Construction and characterisation of the varieties of the third row of the Freudenthal-Tits magic square

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Abstract

We characterise the varieties appearing in the third row of the Freudenthal-Tits magic square over an arbitrary field, in both the split and non-split version, as originally presented by Jacques Tits in his Habilitation thesis. In particular, we characterise the variety related to the 56-dimensional module of a Chevalley group of exceptional type E_7 over an arbitrary field. We use an elementary axiom system which is the natural continuation of the one characterising the varieties of the second row of the magic square. We provide an explicit common construction of all characterised varieties as the quadratic Zariski closure of the image of a newly defined affine dual polar Veronese map. We also provide a construction of each of these varieties as the common null set of quadratic forms.

`MSC 2010 Classification: 51E24; 51B25; 20E42.`

`Key words: Lie incidence geometry, dual polar spaces, Veronese variety, buildings`

Data statement: Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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1 Introduction

The main achievement of this paper is a uniform description (both an axiomatic characterisation and an explicit common construction) of certain Grassmannian varieties, half spin varieties, dual polar Veronese varieties and the exceptional variety in 55-dimensional projective space related to the 56-dimensional module of the exceptional Chevalley group of type E_7 over an arbitrary field. These varieties are exactly the varieties of the split third row of the Freudenthal-Tits Magic Square (FTMS). Our proof uses a new powerful local-to-global result of Lie incidence geometries, which may be of independent interest, and can be viewed as a second main result of this paper.

Below we give a brief introduction of the FTMS and the history of characterising its varieties, the choice of the axioms, the nature of the constructions and the techniques used in this paper.

1.1 The Freudenthal-Tits magic square

In 1954 Jacques Tits published the first version of what later would be called the Freudenthal-Tits Magic Square (FTMS). This somewhat lesser known version emphasises mainly the geometries in their natural occurrence in projective space; in an algebraic-differential geometric setting one could rightfully call them varieties. Every cell contains two geometries: a "basic" one, and its "complexification". This way one obtains two 4×4 tables of representations of geometries, which are referred to today as the *non-split version* and the *split version*, respectively.

The first cell of the second row consists of the Veronese embedding of a Pappian projective plane—the image of the plane under the standard Veronese map. Mazzocca and Melone [23] proposed in 1984 a simple axiom system to characterise the finite such varieties. These axioms were based on the properties of the varieties as algebraic-differential varieties, in particular with regard to the images under the Veronese map of the lines of the projective plane, which yields a system of conics covering the variety. Interestingly, when we replace the "conics" with "(non-degenerate) quadrics of maximal Witt index" in these axioms, the axioms coincide with the basic geometric properties of Severi varieties over an algebraically closed field as deduced by Zak when he proved the Hartshorne conjecture [35]. Even more interestingly, it follows from the main result of [27] that, after this deduction, one can carry out the most substantial and major part of the classification of the Severi varieties in an elementary way, without any reference to differential or algebraic geometry. This also yielded a characterisation of the analogues of the Severi varieties over an arbitrary field, and these are precisely the varieties of the second row of the split version of the FTMS, thus giving rise to a far-reaching generalisation of the first 1984 results of Mazzocca and Melone. It is remarkable that, although allowing quadrics of *any* maximal Witt index, only those of Witt index ≤ 5 lead to the examples. This is the geometric counterpart of the fact that split composition algebras, which index the columns of the FTMS, have dimension ≤ 8 over their center. Since the resulting axiom system for the Severi varieties over an arbitrary field does not make any reference to algebraic-geometric notions, we will use the name "abstract variety" for such point sets in a projective space over an arbitrary field.

A highly similiar situation occurs for the non-split version of the second row of the FTMS. Indeed, these varieties were characterised in [22] by replacing "quadrics of maximal Witt index" with "quadrics of Witt index 1" (i.e., minimal Witt index, this time reflecting the dichotomy of the composition algebras either being division or split). In fact, recently, the first three authors showed in [18] that, using non-degenerate quadrics of arbitrary (even non-uniform) Witt index in the axioms, no more examples arise. This yields a unified axiom system for all (abstract) varieties of *both* the split and non-split version of the second row of the FTMS.

1.2 The characterisation of the third row: axioms and tools

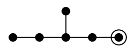
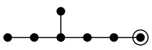
The present paper presents a similar approach to the third row of the FTMS: We characterise the (abstract) varieties in the split and non-split version of the third row of the FTMS over an arbitrary field (see Theorem 3.1). The axioms have the same spirit as those for the second row: They emphasise the differential-geometric properties of the varieties and the occurrence of an abundance of quadrics in subspaces. For the exact axioms and their motivation, see Section 2.3.

Since the point-residuals of the varieties of the third row of the FTMS, that is, the incidence geometric analogue of the geometry induced in the tangent space at a point (see also Definition 2.1), are those of the second row of the FTMS, it will come as no surprise that the characterisation of the second row of the FTMS plays a crucial role in the proof. However, things are not that simple. One only obtains very partial information about the point-residuals, and certainly not enough to immediately be able to apply the known characterisations. We summarise the crucial tools we used.

- Firstly, we take advantage of the fact that the characterisation of the varieties in the second row, as performed in [27] and [18], was itself carried out in a rough inductive scheme, where information got lost when the parameters went down. Hence there was already a need to prove things in various more general settings.
- Secondly, in the last few years, we developed some theory of so-called *lacunary parapolar spaces*, which aimed at characterising essentially the abstract geometries of the FTMS, mainly in its split version and which turns out to be a very powerful tool.
- The third source of arguments and proof techniques is a particular nice new technique that we introduce, namely the characterisation of all abstract geometries related to the varieties of the 3×3 South-East corner of the split FTMS as parapolar spaces

with hyperbolic symplecta and satisfying a simple condition on only one of its singular subspaces. We regard it as our second main result (see Theorem 3.2).

1.3 Constructions of the varieties of the third row

We present a new and unified construction of the varieties of the third row of the FTMS as the projective closure of the image under a kind of "affine dual polar Veronese map" (see Definition 10.1) using a quadratic alternative algebra. This construction is intimately related to a description of these varieties as the common null set of a number of explicitly defined quadratic forms, as was done for the E_7 case by Vavilov & Luzgarev [33]. However, we make these quadratic forms very explicit by using the combinatorics of the Schläfli graph and the Gosset graph, which are the 1-skeleta of the 2_{21} polytope  and the 3_{21} polytope , respectively. Doing that we slightly improve one of the results in [33] by reducing the number of quadratic forms needed to describe the E_7 variety. The eventual verification of the axioms is done using the description of the varieties as the common null set of a number of quadratic forms, cf. Theorem 10.37.

In Section 10.6 we mention some consequences of our constructions for the varieties of the second row; most notably we provide an elegant construction for the Cartan variety $\mathscr{E}_6(\mathbb{K})$.

1.4 Outline of the paper

We head off in Section 2 by introducing the class of abstract (Lagrangian) varieties we will classify, and by providing the necessary background, in particular, on parapolar spaces and Lie incidence geometries. These form an abstract class of point-line geometries underpinning the varieties of the FTMS. We conclude Section 2 with a brief introduction to the geometries which appear in this paper.

In Section 3 we state our main results. The first one, **Theorem 3.1**, characterizes the varieties in the third row of the FTMS as the abstract Lagrangian varieties introduced in Section 2.

Our approach to the proof of that characterization is local-to-global, recognising geometries from their local structure. Our second main result, **Theorem 3.2**, also stated in Section 3, is a new powerful local characterisation of a wide class of Lie incidence geometries. The proof of Theorem 3.2 is the content of Section 4.

After recalling some relevant earlier work on the second row of the FTMS in Section 5 we embark on our proof with the following general observations. In Section 6.1 we explain how the abstract varieties can be viewed as parapolar spaces. In order to recognise the varieties, we show in Section 6.2 that, except in two small cases, the varieties in the third row of the FTMS are the universal embeddings of the corresponding Lie incidence geometries, meaning that all other embeddings of a given variety are a quotient of it (cf. Proposition 6.7). We conclude Section 6 with a result on point-residuals, which allows us to invoke the results of Section 5.

The actual characterisation is preluded by Section 6.4, where we formulate a standing hypotheses for the rest of the paper. We split the characterisation proof in three parts.

(1) The case where the involved quadrics have Witt index 2 (later on we refer to this case as the *ovoidal* case, see Definition 2.2) is dealt with in Section 7 and concerns dual polar spaces (cf. Proposition 7.12). The proof hinges on the fact that the point-residuals are Veronese representation of a projective plane over a quadratic alternative division algebra, see Lemma 7.10, and in Theorem 7.1 we prove a new characterisation of these Veronese varieties by substantially relaxing one of the axioms.
(2) In Section 8 a generalisation of arguments on characterisation results for $\mathscr{S}_{1,2}(\mathbb{K})$ or $\mathscr{S}_{1,3}(\mathbb{K})$ from [26] is carried out. Combined with the local recognition results from Section 4 this leads to characterisations of the varieties in the conclusion of Theorem 3.1: the Grassmannian embedding of $\mathsf{A}_{5,3}(\mathbb{K})$ in $\mathbb{P}^{19}(\mathbb{K})$ in Proposition 8.10, the spinor embedding $\mathscr{HS}_6(\mathbb{K})$ of $\mathsf{D}_{6,6}(\mathbb{K})$ in Proposition 8.11 and finally the exceptional variety $\mathscr{E}_7(\mathbb{K})$ related to $\mathsf{E}_{7,7}(\mathbb{K})$ in Proposition 8.15.
(3) We conclude the characterisation result by eliminating the remaining parameter sets in Section 9.

In our final Section 10 we construct the abstract varieties of the conclusion of Theorem 3.1, as explained above in Subsection 1.3.

2 Definitions and notation

Of central importance in this paper are a class of point sets in a projective space, equipped with a family of quadrics, which we now introduce. To that end, we first settle the notation regarding projective spaces and quadrics.

2.1 Notation

Henceforth, $\mathbb{K}$ is a (commutative) field. We denote by $\mathbb{P}^n(\mathbb{K})$ the n-dimensional projective space over $\mathbb{K}$, for a non-zero cardinal number n. The subspace generated by a family $\mathscr{F}$ of subsets of points is denoted by $\langle S \mid S \in \mathscr{F} \rangle$.

A *non-degenerate quadric* Q in $\mathbb{P}^n(\mathbb{K})$, $n \in \mathbb{N}$, is the null set of an irreducible quadratic homogeneous polynomial in the (homogeneous) coordinates of points of $\mathbb{P}^n(\mathbb{K})$. The *projective index* of Q is the (common) projective dimension of the maximal subspaces of $\mathbb{P}^n(\mathbb{K})$ entirely contained in Q; the *Witt index* is the projective index plus one. A *tangent line* to Q (at a point $x \in Q$) is a line in $\mathbb{P}^n(\mathbb{K})$ which has either only x or all its points in Q. The union of the set of tangent lines to Q at one of its points x is a hyperplane of $\mathbb{P}^n(\mathbb{K})$, denoted by $T_x(Q)$.

An *ovoid* O of $\mathbb{P}^n(\mathbb{K})$ is a spanning point set of $\mathbb{P}^n(\mathbb{K})$ which behaves like (and generalises the notion of) a quadric of projective index 0: each line of $\mathbb{P}^n(\mathbb{K})$ intersects O in at most two points, and the union of the set of tangent lines (defined as above) at each point is a hyperplane of $\mathbb{P}^n(\mathbb{K})$. If $n = 2$, an ovoid is more specifically called an *oval*.

2.2 Abstract varieties with parameters D, I

The varieties that we will encounter have in common that they admit a representation in projective space as a set of points, equipped with a set of subspaces, each of which intersecting the point set in a quadric of a certain Witt index (see above). Depending on the variety, some additional properties will be satisfied in terms of the points and quadrics. We encounter two such sets of additional properties, making it economical and worthwhile to introduce these point-quadric geometries in a general perspective.

Suppose $N \in \mathbb{N} \cup \{\infty\}$ and let D, I be integers with $0 \leq I \leq \lfloor \frac{D}{2} \rfloor$, $D \geq 1$. Let W be a spanning point set of $\mathbb{P}^N(\mathbb{K})$ and let Ω be a collection of $(D+1)$-spaces of $\mathbb{P}^N(\mathbb{K})$ with $|\Omega| \geq 2$ and such that, for any $\omega \in \Omega$, the intersection $\omega \cap W =: W(\omega)$ is either, if $I > 0$, a non-degenerate quadric of projective index I (i.e., Witt index $I+1$) generating ω, or, if $I = 0$, an ovoid generating ω. Moreover, we require $W \subseteq \bigcup_{\omega \in \Omega} \omega$. The pair (W, Ω) is called an *abstract variety (with parameters D, I)*. Of course, this gets more interesting when we add certain properties that have to be satisfied. Regardless of these, we will use the following terminology.

- A quadric $W(\omega)$, with $\omega \in \Omega$, is called a *symp* in case $I > 0$ (inspired by the terminology of parapolar spaces, see Section 2.4) and an *ovoid* in case $I = 0$. Each member of Ω will be called a *host space* (because it "hosts" a symp or an ovoid).
- A subspace S of $\mathbb{P}^N(\mathbb{K})$ is called *singular* if $S \subseteq W$; the set of singular lines is denoted by $\mathscr{L}$. Two points of W are called *collinear* if they are on a common singular line.
- For any $\omega \in \Omega$ and any point $p \in W(\omega)$, the tangent space $T_p(W(\omega))$ at p to $W(\omega)$ is denoted by $T_p(\omega)$. For each point $p \in W$ we denote by $T_p(W)$ (or simply T_p if W is clear from the context) the subspace $\langle \{T_p(\omega) \mid p \in \omega \in \Omega\} \cup \{L \mid p \in L \in \mathscr{L}\} \rangle$. This is the *tangent space of W at p*, a notion which, in our case, coincides with the classical notion of tangent space as a, for instance real or complex, variety. Note that, in case each singular line through p is contained in a member of Ω, then $T_p(W) = \langle T_p(\omega) \mid p \in \omega \in \Omega \rangle = \langle L \mid p \in L \in \mathscr{L} \rangle$.
- Two abstract varieties (W, Ω) and (W', Ω') spanning $\mathbb{P}^N(\mathbb{K})$ and $\mathbb{P}^{N'}(\mathbb{K}')$, respectively (where $\mathbb{K}'$ is a field) are *isomorphic* if there is a (bijective) collineation $\sigma : \mathbb{P}^N(\mathbb{K}) \to \mathbb{P}^{N'}(\mathbb{K}')$ mapping W to W' and Ω to Ω'. Note that the latter implies that, for each host space $\omega \in \Omega$, σ restricted to $W(\omega)$ gives an isomorphism of quadrics, and hence the parameters of (W, Ω) and (W', Ω), if isomorphic, are necessarily the same. Also, in this case $N = N'$ and $\mathbb{K} \cong \mathbb{K}'$.
- The abstract variety (W, Ω) is called *irreducible* if Ω is not the union of two of its subsets Ω_1, Ω_2 such that $\bigcup_{w \in \Omega_1} \omega$ and $\bigcup_{w \in \Omega_2} \omega$ are disjoint subsets of $\mathbb{P}^N(\mathbb{K})$.

Given a point $p \in W$, it will be highly useful to look at the induced geometry of (W, Ω) consisting of lines and quadrics through p, which has a natural representation in the tangent space $T_p(W)$. We speak of the *point-residual of* (W, Ω) *at* p. It only makes sense to consider this provided that $I > 0$ and $D > 2$ (if $I = 0$ of $D = 2$, a quadrics $W(\omega)$ with $\omega \in \Omega$ has either no or exactly 2 singular lines through p,respectively). We have the following definition.

Definition 2.1 The *residue* $\mathrm{Res}_W(p)$ *of* (W, Ω) *at* p is the pair (W_p, Ω_p), where W_p and Ω_p are defined as follows. Take any hyperplane H_p of $T_p(W)$ not containing p. Let W_p denote the set of points of $H_p \cap W$ collinear with p, and let Ω_p be the collection of $(D-1)$-spaces $\{T_p(\omega) \cap H_p \mid p \in \omega \in \Omega\}$.

Then (W_p, Ω_p) is an abstract variety of type $D-2$ and index $I-1$ in $\mathbb{P}^{N'}(\mathbb{K})$, where $N' = \dim H_p$. Indeed, each host space ω of Ω containing p shares $T_p(\omega)$ with $T_p(W)$ and hence intersects H_p in a subspace of dimension $D-1$ and W_p in a quadric of projective index $I-1$. Clearly, the isomorphism type of (W_p, Ω_p) does not depend on the choice of H_p.

2.3 The axioms and their motivation

We are ready to define some special types of abstract varieties, namely the *abstract Lagrangian varieties*, *the abstract Veronese varieties* and variations thereof. It is precisely the former that we will classify, and the latter are their residues, and will play a crucial role in the proof.

Let (Y, Υ) be an irreducible abstract variety with parameters D and I in $\mathbb{P}^N(\mathbb{K})$, where $N \in \mathbb{N} \cup \{\infty\}$. We set $d := D-2$ and $w := I-1$. Recall from the previous subsection that the tangent space $T_y(Y)$ at a point $y \in Y$ is given by $\langle \{T_p(\omega) \mid p \in \omega \in \Omega\} \cup \{L \mid p \in L \in \mathscr{L}\} \rangle$.

