PARAPOLAR SPACES OF INFINITE RANK

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Abstract. We show that in many classical characterization theorems involving parapolar spaces, we can lift the assumption of having only symplecta of finite rank. At the same time, we present an example of a parapolar space of infinite symplectic rank which shows that this is not possible in all characterizations.

Keywords: Spherical buildings, polar spaces, parapolar spaces.

AMS Classification: 51E24

1. Introduction

Buildings, sometimes also called Tits-buildings, were introduced by Jacques Tits [13] and provide a geometric interpretation of semi-simple groups of algebraic origin (semi-simple algebraic groups, classical groups, groups of mixed type, (twisted) Chevalley groups). These buildings can be rather complicated combinatorial structures; however, the properties of *spherical buildings* can be made more accessible using associated point-line geometries, the so called *Lie incidence geometries* [4]. Classical examples are projective spaces, polar spaces and Grassmannians thereof.

About 45 years ago, Bruce Cooperstein [5] initiated the study of *parapolar spaces*, which basically are point-line geometries in which the convex closure of two points at distance two is either a 2-path, or a non-degenerate polar space, which is then called a *symp* (Definition 2.11). However, unlike the fact that an irreducible spherical building of rank at least 3 automatically corresponds to a semi-simple group of algebraic origin, a parapolar space, even of constant symplectic rank at least 3 or 4, is not known to automatically arise from a spherical building. Consequently, ever since the birth of the parapolar spaces, many efforts have been made to characterize certain Lie incidence geometries as parapolar spaces satisfying certain regularity conditions, see for example [2], [8] and [11] and Chapters 13–18 in [12]. For a more recent one, see [6] and [7].

Most of these characterization theorems assume, either implicitly or explicitly, that the symps of the parapolar space have finite rank, or even stronger, that the singular subspaces of the parapolar space are finite dimensional. In this short paper, we zoom in on some classification theorems, and prove that the finite-dimensionality assumption follows from the other regularity conditions. At the same time, we provide an example of a parapolar space in which all symps have infinite rank, and where any two symps intersect in an infinite dimensional singular subspace. This proves that the finite rank assumptions of [6] and [7] are necessary. One of the reasons why we are able to dispense with the finite-dimensionality is, among others, the fact that we have now to our disposal the recent characterization of so-called *lacunary* parapolar spaces, from which we borrow the essential Proposition 2.16.

One of the motivations to do this job, is because the Main Theorem is needed in forthcoming papers classifying certain strong parapolar spaces, the so-called *Jordan spaces*. These Jordan spaces on their turn help to classify certain *Tits sets*, which are the rank 1 analogues of *Tits polygons* defined in [9].

Formulation of results. All necessary terminology regarding parapolar spaces can be found in Section 2. For any terminology regarding Lie incidence geometries, we refer to [12]. In [8, Theorem 2], Kasikova and Shult prove the following characterization result.

Proposition 1.1. *Let $\Delta = (\mathscr{P}, \mathscr{L})$ be a strong parapolar space that satisfies the following axioms:*

- (H'_1) *every point p is collinear to at least one point of each symp,*
- (H'_2) *for each point p, the set $\{q \in \mathscr{P} \mid d_\Delta(q,p) \leq 2\}$ forms a proper subspace of Δ,*
- (F') *if no symp has rank two, all singular subspaces are finite dimensional.*

Then Δ is either a Lie incidence geometry of type $\mathsf{(B|C)}_{3,3}$, $\mathsf{A}_{5,3}$, $\mathsf{D}_{6,6}$ or $\mathsf{E}_{7,7}$ or the direct product of a line and a polar space of arbitrary rank at least 2.

The Main Theorem of the present paper states that (F') automatically follows from (H'_1) and (H'_2).

Main Theorem. *Let Δ be a strong parapolar space that satisfies (H'_1) and (H'_2), then (F') holds too. In particular, Δ is one of the geometries obtained in* Proposition 1.1.

There is a range of other characterization theorems of which the proof makes use of Proposition 1.1. Also in these characterizations, one must no longer require that singular subspaces have finite dimension. We gather some of these results in Section 4.

Any strong parapolar space that satisfies Axioms (H'_1) and (H'_2) also has the property:

(L_0) No two symps intersect in exactly a point.

In [6], all parapolar spaces that satisfy (L_0) were classified, under the extra assumption that all symps have finite rank. In the light of the Main Theorem, it is hence natural to ask whether this latter assumption automatically follows from (L_0). It turns out that this is not the case. In Section 5, we construct an example of a parapolar space that satisfies (L_0), but where every symp has infinite rank.

2. Preliminaries

2.1. **Partial linear spaces.** Throughout, we will work with incidence structures called *partial linear spaces*. In this subsection, we introduce the general definitions we will need.

Definition 2.1. A *point-line space* is a pair $\Delta = (\mathscr{P}, \mathscr{L})$ with $\mathscr{P}$ a set and $\mathscr{L}$ a set of subsets of $\mathscr{P}$. The elements of $\mathscr{P}$ are called *points*, the members of $\mathscr{L}$ are called *lines*. If $p \in \mathscr{P}$ and $l \in \mathscr{L}$ with $p \in l$, we say that the point p *lies on* the line l, and the line l *contains* the point p, or *goes through* p. If two points p and q are contained in a common line, they are called *collinear*, denoted $p \perp q$. If they are not contained in a common line, we say that they are *noncollinear*. For any point p and any subset $P \subset \mathscr{P}$, we denote

$$p^{\perp} := \{q \in \mathscr{P} \mid q \perp p\} \text{ and } P^{\perp} := \bigcap_{p \in P} p^{\perp}.$$

A *partial linear space* is a point-line space in which every line contains at least three points, and where there is a unique line through every pair of distinct collinear points p and q, which is then denoted with pq.

