resp. $[\mathcal{P}_1^{*1}, \mathcal{T}_1^1]$. In $R(b_1)$, we have $\mathcal{P}_1^1 \perp \mathcal{T}_1^1 \perp \mathcal{P}_1^{*1}$. Now if $\mathcal{P}_1^1 \perp \mathcal{P}_1^{*1}$, then by the foregoing, $\mathcal{R}_1^1 = \mathcal{R}_1^{*1} \in \mathrm{cl}(b, \mathcal{P}, \mathcal{M}) \cap \mathrm{cl}(b, \mathcal{P}^*, \mathcal{L})$. But $\mathcal{R}_1^1$ is not in $\mathrm{cl}(b, \mathcal{P}_k, \mathcal{L}_{2k})$, a contradiction. Hence $\mathcal{P}_1^1$ and $\mathcal{P}_1^{*1}$ are opposite in $R(b_1)$. Let, with the same notation as above, q be such that $m\, I\, q\, I\, \ell$ in $\mathcal{W}_\infty$ and let p_1^*, resp. ℓ_1, q_1, m_1, p_1 be the sectorpanel with source b_1 and parallel to p^*, resp. ℓ, q, m, p. Since $p^*\, I\, \ell\, I\, q\, I\, m\, I\, p$ in $\mathcal{W}_\infty$, the elements of $R(b_1)$ they represent form a chain of five consecutive incident varieties and the extremeties are opposite, hence all these elements are pairwise distinct. Hence all sectors bounded by the union of two consecutive elements of $(p_1^*, \ell_1, q_1, m_1, p_1)$ meet one another in