Definition 2.2 We call (Y, Υ) an *abstract Lagrangian variety (ALV) (of type d and index w)* if the following hold:

(ALV1) For any pair of points p and q of Y either $\{p, q\}$ lies in at least one element of Υ, denoted by $[p, q]$ if unique, or $T_p(Y) \cap T_q(Y) = \emptyset$, and the latter situation occurs for at least one pair of points of Y.

(ALV2) If $\upsilon_1, \upsilon_2 \in \Upsilon$, with $\upsilon_1 \neq \upsilon_2$, then $\upsilon_1 \cap \upsilon_2 \subset Y$.

(ALV3) If $y \in Y$, then $\dim T_y(Y) \leq 3d+3$.

Comments. Recall that the characterisation of the second row of the FTMS (see Axioms (AVV1), (AVV2) and (AVV3) below) is based on the properties of Severi varieties of dimension $2d$ in a projective space of dimension $3d+2$. More exactly, the fact that every pair of points is contained in a d-dimensional quadric (this is (AVV1)), the fact that the variety is smooth ((AVV2) is a consequence of this), and the fact that the variety has dimension $2d$ ((AVV3) actually expresses that the dimension is at most $2d$). In the same vain, the axioms of abstract Lagrangian varieties are based on similar properties of the real and complex varieties in the third row of the FTMS. More exactly the abundance of $(d+1)$-dimensional quadrics (first part of (ALV1)), the $(3d+2)$-dimensionality of the variety ((ALV3) is a weakening of this), and the fact that the variety is smooth (again, this is (ALV2), but also the second part of (ALV1)). Note that (ALV1) also takes into account that there exist points at distance 3 from each other.

Some terminology. If $w = 0$ and $d > 0$, then we say that the ALV is of *ovoidal type*; if $w = \frac{d}{2}$ then we say that the ALV is of *hyperbolic type*. This terminology stems from the fact that in the ovoidal case, each point residue of an ALV yields a variety consisting of a system of quadrics of Witt index 1, and the latter are instances of ovoids. In the hyperbolic case, the symps are hyperbolic quadrics.

Using the same values for d, w as above, consider an abstract variety (X, Ξ) with parameters (d, w) in $\mathbb{P}^M(\mathbb{K})$, $M \in \mathbb{N} \cup \{\infty\}$. Consider the following axioms and their variants, which will enable us to describe the properties of the point-residuals of the ALVs. These properties are deduced in Section 6.3.

(AVV1) Any pair of points p and q of X lies in at least one element of Ξ, denoted by $[p, q]$ if unique.
(AVV1′) Any pair of points p and q of X with $\langle p, q\rangle \nsubseteq X$ lies in at least one element of Ξ, denoted by $[p, q]$ if unique.
(AVV2) For all $\xi_1, \xi_2 \in \Xi$, with $\xi_1 \neq \xi_2$, we have $\xi_1 \cap \xi_2 \subset X$.
(AVV3) For all $x \in X$, we have $\dim T_x \leq 2d$.
(AVV3′) There is a subset $\partial\Xi$ of Ξ of cardinality at least $|\xi|$, with $\xi \in \Xi$ arbitrary, such that for each $x \in \partial X := \bigcup_{\xi \in \partial\Xi} X(\xi)$, we have $\dim T_x \leq 2d$. Moreover, the set of host spaces in $\partial\Xi$ containing x also has cardinality at least $|\xi|$. The members of ∂X are called *differential* points, and those of $\partial\Xi$ *differential* host spaces of Ξ.

Definition 2.3 An abstract variety (X, Ξ) with parameters (d, w) is called an (a, b)-*abstract Veronese variety ((a, b)-AVV) of type d and index w* if axioms (AVVa), (AVV2) and (AVVb) hold, with $a \in \{1, 1'\}$ and $b \in \{3, 3'\}$; it is called an (a)-*abstract Veronese variety of type d and index w* if axioms (AVVa) and (AVV2) hold, with $a \in \{1, 1'\}$. Note that in the latter case we merely express that axioms (AVV3) or (AVV3') do not *necessarily* hold true, rather than requiring they do not hold. Finally, we abbreviate $(1, 3)$-AVV to AVV.

2.4 Point-line geometries and parapolar spaces

Again, suppose $I > 0$, and recall that $\mathscr{L}$ denotes the set of singular lines of W. Then the pair $(W, \mathscr{L})$ is a point-line geometry which, at least in the cases that we will encounter, will be a parapolar space (cf. Corollary 6.5). Hence we introduce that concept formally.

A *point-line geometry* Δ is a pair $\Delta = (\mathscr{P}, \mathscr{M})$ where $\mathscr{P}$ is a set of points and $\mathscr{M}$ a non-empty set of subsets of $\mathscr{P}$, which are called *lines*. A *subspace* S of Δ is a subset of $\mathscr{P}$ with the property that each line not contained in S intersects S in at most one point. *Collinearity* between points corresponds to being contained in a common line (not necessarily unique), and we denote this by the symbol $\perp$. The set of points equal or collinear to a point $p \in \mathscr{P}$ is denoted by $p^\perp$. The *collinearity graph* of Δ is the graph on $\mathscr{P}$ with collinearity as adjacency relation. The *distance* $\delta(p, q)$ between two points $p, q \in \mathscr{P}$ is the distance between p and q in the collinearity graph (possibly $\delta(x, y) = \infty$ if there is no path between them). A path between p and q of length $\delta(p, q)$ is called a *shortest path*. The diameter of Δ is the diameter of its collinearity graph. We say that Δ is *connected* if for every two points p, q of $\mathscr{P}$, $\delta(p, q) < \infty$. A subspace $S \subseteq \mathscr{P}$ is called *convex* if all shortest paths between points $p, q \in S$ are contained in S. The *convex* *subspace closure* of a set $S \subseteq \mathscr{P}$ is the intersection of all convex subspaces containing S (this is well defined since $\mathscr{P}$ is a convex subspace itself).

Before moving on to the viewpoint of parapolar spaces, we need to consider each host space as a convex subspace of $(W, \mathscr{L})$ isomorphic to a so-called *polar space* (for a precise definition and background see Section 7.4 of [3]). Indeed, for each $\omega \in \Omega$ (recall that we suppose $I > 0$), $W(\omega)$ is an instance of a polar space, that is, a point-line geometry $(\mathscr{P}', \mathscr{L}')$ in which, apart from three non-degeneracy axioms, the *one-or-all axiom* holds: *Each point $p \in \mathscr{P}'$ is collinear to either exactly one or all points of any given line $L \in \mathscr{L}'$.* We will later on (cf. Lemma 6.2) show that, in our setting, for each host space ω, the quadric $W(\omega)$ is the convex subspace closure of any pair of its non-collinear points.

Definition 2.4 A connected point-line geometry $\Delta = (\mathscr{P}, \mathscr{M})$ is a *parapolar space* if for every pair of non-collinear points p and q in $\mathscr{P}$, with $|p^\perp \cap q^\perp| > 1$, the convex subspace closure of $\{p, q\}$ is a polar space, called a *symplecton* (a *symp* for short); moreover, each line of $\mathscr{L}$ has to be contained in a symplecton and no symplecton contains all points of X.

Let $\Delta = (\mathscr{P}, \mathscr{M})$ be a parapolar space. Then Δ is called *strong* if there are no pairs of points $p, q \in \mathscr{P}$ with $|p^\perp \cap q^\perp| = 1$. We say that Δ has *(constant) symplectic rank* r if all its symps have rank r, meaning that the maximal singular subspaces on the symps have projective dimension $r-1$ (in case a symp is a quadric, then r is the Witt index). We will not need parapolar spaces with non-constant symplectic rank. In general, the singular subspaces of a parapolar space are not necessarily projective if there are symps of rank 2, however, we will in this paper only encounter parapolar spaces which are embedded in a projective space and hence their singular subspaces are projective anyhow. Hence we may use the simplest version of the definition of a point-residual:

Definition 2.5 Let $\Delta = (\mathscr{P}, \mathscr{M})$ be a parapolar space whose singular subspaces are projective. Then for a point $p \in \mathscr{P}$, the *point-residual* $\mathrm{Res}_\Delta(p) = (\mathscr{P}_p, \mathscr{M}_p)$ *of* Δ *at* p is defined as follows. The set $\mathscr{P}_p$ consists of all lines belonging to $\mathscr{M}$ containing p, and the set $\mathscr{M}_p$ consists of all singular (projective) planes of $\mathscr{P}$ containing p.

Let Δ be a parapolar space whose singular subspaces are projective. We call Δ *locally connected* if for each point $p \in \mathscr{P}$, the residue $\mathrm{Res}_\Delta(p)$ is connected. Note that a strong parapolar space of symplectic rank r with $r \geq 3$ is automatically locally connected. If Δ is locally connected and has constant symplectic rank $r \geq 3$, then each of its point-residuals $\mathrm{Res}_\Delta(p)$ with $p \in \mathscr{P}$ is a strong parapolar space of constant symplectic rank $r-1$.

2.5 Description of the geometries

The main result of the paper is Theorem 3.1. The conclusion contains certain representations of certain parapolar spaces. The second main result is Theorem 3.2; its conclusion contains certain parapolar spaces. In this section we give a brief overview of these point-line geometries, which are certain *Lie incidence geometries*, i.e., parapolar spaces related to spherical buildings. We explain in detail the representations (as Veronese varieties) in Section 10. The latter contains a new construction of these varieties.

We assume the reader is familiar with the notion of a spherical building, see [30]. Let Δ be a spherical building, not necessarily irreducible, of rank n and type set S, and let $J \subseteq S$. Then we define a point-line geometry $\Gamma = (\mathscr{P}, \mathscr{M})$ as follows. The point set $\mathscr{P}$ is just the set of flags of Δ of type J; the set $\mathscr{M}$ of lines corresponds to the set of flags of type $S \setminus \{s\}$, with $s \in J$: With each flag F' of type $S \setminus \{s\}$, with $s \in J$, we associate the set of flags F of type J such that $F \cup F'$ is a chamber. The geometry Γ is called a *Lie incidence geometry*. For instance, if Δ has type A_n, and $J = \{1\}$ (using Bourbaki labelling), then Γ is the point-line geometry of a projective space. If X_n is the Coxeter type of Δ and Γ is defined using $J \subseteq S$ as above, then we say that Γ has *type* $\mathsf{X}_{n,J}$ and we write $\mathsf{X}_{n,j}$ if $J = \{j\}$. If there is a unique underlying algebraic structure $\mathbb{A}$ that determines Δ as Lie incidence geometry of type $\mathsf{X}_{n,J}$, then we write Δ as $\mathsf{X}_{n,J}(\mathbb{A})$; if not then we write $\mathsf{X}_{n,J}(*)$; for instance, a Pappian projective plane is referred to as $\mathsf{A}_{2,1}(\mathbb{K})$, where $\mathbb{K}$ is a field, whereas an arbitrary projective plane is denoted by $\mathsf{A}_{2,1}(*)$.

Most Lie incidence geometries are parapolar spaces (see Chapter 10 in [2]), in particular, if, $|J| = 1$ and the corresponding spherical building is irreducible, then we either have a projective space, a polar space, or a parapolar space. We review some examples relevant for this paper. Let $\mathbb{L}$ denote a skew field and $\mathbb{K}$ a field. A *(full) embedding* of a point-line geometry $(\mathscr{P}, \mathscr{M})$ into some projective space $\mathbb{P}(V)$ (with V some vector space over $\mathbb{L}$) is an identification of $\mathscr{P}$ with a spanning subset of points of $\mathbb{P}(V)$ such that the members of $\mathscr{M}$ get identified with (full) lines of $\mathbb{P}(V)$.

- The k-Grassmannian of n-dimensional projective space $\mathsf{A}_{n,k}(\mathbb{L})$ (also known as the Grassmannian of all k-spaces of an $(n+1)$-dimensional vector space over $\mathbb{L}$). The k-Grassmann coordinates define a full embedding denoted by $\mathscr{G}_{n+1,k}(\mathbb{L})$.
- The half spin geometry $\mathsf{D}_{n,n}(\mathbb{K})$ of rank n. A full embedding of this geometry is given by the spinor embedding, see [5].
- The exceptional geometries $\mathsf{E}_{i,i}(\mathbb{K})$ with $i \in \{6, 7\}$. These have a unique full embedding in $\mathbb{P}^{26}(\mathbb{K})$ and $\mathbb{P}^{55}(\mathbb{K})$, for $i = 6, 7$, respectively, see [24]. We call these embeddings the exceptional varieties $\mathscr{E}_i(\mathbb{K})$, $i = 6, 7$.
- Direct products of projective spaces, for instance $\mathsf{A}_{2,1}(*) \times \mathsf{A}_{2,1}(*)$. In case the involved projective spaces are defined over the same fields, they have a standard embedding in a projective space, known as *Segre variety*. We denote the Segre variety related to the direct product space $\mathsf{A}_{i_1,1}(\mathbb{K}) \times \mathsf{A}_{i_2,1}(\mathbb{K}) \times \cdots \times \mathsf{A}_{i_k,1}(\mathbb{K})$ by $\mathscr{S}_{i_1,i_2,\ldots,i_k}(\mathbb{K})$.
- Dual polar spaces $\mathsf{B}_{n,n}(*)$ and $\mathsf{C}_{n,n}(*)$. As simplicial complexes, buildings of type B_n and C_n are the same. The distinction in notation, however, is useful when algebraic considerations come into play (root groups and related root systems, split and non-split semisimple algebraic groups). We will follow this logic with our notation of certain (dual) polar spaces.

  Let $\mathbb{A}$ be an alternative division algebra over the field $\mathbb{K}$. Then there is a unique building of type B_3 (or C_3) with the property that the residues corresponding to projective planes are defined over $\mathbb{A}$, and the residues corresponding to generalized quadrangles (which are polar spaces of rank 2) are determined by the anisotropic quadratic form given by the norm of $\mathbb{A}$ over $\mathbb{K}$, see [30]. We denote the corresponding dual polar space by $\mathsf{C}_{3,3}(\mathbb{K},\mathbb{A})$. Note that, if $\mathbb{A}$ is non-associative, then $\mathsf{C}_{3,1}(\mathbb{K},\mathbb{A})$ is a non-embeddable polar space in the sense of [30]. Setting $d = \dim_{\mathbb{K}} \mathbb{A}$, it follows from Theorem 5.8 of [16] that $\mathsf{C}_{3,3}(\mathbb{K},\mathbb{A})$ has a unique full embedding in $\mathbb{P}^{6d+7}(\mathbb{K})$, which we call the Veronese representation and denote it by $\mathscr{V}(\mathbb{K},\mathbb{A})$. Note that, in principle, d could be infinite. However, our hypothesis will imply that we are only concerned with finite d (and then d is a power of 2).

We will provide a new explicit construction of the representations of the geometries appearing in the conclusion of our first main result in Section 10. For this reason, we have not given a precise description of these embeddings in the previous paragraphs.

3 Main Results

Again, let $\mathbb{K}$ be an arbitrary (commutative) field. Consider integers d, w with $0 \leq w \leq \lfloor \frac{d}{2} \rfloor$.

Theorem 3.1 *An abstract Lagrangian variety (Y,Υ) of type d and index w in $\mathbb{P}^N(\mathbb{K})$ is either of ovoidal type or of hyperbolic type; also $d \in \{0,1,2,4,8\}$ unless $\operatorname{char}\mathbb{K} = 2$ in the ovoidal case. In every case $N = 6d+7$. More precisely:*

- *(i) If $d = 0$, Y is isomorphic to the Segre variety $\mathscr{S}_{1,1,1}(\mathbb{K})$ in $\mathbb{P}^7(\mathbb{K})$;*
- *(ii) If (Y,Υ) is ovoidal and $d > 0$, Y is the Veronese representation $\mathscr{V}(\mathbb{K},\mathbb{A})$ in $\mathbb{P}^{6d+7}(\mathbb{K})$ of a dual polar space $\mathsf{C}_{3,3}(\mathbb{K},\mathbb{A})$ over a quadratic alternative division algebra $\mathbb{A}$ over $\mathbb{K}$ with $\dim_{\mathbb{K}} \mathbb{A} = d$; in particular, d is a power of 2, and $d \leq 8$ if $\operatorname{char}\mathbb{K} \neq 2$;*
- *(iii) If (Y,Υ) is not ovoidal and $d > 0$, then it is hyperbolic and Y is isomorphic to either the plane Grassmannian variety $\mathscr{G}_{6,3}(\mathbb{K})$ in $\mathbb{P}^{19}(\mathbb{K})$ related to the Lie incidence geometry $\mathsf{A}_{5,3}(\mathbb{K})$ ($d = w = 2$), the spinor embedding $\mathscr{HS}_6(\mathbb{K})$ in $\mathbb{P}^{31}(\mathbb{K})$ of the half spin geometry $\mathsf{D}_{6,6}(\mathbb{K})$ ($d = 4$, $w = 2$), or the exceptional variety $\mathscr{E}_7(\mathbb{K})$ in $\mathbb{P}^{55}(\mathbb{K})$ related to the Lie incidence geometry $\mathsf{E}_{7,7}(\mathbb{K})$ ($d = 8$, $w = 4$).*

In all cases, the host spaces are the subspaces generated by the symps of the corresponding parapolar space.