Example 2.2. Let V be a vector space of dimension at least 3. Let $\mathscr{P}$ be the set of 1-spaces of V, and let $\mathscr{L}$ be the set of 2-spaces of V, each of them regarded as the set of 1-spaces it contains. Then $(\mathscr{P}, \mathscr{L})$ is called a *projective space (of dimension* $\dim V - 1$*)*.

Definition 2.3. Let $\Delta = (\mathscr{P}, \mathscr{L})$ be a partial linear space.

(1) A *path of length* n in Δ from point x to point y is a sequence $(x = p_0, p_1, \dots, p_{n-1}, p_n = y)$ of points of Δ such that $p_{i-1} \perp p_i$ for all $i \in \{1, \dots, n-1\}$. It is called a *geodesic* when there exist no paths of Δ from x to y of length strictly smaller than n, in which case *the distance* between x and y in Δ is defined to be n, notation $d_\Delta(x, y) = n$.

(2) The partial linear space Δ is called *connected* when for any two points x and y, there is a path (of finite length) from x to y. If moreover the set $\{d_\Delta(x, y) \mid x, y \in \mathscr{P}\}$ has a supremum in $\mathbb{N}$, this supremum is called the *diameter* of Δ.

(3) A subset S of $\mathscr{P}$ is called a *subspace* of Δ when every line l of $\mathscr{L}$ that contains at least two points of S, is contained in S. A subspace that intersects every line in at least a point, is called a *hyperplane*. A subspace

is called *convex* if it contains all points on every geodesic that connects any two points in S. We usually regard subspaces of Δ in the obvious way as subgeometries of Δ.

(4) A subspace S in which all points are collinear, or equivalently, for which $S \subseteq S^{\perp}$, is called a *singular subspace*. If S is moreover not contained in any other singular subspace, it is called a *maximal singular subspace*. A singular subspace is called *projective* if, as a subgeometry, it is a projective space (cf. Example 2.2). Note that every singular subspace is convex.

(5) For a subset P of $\mathscr{P}$, the *subspace generated by* P is denoted $\langle P \rangle_\Delta$ and is defined to be the intersection of all subspaces containing P. The *convex closure of* P is denoted $\langle\langle P \rangle\rangle_\Delta$ and is defined to be the intersection of all convex subspaces that contain P. A subspace generated by three mutually collinear points, not on a common line, is called a *plane*. Note that, in general, this is not necessarily a singular subspace; however we will only deal with geometries satisfying Axiom (PP_3) (see below), which implies that subspaces generated by pairwise collinear points are singular; in particular planes will be singular subspaces.

2.2. **Polar spaces.** We recall the definition of a polar space, and gather some basic properties. Since we insist on including infinite rank, we take the viewpoint of Buekenhout–Shult [1]. All results in this section are well known.

Definition 2.4. A *polar space* is a partial linear space in which every point is collinear to one or all points of every line. It is called *non-degenerate* if no point is collinear to all other points.

Lemma 2.5 (Theorem 7.3.6 and Lemma 7.3.8 of [12])**.** *Let* Γ *be a non-degenerate polar space. Every singular subspace of* Γ *is projective. Either every maximal singular subspace of* Γ *has finite dimension*[1] n*, in which case we say that* Γ *has rank* $n+1$*, or every maximal singular subspace of* Γ *is infinite dimensional, in which case we say that* Γ *has infinite rank.*

Remark 2.6. If a partial linear space contains no points, or contains at least two points but no lines, it is automatically a (non-degenerate) polar space, of rank 0 or rank 1, respectively.

Lemma 2.7 (Lemma 7.5.2 of [12])**.** *If a non-degenerate polar space contains lines, it is the convex closure of any two noncollinear points contained in it.*

Definition 2.8. Let Γ be a non-degenerate polar space and let N be a singular subspace of Γ. We define $\mathsf{Res}_\Gamma(N)$ to be the point-line space $(\mathscr{P}, \mathscr{L})$ with

[1] we will always work with projective dimensions

$$\mathscr{P} := \{\text{singular subspaces } K \text{ of } S \text{ with } N \subset K \text{ and } \mathsf{codim}_K(N) = 1\},$$
$$\mathscr{L} := \{\text{singular subspaces } L \text{ of } S \text{ with } N \subset L \text{ and } \mathsf{codim}_L(N) = 2\},$$

where any element of $\mathscr{L}$ is identified with the set of elements of $\mathscr{P}$ contained in it.

Lemma 2.9 ([10])**.** *Let Γ be a polar space and let N be a singular subspace of Γ. The point-line space $\mathsf{Res}_\Gamma(N)$ is a polar space, which is non-degenerate if and only if $N^{\perp\perp} = N$.*

If the polar space Γ is non-degenerate and has finite rank, then the polar spaces $\mathsf{Res}_\Gamma(N)$ are non-degenerate for all singular subspaces N. If Γ is non-degenerate and has infinite rank, this is no longer the case. The following lemma ensures that we can still find a lot of such singular subspaces.