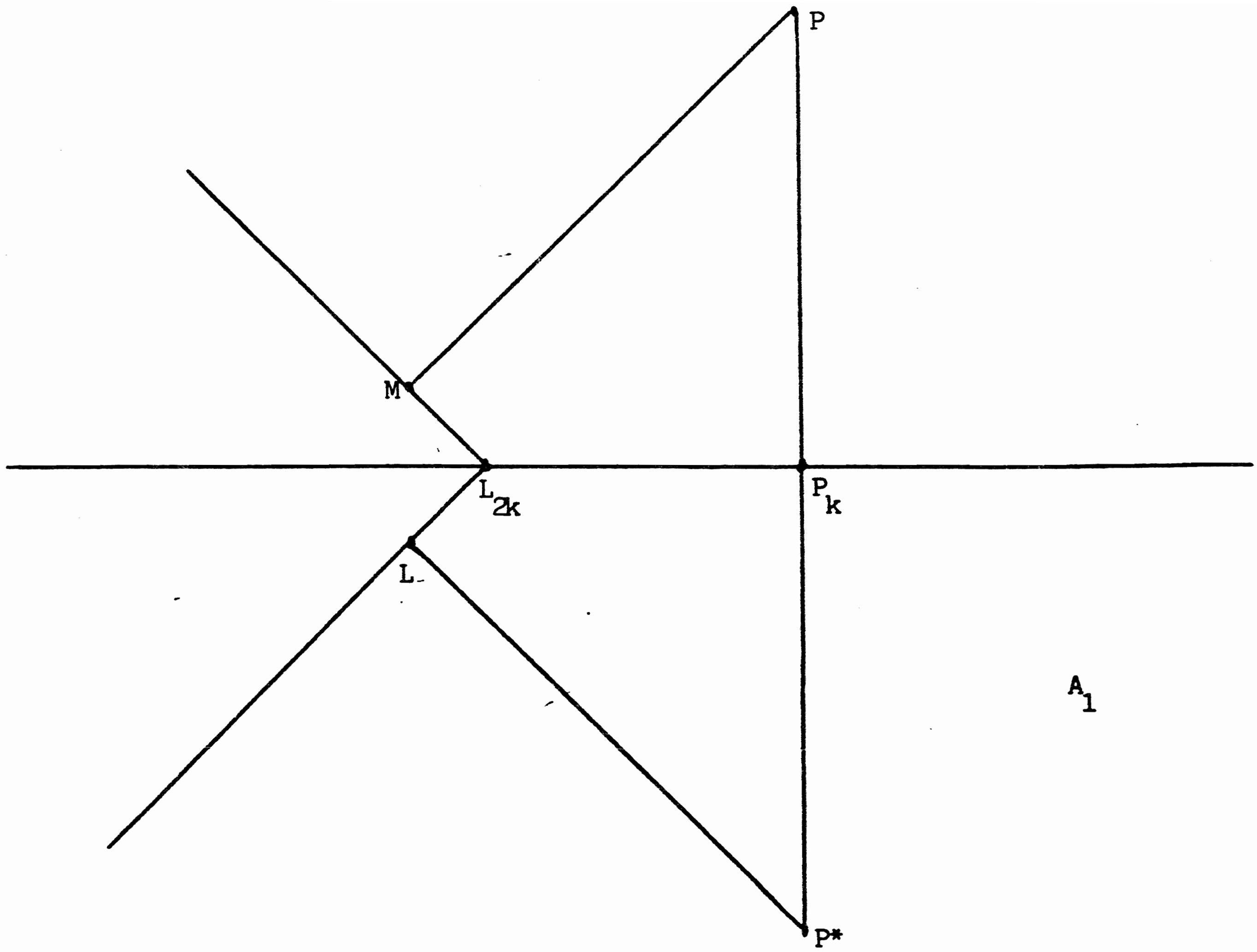


FIGURE 8

at most a sectorpanel. By result 2.2.10, there exists a half apartment containing p_1^*, ℓ_1, q_1, m_1 and p_1. Let Σ_1 be an apartment containing that half apartment. Note that Σ_1 already contains $\mathcal{P}$, $\mathcal{P}^*$ and $\mathcal{P}_k$. We show that Σ_1 also contains $\mathcal{L}$ and $\mathcal{M}$. Let Q_o^1 be the point of $R(b_o)$ represented by q_o (which is parallel to q and has source b_o). We use the notations of **(3)**. Suppose $Q_o^1 = \mathcal{P}_o^{*1}$. We have : $\mathcal{P}_o^1 \neq \mathcal{P}_o^{*1} \neq Q_o^{*1} \neq \mathcal{P}_o^1$ (Q_o^{*1} as on figure 7); $\mathcal{M}_o^1$ is incident with $\mathcal{P}_o^1$ and with $Q_o^1 = \mathcal{P}_o^{*1}$; $\mathcal{M}_o^1$ is not incident with Q_o^{*1} otherwise $\mathcal{J}_o^1 \perp \mathcal{M}_o^1$, contradicting the result in **(4)**.

Hence $\mathcal{P}_o^1 \perp \mathcal{P}_o^{*1} \perp Q_o^{*1} \perp \mathcal{P}_o^1$ forms a triangle in $R(b_o)$. Hence $Q_o^1 \neq \mathcal{P}_o^{*1}$. Also, Q_o^1 is not collinear with Q_o^{*1}, otherwise $\mathcal{P}_o^1 \perp Q_o^{*1} \perp Q_o^1 \perp \mathcal{P}_o^1$ forms a triangle in $R(b_o)$ (use the fact that $\mathcal{J}_o^1$ and $\mathcal{M}_o^1$ are opposite in $R(b_o)$. As above, this implies the existence of an apartment Σ' containing q_o, ℓ_o, p_o^* and $[b_o, b_1]$. Hence Σ' also contains $\mathcal{P}^*$, q_1 and $\mathcal{L}$. This implies that $\mathcal{L} \in cl(q_1, \mathcal{P}^*)$. Similarly $\mathcal{M} \in cl(q_1, \mathcal{P})$. Hence Σ_1 contains $\mathcal{L}$ and $\mathcal{M}$. We picture Σ_1 in figure 8. It will play a crucial role in the last two steps of the proof of (GQ2).

(5). We use the same notation as in **(4)**. We show that $\mathcal{M}$ is unique i.e. let $\mathcal{M}^* \in \mathcal{L}(V_n^b)$ with $\mathcal{P}\, I\, \mathcal{M}^* \perp \mathcal{L}$, then $\mathcal{M} = \mathcal{M}^*$. Since $u_2(\mathcal{P},\mathcal{L}) = 2k$, $u(\mathcal{L},\mathcal{M}^*) \leq 2k$. There are two possibilities.