Conversely, each variety mentioned in (i), (ii) and (iii) above is an abstract Lagrangian variety, if furnished with the subspaces generated by the symps as host spaces.

Proof In Section 9, more precisely Propositions 9.1, 9.3, 9.7, 9.11 and 9.12, we restrict the parameters of an abstract Lagrangian variety to those that really occur. Those are $w = 0, d > 0$ (cf. Theorem 7.1), $w = d = 0$ (cf. Proposition 8.1), $w = 1, d = 2$ (cf. Proposition 8.10), $w = 2, d = 4$ (cf, Proposition 8.11) and, finally, $w = 4, d = 8$ (cf. Proposition 8.15). In Theorems 10.37 and 10.39 we verify that the varieties in (*i*), (*ii*) and (*iii*) satisfy the axioms of an abstract Lagrangian variety. □

Our approach will exploit the structure of the residue (Y_y, Υ_y) of points $y \in Y$ with the property that not all points in Y are in a common host space with y. Ideally, we wish to show that this is an AVV of type d and index w (cf. Definition 2.3), as these have been classified in [18], see Theorem 5.1.

Knowing the structure of the residue in such points $y \in Y$ is a key element to determine the global structure of (Y, Υ). The crux of the proof however lies in extracting even more from local information. Indeed, if $w > 0$ and $d > 0$, we will show that (Y, Υ) is a strong (and hence locally connected if the symplectic rank r is at least 3) parapolar space, with hyperbolic symps. For such parapolar spaces, we were able to determine powerful local recognition results (see Section 4) that can be used in more general settings than these, but already here they prove their value. As a corollary of these results, we have the following theorem, which we will strictly speaking not fully need but it showcases the beauty and the strength of the results of Section 4.

Theorem 3.2 *Let Δ be a parapolar space of constant symplectic rank $r \geq 2$ all symps of which are hyperbolic and all singular subspaces of which are projective. Assume Δ is locally connected if $r \geq 3$ and strong if $r = 2$. If there exists a singular subspace of dimension $r - 2$ contained in exactly two (maximal) singular subspaces such that the sum of their dimensions is at most $2r$, then Δ is one of* $\mathsf{A}_{1,1}(*) \times \mathsf{A}_{2,1}(*)$, $\mathsf{A}_{1,1}(*) \times \mathsf{A}_{3,1}(\mathbb{L})$, $\mathsf{A}_{2,1}(*) \times \mathsf{A}_{2,1}(*)$, $\mathsf{A}_{4,2}(\mathbb{L})$, $\mathsf{A}_{5,2}(\mathbb{L})$, $\mathsf{A}_{5,3}(\mathbb{L})$, $\mathsf{D}_{5,5}(\mathbb{K})$, $\mathsf{D}_{6,6}(\mathbb{K})$, $\mathsf{E}_{6,1}(\mathbb{K})$, $\mathsf{E}_{7,7}(\mathbb{K})$, $\mathsf{E}_{6,2}(\mathbb{K})$, $\mathsf{E}_{7,1}(\mathbb{K})$, $\mathsf{E}_{8,8}(\mathbb{K})$, *for some skew field $\mathbb{L}$ and some field $\mathbb{K}$.*

In the next section, we start with proving these local recognition results for parapolar spaces, in particular, we show Theorem 3.2.

4 Local recognition results

In this section we prove some useful local recognition results in the following style:

Suppose all symps of a parapolar space Δ of constant symplectic rank r are hyperbolic, and all singular subspaces are projective. If some singular subspace U of dimension $r - 2$ is contained in exactly two maximal singular subspaces, say of dimension d_1 and d_2, and $d_1 + d_2 \leq 2r$, then Δ is known.

See Corollary 4.4, and Theorem 3.2 for the exact conclusions. In order to tackle this problem in a systematic way, we introduce the *haircut condition (H)* on a singular subspace S of a parapolar space Δ with set of symps Ξ below. This peculiar terminology goes back to Shult [29] who used it as a generalisation of a property discovered by Cohen and Cooperstein in the 1980s [6, 12, 8].

(H) Whenever some $\xi \in \Xi$ with $2 + \dim S = \operatorname{rk} \xi$ contains S, and $x \notin \xi$ is a point such that $S \subseteq x^{\perp}$, then $S \subsetneq x^{\perp} \cap \xi$.

If each singular subspace of Δ satisfies (H), then we say that Δ satisfies (H). Our above recognition result will now follow from the following local-to-global result:

Suppose all symps of a locally connected parapolar space Δ with set of symps Ξ of constant symplectic rank r are hyperbolic. If some singular subspace of dimension $r - 2$ satisfies (H)*, then Δ satisfies* (H).

First an observation:

Lemma 4.1 *Let Δ be a parapolar space of constant symplectic rank $r \geq 2$. Then two distinct maximal singular subspaces M_1 and M_2 intersect in a subspace of dimension at most $r-2$.*

Proof Suppose for a contradiction that $S := M_1 \cap M_2$ is a subspace with $\dim S \geq r-1$. Let x_1, x_2 be arbitrary points of $M_1 \setminus S$ and $M_2 \setminus S$. Suppose x_1, x_2 are not collinear. Then since $S \subseteq x_1^\perp \cap x_2^\perp$ and S contains a line, there is a unique symp $\xi(x_1, x_2)$ containing $\langle x_1, S\rangle$ and $\langle x_2, S\rangle$. As the latter have dimension at least r, this contradicts the fact that the symps of Δ have rank r. So x_1 and x_2 are collinear and hence $\langle M_1, M_2\rangle$ is a singular subspace of Δ, contradicting the maximality of M_1 and M_2. □

We start with the case $r = 2$, which carries the crux of the argument.

Proposition 4.2 *Let Δ be a strong parapolar space of constant symplectic rank* 2 *all symps of which are hyperbolic and all singular subspaces of which are projective. Then the following are equivalent.*

(*i*) Δ *satisfies* (H).
(*ii*) Δ *is isomorphic to the Cartesian product* $\Pi \times \Pi'$ *of two projective spaces.*
(*iii*) *Some point satisfies* (H).
(*iv*) *There exists a point contained in exactly two maximal singular subspaces* Π *and* Π'.

Proof Lemma 4.2 of [10] shows $(i) \Rightarrow (ii) \Rightarrow (iii)$. The next claim in particular implies $(iii) \Rightarrow (iv)$.

Claim 1. A point x satisfies (H) if and only if it is contained in exactly two maximal singular subspaces (and this property we will denote by (H')).
Suppose first that x satisfies (H). Clearly x is contained in at least two maximal singular subspaces, so suppose for a contradiction that x is contained in three maximal singular subspaces Π_i, $i = 1, 2, 3$, which intersect each other pairwise in the point x by Lemma 4.1 and $r = 2$. Then, picking arbitrary $x_i \in \Pi_i \setminus \{x\}$, the point x_1 would be collinear to only the point x of the hyperbolic symp $\xi(x_2, x_3)$ since x_1 is collinear to neither x_2 nor x_3 by maximality of Π_1 and Lemma 4.1. This contradicts the fact that x satisfies (H). Conversely, if x is contained in exactly two maximal singular subspaces Π and Π' then, since every point collinear with x belongs to either Π or Π' and every symp through x contains a line of Π and one of Π', it is clear that x satisfies (H).

We now show $(iv) \Rightarrow (i)$. So, let $x \in X$ be contained in exactly two maximal singular subspaces Π and Π'. As above, $\Pi \cap \Pi' = \{x\}$. Also, if both Π and Π' were lines, then each symp through x would coincide with the symp ξ containing $\Pi \cup \Pi'$. Connectivity and strongness now readily imply that ξ is the unique symp of Δ, contradicting the definition of parapolar spaces.

Claim 2. Each point y of Π *satisfies* (H').
Suppose first that Π' is a line. Then each symp through xy contains Π' and hence is unique, so by strongness it follows that there is only one line through y not contained in Π.

Next, suppose that Π' is at least a plane, so we can choose points $z, z' \in \Pi' \setminus \{x\}$ with $z' \notin xz$. The symps $\xi(y,z)$ and $\xi(y,z')$ contain unique lines L and L', respectively, with $z \in L$, $z' \in L'$ and $x \notin L \cup L'$. There is also a symp ζ containing L and zz', and let M' be the line in ζ containing z' and distinct from zz'.

We show that $L' = M'$. Indeed, suppose not. The symp η containing M' and x has a line M in common with Π. But $M \neq xy$, since, if $M = xy$, then $[y, z'] = \eta$ and z' would be contained in three lines of η (namely M', L' and xz'), a contradiction. Now, there is a unique point u on L collinear to y; there is a unique point v' on M' collinear to u, and there is a unique point $v \in M$ collinear to v'.

Select any y_* on $xy \setminus \{x, y\}$. Set $u_* = L \cap y_*^\perp$, $v'_* = M' \cap u_*^\perp$, and $v_* = M \cap v'^\perp_*$. Since Π is a projective space, $yv \cap y_*v_*$ is a unique point s. Noting that v and u are not collinear as otherwise $\langle M, xy\rangle \subseteq [y, z]$, they determine a unique symp containing y and v', and so s is collinear to a unique point t of uv'. Likewise, s is collinear to a unique point t_* of $u_*v'_*$. Since s is not contained in the symp ζ (otherwise, $\langle x, z, z'\rangle \subseteq \zeta$), and since the points t and t_* are distinct, they are collinear and s is collinear to all points of tt_*. But tt_* intersects zz' in some point w, which is then collinear to the line xs, implying that Π is not a maximal singular subspace, a contradiction. We conclude that $L' = M'$.

Since now y is collinear to the points $u \in L$ and $v' \in M' = L'$, then since $u, v' \in \zeta$ we deduce that $u \perp v'$ and so u, v', y are contained in a unique plane π'_y containing y, with $\pi'_y \cap \Pi = \{y\}$. Collinearity defines a bijection from the line zz' to the line uv'; hence "being contained in the same symp with xy" defines a bijection from the set of lines of $\pi'_x = \langle x, z, z'\rangle$ through x to the set of lines of π'_y through y. Varying π'_x in Π', we obtain that "being contained in the same symp with xy" is a bijective collineation between the residue $\mathrm{Res}_{\Pi'}(x)$ and the set of lines of Δ through y, but not in Π. This implies that all such lines are contained in a singular subspace Π'_y (with $\dim \Pi'_y = \dim \Pi'$), and so y satisfies (H$'$).

Claim 3. Every point of Δ *satisfies* (H$'$).
Indeed, by Claim 2, and interchanging the roles of Π and Π' if needed, every point collinear to x satisfies (H$'$). By connectivity, all points do.

The proposition now follows using Claim 1. □

The next result is our most general local recognition result for parapolar spaces of constant symplectic rank $r \geq 3$.

Theorem 4.3 *Let* Δ *be a locally connected parapolar space of constant symplectic rank* $r \geq 3$ *all symps of which are hyperbolic. Then the following are equivalent.*

(i) Δ *satisfies* (H).
(ii) Some singular subspace of dimension $r-2$ *satisfies* (H).
(iii) There exists a singular subspace of dimension $r-2$ *which is contained in exactly two maximal singular subspaces.*

Proof The implication *(i)* ⇒ *(ii)* is trivial. Suppose some singular subspace U of dimension $r-2$ satisfies (H). Suppose also, for a contradiction, that U is contained in (at least) three maximal singular subspaces Π_i, $i = 1, 2, 3$. Then there exist points $x_i \in \Pi_i \setminus (\Pi_j \cup \Pi_k)$, $\{i, j, k\} = \{1, 2, 3\}$. It follows that the point x_1 is collinear to all points of U and does not belong to the symp $\xi(x_2, x_3)$ (since the latter is hyperbolic and U is contained in the generators $\langle U, x_2\rangle$ and $\langle U, x_3\rangle$). Since U satisfies (H), we may assume without loss of generality that x_1 is collinear to all points of $\langle U, x_2\rangle$, and hence to x_2, a contradiction. Hence we have shown the implication $(ii) \Rightarrow (iii)$. We now show $(iii) \Rightarrow (i)$, and proceed by strong induction on r (the base case $r = 3$ is included in the induction argument).

So let U be a subspace of dimension $r - 2$, contained in two maximal singular subspaces (of Δ). Pick a point $x \in U$. Then, in $\Delta_x := \mathrm{Res}_\Delta(x)$, the subspace U_x is also contained in two maximal singular subspaces (of Δ_x). Since Δ is locally connected, $\mathrm{Res}_\Delta(x)$ is a parapolar space. Also, $\mathrm{Res}_\Delta(x)$ is strong and all of its singular subspaces are projective. Hence we can either apply induction (if $r > 3$) or Proposition 4.2 (if $r = 3$) and conclude that Δ_x satisfies (H).

Now let $y \perp x$. We can select a symp containing xy and a singular subspace U' of dimension $r - 2$ in that symp, containing xy.

Claim $(*)$: The subspace U' satisfies (H).
Indeed, let u be a point collinear to all points of U', and let ξ be a symp containing U' but not u. In Δ_x, the point u_x corresponding to xu is collinear to all points of some generator of the symp ξ_x corresponding to ξ, because Δ_x satisfies (H). This implies that u is collinear to all points of some generator of ξ, and so the claim follows.

Now we can interchange the roles of U and U' and of x and y, and as before, this implies by induction or Proposition 4.2 that Δ_y satisfies (H). A connectivity argument implies that for all points z, the point-residual Δ_z satisfies (H). Then Claim $(*)$ applied to any singular subspace of dimension $r - 2$ of Δ, and every point contained in it, implies that Δ satisfies (H). □

Some consequences of the previous theorem.

Corollary 4.4 *Let Δ be a strong parapolar space of constant symplectic rank $r \geq 2$, all symps of which are hyperbolic and all singular subspaces of which are projective. If there exists a singular subspace of dimension $r - 2$ contained in exactly two (maximal) singular subspaces S_1 and S_2, say of dimensions d_1 and d_2, with $d_1 + d_2 \leq 2r$, then the following hold where $\mathbb{L}$ is some skew field and $\mathbb{K}$ is some field.*

(1) *If $d_1 = d_2 = r$, then either $\Delta \cong \mathsf{A}_{2,1}(*) \times \mathsf{A}_{2,1}(*)$, or $\Delta \cong \mathsf{A}_{5,3}(\mathbb{L})$.*
(2) *If $d_1 = r - 1$ and $d_2 = r + 1$, then either $\Delta \cong \mathsf{A}_{1,1}(*) \times \mathsf{A}_{3,1}(\mathbb{L})$, or $\Delta \cong \mathsf{A}_{5,2}(\mathbb{L})$, or $\Delta \cong \mathsf{D}_{6,6}(\mathbb{K})$.*
(3) *If $d_1 = r - 1$ and $d_2 = r$, then either $\Delta \cong \mathsf{A}_{1,1}(*) \times \mathsf{A}_{2,1}(*)$, or $\Delta \cong \mathsf{A}_{4,2}(\mathbb{L})$, or $\Delta \cong \mathsf{D}_{5,5}(\mathbb{K})$, or $\Delta \cong \mathsf{E}_{6,1}(\mathbb{K})$, or $\Delta \cong \mathsf{E}_{7,7}(\mathbb{K})$.*

Proof If $r = 2$, then it follows from Proposition 4.2 that Δ is the Cartesian product $S_1 \times S_2$ of two projective spaces S_1, S_2 of respective dimensions, say $d_1, d_2 \geq 1$. Since $d_1 + d_2 \leq 4$, there are exactly three possibilities, all of which are listed above. If $r \geq 3$, then recalling that in this case strongness implies locally connected, it follows from Theorem 4.3 that Δ satisfies (H). Note that the singular subspaces of Δ are finite-dimensional, which follows from an easy inductive argument and the fact that (H) is a residual property, and in case of constant symplectic rank 2, (H) is equivalent to being a direct product space (cf. Proposition 4.2). The result then follows from Theorem 15.4.5 in [28]. Alternatively, it also follows from the classification of parapolar spaces satisfying the Haircut Axiom (H) in [10]. $\square$

Proof of Theorem 3.2 Either one can argue as in the proof of Corollary 4.4 using the alternative argument which relies on the revised Haircut Theorem in [10], or one argues as follows. If the parapolar space is strong, then the assertion follows from Corollary 4.4. If not then we consider its point-residues, which are automatically strong and also satisfy the hypotheses. Therefore, each one is isomorphic to a parapolar space in one of the three cases of Corollary 4.4. A standard inductive argument (on the distance between points) using connectivity shows that all point-residues are isomorphic. Since we assume Δ not to be strong, the diameter of such residue is at least 3. This leaves us with the possibilities $\mathsf{A}_{5,3}(\mathbb{L})$, $\mathsf{D}_{6,6}(\mathbb{K})$ and $\mathsf{E}_{7,7}(\mathbb{K})$. Theorem 2.1 in [9] leads to the assertion $\Delta \cong \mathsf{E}_{6,2}(\mathbb{K})$, $\mathsf{E}_{7,1}(\mathbb{K})$, or $\mathsf{E}_{8,8}(\mathbb{K})$, respectively. $\square$

5 Some known classification results

5.1 Abstract Veronese varieties and relatives

For ease of reference, we collect some useful classification results of earlier papers. We phrase them in the current terminology.