Lemma 2.10 ([10])**.** *Let Γ be a non-degenerate polar space of infinite rank and let M be a maximal singular subspace of Γ. For every $k \in \mathbb{N}$, there is a singular subspace N_k of Γ contained in M with $\mathsf{codim}_M(N_k) = k$ and $N_k^{\perp\perp} = N_k$. For such N_k, the polar space $\mathsf{Res}_\Gamma(N_k)$ is non-degenerate and has rank k.*

2.3. **Parapolar spaces.** We also recall the definition of a parapolar space, and state some corollaries of classification theorems that will be useful later on.

Definition 2.11. A *parapolar space* Δ is a connected partial linear space, which is not a polar space, and which satisfies the following two axioms.

(PP_1) For points p and q with $d_\Delta(p,q) = 2$ and $|p^\perp \cap q^\perp| \geq 2$, the convex subspace $\langle\langle p,q\rangle\rangle_\Delta$ is a non-degenerate polar space. Any subspace that can be obtained like this is called a *symp* of Δ (which is short for *symplecton*).
(PP_2) Every line of Δ is contained in a symp of Δ.

A pair of points p and q with $d_\Delta(p,q) = 2$ is called *special* if $|p^\perp \cap q^\perp| = 1$ and symplectic if $|p^\perp \cap q^\perp| \geq 2$. A parapolar space is called *strong* when it contains no pair of special points.

Remark 2.12. In the definition of parapolar spaces, one often also adds the following axiom:

(PP_3) Every point is collinear to zero, one or all points of any line.

This however automatically follows from (PP_1), which is why we do not explicitly include it in the axioms.

We will need the following lemma.

Lemma 2.13. *Let Δ be a partial linear space in which the convex closure of any two noncollinear points is a non-degenerate polar space (of rank at least two). Then either all points of Δ are mutually collinear, or Δ is a polar space, or Δ is a strong parapolar space of diameter* 2*.*

Proof. Suppose that not all points of Δ are mutually collinear. Then there are points x_1 and x_2 which are noncollinear. Suppose that some point x is collinear to all points of Δ. Then it is in particular collinear to x_1 and x_2, and hence contained in $\langle\langle x_1, x_2\rangle\rangle_\Delta$. By assumption, this is a non-degenerate polar space, so there is some point in $\langle\langle x_1, x_2\rangle\rangle_\Delta$ which is noncollinear to x, a contradiction.

It is clear that Δ is connected and that (PP_1) holds. It hence suffices to check (PP_2). To that end, let l be any line of Δ, and let p and x be two distinct points on l. By the arguments in the previous paragraph, there exists some point q that is noncollinear to x. The convex closure $\xi := \langle\langle x, q\rangle\rangle_\Delta$ is a non-degenerate polar space of rank at least two, which of course contains x. Let p_1 and p_2 be two points of ξ collinear to x such that p_1 is not collinear to p_2. If p is collinear to both p_1 and p_2, the point p is contained in a geodesic connecting p_1 and p_2, implying that p is contained in ξ, and in particular that $l = xp$ is indeed contained in some symp. We may therefore assume without loss of generality that p is not collinear to p_1. Then $\langle\langle p, p_1\rangle\rangle_\Delta$ is a non-degenerate polar space which contains x and p, and hence also l. This concludes the proof. □

Definition 2.14. A parapolar space has *symplectic rank at least d* (for $d \in \mathbb{N}$) when every symp has rank at least[2] d. It has *infinite symplectic rank* when every symp has infinite rank.

Lemma 2.15 (Lemma 4.1 of [6])**.** *In a parapolar space of symplectic rank at least $d \in \mathbb{N}$, every singular subspace of dimension $d-1$ is contained in a symp. If $d \geq 3$, every singular subspace is projective.*

Proposition 2.16 (Lemma 6.9 of [7] and Lemma 7.14 of [6])**.** *Let Δ be a strong parapolar space where every symp has finite rank at least three.*

(1) *If the intersection of any two symps contains at least a point, and there is a line which is contained in two symps, then every symp of Δ has rank at most* 5*.*
(2) *If the intersection of any two symps is never exactly a point, then every singular subspace of Δ has finite dimension.*

[2]Every symp of infinite rank has of course rank at least d.

3. Proof of the Main Theorem

In this section, we prove the Main Theorem. To that end, we denote with Δ a strong parapolar space of symplectic rank at least three that satisfies Axioms (H_1') and (H_2'). We aim to prove that Δ satisfies the following property:

(F′) Every singular subspace is finite dimensional.

In [8], Kasikova and Shult characterize all strong parapolar spaces of symplectic rank at least three that satisfy axioms (H_1') and (H_2'), and (F′). However, for many of the results they gather on the way, they do not use (F′). We start by gathering two of these results that will still prove to be useful.

Lemma 3.1 (Theorem 17.2.6 of [8]). *No two symps intersect in exactly one point.*

Lemma 3.2 (Proof of Theorem 17.2.9 of [8]). *Let ξ_1 and ξ_2 be two symps intersecting in a singular subspace M of dimension at least 2. For any point $x_1 \in \xi_1 \setminus M$, with $M \not\subseteq x_1^\perp$, we find $x_2 \in \xi_2 \setminus M$ such that $x_1 \perp x_2$.*

We can use this to obtain the following properties.