(i) Suppose $u(\mathcal{L},\mathcal{M}^*) = 2k$. Consider again $V_{n-2k}^{b_o}$ and let $\mathcal{P}_o$ be as in **(3)**. Every sector containing $b, \mathcal{P}$ and $\mathcal{M}^*$, contains b_o and $\mathcal{P}$ and hence $cl(\mathcal{P}, b_o) \ni \mathcal{P}_o$. Hence $\mathcal{P}_o$ is incident with $\mathcal{M}^*$ in $V_{n-2k}^{b_o}$. Similarly one shows $\mathcal{M}^* \perp \mathcal{L}$ in $V_{n-2k}^{b_o}$. By induction (or by **(1)**), $\mathcal{M} = \mathcal{M}^*$ in $V_{n-2k}^{b_o}$ and hence also in V_n^b.

(ii) Suppose $u(\mathcal{L},\mathcal{M}^*) = 2k^* < 2k$. Let $b_o^* = \Pi_{2k^*}(\mathcal{L})$ and consider $V_{n-2k^*}^{b_o^*}$. Let Q_o be the vertex on $[\mathcal{P},\mathcal{M}]$ on distance $2k^*$

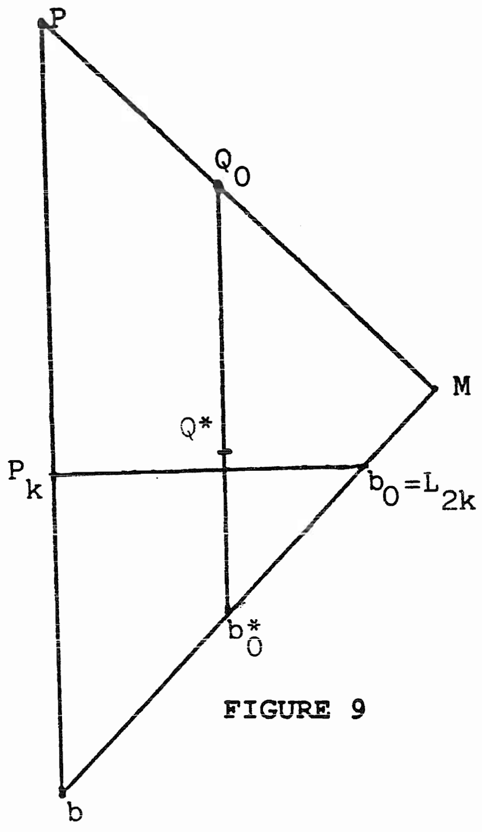


FIGURE 9

from $\mathcal{P}$ (see figure 9). Let Q^* be the vertex on $[b_0^*, Q_0]$ on d-distance $k-k^*+1$ from b_0^* (same figure). Note that Q^* does not lie on the wall w in Σ_1 supporting $[\mathcal{L}_{2k}, \mathcal{P}_k]$. As above, Q_0 is incident with both $\mathcal{M}$ and $\mathcal{M}^*$ in $V^{b_0^*}_{n-2k^*}$. Let u^* denote the valuation map in $V^{b_0^*}_{n-2k^*}$, then we have $u^*(\mathcal{L},\mathcal{M}^*) = 0$ and $u^*(\mathcal{L},\mathcal{M}) = 2k-2k^* > 0$, hence $u^*(\mathcal{M},\mathcal{M}^*) = 0$. By lemma 2.3.5, there exists an apartment $\Sigma_{\mathcal{M}}$ containing b_0^*, $\mathcal{M}$, $\mathcal{M}^*$ and Q_0. Similarly, there exists an apartment $\Sigma_{\mathcal{L}}$ containing b_0^*, $\mathcal{L}$ and $\mathcal{M}^*$. By looking in $\Sigma_{\mathcal{M}}$, one sees $Q^* \in \mathrm{cl}(\mathcal{L}_{2k}, \mathcal{M}^*)$. Since $\Sigma_{\mathcal{L}}$ contains $\mathcal{L}_{2k}$ and $\mathcal{M}^*$, it also contains Q^*. Looking in $\Sigma_{\mathcal{M}}$, we see $d_\Delta(Q^*.\mathcal{M}) = d_\Delta(Q^*,\mathcal{M}^*)$. But in $\Sigma_{\mathcal{L}}$, we see $d_\Delta(Q^*,\mathcal{M}^*) = d_\Delta(Q^*,\mathcal{L})$. Next, figure 9 tells us $Q^* \in \mathrm{cl}(\mathcal{P}_k, \mathcal{L}_{2k}, \mathcal{P}, \mathcal{M})$ and hence $Q^* \in \Sigma_1$. But we just showed that $d_\Delta(Q^*,\mathcal{M}) = d_\Delta(Q^*,\mathcal{L})$ and so $Q^* \in w$, a contradiction.