Theorem 5.1 (Theorem 1.2 of [18]) *An AVV of type d in $\mathbb{P}^N(\mathbb{K})$ is projectively equivalent to one of the following:*

($d=1$) *The quadric Veronese variety $\mathscr{V}_2(\mathbb{K})$, and then $N=5$;*
($d=2$) *the Segre variety $\mathscr{S}_{1,2}(\mathbb{K})$ ($N=5$), $\mathscr{S}_{1,3}(\mathbb{K})$ ($N=7$) or $\mathscr{S}_{2,2}(\mathbb{K})$ ($N=8$);*
($d=4$) *the line Grassmannian variety $\mathscr{G}_{5,2}(\mathbb{K})$ ($N=9$) or $\mathscr{G}_{6,2}(\mathbb{K})$ ($N=14$);*
($d=6$) *the half-spin variety $\mathscr{HS}_5(\mathbb{K})$, and then $N=15$;*
($d=8$) *the (Cartan) variety $\mathscr{E}_6(\mathbb{K})$, and then $N=26$;*
($d=2^\ell$) *the Veronese variety $\mathscr{V}_2(\mathbb{K},\mathbb{A})$, for some d-dimensional quadratic alternative division algebra $\mathbb{A}$ over $\mathbb{K}$. Moreover, if the characteristic of the underlying field $\mathbb{K}$ is not 2, then $d\in\{1,2,4,8\}$. Here, $N=3d+2$.*

Note that the case $d=1$ is also included in the last case, $d=2^\ell$. We repeat it though, as it fits in the two series, the first one with quadrics of maximal projective index (the first five items), the second one with quadrics of projective index 1 (the sixth item).

Lemma 5.2 (Lemma 5.1 and Proposition 5.2 of [27]) *Let (X,Ξ) be a (1)-AVV of type 2 and index 1 in $\mathbb{P}^7(\mathbb{K})$. Then (X,Ξ) is isomorphic to a Segre variety $\mathscr{S}_{1,i}(\mathbb{K})$, $i\in\{2,3\}$.*

Proposition 5.3 (Proposition 4.5 of [25]) *If $\mathbb{K} \not\cong \mathbb{F}_2$, then every $(1')$-AVV of type 1 and index 0 contained in $\mathbb{P}^5(\mathbb{K})$ is isomorphic to $\mathscr{V}_2(\mathbb{K})$. If $\mathbb{K} \cong \mathbb{F}_2$, then every $(1')$-AVV of type 1 and index 0 contained in $\mathbb{P}^5(\mathbb{K})$ has at most nine conics.*

5.2 Lacunary parapolar spaces

Definition 5.4 Let $k \in \mathbb{Z}_{\geq -1}$. A parapolar space is called *k-lacunary* if k-dimensional singular subspaces never occur as the intersection of two symplecta, and all symplecta contain k-dimensional singular subspaces.

In [20] and [19], k-lacunary parapolar spaces have been classified for $k = -1$ and $k \geq 0$, respectively. At several points in the proof we will use the classification of (-1)- or 0-lacunary parapolar spaces. We extract from the Main Result of [19] the results that we will need, restricting our attention to strong parapolar spaces embedded in a projective space over a field $\mathbb{K}$.

Lemma 5.5 *Let $\Gamma = (X, \mathscr{L})$ be a strong (-1)-lacunary parapolar space whose points are points of a projective space $\mathbb{P}$ over a field $\mathbb{K}$, whose lines are lines of $\mathbb{P}$ and whose symplecta are all isomorphic to each other. Then $\Gamma = (X, \mathscr{L})$ is, as a point-line geometry, isomorphic to either a Segre variety $\mathscr{S}_{n,2}(\mathbb{K})$ with $n \in \{1, 2\}$, a line Grassmannian variety $\mathscr{G}_{n,1}(\mathbb{K})$ with $n \in \{4, 5\}$, or to the Cartan variety $\mathscr{E}_{6,1}(\mathbb{K})$. In particular, the symps of Γ are all hyperbolic quadrics.*

Lemma 5.6 *Let $\Gamma = (X, \mathscr{L})$ be a strong 0-lacunary parapolar space whose points are points of a projective space $\mathbb{P}$ over a field $\mathbb{K}$, whose lines are lines of $\mathbb{P}$ and whose symplecta are all isomorphic to each other. Then the symps of Γ are all hyperbolic quadrics. Moreover, if these quadrics all have projective index 1, then $\Gamma = (X, \mathscr{L})$ is, as a point-line geometry, isomorphic to a Segre variety $\mathscr{S}_{1,n}(\mathbb{K})$, for some $n \in \mathbb{N}$ with $n \geq 2$, or the direct product of a line and a hyperbolic quadric of projective index n, for some $n \in \mathbb{N}$ with $n \geq 2$.*

6 General observations for the proof of the main theorem

6.1 Properties of ALV and AVV as parapolar spaces

Suppose that (W, Ω) is either a $(1')$-AVV of type d and index w or an ALV of type $d-2$ and index $w-1$ in $\mathbb{P}^N(\mathbb{K})$; so each host space intersects W in a non-degenerate quadric spanning $\mathbb{P}^{d+1}(\mathbb{K})$ and has w-dimensional subspaces as maximal isotropic subspaces. We record general properties holding for both types of abstract varieties.

Lemma 6.1 *Let L_1 and L_2 be two singular lines of (W, Ω) sharing a point y. Then either there is a unique host space containing $L_1 \cup L_2$, or, L_1 and L_2 generate a singular plane π. In the latter case, if $w \geq 2$, then there is a host space containing π.*

Proof For (1′)-AVVs, the first statement is proved in Lemma 3.3 of [18] and the second statement in Lemma 3.11 of [18]. The same proof holds for ALVs since, when looking in $y^\perp$, axiom (ALV1) implies axiom (AVV1′), and (ALV2) and (AVV2) coincide anyhow. □

If two singular lines L_1 and L_2, which share a point, are contained in a unique host space, then we denote the latter by $[L_1, L_2]$.

As a consequence, we have:

Lemma 6.2 *For $y \in W$ and $\omega \in \Omega$ with $y \notin \omega$, the set $y^\perp \cap \omega$ is a singular subspace.*

Proof Suppose y_1, y_2 are points in ω collinear to y (so $y_1, y_2 \in W$). By Lemma 6.1, the singular lines yy_1 and yy_2 are either contained in a unique host space ω', or y_1y_2 is singular. In the first case, $\omega \cap \omega' \subseteq W$ by the second axiom, and hence also in this case, y_1y_2 is singular. □

Lemma 6.2 allows for a higher-dimensional version of Lemma 6.1.

Lemma 6.3 *Let Π_1 and Π_2 be two singular k-spaces of (W, Ω) sharing a $(k-1)$-space, $k \geq 1$. Then either there is a unique host space containing $\Pi_1 \cup \Pi_2$, or, Π_1 and Π_2 generate a singular $(k+1)$-space Π. If $w < k$ then the first option is not possible; moreover, if $w \geq k+1$ then each singular $(k+1)$-space is contained in a host space.*

Proof In case (W, Ω) is a hyperbolic AVV, this is proved in Lemmas 4.4 and 4.5 of [27]. Exactly the same proofs hold in the current context. □

Lemma 6.4 *For any $x, y \in W$, there is a finite number n and a sequence $(\omega_1, ..., \omega_n)$ in Ω such that $x \in \omega_1$, $y \in \omega_n$ and $\omega_i \cap \omega_{i+1} \neq \emptyset$ for all $i \in \{1, \ldots, n-1\}$.*

Proof If (W, Ω) is an (1)-AVV, this follows immediately from (AVV1). So suppose (W, Ω) is an ALV. Define Ω_1 as the set of all host spaces containing x and Ω_2 as the set of all $\omega \in \Omega$ such that there is a finite m and host spaces $\omega_1, ..., \omega_m$ with $\omega = \omega_1$, $y \in \omega_m$ and $\omega_i \cap \omega_{i+1}$ non-empty for all $i \in \{1, \ldots, m-1\}$. Since (W, Ω) is irreducible, there is a $\omega \in \Omega_1 \cap \Omega_2$, showing the result. □

Corollary 6.5 *If (W, Ω) is either a (1)-AVV of type d and index w or an ALV of type $d-2$ and index $w-1$ in $\mathbb{P}^N(\mathbb{K})$ and $w > 0$, then $(W, \mathscr{L})$ is a strong parapolar space of constant symplectic rank w.*

Proof We verify the axioms (see Definition 2.4). The fact that $(W, \mathscr{L})$ is connected follows from Lemma 6.4, $w > 0$ and (AVV2) or (ALV2). Moreover, if $p, q \in W$ are non-collinear points with $|p^\perp \cap q^\perp| > 1$, then it again follows from (AVV1) or (ALV1) that there is a host space ω containing p and q. Moreover, Lemma 6.2 implies that the symp $W(\omega)$ is the convex closure subspace of any pair of its non-collinear points (noting that the only proper convex closure subspaces of $W(\omega)$ are its singular subspaces). Thirdly, it is again (AVV1) and (ALV1) that make sure that each line of $\mathscr{L}$ is contained in a symp. Finally, the fact that $d+1 < N$ and that W is a spanning point set of $\mathbb{P}^N(\mathbb{K})$ imply that there is no symp containing all points of W. □

Lemma 6.6 *For each $x \in W$ we can find $\omega \in \Omega$ not containing x.*

Proof Suppose for a contradiction that all host spaces contain x. Let ω_1, ω_2 be two distinct host spaces (recall that $|\Omega| \geq 2$). Let y_1 be a point in $W(\omega_1)$ not collinear to x. By Lemma 6.2, there is a point $y_2 \in W(\omega_2)$ which is collinear to neither x nor y_1 (noting that $W(\omega_2) \setminus x^\perp$ contains a pair of non-collinear points). By assumption, $[y_1, y_2]$ contains x, but then the second axiom (i.e., (AVV2) or (ALV2)) implies that $\omega_1 = [y_1, x] = [y_1, y_2] = [x, y_2] = \omega_2$, a contradiction. □

6.2 Embeddings

One important step in our proof is to show that, once we pinned down the isomorphism type of the abstract geometry $(Y, \mathscr{L})$, where $\mathscr{L}$ is the set of singular lines and Y a spanning point set of $\mathbb{P}^N(\mathbb{K})$, there is a projectively unique representation (or full embedding) of $(Y, \mathscr{L})$ which satisfies the axioms (ALV1), (ALV2) and (ALV3). This will be achieved in three steps. First we refer to Theorems 10.37 and 10.39. These theorems establish a full embedding of $(Y, \mathscr{L})$, say in $\mathbb{P}^M(\mathbb{K})$, that satisfies the said axioms. Secondly, except if, only in the ovoidal case, the ground field $\mathbb{K}$ has exactly two elements, then that embedding is projectively unique in $\mathbb{P}^j(\mathbb{K})$, for $j \geq M$, and it is universal. Thirdly, we show that $N \geq M$. For $|\mathbb{K}| = 2$ in the ovoidal case, we show (later) that the embedding occurring in Theorem 10.37 is the projectively unique one in the given dimension that satisfies the axioms (ALV1), (ALV2) and (ALV3). We here show the second step.

Proposition 6.7

(S) *The unique (full) embedding of $\mathsf{A}_1(\mathbb{K}) \times \mathsf{A}_1(\mathbb{K}) \times \mathsf{A}_1(\mathbb{K})$ in $\mathbb{P}^7(\mathbb{K})$ is the Segre variety $\mathscr{S}_{1,1,1}(\mathbb{K})$;*

(O) *The unique (full) embedding of the dual polar space $\mathsf{C}_{3,3}(\mathbb{K}, \mathbb{A})$ in $\mathbb{P}^{6d+7}(\mathbb{K})$, where $|\mathbb{K}| > 2$ and $\mathbb{A}$ is a d-dimensional quadratic alternative division algebra over $\mathbb{K}$, is the Veronese representation $\mathscr{V}(\mathbb{K}, \mathbb{A})$.*

(H) *The unique (full) embedding of the Lie incidence geometries $\mathsf{A}_{5,3}(\mathbb{K})$, $\mathsf{D}_{6,6}(\mathbb{K})$ and $\mathsf{E}_{7,7}(\mathbb{K})$ in $\mathbb{P}^{19}(\mathbb{K})$, $\mathbb{P}^{31}(\mathbb{K})$ and $\mathbb{P}^{55}(\mathbb{K})$, respectively, are the plane Grassmannian variety $\mathscr{G}_{6,3}(\mathbb{K})$, the spinor embedding $\mathscr{HS}_6(\mathbb{K})$ and the exceptional variety $\mathscr{E}_7(\mathbb{K})$.*

Proof For $\mathsf{A}_1(\mathbb{K}) \times \mathsf{A}_1(\mathbb{K}) \times \mathsf{A}_1(\mathbb{K})$, this is obvious, noting that $\mathbb{P}^7(\mathbb{K})$ is generated by two hyperbolic quadrics in disjoint 3-spaces. For Case (O), $|\mathbb{K}| \neq 2$, this is Theorem 5.8 in [16]. Case (H) follows from the main results in [34] (for $\mathsf{A}_{5,3}(\mathbb{K})$ and $\mathsf{D}_{6,6}(\mathbb{K})$), and [24] (for $\mathscr{E}_7(\mathbb{K})$). □

6.3 The residue of a point $a \in Y$ having a point $e \in Y$ at distance 3

Let (Y, Υ) be an ALV of type d and index w. Let $a \in Y$ be a point such that there is a point $e \in Y$ at distance 3 from a; the existence of such a pair of points is guaranteed by Axiom (ALV1) and Lemma 6.4. We show that the residue (Y_a, Υ_a) (cf. Definition 2.1) is a $(1', 3')$-AVV of type d and index w.

Consider a path $a \perp b \perp c \perp e$ of length 3 between a and e. Set $W_{a,c} := a^\perp \cap c^\perp$ and likewise $W_{b,e} := b^\perp \cap e^\perp$, and note that these sets are contained in the subspaces $[a, c]$ and $[b, e]$, respectively. Recall the definition of $T_p(Y_a)$ as given in Subsection 2.2.

Lemma 6.8 *The point $p \in Y_a$ corresponding to the line ab satisfies* $\dim T_p(Y_a) \leq 2d$.

Proof It suffices to show $\alpha := \dim(T_a(Y) \cap T_b(Y)) \leq 2d + 1$. By (ALV1), $T_a(Y) \cap T_e(Y) = \emptyset$; and by (ALV3), $\dim T_a(Y) \leq 3d + 3$. Since $\dim(T_b(Y) \cap T_e(Y)) \geq d + 1$, we obtain $3d + 3 \geq \dim T_b(Y) \geq d + 1 + \alpha + 1$ and therefore $\alpha \leq 2d + 1$. □

Lemma 6.9 *Let $c' \in W_{b,e}$ be arbitrary and consider $\upsilon := [a, c']$. Then $\upsilon \cap W_{b,e} = \{c'\}$. Moreover, for each point $p \in Y_a$ corresponding to a singular line ab' in υ, we have* $\dim T_p(Y_a) \leq 2d$.

Proof If $\upsilon \cap W_{b,e}$ contained a line L through c', then L would contain a point of $T_a(\upsilon)$, whereas $L \subseteq T_e(Y)$ and $T_a(Y) \cap T_e(Y)$ is empty by (ALV3). So $\upsilon \cap W_{b,e} = \{c'\}$ indeed.

Now let b' be a point of $a^\perp \cap c'^\perp$. Then $a \perp b' \perp c' \perp e$ is a path of length 3 between a and e and hence we can apply Lemma 6.8 with the line ab' in the role of ab, from which the second assertion follows. □

Lemma 6.10 *The residue* $\mathrm{Res}_Y(a) = (Y_a, \Upsilon_a)$ *is a $(1', 3')$-AVV of type d and index w; moreover, if $w > 0$ then it is actually a $(1, 3')$-AVV.*

Proof By Lemma 6.9 we have $|\Upsilon_a| \geq 2$. The fact that (AVV1′) and (AVV2) are satisfied follows immediately from (ALV1) and (ALV2); and if $w > 0$ then also (AVV1) holds by Lemma 6.3. Defining $\partial\Upsilon_a$ as the set of members of Υ_a corresponding to the host spaces $\upsilon \in \Upsilon$ with the properties that $a \in \upsilon$ and there exists $e_* \in Y$ with $e_*^\perp \cap \upsilon \neq \emptyset$ and $T_{e_*} \cap T_a = \emptyset$, (AVV3′) holds by Lemma 6.9. □

In the sequel we will hence study such AVVs, and for ease of notation we put $X := Y_a$ and $\Xi := \Upsilon_a$. We note the following corollary.