Lemma 3.3. *Let ξ be a symp and x a point not in ξ, then $x^\perp \cap \xi$ is either a point or a maximal singular subspace of ξ.*

Proof. It is well known that $M := x^\perp \cap \xi$ is a singular subspace of ξ. Moreover, Axiom (H_1') ensures that M is not empty. Suppose for a contradiction that M contains a line, but is not a maximal singular subspace of ξ. There exists some point y in $M^\perp \cap \xi \setminus M$. The set $x^\perp \cap y^\perp$ contains M, implying that x and y are symplectic. The intersection $\xi \cap \langle\langle x, y\rangle\rangle_\Delta$ contains y and M and is hence a subspace of dimension at least two. Applying Lemma 3.2 with $x, y, \langle\langle x, y\rangle\rangle_\Delta, \xi$ taking up the role of x_1, ξ_1, ξ_2, then yields some point in $\xi \setminus (\xi \cap \langle\langle x, y\rangle\rangle_\Delta)$ which is collinear to x; but $M = x^\perp \cap \xi$ is contained in $\xi \cap \langle\langle x, y\rangle\rangle_\Delta$, so we find a contradiction. □

Lemma 3.4. *Two different symps intersect either trivially, in a line or in a subspace which is a maximal singular subspace of both symps.*

Proof. Let ξ_1 and ξ_2 be two symps. It is well known that $\xi_1 \cap \xi_2$ is a singular subspace. If $\xi_1 \cap \xi_2 \neq \emptyset$, then it follows from Lemma 3.1 that $M := \xi_1 \cap \xi_2$ contains a line. Suppose that M is a subspace of dimension at least two. Take $y \in M$ and $x \in \xi_2$ such that x and y are not collinear. It follows from Lemma 3.3 that $M_1 := x^\perp \cap \xi_1$ is a maximal singular subspace of ξ_1. The point y is contained in ξ_1, so $y^\perp \cap M_1$ is a hyperplane of M_1. Each point of this hyperplane is contained in $x^\perp \cap y^\perp \subset \langle\langle x, y\rangle\rangle_\Delta = \xi_2$. We hence find that $y^\perp \cap M_1$ is contained in $\xi_1 \cap \xi_2 = M$. It is at the same time a hyperplane of M, implying that M is a maximal singular subspace of ξ_1. □

Lemma 3.5. *For any two symps ξ and ξ', there exists a sequence of symps $(\xi_0, \xi_1, \dots, \xi_n)$ such that $\xi_0 = \xi$ and $\xi_n = \xi'$ and such that, for every $i \in \{1, \dots, n\}$, the intersection $\xi_{i-1} \cap \xi_i$ is a maximal singular subspace of both ξ_{i-1} and ξ_i.*

Proof. Let ξ and ξ' be any two symps. First suppose that $\xi \cap \xi'$ contains a plane. It follows from Lemma 3.4 that either $\xi = \xi'$ or that $\xi \cap \xi'$ is a maximal singular subspace of both ξ and ξ', in both cases, the claim follows immediately. Next, suppose that $\xi \cap \xi'$ is a line l. The symp ξ has rank at least three, so there is some point $x \in \xi \cap l^\perp$, with $x \notin l$. By Lemma 3.3, the set $x^\perp \cap \xi'$ is a maximal singular subspace M of ξ' that contains l. Since ξ' also has rank at least three, there exists some $x' \in \xi' \cap l^\perp$, with $x' \notin M$. The points x and x' are symplectic, and the symp $\langle\langle x, x' \rangle\rangle_\Delta$ intersects both ξ and ξ' in at least a plane. The claim then follows from the arguments above. Finally, assume that $\xi \cap \xi'$ does not contain a line. It follows from Lemma 3.4 that the two symps intersect trivially. Take $x \in \xi$. By Axiom (H_1'), there is some point $x' \in x^\perp \cap \xi'$. The line xx' is moreover contained in some symp, which, by Lemma 3.4, intersects both ξ and ξ' in at least a line. The claim again follows from the arguments above. □

Our next aim is to prove that there exists some symp of finite rank.

Definition 3.6. Let N be a singular subspace of Δ. We define $\mathsf{Res}_\Delta(N)$ to be the point-line space $(\mathscr{P}, \mathscr{L})$ with

$$\mathscr{P} := \{\text{singular subspaces } K \text{ of } \Delta \text{ with } N \subset K \text{ and } \mathsf{codim}_K(N) = 1\},$$
$$\mathscr{L} := \{\text{singular subspaces } L \text{ of } \Delta \text{ with } N \subset L \text{ and } \mathsf{codim}_L(N) = 2\},$$

where again, any element of $\mathscr{L}$ is identified with the set of elements of $\mathscr{P}$ contained in it. It is clear that $\mathsf{Res}_\Delta(N)$ forms a partial linear space.

Remark 3.7. Let ξ be a symp and N be a singular subspace of Δ contained in ξ, denote with $N^{\perp_\xi} := N^\perp \cap \xi$. The point-line geometry $\mathsf{Res}_\xi(N)$, as defined in Definition 2.8, is a subspace of $\mathsf{Res}_\Delta(N)$. By Lemma 2.9, it is a polar space, which is non-degenerate if and only if $N = N^{\perp_\xi \perp_\xi}$.