Hence $\mathcal{M}^* = \mathcal{M}$.

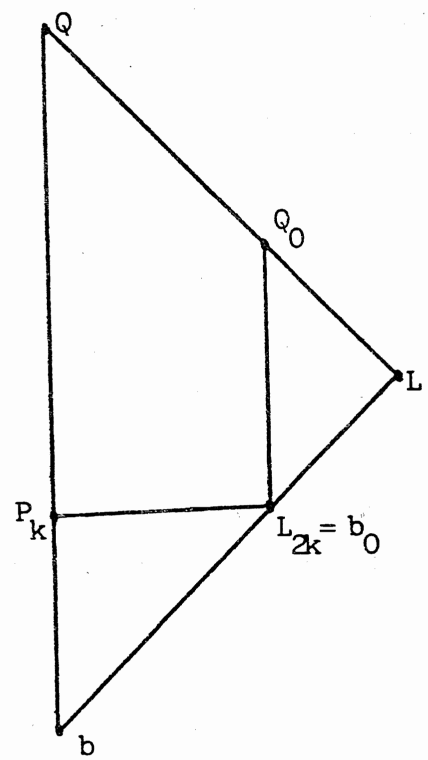


FIGURE 10

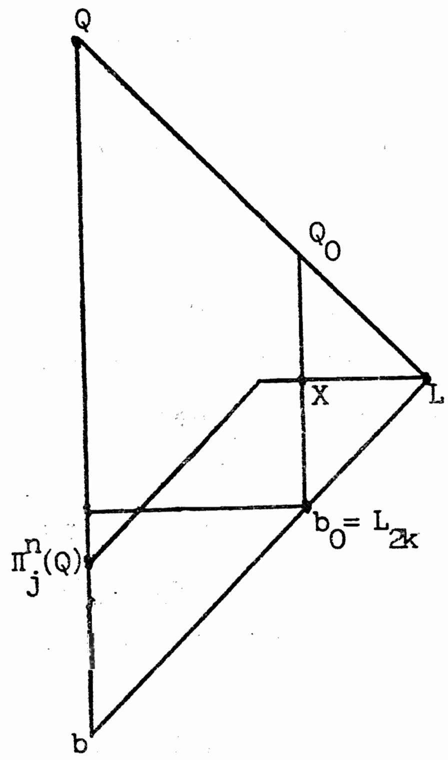


FIGURE 11

(6). We use the same notation as in (5). there remains to show that, if $Q \in \mathcal{P}(\mathcal{V}_n^b)$ is incident with $\mathcal{L}$ and $\mathcal{M}$, then $u(\mathcal{P},Q) = 0$. So let $Q\ I\ \mathcal{L},\mathcal{M}$ and suppose $u(\mathcal{P},Q) = j > 0$. Now $j \le k$ since $u_1(\mathcal{P},\mathcal{L}) = k$. There are two distinct cases to consider.

(i) Suppose $j = k$. Consider $\mathcal{V}^{b_0}_{n-2k}$ again. In Σ_1, there exists a point $Q_0^* \in \mathcal{P}(\mathcal{V}^{b_0}_{n-2k})$ lying on w and incident with $\mathcal{M}$ and $\mathcal{L}$ in $\mathcal{V}^{b_0}_{n-2k}$. The point $Q_0' \in \mathcal{P}(\mathcal{V}^{b_0}_{n-2k})$ defined as on figure 10 is also incident with $\mathcal{L}$ and $\mathcal{M}$ in $\mathcal{V}^{b_0}_{n-2k}$ and we clearly have $u'(Q_0^*,Q_0') = 0$ (u' is the valuation map in $\mathcal{V}^{b_0}_{n-2k}$) and $u'(\mathcal{L},\mathcal{M}) = 0$. This contradicts the induction hypothesis by [8,property(2.22)].