Corollary 6.11 *Let (Y, Υ) be an ALV of type d and index $w \geq 1$. Let $a \in Y$ and suppose there exists $e \in Y$ with $T_a(Y) \cap T_e(Y) = \emptyset$. If each line $L \ni a$ contains a point b with $T_b(Y) \cap T_e(Y) \neq \emptyset$, then the point-residual (Y_a, Υ_a) is an abstract Veronese variety.*

Proof This follows from Lemmas 6.8 and 6.10. □

The previous results are crucial for the start of the proof of our Main Result; the next proposition provides a standard way to finish the hyperbolic cases.

Proposition 6.12 *Let Δ be one of the parapolar spaces $\mathsf{A}_{5,3}(\mathbb{K})$, $\mathsf{D}_{6,6}(\mathbb{K})$ or $\mathsf{E}_{7,7}(\mathbb{K})$. Suppose the point-line geometry $(Y, \mathscr{L})$ related to an ALV (Y, Υ) of type d and index w is isomorphic to Δ. Then Y is projectively unique and isomorphic to the universal embedding of Δ.*

Proof It is obvious that (d, w) is either $(2,1), (4,2)$, or $(8,4)$, depending on $\Delta \cong \mathsf{A}_{5,3}(\mathbb{K})$, $\mathsf{D}_{6,6}(\mathbb{K})$ or $\mathsf{E}_{7,7}(\mathbb{K})$, respectively. Consider any point $a \in Y$. Since in Δ, no point is at distance at most 2 of all others, Corollary 6.11 implies that (Y_a, Υ_a) is an AVV of type d and index w, and its related point-line geometry is isomorphic to $\mathsf{A}_{2,1}(\mathbb{K}) \times \mathsf{A}_{2,1}(\mathbb{K})$, $\mathsf{A}_{5,2}(\mathbb{K})$, or $\mathsf{E}_{6,1}(\mathbb{K})$, respectively. It follows from the Main Result of [27] that Y_a is isomorphic to $\mathscr{S}_{2,2}(\mathbb{K})$, $\mathscr{G}_{6,2}(\mathbb{K})$, or $\mathscr{E}_6(\mathbb{K})$, respectively, living in a projective space of dimension $3d+2$. It follows that $\dim T_a(Y) = 3d+3$. Consideration of a point $e \in Y$ with $T_a(Y) \cap T_e(Y) = \emptyset$ yields $\dim Y \geq 6d+7$. Now the assertion follows from Proposition 6.7. □

6.4 Standing Hypotheses

We now start the proof of Theorem 3.1. We let (Y, Υ) be an abstract Lagrangian variety of type d and index w. We consider the point-residual $(Y_a, \Upsilon_a) = (X, \Xi)$ of (Y, Υ) at a point $a \in Y$ for which there exist points $b, c, e \in Y$ with $a \perp b \perp c \perp e$ and $T_a(Y) \cap T_e(Y) = \emptyset$. It is a $(1, 3')$-AVV of type d and index w, if $w > 0$, by Lemma 6.10, and otherwise it is a $(1', 3')$-AVV of type d and index 0. We keep denoting the set of singular lines of Y by $\mathscr{L}$. We will adopt these hypotheses and this notation in Sections 7, 8 and 9, except for Subsections 7.1 and 8.4.

7 Ovoidal case—dual polar spaces ($w = 0, d > 0$)

Let (Y, Υ) be an ALV of type $d \geq 1$ and index 0. The Standing Hypotheses 6.4 yield a $(1', 3')$-AVV $(Y_a, \Upsilon_a) = (X, \Xi)$, which is of type $d \geq 1$ and index 0 (recall that the intersections of host spaces with X are called ovoids, regardless of d, although if $d = 1$ we will more accurately call them ovals). However, we will prove a slightly stronger result by introducing a considerable weakening of Axiom (AVV3$'$). Namely, we only require the dimension of the tangent space to be bounded by $2d$ for the points on one ovoid. Since this might be of independent interest, we state and prove it independently in the next subsection.

7.1 A characterisation of Veronese varieties

As explained in the previous paragraph, we temporarily abandon the Standing Hypotheses 6.4 in this subsection. We show the following characterisation of the Veronese varieties $\mathscr{V}_2(\mathbb{K},\mathbb{A})$, where $\mathbb{A}$ is a quadratic alternative division algebra over the field $\mathbb{K}$.

Theorem 7.1 *Let (X,Ξ) be a $(1')$-abstract Veronese variety of type $d \geq 1$ and index 0 in (possibly a subspace of) $\mathbb{P}^{3d+2}(\mathbb{K})$, such that $\dim T_x \leq 2d$ for all points x of a certain ovoid O. Then (X,Ξ) is isomorphic to a Veronese variety $\mathscr{V}_2(\mathbb{K},\mathbb{A})$, for some quadratic alternative division algebra $\mathbb{A}$ over $\mathbb{K}$ with $\dim_{\mathbb{K}} \mathbb{A} = d$.*

We prove Theorem 7.1 in a sequence of lemmas, first getting rid of the finite case. Strictly speaking we only need to treat the cases where $|\mathbb{K}| < 5$ separately (this manifests itself in the proof of Lemma 7.4), but our approach works for all finite fields. Note that each point x is contained in at least two ovoids, which implies $\dim T_x(X) = 2d$ as soon as $\dim T_x(X) \leq 2d$.

Throughout Subsection 7.1 we adopt the notation of Theorem 7.1. In particular, O is a fixed ovoid of a (1)-AVV (X,Ξ) of type $d \geq 1$ and index 0 in (possibly a subspace of) $\mathbb{P}^{3d+2}(\mathbb{K})$ and for each point x of O holds $\dim T_x \leq 2d$.

7.1.1 The finite case

Suppose $\mathbb{K} = \mathbb{F}_q$, the finite field with q elements. This implies that $d \in \{1,2\}$ [17, p.48].

Lemma 7.2 *There are no singular lines in X and each pair of ovoids has a non-trivial intersection, giving (X,Ξ) (viewed as an abstract geometry) the structure of a projective plane.*

Proof We aim to show that there are no singular subspaces of dimension at least 1. Note that Lemma 6.1 implies that distinct maximal singular subspaces are disjoint, so in particular, if singular lines share a point, they are contained in a singular plane, etc.

Claim 1. There is no singular subspace of dimension at least 2.
Indeed, assume for a contradiction that S is a singular plane. Select a point z not contained in the maximal singular subspace containing S. Then counting the number of points on ovoids containing z and a point of S (note that no point of S is collinear to z) we obtain $|X| \geq 1+q^d(q^2+q+1)$, so $|X| \geq q^{2d}+q^{d+1}+q^d+1$ as $d \leq 2$. Now select $x \in O$ and let $O' \in \Xi$ be an ovoid not containing x (which exists by Lemma 6.6). If x is not contained in any singular line, then the tangent spaces at x of the ovoids $X([x,y])$, with $y \in O'$ fill the whole space $T_x(X)$ (indeed the number of points contained in these tangent spaces is $(q^d+1)(\frac{q^{d+1}-1}{q-1}-1)+1)$, and so (AVV2) implies that $|X| = q^{2d}+q^d+1$, a contradiction. Next, suppose x is contained in a maximal singular subspace S_x of dimension at least 1. As in the previous case, we consider ovoids determined by x and points of O'.

Let t denote the number of tangent spaces in $T_x(X)$ different from S_x. With a similar reasoning as above we obtain $t(\frac{q^{d+1}-1}{q-1} - 1) + q + 1 \leq \frac{q^{2d+1}-1}{q-1}$ hence $t \leq q^d$. Recalling that maximal singular subspaces do not intersect non-trivially, we hence obtain $|X| \leq q^{2d}+|S_x|$. This implies that $|S_x| \geq q^{1+d} + q^d + 1$, so $\dim S_x > d$, but then S_x does not fit in $T_x(X)$ without violating (AVV2), a contradiction. Claim 1 is proved.

Claim 2. If $d = 2$*, then there are no nontrivial singular subspaces.*
Indeed, assume there is a nontrivial maximal singular subspace L. By Claim 1 we may assume that L is a line. The number of points on ovoids containing a fixed point $z \in X \setminus L$ and a variable point $y \in L$ is $(q+1)q^2 + 1$. Comparing this with the number of points on ovoids containing z and a variable point (not collinear to z) on a fixed ovoid not containing z computed above, we conclude that there exists an ovoid on z disjoint from L. Now there are two possibilities.

Some point x *of* O *is contained in a singular line* L'*.* Then by the above we may select an ovoid O' disjoint from L'. Then no point of O' is collinear to x for this would yield a singular plane. But then the tangent planes to the ovoids containing x and a point of O' already fill $T_x(X)$, leaving no room for L', a contradiction.

No point of O *is contained in a singular line.* Then considering $x \in O$ and an ovoid O' not containing x, we count, as before, $|X| = q^4 + q^2 + 1$. Pick $y \in L$. Let α be the number of ovoids containing y. Then $|X| = \alpha q^2 + q + 1$, a contradiction.

Claim 2 is proved.

Claim 3. If $d = 1$*, then there are no nontrivial singular subspaces.*
Indeed, consider a point $x \in O$ and an oval $O' \not\ni x$. If some singular line L joins x with a point y of O', then L together with the tangent lines at x of the ovals joining x with the points of $O' \setminus \{y\}$, fill T_x and so $|X| = q^2 + q + 1$. If there is no singular line on x, then the same conclusion holds. Since every pair of points is either on an oval, or on a singular line, and both have size $q + 1$, we see that X, viewed as a point-line geometry where the line set $\mathscr{L}$ consists of the ovals and the singular lines, is a projective plane of order q. Indeed, if two elements of $\mathscr{L}$ were disjoint we would obtain $|X| > q^2 + q + 1$, a contradiction.

Now assume for a contradiction that there is some singular line L (and note that there can only be one since by the above paragraph they pairwise intersect and such an intersection would lead to a singular plane, a contradiction). Consider a point x in O not on L. Clearly, $\langle X \rangle = \langle T_x, L \rangle$ and hence $\dim\langle X \rangle = 4$. Projecting $X \setminus O$ from $\langle O \rangle$ onto a complementary subspace in $\langle X \rangle$, we see that the points of two ovals intersecting O in the same point project onto the same set of q points, yielding q singular lines, a contradiction. Claim 3 is proved.

Hence we have shown that there are no singular subspaces of dimension at least 1. Moreover, a similar counting argument as before then shows $|X| = q^{2d} + q^d + 1$, implying that (X, Ξ) is indeed a projective plane. □

Lemma 7.3 *If* $|\mathbb{K}| < \infty$*, then* (X, Ξ) *is isomorphic to a Veronese variety* $\mathscr{V}_2(\mathbb{K}, \mathbb{A})$*, for either* $\mathbb{A} = \mathbb{K}$ *or* $\mathbb{A}$ *a quadratic extension of* $\mathbb{K}$*.*

Proof By Lemma 7.2, (X, Ξ) is a (1)-AVV which moreover has the structure of a projective plane, i.e., each two ovoids have a non-trivial intersection. Such varieties have been studied in [21], Main Result 4.3 of which asserts that (X, Ξ) is indeed isomorphic to $\mathscr{V}_2(\mathbb{F}_q, \mathbb{F}_{q^d})$ if $q > 2$, and, if $q = 2$, it is either isomorphic to $\mathscr{V}_2(\mathbb{F}_q, \mathbb{F}_{q^d})$ or to a member of a restricted list of additional possibilities, each of which we will now rule out. Taking into account that by assumption $\dim\langle X\rangle \leq 3d+2$, only one additional possibility remains for each value of d:

($d = 1$) *Six points of X form a frame of a 4-space S and the seventh point of X lies outside S and forms a basis with any five points of $S \cap X$.*
Let x be a point of O contained in S and let z be the unique point of X not contained in S. Let O' be the oval determined by x and z and denote by y the unique point on O' distinct from x and z. Since the two ovals containing x distinct from O' belong to S, also $T_x(X)$ belongs to S. But then $\langle O'\rangle = \langle T_x(O), y\rangle \subseteq S$, a contradiction. So this additional possibility is ruled out.

There are a few things to be said before discussing the second alternative, which occurs for $d = 2$. Firstly, an ovoid of $\mathbb{P}^3(\mathbb{F}_2)$ coincides with a *frame* of $\mathbb{P}^3(\mathbb{F}_2)$, i.e., a set of 5 points no 4 of which are contained in a plane. Moreover, four points p_1, p_2, p_3, p_4 of such a frame determine the frame uniquely, as its fifth point is given by $p_1 + p_2 + p_3 + p_4$. A *pseudo-embedding* of the projective plane $\mathbb{P}^2(\mathbb{F}_4)$ is given by identifying its points to points of a certain projective space $\mathbb{P}^n(\mathbb{F}_2)$, with $n \geq 4$, such that its lines get identified with frames in 3-spaces. Such embeddings were introduced and studied by De Bruyn [14, 15]. He obtained that the universal pseudo-embedding $\mathscr{M}$ of $\mathbb{P}^2(\mathbb{F}_4)$ lives in $\mathbb{P}^{10}(\mathbb{F}_2)$ [15, Proposition 4.1] and an explicit (coordinate) construction [14, Theorem 1.1]. A geometric construction, using a basis of $\mathbb{P}^{10}(\mathbb{F}_2)$, was given in [21, Section 7.3.2], where it arose as the universal embedding of an AVV-like set (X', Ξ'), which satisfies (using our notation) (AVV1), (AVV2) and the additional property that each two members of Ξ' share a point of X'; whence the connection with the current situation.

($d = 2$) *X arises as the (injective) projection of the universal pseudo-embedding $\mathscr{M} = (X', \Xi')$ of $\mathbb{P}^2(\mathbb{F}_4)$ (where the members of Ξ' are the 3-spaces corresponding to lines of $\mathbb{P}^2(\mathbb{F}_4)$.)*
To obtain our variety (X, Ξ), we consider the projection ρ from (X', Ξ') from an "admissible" line M', meaning that the projection of (X', Ξ') from M' is not only required to be injective but also to preserve property (AVV2). In $\mathscr{M}$, it is known that all points $x' \in X'$ are such that $\dim T_{x'}(X') = 6$. Now, if x, y, z are the three points of O, then the only way to obtain $\dim T_x(X) = \dim T_y(X) = \dim T_z(X) = 4$ is to choose M' in $T_{\rho^{-1}(x)}(X') \cap T_{\rho^{-1}(y)}(X') \cap T_{\rho^{-1}(z)}(X')$. However, by Lemma 7.9 of [21], there is only one line M contained in this intersection, and the projection of (X', Ξ') from M yields $\mathscr{V}_2(\mathbb{F}_q, \mathbb{F}_{q^2})$. This also excludes the existence of other possibilities than $\mathscr{V}_2(\mathbb{F}_q, \mathbb{F}_{q^2})$, at least in our current setting.

We conclude that (X, Ξ) is indeed isomorphic to $\mathscr{V}_2(\mathbb{F}_q, \mathbb{F}_{q^d})$. $\square$

7.1.2 The infinite case

Suppose $|\mathbb{K}| = \infty$. We will consider the projection ρ of $X \setminus O$ from O onto a complementary subspace Π (which has dimension at most $2d$ since, by assumption, $\dim\langle X\rangle \leq 3d+2$). We introduce some notation. If O_i, with i in some index set, is an ovoid meeting O in a point p_i, then we denote by P_i the projective d-space $\rho(\langle O_i\rangle)$. Then the projection $\rho(T_{p_i}(O_i))$ is a hyperplane of P_i which we denote by T_i. Since $\dim(T_{p_i}(X)) = 2d$, T_i also coincides with $\rho(T_{p_i}(X))$. The affine d-space $P_i \setminus T_i$ is denoted by A_i and coincides with $\rho(O_i \setminus \{p_i\})$.

Lemma 7.4 *Consider distinct ovoids O_1 and O_2 and pairwise distinct points p_1, p_2, p such that $\{p_i\} = O \cap O_i$, $i = 1, 2$, and $\{p\} = O_1 \cap O_2$. Then* $\dim(P_1 \cap P_2) = 0$.