We aim to construct singular subspaces N of Δ for which $\mathsf{Res}_\Delta(N)$ forms a parapolar space. From now on we assume, for a contradiction, that there is no symp of finite rank.

Notation 3.8. Fix some symp ξ. Using Lemma 3.5, we find a maximal singular subspace M of ξ that is contained in at least one other symp. Take $k \in \mathbb{N}_{\geq 3}$, and use Lemma 2.10 to find a singular subspace N_k in M for which $\mathsf{codim}_M(N_k) = k$ and $\mathsf{Res}_\xi(N_k)$ is a non-degenerate polar space of rank k.

Lemma 3.9. *For any symp ξ' of Δ that contains N_k, the subspace $\mathsf{Res}_{\xi'}(N_k)$ of $\mathsf{Res}_\Delta(N_k)$ is a non-degenerate polar space of rank k.*

Proof. If $\xi = \xi'$, the claim holds trivially, so suppose that this is not the case. Denote with M' the intersection of ξ and ξ'. The subspace N_k is contained in M', so using Lemma 3.4, we obtain that M' is a maximal singular subspace of both ξ and ξ'. The subspace M' induces a maximal singular subspace in $\mathsf{Res}_\xi(N_k)$, which implies that $\mathsf{codim}_{M'}(N_k) = k$. It follows from Lemma 2.9 that, in order to show that $\mathsf{Res}_{\xi'}(N_k)$ is a non-degenerate polar space of rank k, it suffices to prove that $N_k = N_k^{\perp_{\xi'}\perp_{\xi'}}$.

It is clear that $N_k \subseteq N_k^{\perp_{\xi'}\perp_{\xi'}}$, so we prove the opposite inclusion. The subspace M' is contained in $N_k^{\perp_{\xi'}}$, which implies that $N_k^{\perp_{\xi'}\perp_{\xi'}} \subseteq M'^{\perp_{\xi'}} = M'$, where the latter equality holds because M' is a maximal singular subspace of ξ'. Take $p \in M' \setminus N_k$. The polar space $\mathsf{Res}_\xi(N_k)$ is non-degenerate, so $N_k^{\perp_\xi\perp_\xi} = N_k$, and in particular, there exists some point $x \in N_k^{\perp_\xi}$ which is not collinear to p. The set $x^\perp \cap M'$ is a hyperplane N' of M' that contains N_k but not p. By Lemma 3.3, the set $x^\perp \cap \xi'$ is a maximal singular subspace M_x of ξ' with $M' \cap M_x = N'$. For any point $x' \in M_x \setminus N'$, the set $x'^\perp \cap M'$ contains N'. Since M' is a maximal singular subspace of ξ', and x' is not contained in M', this implies that $x'^\perp \cap M' = N'$. The point x' is hence a point of ξ' which is noncollinear to p and collinear to all points of N_k, which implies that $p \notin N_k^{\perp_{\xi'}\perp_{\xi'}}$. □

Lemma 3.10. *The partial linear space* $\mathsf{Res}_\Delta(N_k)$ *is a strong parapolar space of diameter two in which all symps have rank* k*, and any two symps intersect in a singular subspace of dimension* $k-1$*.*

Proof. We first prove that the convex closure in $\mathsf{Res}_\Delta(N_k)$ of any two noncollinear points of $\mathsf{Res}_\Delta(N_k)$ forms a non-degenerate polar space of rank k. To that end, let K_1 and K_2 be any two such points of $\mathsf{Res}_\Delta(N_k)$. By definition, both K_1 and K_2 are singular subspaces of Δ that contain N_k with $\mathsf{codim}_{N_k}(K_1) = \mathsf{codim}_{N_k}(K_2) = 1$. Let $x_i \in K_i \setminus \{N_k\}$, $i = 1, 2$, be arbitrary. Then $K_i = \langle x_i, N_k\rangle_\Delta$. It follows immediately that x_1 and x_2 are collinear in Δ if and only if K_1 and K_2 are collinear in $\mathsf{Res}_\Delta(N_k)$. We hence find that x_1 and x_2 are not collinear. We have that $N_k \subseteq x_1^\perp \cap x_2^\perp$, which implies that x_1 and x_2 are symplectic in Δ, and that the symp $\langle\!\langle x_1, x_2\rangle\!\rangle_\Delta$ of Δ contains N_k. The subspace $\mathsf{Res}_{\langle\!\langle x_1,x_2\rangle\!\rangle}(N_k)$ is, by Lemma 3.9, a non-degenerate polar space of rank k which of course contains K_1 and K_2. It is clear that $\langle\!\langle K_1, K_2\rangle\!\rangle_{Res_\Delta(N_k)}$ is contained in $\mathsf{Res}_{\langle\!\langle x_1,x_2\rangle\!\rangle_\Delta}(N_k)$. Using Lemma 2.7, one obtains that equality holds.