(ii) Suppose $0 < j < k$. Let Q_0' be as above. We have $u(\mathcal{P}^*,Q) = j > 0$ and hence there exists a vertex $X \in [b_0,Q_0']$ not lying on w and belonging to $\mathrm{cl}(b,Q,\mathcal{M}) \cap \mathrm{cl}(b,Q,\mathcal{L}) \cap \mathrm{cl}(b,\mathcal{P}^*,\mathcal{L})$ (see figure 11). Looking in $\mathrm{cl}(b,\mathcal{P}^*.\mathcal{L})$, we see $X \in \mathrm{cl}(b_0,\mathcal{P}^*) \subseteq \Sigma_1$. But as in (5)(ii), we have $d_\Delta(X,\mathcal{L}) = d_\Delta(X,\mathcal{M})$ and so $X \in w$, a contradiction.

This completes the proof of (GQ2) and of the main theorem.

Q.E.D.

4.4. Consequences.

The next two corollaries follow immediately from the main theorem and a theorem by M.A.Ronan [10].

COROLLARY 4.4.1. *For every thick generalized quadrangle* b, *there exists an HQ-Artmann-sequence* $(\mathcal{V}_n, \Pi_n^{n+1})_{n \in \mathbb{N}}$ *such that* b *is the base geometry of* $\mathcal{V}_1$.

This is the analogue of Artmann's theorem [1] for projective planes and projective Hjelmslev planes of level n. In particular we have :

COROLLARY 4.4.2. *For every thick generalized quadrangle $\mathcal{L}$ and every positive integer n, there exists a level n HQ having as canonical 1-image a level 1 HQ with base geometry $\mathcal{L}$.*

5. A CHARACTERIZATION OF AFFINE BUILDINGS OF TYPE $\tilde{C}_2$.

THEOREM 5.1. *Every affine building Δ of type $\tilde{C}_2$ is completely and unambigously determined by any special vertex b and the corresponding Hjelmslev-quadrangles of level n. More exactly, given Δ and b, then the HQ's defined by the floors in Δ with basement b as in the previous section form an HQ-Artmann-sequence $\mathcal{V}(\Delta,b)$ such that the building $\Delta(\mathcal{V}(\Delta,b))$ (see 3.3) is canonically isomorphic to Δ and this canonical isomorphism maps b to the unique element of the level 0 HQ of $v(\Delta,b)$.*

PROOF. We use the notation of 4.1. By the construction of $\mathcal{V}_n^b$, it is clear that $\mathcal{V}_{n-1}^b$ is isomorphic to the canonical $(n-1)$-image of $\mathcal{V}_n$. Hence $\mathcal{V}(\Delta,b) = (\mathcal{V}_n,\Pi_n^{n+1})_{n\in\mathbf{N}}$ is an HQ-Artmann-sequence. Now let x be any vertex of Δ. Set $n=d(x,b)$ and $j\in\mathbf{N}$ with $n^2+(n-j)^2 = d_\Delta^2(x,b)$ and $j\leq n$. Considering an apartment containing b and x, one sees that, if $j=0$, then x is a point of $\mathcal{V}_n^b$; if $0<j<n$, then x is an affine line in a certain unique j-point-neighbourhood of $\mathcal{V}_n^b$ and if $j=n$, then x is a line of $\mathcal{V}_n^b$. Every point, every line in every point-neighbourhood and every line of every HQ of $\mathcal{V}(\Delta,b)$ can be reached this way. Hence this defines a bijection Ψ : $\mathrm{Ve}(\Delta) \to \mathrm{Ve}(\Delta(v(\Delta,b)))$ mapping b to the unique element of $\mathcal{V}_0^b$. It is easy to check that Ψ and its inverse preserve adjacency (similarly to [15,propsition(5.1.5)], the case $\tilde{A}_2$). Q.E.D.

We keep the notation $\mathcal{V}(\Delta,b)$ of the previous theorem and will now show that every HQ-Artmann-sequence $\mathcal{V}$ is equivalent to $\mathcal{V}(\Delta(\mathcal{V}),b))$, for some well defined special vertex b of $\Delta(\mathcal{V})$. This is in fact the converse of theorem 5.1.