Proof Note that $\rho(p) \in A_1 \cap A_2$. Suppose for a contradiction that $\dim(P_1 \cap P_2) \geq 1$ and let L be a line in $P_1 \cap P_2$ containing $\rho(p)$. Then $\Pi' := \langle O, \rho^{-1}(L)\rangle$ has dimension $d+3$ and since $\dim\langle O_i, O\rangle = 2d+2$ and $\dim\langle O_i\rangle = d+1$, we obtain that $\pi_i := \Pi' \cap \langle O_i\rangle$ is a plane intersecting O_i in an oval o_i containing p_i and p. Let $q_i \in o_i$ be arbitrary and let L_i be the line $\langle p_i, q_i\rangle$ if $q_i \neq p_i$, and otherwise L_i is the tangent to o_i at p_i. Let M_i be a line in π_i not containing p_i. Consider the projectivity $\sigma_i : o_i \to L$ defined by the composition of the perspectivities $q_i \mapsto L_i \mapsto r_i = L_i \cap M_i \mapsto \rho(r_i) = \rho(L_i)$. Thus $\sigma := \sigma_2^{-1} \circ \sigma_1 : o_1 \to o_2$ is a projectivity fixing p. Note that, if $q_1 \in o_1 \setminus \{p, p_1\}$, then the line $\langle q_1, \sigma(q_1)\rangle$ is contained in the subspace $\langle O, \rho(\langle p_1, q_1\rangle)\rangle$ and hence intersects $\langle O\rangle$ in a unique point. Consequently, if $\sigma(q_1) \neq p_2$, then the line $\langle q_1, \sigma(q_1)\rangle$ is singular. Since $|\mathbb{K}| > 4$, there are at least three such singular lines which, by Lemma A.3 of [21], are transversals of the rational normal cubic scroll $\mathscr{S}$ determined by o_1 and o_2 (see also Appendix A of [21]). Clearly, also the unique line meeting all transversals of $\mathscr{S}$ (the axis of $\mathscr{S}$), is a singular line. Recalling that maximal singular subspaces are disjoint, it follows that $\langle \mathscr{S}\rangle = \langle o_1, o_2\rangle$ is singular, a contradiction. □

Lemma 7.5 *There is no singular line intersecting O. Consequently, ρ is injective on $X \setminus O$.*

Proof Assume L is a singular line intersecting O in a point p. Consider points $q \in L \setminus \{p\}$ and $p' \in O \setminus \{p\}$. Then the line $\langle p', q\rangle$ is not singular by Lemma 6.2. Let $O_1 = X([q, p'])$ and consider a point $r \in O_1 \setminus \{q, p'\}$. Likewise, p and r determine an ovoid O_2. Then we obtain that $\rho(q) \in T_2$ (recall that $T_2 = T_{p_2}(X)$) and $\rho(r) \in A_2$. But $\rho(q)$ and $\rho(r)$ also belong to A_1, contradicting Lemma 7.4.

Now suppose that x_1, x_2 are two points of $X \setminus O$ with $\rho(x_1) = \rho(x_2)$. Then (AVV2) implies that the line $\langle x_1, x_2\rangle$ is singular and meets O, contradicting the above. □

Lemma 7.6 *Two ovoids O_i, $i = 1, 2$, which intersect O in distinct points p_1, p_2, respectively, intersect each other. Also, $T_1 \cap P_2 = \emptyset = P_1 \cap T_2$.*

Proof Suppose O_1 and O_2 intersect O in points p_1 and p_2, respectively. Recalling that $\dim \Pi \leq 2d$, P_1 and P_2 share a point x. Suppose first that $x \in A_1 \cap A_2$. By Lemma 7.5, ρ is injective on $X \setminus O$ and hence $O_1 \cap O_2$ coincides with $\rho^{-1}(x)$. So we may assume, without loss of generality, that $x \in T_1 \cap P_2$. Consider an ovoid O_1' through p_1 and a point r in $O_2 \setminus \{p_2\}$ such that $\rho(r) \neq x$. Conform our notation, we then have $x \in T_1 = T_1'$, and therefore $\langle x, \rho(r)\rangle \subseteq P_1' \cap P_2$, a contradiction to Lemma 7.4. $\square$

Lemma 7.7 *If O_1 and O_2 intersect O in distinct points p_1 and p_2, respectively, then $T_1 \cap T_2 = \emptyset$ and $\langle T_1, T_2\rangle \cap \rho(X \setminus O) = \emptyset$. Consequently, there are no singular lines.*

Proof The first statement follows immediately from Lemma 7.6. Suppose there is a point $p \in \langle T_1, T_2\rangle \cap \rho(X \setminus O)$. Consider the ovoid O_2' containing p_2 and $p' = \rho^{-1}(p)$ (recall that ρ is injective on $X \setminus O$). Then A_2' belongs to $\langle T_1, T_2\rangle$ and hence, by a dimension argument, meets T_1 in a point t_1, which then belongs to $T_1 \cap P_2'$, contradicting the second assertion of Lemma 7.6.

Now suppose L is a singular line. Then by the above, $\dim\langle T_1, T_2\rangle = 2d-1$ and $\dim \Pi = 2d$, so $\rho(L) \cap \langle T_1, T_2\rangle \neq \emptyset$, contradicting the above. $\square$

Lemma 7.8 *Each pair of ovoids intersect in a point.*

Proof By Lemma 7.6, it suffices to show that each ovoid intersects O in a point. Let O' be an ovoid different from O. Take distinct points $p, p' \in O$ and a point $q \in O'$. By Lemma 7.7, we may put $O_1 := X([p, q])$ and $O_2 := X([p', q])$. By Lemmas 7.6 and 7.7, the map $\psi : O_2 \setminus \{q\} \to \Xi_p \setminus \{O_1\} : r \mapsto [p, r]$, where Ξ_p denotes the subset of Υ whose members contain p, is a bijection.

Consider the projection ρ_1 of $X \setminus O_1$ from O_1 onto a complementary subspace Π_1 of O_1. Let $T = \rho_1(T_p(O))$, $A = \rho_1(O \setminus \{p\})$, $T_2 = \rho_1(T_q(O_2))$ and $A_2 = \rho_1(O_2 \setminus \{q\})$. If $t \in T \cap T_2$, then $\langle \rho_1(p'), t\rangle \setminus \{t\} \subseteq A \cap A_2$, leading to singular lines (cf. last paragraph of the proof of 7.5), contradicting Lemma 7.7. So $T \cap T_2 = \emptyset$ and hence, by a dimension argument, $\langle T, T_2\rangle$ is a hyperplane of Π_1. The bijectivity of ψ, together with the fact that $T = \rho_1(T_p)$ since $\dim T_p = 2d$, implies $\rho_1(X \setminus O_1) = \Pi_1 \setminus \langle T, T_2\rangle$. Let $T' = \rho_1(T_q(O'))$ and $A' = \rho_1(O' \setminus \{q\})$. Then $A' \subseteq \rho_1(X \setminus O_1)$, hence $T' \subseteq \langle T, T_2\rangle$. Similarly as earlier in this paragraph, we deduce that $T \cap T' = \emptyset$ (now using an ovoid O_2' containing p and some point $q' \in O' \setminus \{q\}$). Then, as A and A' are both contained in $\rho_1(X \setminus O_1) = \Pi_1 \setminus \langle T, T'\rangle$, we have $A \cap A' \neq \emptyset$. As before, the absence of singular lines implies that $O \cap O' \neq \emptyset$. $\square$

7.1.3 Conclusion

Proof of Theorem 7.1 If $|\mathbb{K}| < \infty$ this was proved in Lemma 7.3, so suppose $|\mathbb{K}| = \infty$. By Lemmas 7.7 and 7.8, (X, Ξ) is a projective plane satisfying (AVV1) and (AVV2), so we can again apply the Main Result 4.3 of [21], which asserts that (X, Ξ) is indeed isomorphic to $\mathscr{V}_2(\mathbb{K}, \mathbb{A})$ where $\mathbb{A}$ is a quadratic alternative algebra over $\mathbb{K}$ with $\dim_{\mathbb{K}} \mathbb{A} = d$. $\square$

7.2 Proof of ovoidal case

We again assume the Standing Hypotheses 6.4. Recall that we assume that (Y, Υ) is an ALV of type $d \geq 1$ and index 0. The previous section has the following consequence.

Corollary 7.9 *The residue of (Y, Υ) at every point a' admitting a point at distance 3 from a' in the collinearity graph of (Y, Υ) is a Veronese representation of a projective plane over a quadratic alternative division algebra.*

Proof The said residue is a $(1', 3')$-AVV by our Standing Hypotheses 6.4. The conclusion now follows from Theorem 7.1. □

Lemma 7.10 *The residue at every point is a Veronese representation of a projective plane over a quadratic alternative division algebra $\mathbb{A}$. In particular, $\dim T_y = 3 + 3\dim_{\mathbb{K}} \mathbb{A}$ for each $y \in Y$.*

Proof By Lemma 6.4 and Corollary 7.9 it suffices to prove that an arbitrary point v collinear with a admits a point at distance 3 from v in the collinearity graph of (Y, Υ). Suppose for a contradiction that v does not admit a point at distance 3. Then $\delta(v, e) = 2$ and by potentially rechoosing c in $[b, e]$ we may assume that $\delta(v, c) = 2$. Consider the tangent spaces T_v and T_c. Since $\dim\langle T_v \cap T_a\rangle = 2d + 1$ (by Corollary 7.9), $\dim\langle T_v \cap T_e\rangle \geq d + 1$, and $T_a \cap T_e = \emptyset$, we have $3d + 3 \geq \dim T_v \geq \dim\langle T_v \cap T_a, T_v \cap T_e\rangle = \dim\langle T_v \cap T_a\rangle + \dim\langle T_v \cap T_e\rangle + 1 \geq 3d + 3$. This yields $T_v = \langle T_v \cap T_a, T_v \cap T_e\rangle$. Similarly, $T_c = \langle T_c \cap T_a, T_c \cap T_e\rangle$. Hence by Corollary 7.9, we have $(T_v \cap T_a) \cap (T_c \cap T_a) = \emptyset$ and $(T_c \cap T_e) \cap (T_v \cap T_e) = \emptyset$. Since $\delta(v, c) = 2$ there exists $q \in T_v \cap T_c$ and by the above $q \notin T_a \cup T_e$.

Hence, q is the intersection of two uniquely determined lines $\langle c_e, c_a\rangle$ and $\langle v_e, v_a\rangle$, with $c_e \in T_c \cap T_e, c_a \in T_c \cap T_a$, $v_a \in T_v \cap T_a$ and $v_e \in T_v \cap T_e$. However, then the lines $\langle v_a, c_a\rangle$ and $\langle v_e, c_e\rangle$ intersect in a point p belonging to $T_a \cap T_e$, a contradiction. □

Lemma 7.11 *The point-line geometry $(Y, \mathscr{L})$ associated to (Y, Υ) is a 0-lacunary parapolar space of uniform symplectic rank 2.*

Proof Suppose $\upsilon_1, \upsilon_2 \in \Upsilon$ share a point $y \in Y$. Then $\mathrm{Res}_Y(y)$ is a projective plane by Lemma 7.10 and hence υ_1 and υ_2 share at least a line. □

Proposition 7.12 *Let (Y, Υ) be an abstract Lagrangian variety of type $d \geq 1$ and index 0. Then Y is isomorphic to the Veronese representation $\mathscr{V}(\mathbb{K}, \mathbb{A})$ in $\mathbb{P}^{6d+7}(\mathbb{K})$ of a dual polar space $\mathsf{C}_{3,3}(\mathbb{K}, \mathbb{A})$ over a quadratic alternative division algebra $\mathbb{A}$ over $\mathbb{K}$ with $\dim_{\mathbb{K}} \mathbb{A} = d$.*

Proof Using Lemma 7.11 and the classification of 0-lacunary parapolar spaces in [19], combined with Lemma 7.10, we obtain that $(Y, \mathscr{L})$ is a dual polar space of rank 3 isomorphic to $\mathsf{C}_{3,3}(\mathbb{K}, \mathbb{A})$ (in view of each point-residual being isomorphic to a projective plane over a quadratic alternative division algebra $\mathbb{A}$ and each symp being isomorphic to an orthogonal quadrangle over $\mathbb{K}$). By Lemma 7.10 and Axiom (ALV1), $N \geq 7 + 6\dim_{\mathbb{K}} \mathbb{A}$. The assertion for $|\mathbb{K}| \neq 2$ now follows from Proposition 6.7.

Now let $\mathbb{K} = \mathbb{F}_2$. By Theorem 10.37, it suffices to show that (Y, Υ) is projectively unique. The point-line geometry $(Y, \mathscr{L})$ is either the dual polar space $\mathsf{C}_{3,3}(\mathbb{F}_2, \mathbb{F}_2)$ or $\mathsf{C}_{3,3}(\mathbb{F}_2, \mathbb{F}_4)$, and it is embedded in (and spans) $\mathbb{P}^N(\mathbb{K})$, $N \geq 6d + 7$, with $d = 1, 2$, respectively. Note that $(Y, \mathscr{L})$ has diameter 3. Let $Y \subseteq \mathbb{P}^m(\mathbb{F}_2)$ be an arbitrary embedding of $(Y, \mathscr{L})$ into the projective space $\mathbb{P}^m(\mathbb{F}_2)$, with $m \in \mathbb{N}$. We pick points x and y at distance 3 from one another. Let $T_x(Y)$ and $T_y(Y)$ be the subspaces generated by all lines on x and all lines on y, respectively. Lemma 5.7(1) of [16] yields $\mathbb{P}^m(\mathbb{F}_2) = \langle T_x(Y), T_y(Y)\rangle$. Applied to the embedding corresponding to (Y, Υ), we conclude that $N = 6d + 7$.

Since $(Y, \mathscr{L})$ is a geometry with three points per line, and it admits at least one embedding in a projective space over $\mathbb{F}_2$ (namely, $\mathscr{V}(\mathbb{F}_2, \mathbb{F}_m)$, $m = 2, 4$), it admits a universal embedding $\mathscr{E}_{m/2}$, and Y is a projection, or quotient, of $\mathscr{E}_{m/2}$, see for instance [13]. It also follows from *loc. cit.* that the dimension of the ambient projective space of $\mathscr{E}_d$ is equal to $7d + 7$, $d \in \{1, 2\}$.

First let $d = 1$. Consider the universal embedding $\mathscr{E}_1$ in $\mathbb{P}^{14}(\mathbb{F}_2)$. With similar notation as above, the subspaces $T_x(\mathscr{E}_1)$ and $T_y(\mathscr{E}_1)$ generate $\mathbb{P}^{14}(\mathbb{F}_2)$. Note that $T_x(\mathscr{E}_1)$ is generated by seven lines, so $\dim T_x(\mathscr{E}_1) = \dim T_y(\mathscr{E}_1) \leq 7$. It follows that $\dim T_x(\mathscr{E}_1) = \dim T_y(\mathscr{E}_1) = 7$ and $T_x(\mathscr{E}_1) \cap T_y(\mathscr{E}_1)$ is a point c. Since $\dim T_z(Y) = 6$ for each point $z \in Y$ by Lemma 7.10, it follows that (Y, Υ) is obtained from $\mathscr{E}_1$ by projecting from c (and c is contained in $T_z(\mathscr{E}_1)$, for every point $z \in \mathscr{E}_1$). Hence (Y, Υ) is projectively unique.

Now let $d = 2$. Consider the universal embedding $\mathscr{E}_2$ in $\mathbb{P}^{21}(\mathbb{F}_2)$. With the same notation as before, we claim that $\dim T_x(\mathscr{E}_2) = 11$, for each point $x \in \mathscr{E}_2$. Indeed, by our claim above, we have $\langle T_x(\mathscr{E}_2), T_y(\mathscr{E}_2)\rangle = \mathbb{P}^{21}(\mathbb{F}_2)$. Since the universal embedding admits the full (point-transitive) automorphism group of the geometry, this implies $\dim T_x(\mathscr{E}_2) = \dim T_y(\mathscr{E}_2) \geq 10$. By Paragraph 7.3 of [21], the residue at x admits an embedding in a projective space of dimension at most 10, so it follows that $\dim T_x(\mathscr{E}_2) \in \{10, 11\}$. Since the stabilizer of a point in the full automorphism group of the abstract geometry $(Y, \mathscr{L})$ is the full automorphism group of the corresponding point-residual, we have $\dim T_x(\mathscr{E}_2) = \dim T_y(\mathscr{E}_2) = 11$ (indeed, if $\dim T_x(\mathscr{E}_2)$ were equal to 10, then the residue at x would be embedded in $\mathbb{P}^9(\mathbb{F}_2)$, and hence arises from its universal embedding in $\mathbb{P}^{10}$ by projecting from a point; the results of Paragraph 7.3.2 of [21] show that no such embedding admits the full automorphism group). So $T_x(\mathscr{E}_2) \cap T_y(\mathscr{E}_2)$ is a line L. Similarly as for the case $d = 1$, since $\dim T_z(Y) = 9$ for all $z \in Y$ by Lemma 7.10, we now conclude that L is the intersection of all tangent spaces, (Y, Υ) is the projection of $\mathscr{E}_2$ from L and (Y, Υ) is projectively unique. □

8 Hyperbolic case ($w = \frac{d}{2}$)

If $w \geq 1$, then by the Standing Hypotheses 6.4 and Lemma 6.10, the point-residual $(Y_a, \Upsilon_a) = (X, \Xi)$ is a $(1, 3')$-AVV of type d and index w in $\mathbb{P}^M(\mathbb{K})$ for $M \leq 3d + 2$

(and recall the notation $\partial\Xi$, the set of differential host spaces of Ξ, and ∂X, the set of differential points of X, from Axiom (AVV3$'$)). Our aim is to use Proposition 6.12. Since we have hyperbolic symps, we can use Corollary 4.4. Hence it suffices to show that there exists some singular subspace of dimension w contained in exactly two maximal singular subspaces of prescribed well-defined dimensions. We split up our analysis according to the value of w.

We first treat the case $w = 0$ (and hence also $d = 0$), which is an extreme ovoidal case.