We have picked the subspace N_k of Δ in such a way that it is contained in two distinct symps of Δ. As a consequence, $\mathsf{Res}_\Delta(N_k)$ contains points that are not collinear to each other, and is not a polar space. We can hence conclude using Lemma 2.13 that $\mathsf{Res}_\Delta(N_k)$ is a strong parapolar space of diameter two. All symps have rank k. By Lemma 3.3, any two symps of Δ that contain N_k intersect in a maximal singular subspace of both symps, so this translates to the fact that every two symps of $\mathsf{Res}_\Delta(N_k)$ intersect in a maximal singular subspace of both symps, which in this case, is a singular subspace of dimension $k-1$. □

Proposition 3.11. *The parapolar space Δ contains some symp of finite rank.*

Proof. Suppose this is not the case. For $k \geq 6$, it then follows from Lemma 3.10 that $\mathsf{Res}_\Delta(N_k)$ is a parapolar space which satisfies the assumptions of Proposition 2.16 but where all symps have rank $k > 5$, a contradiction. □

Using the previous proposition, we find some symp of Δ that has finite rank d. Then $d \geq 3$ by assumption.

Lemma 3.12. *All symps of Δ have rank d.*

Proof. By assumption, there exists some symp ξ of rank d. Let ξ' be any other symp for which $\xi \cap \xi'$ is a maximal singular subspace of both ξ and ξ'. Then ξ' has a maximal singular subspace of dimension $d-1$, which implies that it has rank d. The claim now follows directly from Lemma 3.5. □

Proposition 2.16 now finishes the proof of the Main Theorem.

4. Corollaries of the Main Theorem

Our first consequence improves a theorem of Shult, namely Theorem 2 of [11].

Corollary 4.1. *Let Δ be a parapolar space of diameter 3 and symplectic rank at least 3, that satisfies*

(H$_1$) *no point is collinear to exactly one point of a symp.*

Then Δ is either a Lie incidence geometry of type $\mathsf{F}_{4,1}$, $\mathsf{E}_{6,2}$, $\mathsf{E}_{7,1}$ or $\mathsf{E}_{8,8}$ or the line Grassmannian of a non-degenerate polar space of arbitrary rank at least 4.

Proof. In [11], Shult classifies these geometries Δ under the extra condition that, if Δ has no symp of rank three, all singular subspaces have finite dimension. So if Δ contains a symp of rank three, the claim follows. Suppose this is not the case. Let M be any singular subspace of Δ, and take $p \in M$. In [8], it is proved that the residue $\mathsf{Res}_\Delta(p)$ is a strong parapolar space of symplectic rank at least 3 satisfying Assumptions (H$_1'$) and (H$_2'$). Our Main Theorem then proves that $\mathsf{Res}_\Delta(p)$ is finite dimensional, implying that M itself is also finite dimensional. □

The next corollary improves a result of Cohen & Ivanyos, namely Theorem 1 of [3]. The proof is similar to the one of Corollary 4.1 and we omit the details. The original statement, along with all relevant definitions, can be found in [3]. We content ourselves with describing the point-line space $\mathscr{E}(\mathbb{P}, \mathbb{H})$.

Example 4.2. Let $\mathbb{P}$ be projective space and $\mathbb{H}$ be a set of hyperplanes of $\mathbb{P}$ such that $\mathbb{H}$ forms a subspace of the dual of $\mathbb{P}$ and $\cap_{H\in\mathbb{H}} H = \emptyset$.

The partial linear space $\mathscr{E}(\mathbb{P},\mathbb{H})$ is defined as follows. The point set is the set $\{(p,H) \in \mathbb{P}\times\mathbb{H} \,|\, p \in H\}$. The line set consists of two types: subsets of the form $\{(p,H) \,|\, p \in l\}$ where l is a line of $\mathbb{P}$ that is contained in H, and subsets of the form $\{(p,H) \,|\, K \subset H\}$ where K is a codimension-2 subspace of $\mathbb{P}$ that contains p for which there are at least two elements of $\mathbb{H}$ containing it.

If $\mathbb{P}$ is finite dimensional, the set $\mathbb{H}$ necessarily consists of all hyperplanes of $\mathbb{P}$, and $\mathscr{E}(\mathbb{P},\mathbb{H})$ is a Lie incidence geometry of type $\mathsf{A}_{n,\{1,n\}}$.

Corollary 4.3. *Let Δ be a non-degenerate root filtration space. Then Δ is either a generalized hexagon, a Lie incidence geometry of type* $\mathsf{F}_{4,1}$, $\mathsf{E}_{6,2}$, $\mathsf{E}_{7,1}$ *or* $\mathsf{E}_{8,8}$*, the line Grassmannian of a non-degenerate polar space of arbitrary rank at least* 3*, or a point-line space* $\mathscr{E}(\mathbb{P},\mathbb{H})$.

5. Example of a parapolar space of infinite symplectic rank

In this section, we aim to prove the following proposition.

Proposition 5.1. *There exists a strong parapolar space of diameter two such that all symps are infinite dimensional, and every two symps intersect in an infinite dimensional projective subspace.*

We will construct such an example Ω using a free construction process. Hence the construction is inductive: we start with some partial linear space $\Omega_0 = (\mathscr{P}_0, \mathscr{L}_0)$, and in every step $i \in \mathbb{N}_{>0}$, we add some points and lines to Ω_{i-1} to obtain a new partial linear space $\Omega_i = (\mathscr{P}_i, \mathscr{L}_i)$. That way, we obtain a chain

$$\Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega_n \subset \dots .$$

We will then use this chain to construct a new partial linear space, namely $\bigcup_{i\in\mathbb{N}} \Omega_i$ which will satisfy the desired properties. Here, the union is just the direct limit. Each Ω_i will also be defined as a union of two nondisjoint partial linear spaces. In order to obtain a partial linear space, we need some necessary conditions.