THEOREM 5.2. *Every HQ-Artmann-sequence $\mathcal{V}$ is equivalent to some HQ-Artmann-sequence arising from some affine building of type $\tilde{C}_2$. More exactly, given $\mathcal{V}$, let b be the vertex of $\Delta(\mathcal{V})$ corresponding to the unique element of the level 0 HQ of $\mathcal{V}$, then $\mathcal{V}$ is equivalent to $\mathcal{V}(\Delta(\mathcal{V}),b))$.*

PROOF. Let $\mathcal{V} = (\mathcal{V}_n, \nabla_n^{n+1})_{n\in\mathbb{N}}$ be an HQ-Artmann-sequence. We denote the partition maps of all point-neighbourhoods also by ∇_k^j (for the right indices). Consider the affine building $\Delta(\mathcal{V})$ of type $\tilde{C}_2$. We will use here again the notation $\mathbf{B}_n^j$ (see section 3.3). Let b be the unique element of $\mathbf{B}_0^0$ and suppose $c \in \mathbf{B}_n^j$ for certain n and j. By construction of $\Delta(\mathcal{V})$, one easily verifies $d(b,c) = n$. Consider the path $\gamma = (c, \nabla_{n-1}^{n}(c), \nabla_{n-2}^{n}(c), \ldots, \nabla_1^{n}(c), b)$. It has minimum length and so γ belongs to every apartment Σ through b and c. Now looking in Σ, one can see that the sequence of types of γ completely determines j and n. Hence c belongs to a certain j-point-neighbourhood of $\mathcal{V}_n^b$ and this defines a bijection from the set of affine lines of the j-point-neighbourhoods of $\mathcal{V}_n$ to the set of affine lines of the j-point-neighbourhoods of $\mathcal{V}_n^b$. It is straight forward to verify (similarly to Van Maldeghem [15,proposition 5.2.1]) that this bijection as well as its inverse preserves incidence (lines of 0-point-neighbourhoods are just points), "being component of" and j-point-(resp. line-)neighbourhoods. Hence, with obvious notation, $(\mathcal{V}_n, \nabla_n^{n+1})_{n\in\mathbb{N}}$ is equivalent to $(\mathcal{V}_n^b, \Pi_n^{n+1})_{n\in\mathbb{N}}$, i.e. $\mathcal{V}$ is equivalent to $\mathcal{V}(\Delta(\mathcal{V}),b))$. Q.E.D.

There is an interesting corollary.

COROLLARY 5.3. *Suppose $\mathcal{V}$ is an HQ-Artmann-sequence, then the geometry at infinity of $\Delta(\mathcal{V})$ is isomorphic to the inverse limit of the sequence of base geometries of $\mathcal{V}$ with respect to the partition maps.*

PROOF. This is a direct consequence of lemma 2.3.1 and theorem 5.2. Q.E.D.

A classical example.

We now briefly describe a classical HQ-Artmann-sequence. We will only construct the base geometries.

Let $\mathbf{F}$ be a field with discrete valuation v and valuation ring F_0. Denote by F_n the set of all elements x of $\mathbf{F}$ with $v(x) \geq n$ and set $\mathcal{R}_n = F_0/F_n$. Note that $\mathcal{R}_1$ is exactly the residue field of $\mathbf{F}$. In general, $\mathcal{R}_n$ is a ring, called a *Hjelmslev ring*. Consider the projective Hjelmslev space of projective dimension 4 and of level n over $\mathcal{R}_n$ and denote it by $PH(4,\mathcal{R}_n)$. Consider in $PH(4,\mathcal{R}_n)$ a parabolic Hjelmslev quadric $HQ(4,\mathcal{R}_n)$ containing lines but no planes ("Witt-index 2"). We define an incidence structure $\mathcal{V}_n = (\mathcal{P}(\mathcal{V}_n), \mathcal{L}(\mathcal{V}_n), I)$ as follows. The points (elements of $\mathcal{P}(\mathcal{V}_n)$) are the points of $HQ(4,\mathcal{R}_n)$. A general line is the intersection of two non-neighbouring tangent hyperplanes (defined in the usual way as polar hyperplane of a point of $HQ(4,\mathcal{R}_n)$) and the Hjelmslev quadric. Incidence is the set-theoretical $\in$ or $\ni$. For $n = 1$, this is the classical generalized quadrangle $Q(4,\mathcal{R}_1)$. If $\mathbf{F}$ is complete with respect to v, the inverse limit of these geometries is the classical generalized quadrangle $Q(4,\mathbf{F})$ which is also the geometry at infinity of the associated affine building of type $\widetilde{C}_2$.

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