8.1 Segre product of 3 lines ($w = d = 0$)

Proposition 8.1 *If $w = d = 0$, then (Y, Υ) is isomorphic to $\mathscr{S}_{1,1,1}(\mathbb{K})$.*

Proof Consider two distinct host spaces $\upsilon_1, \upsilon_2 \in \Upsilon$ sharing a point $y \in Y$. Since $\dim T_y(Y) \leq 3$, we obtain that υ_1 and υ_2 share a line. Then the point-line geometry $(Y, \mathscr{L})$ associated to (Y, Υ) is a 0-lacunary parapolar space with hyperbolic symps of rank 2 of diameter at least 3. Lemma 5.6 implies that $(Y, \mathscr{L})$ is isomorphic to $\mathsf{A}_1(\mathbb{K}) \times \mathsf{A}_1(\mathbb{K}) \times \mathsf{A}_1(\mathbb{K})$. Since there exist disjoint host spaces, we have $N \geq 7$. Hence the result follows from Proposition 6.7(S). □

8.2 The plane Grassmannian ($w = 1, d = 2$)

Here, by Corollary 4.4 and Proposition 6.12, it suffices to show that there is a point $x \in X$ contained in exactly two maximal singular subspaces, which are planes. Equivalently, $T_x(X)$ is the union of two singular planes. We accomplish this in a series of lemmas, our first major aim being to exhibit two host spaces intersecting in a point x only.

Lemma 8.2 *For each differential point $x \in \partial X$, there exist $\xi_i \in \partial\Xi$, $i = 1, 2$ with $\xi_1 \cap \xi_2 = \{x\}$. In particular, there are at least four singular lines through x.*

Proof As $x \in \partial X$, there is a host space $\xi \in \partial\Xi$ with $x \in X(\xi)$. We first show that not all members of $\partial\Xi$ containing x contain the same line L of $X(\xi)$. Suppose for a contradiction that they do. We may assume that ξ corresponds to $\upsilon := [a, c] \in \Upsilon$ and the point x to the line ab of Y. Also, L corresponds to some plane π containing ab. Consider the grid $G := b^\perp \cap e^\perp$. Let c' be any point of G collinear to c. Then $[a, c'] \in \Upsilon$ corresponds to a host space ξ^* containing x. By Lemma 6.9, $\xi^* \in \partial\Xi$. Our assumption implies that ξ^* also contains L, i.e., $[a, c']$ contains π. Hence $c'^\perp \cap \pi$ is a line K'. Set $c^\perp \cap \pi = K$. We claim that $K = K'$. Indeed, suppose not, then there exists a point $f \in K' \setminus K$ collinear to c', and not to c. By (ALV1) and Lemma 6.2, the host space $[c, f] \in \Upsilon$ contains K and hence a, and thus coincides with $[a, c]$. As such, $c' \in f^\perp \cap c^\perp \subseteq [f, c] = [a, c]$, implying that $a^\perp$ contains a point of $cc' \subseteq e^\perp$, contradicting $T_a(Y) \cap T_e(Y) = \emptyset$. The claim follows. Interchanging the roles of c and c', there is also a point $c'' \in G \setminus c^\perp$ collinear to K, implying that $K \subseteq [c, c''] = [e, b]$, again contradicting $T_a(Y) \cap T_e(Y) = \emptyset$.

Let L_1 and L_2 be the two lines of $X(\xi)$ containing x. By the previous paragraph there exist $\xi_i \in \partial\Xi$, $i = 1, 2$, not containing L_{3-i}. If $\xi_i \cap \xi$ is $\{x\}$, for some $i \in \{1, 2\}$, we are done, so assume $L_i \subseteq \xi_i$, $i = 1, 2$. Let M_i be the unique line of ξ_i distinct from L_i and containing x. Again, if $M_1 \neq M_2$, we are done, so suppose $M_1 = M_2$. By (AVV3′), there are at least $|\xi|$ members of $\partial\Xi$ containing x, so there exists $\xi_1' \in \partial\Xi$ containing x with $\xi_1' \notin \{\xi, \xi_1, \xi_2\}$. Then ξ_1' contains at most one line from $\{L_1, L_2, M_1\}$. Hence the other two lines define $\xi_2' \in \{\xi, \xi_1, \xi_2\} \subseteq \partial\Xi$, which then intersects ξ_1' in exactly $\{x\}$. □

As a second major step, we show the existence of a singular plane containing a differential point. This can be achieved by slightly generalising a series of proofs used in [26]. As the statements of almost all lemmas need to be adapted and every proof requires minor tweaks we include them here, as we feel just stating that one can adapt them is prone to errors and puts a burden on the reader.

Standing hypothesis until Lemma 8.7: In the sequel, we suppose for a contradiction that no singular plane contains a differential point. We fix a point $x \in \partial X$ and host spaces $\xi, \xi' \in \partial\Xi$ with $\xi \cap \xi' = \{x\}$ (which exist by Lemma 8.2).

We want to study the projection of $X \setminus \xi$ from ξ onto some $(N-4)$-dimensional subspace F. In order to do so, we first prove some additional lemmas.

Lemma 8.3 *For any $x' \in \partial X$ and any four (distinct) singular lines L_1, L_2, L_3, L_4 containing x', we have $\dim\langle L_1, L_2, L_3, L_4\rangle = 4$ and $[L_1, L_2], [L_3, L_4]$ are host spaces meeting each other in x' only.*

Proof By Lemma 6.1 and since there are no singular planes containing x', there are unique host spaces containing L_1, L_2, and L_3, L_4, respectively. By (AVV2), $[L_1, L_2] \cap [L_3, L_4] = \{x'\}$. □

Lemma 8.4 *Let L_1 and L_2 be two distinct singular lines of X meeting ξ in respective points x_1, x_2. Then $\dim\langle \xi, L_1, L_2\rangle = 5$.*

Proof If $x_1 = x_2$, this follows from Lemma 8.3, so suppose $x_1 \neq x_2$. Assume for a contradiction that $\dim\langle \xi, L_1, L_2\rangle = 4$. If L_1 and L_2 have a point x_{12} in common, then by Lemma 6.2 and $x_{12} \notin \xi$, we obtain that $x_1 \perp x_2$. Therefore $\langle L_1, L_2\rangle$ is a singular plane containing the points $x_1, x_2 \in \partial X$, contradicting our hypothesis. Thus $\langle L_1, L_2\rangle$ is a 3-space, intersecting ξ in a (non-singular) plane π. Take a point $y \in \pi \setminus (X \cup \langle x_1, x_2\rangle)$. Since $y \in \langle L_1, L_2\rangle$, it lies on a line M meeting both L_1 and L_2 in respective points z_1 and z_2, with $z_i \neq x_i$, $i = 1, 2$. So, by (AVV1) and (AVV2), $\{y\} = M \cap \xi \subseteq [z_1, z_2] \cap \xi \subseteq X$, a contradiction. □

Lemma 8.5 *Suppose ξ_1, ξ_2 are distinct members of $\Xi \setminus \{\xi\}$ meeting ξ in a singular line L. Then $\dim\langle \xi, \xi_1, \xi_2\rangle = 7$.*

Proof Set $i = 1, 2$ and put $W_i := \langle \xi, \xi_i \rangle$, and note that $\dim W_i = 5$ since $\xi \cap \xi_i = L$ by (AVV2). Suppose for a contradiction that $\dim(W_1 \cap W_2) \geq 4$. Select a 4-dimensional subspace U contained in $W_1 \cap W_2$ and containing ξ (possibly, $U = W_1 \cap W_2$). Let $M_i \subseteq X(\xi_i)$ be a singular line disjoint from ξ. Then M_i meets U in a unique point m_i. Denote the unique line of $X(\xi_i)$ containing m_i and distinct from M_i by L_i. As L_i meets L in a unique point x_i, Lemma 8.4 implies that $\langle L_1, L_2, \xi \rangle \subseteq U$ has dimension 5, a contradiction. □

We can now prove the following two important lemmas.

Lemma 8.6 *Let $L = x_1x_2$ be a line of $X(\xi)$. Then* $\dim\langle \xi, T_{x_1}(X), T_{x_2}(X) \rangle = 7$.

Proof By Lemma 8.2, there are two singular lines L_1 and L_1' containing x_1 not in $X(\xi)$. By Lemma 8.3 and $x_1 \in \partial X$, we have $T_{x_1}(X) = \langle T_{x_1}(\xi), L_1, L_1' \rangle$. By Lemma 6.1 and our assumption that no singular plane meets L, $\xi_1 := [L, L_1]$ and $\xi_1' := [L, L_1']$ belong to Ξ. Let L_2 and L_2' be the respective singular lines of ξ_1, ξ_1' containing x_2 distinct from L. Since $\langle L_1, L_2 \rangle = \xi_1$ and $\langle L_1', L_2' \rangle = \xi_1'$, we obtain $\langle \xi, T_{x_1}(X), T_{x_2}(X) \rangle = \langle \xi, \xi_1, \xi_1' \rangle$, which by Lemma 8.5 has dimension 7. □

Lemma 8.7 *Let $x' \in X(\xi)$, then $\langle \xi, T_{x'}(X) \rangle \cap X$ belongs to $X(\xi) \cup x'^{\perp}$.*

Proof Let y be a point of $\langle \xi, T_{x'}(X) \rangle \cap X$. Suppose for a of contradiction that $y \notin X(\xi)$ and that x' is not collinear to y. Set $\xi_y := [x', y]$. Then $\xi_y \subseteq \langle \xi, T_{x'}(X) \rangle$, and hence ξ and ξ_y share a singular line L containing x'. Let M be the unique line of $X(\xi_y)$ containing y and meeting L in a point, say z (note that $z \neq x'$). Then $M \subseteq \langle \xi, T_{x'}(X) \rangle$, which implies $\dim\langle \xi, T_{x'}(X), T_z(X) \rangle \leq 6$, contradicting Lemma 8.6. □

Finally, we are ready to show that there are singular planes containing differential points.

Proposition 8.8 *There is a singular plane containing a point of ∂X.*

Proof Suppose the contrary. Recall that $\xi' \in \partial\Xi$ meets ξ in precisely the point x. It is convenient to rename $\xi_1 := \xi'$ and $x_1 := x$. Let x_2 be a point on $X(\xi)$ collinear to x_1 and put $L = x_1x_2$. Let L_1, L_1' be the unique singular lines of $X(\xi_1)$ through x_1. Let L_2 be the singular line of $[L, L_1]$ not in ξ and containing x_2, and let L_2' be any singular line through x_2, distinct from L_2 and not in ξ (which exists by Lemma 8.2 and $x_2 \in \partial X$). Set $\xi_2 := [L_2, L_2']$. Let F be a subspace of $\langle X \rangle$ complementary to ξ and note that $\dim F = \dim\langle X \rangle - \dim \xi - 1 \leq (3d + 2) - (d + 1) - 1 = 2d = 4$. We project $X \setminus \xi$ from ξ onto F. For $i = 1, 2$, the projection of $X(\xi_i) \setminus x_i^{\perp}$ is an affine plane π_i^* in F, with projective completion π_i, where the line $T_i := \pi_i \setminus \pi_i^*$ is the projection of $T_{x_i}(X)$. By Lemma 8.6, $\dim\langle T_1, T_2 \rangle = 3$ and hence $T_1 \cap T_2$ is empty. We claim that also $\pi_1 \cap T_2 = \emptyset$ (likewise, $\pi_2 \cap T_1 = \emptyset$). Indeed, if not, then there is a point $z \in X(\xi_1) \setminus x_1^{\perp}$ which is contained in $\langle \xi, T_{x_2}(X) \rangle$. By Lemma 8.7 and $z \notin \xi$, we have $z \in x_2^{\perp}$, but then $x_2 \in X(\xi_1)$ by Lemma 6.2, a contradiction. This shows the claim. Consequently, since $\dim F \leq 4$, the affine planes π_1^* and π_2^* share a unique point z (and note that $\dim F = 4$).

The pre-image of z yields points $z_1 \in X(\xi_1) \setminus x_1^\perp$ and $z_2 \in X(\xi_2) \setminus x_2^\perp$ lying in a common 4-space with ξ. We now prove that $z_1 = z_2$. To that end, suppose $z_1 \neq z_2$. Let ξ^* be a host space containing z_1, z_2. Considering $\xi^* \cap \xi$, (AVV2) implies that $\langle z_1, z_2 \rangle$ is a singular line meeting $X(\xi)$ in some point u. First note that $u \notin L$ because otherwise $L \subseteq \xi_1 = [x_1, z_1]$ by Lemma 6.2. Likewise, neither does u belong to the other singular line of ξ through x_2, because then $u \in \xi_2 = [z_2, x_2]$. So u is not collinear to x_2. Since $z \notin T_2$, there is a unique host space ξ_2' containing x_2 and z_1. We claim that $\xi_2' \cap \xi = \{x_2\}$. Suppose that ξ_2' contains a singular line K of ξ. Then z_1 and u are collinear with respective points v_1 and v_2 on K. If $v_1 = v_2$, we obtain a singular plane $\langle z_1, u, v_1 \rangle$ containing a point of ∂X, so $v_1 \neq v_2$. In particular, v_1 and u are non-collinear points of ξ collinear to z_1. By Lemma 6.2, $z_1 \in X(\xi)$, a contradiction. The claim follows. Consequently, the projection of $\xi_2' \setminus \{x_2\}$ coincides with π_2. Since $\langle \pi_1, \pi_2 \rangle = F$, the singular lines in ξ_1 and ξ_2' through z_1 span a 4-dimensional space, which coincides with $T_{z_1}(X)$ since $\dim T_{z_1}(X) \leq 4$ as $z_1 \in \xi_1 \in \partial\Xi$, and which is projected onto F. Consequently, $T_{z_1}(X)$ is disjoint from ξ, contradicting $u \in T_{z_1}(X) \cap \xi$.

Hence we have shown that $z_1 = z_2$. Now let M_i be the singular line in ξ_i containing z_1 and meeting L_i, say in a point m_i, $i = 1, 2$. Noting that $\pi_1^* \cap \pi_2^* = \{z\}$, we have $\xi_1 \cap \xi_2 = \{z_1\}$, so $M_1 \neq M_2$. Let ℓ_1 be the unique point of L_1 collinear to m_2 (recall $L_2 \subseteq [L, L_1]$). If $m_1 = \ell_1$, then $\langle z_1, m_1, m_2 \rangle$ is a singular plane containing $z_1 \in \partial X$ (recall that $\xi_1 \in \partial\Xi$). So $m_1 \neq \ell_1$, and hence $\xi_1 = [z_1, \ell_1]$. By Lemma 6.2, the latter contains M_2, contradicting $\xi_1 \cap \xi_2 = \{z_1\}$. This final contradiction implies that there is a singular plane containing a point of ∂X. ☐

Lemma 8.9 *There is a point $x \in X$ such that $T_x(X) = \pi \cup \pi'$, where π, π' are singular planes meeting each other in the point x.*

Proof By Lemma 8.8, there is a singular plane π containing a point $x \in \partial X$. Lemma 8.2 yields two host spaces $\xi, \xi' \in \partial\Xi$ with $\xi \cap \xi' = \{x\}$. The symps $X(\xi)$ and $X(\xi')$ have respective lines L_x and L_x' sharing only x with π.

Suppose first that there is a third singular line L_x'' meeting π in x only.
If L_x, L_x' and L_x'' are contained in a plane, then this plane is singular by Lemma 6.1. If they are not contained in a plane, then the 3-space they generate contains a line L of π as $\dim T_x \leq 4$. If no pair of $\{L_x, L_x', L_x''\}$ is contained in a singular plane, then the planes $\langle L_x, L_x' \rangle$ and $\langle L_x'', L \rangle$ are distinct and hence, by (AVV2), the line L' they share is singular and hence belongs to $\{L_x, L_x'\}$, and therefore $\langle L_x'', L' \rangle$ is singular after all. So we have a second singular plane π' containing x. If $\pi \cap \pi'$ is not just x, then they determine a singular 3-space Π by Lemma 6.3. Without loss of generality, the lines L_x and L_x' do not belong to Π (since $X(\xi)$ and $X(\xi')$ cannot have two singular lines in Π). Again using $\dim T_x(X) \leq 4$, the plane $\langle L_x, L_x' \rangle$ meets Π in a singular line. Repeated use of Lemma 6.3 implies that $T_x(X)$ is a singular 4-space, a contradiction since $X(\xi)$ contains a pair of non-collinear lines through x. So $\pi \cap \pi' = \{x\}$ and a similar argument shows that $T_x(X) = \pi \cup \pi'$.