Definition 5.2. Let $\Delta_i = (\mathscr{P}_i, \mathscr{L}_i)$, $i = 1, 2$, be a partial linear space with $\mathscr{P}_1 \cap \mathscr{P}_2 \neq \emptyset$. Assume that two points in $\mathscr{P}_1 \cap \mathscr{P}_2$ are contained in some $l_1 \in \mathscr{L}_1$ if and only if they are contained in some $l_2 \in \mathscr{L}_2$, in which case l_1 and l_2 are assumed to coincide. The union $\Delta_1 \cup \Delta_2$ is defined to be $(\mathscr{P}_1 \cup \mathscr{P}_2, \mathscr{L}_1 \cup \mathscr{L}_2)$, which is a partial linear space.

Every partial linear space Ω_i that we will construct, is what we call *a preparapolar space.*

Definition 5.3. A *preparapolar space* Δ is a connected partial linear space in which Axioms (PP_1'), (PP_2) and (PP_3) hold, where (PP_1') is defined as follows.

(PP_1') For points p and q with $d(p,q)=2$, one of the following holds:
- (a) The set $p^\perp \cap q^\perp$ contains a pair of noncollinear points. The convex closure of p and q forms a non-degenerate polar space. Any subspace that can be obtained like this is called a *symp*.
- (b) The set $p^\perp \cap q^\perp$ is a singular subspace. The convex closure of p and q in Δ is $\langle p, p^\perp \cap q^\perp\rangle \cup \langle q, p^\perp \cap q^\perp\rangle$.

Note that every parapolar space is a preparapolar space. Constructing preparapolar spaces is significantly easier than constructing parapolar spaces. One example is the following.

Construction (Construction of Ω_0 and ϕ_0)**.** Let V be a vector space over a countable field k with basis $\{e_i\}_{i\in\mathbb{N}}$. For $j=0,1$ and 2, define V_j be the vector subspace of V generated by $\{e_i\}_{i>2}$ and e_j, and denote with M_j the subspace $\mathbb{P}(V_j)$ of $\mathbb{P}(V)$. Define $\Omega_0 = (\mathscr{P}_0, \mathscr{L}_0)$ to be the partial linear space with is the union of $M_0 \cup M_1 \cup M_2$. One easily checks that Ω_0 is a preparapolar space, but of course not a parapolar space. Note that there exists a bijection $\phi_0 : \mathbb{N} \to \mathscr{P}_0 \times \mathscr{P}_0$.

The preparapolar space Ω_0 is constructed such that it satisfies the following conditions.

(A_1) The point set is countably infinite.
(A_2) There is some field k such that every singular subspace is a projective space over k.

Moreover, Ω_0 contains a singular subspace M of countably infinite dimension (for example M_0), for which the following holds.

(A_M) For every point p, the subspace $p^\perp \cap M$ has finite codimension in M.

Remark 5.4. Any preparapolar space that satisfies (A_M) has only symps of infinite rank, and has diameter at most two.

Starting from a preparapolar space that satisfies the axioms above, we can construct a strictly bigger preparapolar space that still satisfies the same axioms. We make this explicit in the next lemma.

Lemma 5.5. *Let $\Delta = (\mathscr{P}, \mathscr{L})$ be a preparapolar space that satisfies* (A_1), (A_2), *and that contains some singular subspace M of countably infinite dimension for which* (A_M) *holds. Let x and y be noncollinear points of Δ for which x and y are not in a symp of Δ. There exists a partial linear space $\Delta_{x,y}$ for which the following hold.*

(1) *The point-line space $\Delta_{x,y}$ is a preparapolar space.*

(2) *The point-line space* Δ*, and hence also* M*, is a subspace of* $\Delta_{x,y}$.

(3) *The point-line space* $\Delta_{x,y}$ *satisfies* (A_1), (A_2) *and* (A_M).

(4) *For noncollinear points* p *and* q *of* Δ*, the following hold:*

  (a) *if* $\langle\langle p,q\rangle\rangle_\Delta$ *contains* x *and* y*, then* $\langle\langle p,q\rangle\rangle_{\Delta_{x,y}}$ *is a symp,*

  (b) *if* $\langle\langle p,q\rangle\rangle_\Delta$ *does not contain both* x *and* y*, then* $\langle\langle p,q\rangle\rangle_{\Delta_{x,y}} = \langle\langle p,q\rangle\rangle_\Delta$.

Proof. Let Δ, x, y and M be as in the statement. Denote $M_{x,y} := x^\perp \cap y^\perp$, $M_x := \langle x, M_{x,y}\rangle$ and $M_y := \langle y, M_{x,y}\rangle$. Using (A_2), we see that the singular subspaces $M_{x,y}, M_x$ and M_y are projective spaces over some field k, with $\mathsf{codim}_{M_x}(M_{x,y}) = \mathsf{codim}_{M_y}(M_{x,y}) = 1$. Since $M \cap M_{x,y}$ is the intersection of two subspaces of M of finite codimension of M, it has finite codimension in M, implying that the projective subspace $M_{x,y}$ has countable infinite dimension.

By (PP_1'), the convex closure of x and y in Δ is given by $M_x \cup M_y$. We construct a polar space $\Gamma_{x,y}$ which intersects Δ in $M_x \cup M_y$. To that end, let V be a vector space over k with basis $\{e_i\}_{i\in\mathbb{N}}$. Denote with V_x the vector subspace generated by $\{e_{2i}\}_{i\in\mathbb{N}}$, with V_y the vector subspace generated by $\{e_{2i}\}_{i>0}$ and $\{e_1\}$, and with $V_{x,y}$ the intersection $V_x \cap V_y$. Using the arguments in the previous paragraph, it is clear that we can identify $M_{x,y}, M_x$ and M_y with the subspaces $\mathbb{P}(V_{x,y}), \mathbb{P}(V_x)$ and $\mathbb{P}(V_y)$ of $\mathbb{P}(V)$ respectively. Consider the symmetric bilinear form $f : V \times V \to k$ with $f(\sum_i a_i e_i, \sum_i b_i e_i) = \sum_i (a_{2i}b_{2i+1} + b_{2i}a_{2i+1})$. A subspace S of V is called isotropic when $f(s,t) = 0$ for all $s,t \in S$. The point-line space $\Gamma_{x,y}$ with as point set the isotropic 1-dimensional vector subspaces of V, and as line set the isotropic 2-dimensional vector subspaces of V, is a polar space. Note that M_x and M_y are maximal singular subspaces of $\Gamma_{x,y}$ and that we have constructed $\Gamma_{x,y}$ in such a way that it intersects Δ in $M_x \cup M_y$. Moreover, every point of $\Gamma_{x,y}$ is collinear with a subspace of codimension at most one of $M \cap M_{x,y}$.

It is clear that $\Gamma_{x,y}$ and Δ satisfy the conditions of Definition 5.2, we can hence consider the partial linear space $\Delta\cup\Gamma_{x,y}$. It is straightforward to check that $\Delta\cup\Gamma_{x,y}$ is indeed a preparapolar space that satisfies the required properties. ☐

We will apply Lemma 5.5 to inductively construct the chain of preparapolar spaces Ω_n. Along the way, we use the following bijection:

$$f : \mathbb{N}\times\mathbb{N} \to \mathbb{N} : (n_1,n_2) \mapsto \frac{(n_1+n_2)(n_1+n_2+1)}{2} + n_2.$$

Construction (Construction of Ω_n and ϕ_n)**.** Suppose that we have constructed preparapolar spaces $\Omega_0, \Omega_1, \ldots \Omega_{n-1}$ that satisfy (A_1), (A_2) and (A_M), but such that none of these is a parapolar space. Suppose moreover that for all $i = 1\ldots n-1$, the partial linear space $\Omega_i = (\mathscr{P}_i, \mathscr{L}_i)$ contains Ω_{i-1} as a subspace and that $\phi_i : \mathbb{N} \to \mathscr{P}_i \times \mathscr{P}_i$ is a bijection. Set $(n_1,n_2) = f^{-1}(n)$. Note that $n_1 \le n$, which implies that the map ϕ_{n_1} is defined. If the pair $\phi_{n_1}(n_2)$ of $\mathscr{P}_{n-1}\times\mathscr{P}_{n-1}$, is not contained in a symp of Ω_{n-1}, define $(x,y) := \phi_{n_1}(n_2)$. If the pair $\phi_{n_1}(n_2)$ is already contained in a symp of Ω_{n-1}, use the fact that Ω_{n-1} is not a parapolar space to find a pair of points $(x, y) \in \Omega_{n-1} \times \Omega_{n-1}$ that is not in a symp of Ω_n. Use Lemma 5.5 to construct Ω_n from Ω_{n-1} and the points x and y. It follows from Lemma 5.5 that Ω_n is a preparapolar space that contains Ω_{n-1} as a subspace and that satisfies axioms (A_1), (A_2) and (A_M). Let $\phi_n : \mathbb{N} \to \mathscr{P}_n \times \mathscr{P}_n$ be a bijection. No pair of points in $(\mathscr{P}_{n-1} \setminus \langle\langle x, y\rangle\rangle_{\Omega_n}) \times (\mathscr{P}_n \setminus \mathscr{P}_{n-1})$ is contained in a common symp of Ω_n, which implies that Ω_n is not a parapolar space.

Proof of Proposition 5.1. Define $\Omega = \bigcup_{i\in\mathbb{N}} \Omega_i$. Let x and y be any two non-collinear points of Ω. There exists some n_1 for which x and y are contained in Ω_{n_1}. Set $n_2 := \phi_{n_1}^{-1}(x, y)$. By construction, the convex closure of x and y in $\Omega_{f(n_1,n_2)}$ is a non-degenerate polar space S. But then S is the convex closure of x and y in Ω_i for all $i \in \mathbb{N}_{\geq f(n_1,n_2)}$. This implies that the convex closure of x and y in Ω is S. □

Acknowledgement. The authors are grateful to the referee for pointing out some small inaccuracies. The first author is supported by FWO Flanders and the L'Oréal-UNESCO "For Women in Science" program.

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