Next, suppose that there are no other singular lines meeting π in x than L_x and L_x'.
In this case, the symp $X(\xi)$ has a line L in common with π. Consider a point $y \in L$

and note that $y \in \partial X$ as $\xi \in \partial \Xi$. The previous paragraph implies that we may assume that there are also exactly two singular lines L_y and L'_y meeting π exactly in y. Consider $\xi^* := [L_x, L'_x]$ and let z be an arbitrary point in $X(\xi^*) \setminus x^\perp$. Note that $z^\perp \cap \pi = \emptyset$ for no line of $X(\xi^*)$ lies in π. Hence $[z, y] \in \Xi$ and moreover, the symp $X([z, y])$ does not contain a line of π, so it contains L_y and L'_y. Hence $z \in [L_y, L'_y]$. As z was arbitrary we obtain $[L_y, L'_y] = \xi^*$, a contradiction. $\square$

Proposition 8.10 *If $(d, w) = (2, 1)$, then (Y, Υ) is isomorphic to the Grassmannian embedding of $\mathsf{A}_{5,3}(\mathbb{K})$ in $\mathbb{P}^{19}(\mathbb{K})$.*

Proof Combining Lemma 8.9 and (1) of Corollary 4.4, it follows that (Y, Υ) is (as an abstract variety) isomorphic to $\mathsf{A}_{5,3}(\mathbb{K})$. Proposition 6.12 concludes the proof. $\square$

8.3 The spinor embedding of $\mathsf{D}_{6,6}(\mathbb{K})$ $(w = 2, d = 4)$

Proposition 8.11 *If $(d, w) = (4, 2)$, then (Y, Υ) is projectively equivalent to the spinor embedding $\mathscr{HS}_6(\mathbb{K})$ of $\mathsf{D}_{6,6}(\mathbb{K})$.*

Proof Referring to the Standing Hypotheses 6.4, $(Y_a, \Upsilon_a) = (X, \Xi)$ is a $(1, 3')$-AVV in (possibly a subspace of) $\mathbb{P}^{14}(\mathbb{K})$. For every differential point $x \in \partial X$, $\dim T_x(X) \leq 7$. Hence, for such x, the point-residual (X_x, Ξ_x) of (X, Ξ) at x is a (1)-AVV of type 2 and index 1 in (a subspace of) $\mathbb{P}^7(\mathbb{K})$. It follows from Lemma 5.2 that (X_x, Ξ_x) is either $\mathscr{S}_{1,2}(\mathbb{K})$ or $\mathscr{S}_{1,3}(\mathbb{K})$.

Suppose first that (X_x, Ξ_x) is isomorphic to $\mathscr{S}_{1,2}(\mathbb{K})$. Then we find a singular plane in Y through a contained in exactly two maximal singular subspaces of Y, and they have dimensions 3 and 4. Now Corollary 4.4(3) implies that, as an abstract parapolar space, (Y, Υ) is isomorphic to $\mathsf{D}_{5,5}(\mathbb{K})$. However, the latter has diameter 2, and is strong, hence $u^\perp \cap v^\perp \neq \emptyset$ for all $u \neq v \in Y$, contradictory to Axiom (ALV1).

Consequently, (X_x, Ξ_x) is isomorphic to $\mathscr{S}_{1,3}(\mathbb{K})$. Then, similarly as in the previous paragraph, but now using Corollary 4.4(2), we conclude that, as an abstract parapolar space, (Y, Υ) is isomorphic to $\mathsf{D}_{6,6}(\mathbb{K})$. Proposition 6.12 concludes the proof. $\square$

8.4 A reduction lemma

In this paragraph, we prove a general reduction lemma that we will use often in the sequel. Its purpose is to find a point in the residue of a (1)-AVV with a tangent space of small dimension.

We temporarily abandon the Standing Hypotheses 6.4. However, in this general setting, we still use the terminology of *differential points* of a (1)-AVV of type d, meaning points x for which the dimension of the tangent space at x is at most $2d$.

We begin by quoting a lemma that provides conditions guaranteeing the existence of a pair of non-collinear points in the intersection of subspaces with a quadric.

Lemma 8.12 (Lemma 3.13 of [18]) *Let Q be a non-degenerate quadric in $\mathbb{P}^{d+1}(\mathbb{K})$ of projective index w. Consider a subspace D of $\mathbb{P}^{d+1}(\mathbb{K})$, with $\dim D = d+1-w$. Then the following hold.*

(i) The subspace D contains at least two non-collinear points of Q.
(ii) The intersection $D \cap Q$ spans D. Equivalently, for each hyperplane H of D, the complement $D \setminus H$ contains a point of Q.

The next lemma excludes the possibility of having points not collinear with a given point inside its tangent space. The original version, Lemma 3.14 of [18] is in the context of $(1,3)$-AVVs of type $d \geq 1$; however, its proof only uses that $\dim T_x(X) \leq 2d$, i.e., when rephrased as is done below, exactly the same proof holds.

Lemma 8.13 (Lemma 3.14 of [18]) *Suppose (X, Ξ) is a (1)-AVV of type $d \geq 1$. If (distinct) $\xi_1, \xi_2 \in \Xi$ share a point $x \in X$, and $\dim T_x(X) \leq 2d$, then $\langle T_x(\xi_1), T_x(\xi_2)\rangle \cap X \subseteq x^\perp$.*

Lemma 8.14 *Let (X, Ξ) be a (1)-abstract Veronese variety of type $d \geq 3$ and index $w \geq 1$ in $\mathbb{P}^N(\mathbb{K})$, and let $x, y \in X$ be two collinear differential points. Suppose that there exist two symps intersecting in just $\{x\}$ and there exists a symp containing y but not x. Let y_* be the point of (X_x, Ξ_x) corresponding to the line xy. Then $\dim T_{y_*}(X_x) \leq 2d-1-w$.*

Proof The assumption that there exist two host spaces ξ_1, ξ_2 intersecting in just $\{x\}$ implies, since x is differential, that $T_x(X) = \langle T_x(\xi_1), T_x(\xi_2)\rangle$. Now, by Lemma 8.13, all points of X contained in $\langle T_x(\xi_1), T_x(\xi_2)\rangle$ are necessarily collinear to x, which here means that every point of $T_x(X) \cap X$ is collinear to x. Hence $T_x(X) \cap X(\zeta)$ coincides with $x^\perp \cap \zeta$ and so by Lemma 6.2, it is a singular subspace of ζ. We hence deduce that $T_x(X) \cap \zeta$ contains no pair of non-collinear points of $X(\zeta)$; note that this implies that it is contained in $T_y(\zeta)$. Moreover, $\dim(T_x(X) \cap \zeta) \leq d-w$ since Lemma 8.12 asserts that any subspace of dimension at least $d-w+1$ of ζ contains a pair of non-collinear points. So we can choose a subspace S of dimension $w-1$ in $T_y(\zeta) \subseteq T_y(X)$ disjoint from $T_x(X)$. Using that $\dim T_y(X) \leq 2d$, this implies that $\dim(T_y(X) \cap T_x(X)) \leq 2d-w$. Hence $T_{y_*}(X_x) \leq 2d-1-w$. □

8.5 The exceptional variety $\mathscr{E}_7$ ($w = 4, d = 8$)

We are now ready to characterise the exceptional variety $\mathscr{E}_7(\mathbb{K})$ as the only abstract Lagrangian variety of index $w \geq 4$, excluding all other possible abstract Lagrangian varieties with $w \geq 4$.

Proposition 8.15 *If $w \geq 4$, then $w = 4$ and (Y, Υ) is isomorphic to the exceptional variety $\mathscr{E}_7(\mathbb{K})$.*

Proof By the Standing Hypotheses 6.4, the point-residual (X, Ξ) of (Y, Υ) at the point $a \in Y$ is a $(1, 3')$-AVV of type d and index w. Let $x, y \in \partial X$ be collinear and distinct. If every pair of symps containing x intersect in at least a line, then the point-line geometry associated to (X_x, Ξ_x) is a (-1)-lacunary parapolar space with symps of projective index $w - 1 \geq 3$. By Lemma 5.5 (X_x, Ξ_x) is isomorphic to $\mathsf{E}_{6,1}(\mathbb{K})$ (in which case $w = 5$). It follows that the point-line geometry related to (Y, Υ) is a strong parapolar space of symplectic rank 7, satisfying the hypothesis of Corollary 4.4(3); however, there are no parapolar spaces in the list of conclusions with symplectic rank 7, a contradiction.

We conclude that there exist two host spaces $\xi_1, \xi_2 \in \Xi$ with $\xi_1 \cap \xi_2 = \{x\}$. Also, by Lemma 6.6 applied to (X_y, Ξ_y), we find a host space $\zeta \in \Xi$ containing y but not containing x. We have now everything in place to apply Lemma 8.14 and we obtain a point $y_* \in X_x$ with $\dim T_{y_*}(X_x) \leq 2d - 1 - w \leq 2d - 5$.

A dimension argument now yields that every pair of members of Ξ_x containing y_* intersects in at least a line, implying that the corresponding point-residual $((X_x)_{y_*}, (\Xi_x)_{y_*})$ is a (-1)-lacunary parapolar space with symps of projective index $w - 2 \geq 2$. Lemma 5.5 implies that the corresponding point-line geometry is either $\mathsf{A}_{4,2}(\mathbb{K})$, $\mathsf{A}_{5,2}(\mathbb{K})$ (and in both these cases $w = 4$), or $\mathsf{E}_{6,1}(\mathbb{K})$ (in which case $w = 6$). Also as above, these parapolar spaces satisfy the hypotheses of Corollary 4.4 and hence so does the parapolar space related to (Y, Υ). The former leads with Corollary 4.4(3) to $(Y, \mathscr{L}) \cong \mathsf{E}_{7,7}(\mathbb{K})$, and hence to $\mathscr{E}_7(\mathbb{K})$ by Proposition 6.12; the latter two lead to contradictions, using (2) and (3) of Corollary 4.4, respectively. □

9 Remaining parameter values that do not lead to examples

Section 7 and Subsection 8.1 cover the case $w = 0$, so Proposition 8.15 implies we only have to complete the cases $w \in \{1, 2, 3\}$.

9.1 The case $w = 1, d > 2$

We start by excluding $d = 3$. The proof of the following proposition is inspired by the approach taken in [25] to deal with so-called "Lagrangian Veronesean sets", more precisely those of diameter 2 (which do not exist either).

Proposition 9.1 *There is no ALV (Y, Υ) of type 3 and index 1.*

Proof As $d = 3$, each symp of $(X, \Xi) = (Y_a, \Upsilon_a)$ is isomorphic to the parabolic quadric $Q(4, \mathbb{K})$ in $\mathbb{P}^4(\mathbb{K})$; this quadric has lines as its maximal singular subspaces. Our proof distinguishes between $|\mathbb{K}| = 2$ and $|\mathbb{K}| > 2$. This is already visible in our first claim:

Claim: Let $p \in \partial X$ be a differential point of X. If $|\mathbb{K}| > 2$, there are no singular planes in X containing p, and each pair of host spaces through p shares a line; if $|\mathbb{K}| = 2$, then there are at most 9 host spaces through p.

Consider the point-residual (X_p, Ξ_p). Then (X_p, Ξ_p) is a $(1')$-AVV in $\mathbb{P}^5(\mathbb{K})$. Proposition 5.3 implies that, if $|\mathbb{K}| > 2$, then (X_p, Ξ_p) is isomorphic to $\mathscr{V}_2(\mathbb{K})$, and hence has no singular lines. If $|\mathbb{K}| = 2$, then Proposition 5.3 implies that $|\Xi_p| \leq 9$. Both assertions now follow. We now distinguish between the two cases.

Suppose first that $|\mathbb{K}| > 2$.
Let $\xi \in \partial\Xi$ and let p, q be non-collinear points in $X(\xi)$. Let r be a point collinear to q, not contained in ξ, which exists as there are multiple host spaces through q. Then $r \notin p^\perp$, so we can consider $[p, r]$, which intersects ξ in a singular line L by the above claim. Let r' be the unique point on L collinear to r. Then q is collinear to r', for otherwise $r \in r'^\perp \cap q^\perp \subseteq \xi$. As such, the plane $\langle q, r, r'\rangle$ is singular. However, the point q, belonging to ξ, is differential and hence there are no singular planes containing q by our claim above, a contradiction.

Secondly, suppose $|\mathbb{K}| = 2$.
By (AVV3'), the number of members of $\partial\Xi$ containing a differential point $p \in \partial X$ is at least the number of points in a symp, which is 15. This contradicts our claim above. $\square$

In order to rule out ALVs of type $d > 3$ and index 1, we first restrict the dimension.

Lemma 9.2 *Let (X, Ξ) be a $(1')$-AVV of type $d \geq 2$ and index 0 in $\mathbb{P}^N(\mathbb{K})$. Then $N \geq 2d + 4$.*

Proof This is the content of Subsection 6.3 in [18]. There, the $(1')$-AVV (X, Ξ) arises as the point-residual of a more generalized object at a point contained in at least two quadrics of projective index 1. Then the authors showed (though not explicitly stated as such) that the ambient projective space cannot have dimension $2d + 3$ or smaller. $\square$

Proposition 9.3 *There are no abstract Lagrangian varieties of type $d > 3$ and index 1.*

Proof Assume (Y, Υ) is an ALV of type $d > 3$ and index 1. We use the Standing Hypotheses 6.4. Let $p \in \partial X$. Then (X_p, Ξ_p) is a $(1')$-AVV of type $d - 2$, $d \geq 4$ and index 0, in (a subspace of) $\mathbb{P}^{2d-1}(\mathbb{K})$ which is impossible by Lemma 9.2. $\square$

9.2 The case $w = 2$, $d > 4$

Here the case $d = 5$ needs special attention, so we first treat the case $d > 5$.

We will use two results from [18]. The first one can be stated in our terminology as follows.

Lemma 9.4 (Lemma 4.4 of [18]) *Let (X, Ξ) be a (1)-AVV of type d with $d \geq 3$. Suppose $\langle X\rangle \subseteq \mathbb{P}^{2d+3}(\mathbb{K})$. If ξ, ξ_1 are two host spaces intersecting each other in precisely a point p_1, then there is a point z_1 in $X(\xi_1) \setminus p_1^\perp$ collinear to a point z of $X(\xi) \setminus p_1^\perp$.*

The second one is about a slightly more generalized notion compared to (1)-AVV. Basically, it concerns a structure satisfying all axioms of a (1)-AVV of type d, except that the quadrics may have different projective index. Then Lemma 4.5 of [18] guarantees, under certain conditions, the existence of two quadrics with different projective index. In our setting, these conditions lead to a contradiction. That is how we will state it:

Lemma 9.5 (Lemma 4.5 of [18]) *Let (X, Ξ) be a (1)-AVV of type $d \geq 4$ and index 1 in $\mathbb{P}^{2d+3}(\mathbb{K})$. Then the following assumptions lead to a contradiction: There exist $\xi, \xi_1, \xi_2 \in \Xi$ such that $\xi \cap \xi_1$ is a point p_1, $\xi \cap \xi_2$ is a line L_2 and $\xi_1 \cap \xi_2$ contains a point p with $p \notin p_1^\perp \cap L_2^\perp$.*

We combine the previous two lemmas into the following proposition.

Proposition 9.6 *Let (X, Ξ) be a (1)-AVV of type $d \geq 4$ and index 1 in $\mathbb{P}^{2d+3}(\mathbb{K})$. Then the associated point-line geometry is 0-lacunary.*

Proof Assume for a contradiction that two host spaces ξ, ξ_1 intersect in just the point p_1. Then by Lemma 9.4, there is a point $z_1 \in X(\xi_1) \setminus p_1^\perp$ collinear to a point $z \in X(\xi) \setminus p_1^\perp$. Since $z_1^\perp \cap \xi$ is a singular subspace, we find a line L_2 containing z and not contained in $z_1^\perp$. It follows that there is a unique host space ξ_2 containing z_1 and L_2. Clearly $\xi \cap \xi_2 = L_2$ and $z_1 \in \xi_1 \cap \xi_2$. Moreover, $z_1 \notin p_1^\perp \cup L_2^\perp$. Hence Lemma 9.5 leads to a contradiction and the proposition is proved. □

Proposition 9.7 *There are no abstract Lagrangian varieties of type $d > 5$ and index 2.*

Proof The point-residual (X, Ξ) of (Y, Υ) at the point $a \in Y$ (see the Standing Hypotheses 6.4) is a $(1, 3')$-AVV of type d and index 2 in (a subspace of) $\mathbb{P}^{3d+2}(\mathbb{K})$. Select $p \in \partial X$. Then the point-residual (X_p, Ξ_p) of (X, Ξ) at p is a (1)-AVV of type $d' := d-2 > 3$ and index 1 in (a subspace of) $\mathbb{P}^{2d'+3}(\mathbb{K})$. Proposition 9.6 implies that the point-line geometry related to (X_p, Ξ_p) is a 0-lacunary parapolar space whose symps have projective index 1. Lemma 5.6 now yields $d' = 2$, hence $d = 4$, a contradiction. The assertion follows. □

Before handling the case $d = 5$, we report on the content of Section 6.1 of [27]. The main hypothesis of that section is a given AVV of type 5 and index 2. The existence of such object is ruled out and this is done by considering an arbitrary point-residual, call it (X, Ξ) here, which is a (1)-AVV of type 3 and index 1 in $\mathbb{P}^9(\mathbb{K})$. It is also assumed (since it is proved in an earlier section) that the tangent space at each point of the point-residual has dimension at most 7, and then it is shown that the dimension of such space is in fact at most 6. However, the arguments are almost completely local, that is, one argues in a fixed tangent space of dimension 7, and shows this leads to a contradiction. Moreover, doing so, the (global) fact that $X \subseteq \mathbb{P}^9(\mathbb{K})$ is also ignored. Indeed, it can be checked easily that, in case $|\mathbb{K}| > 2$, Lemmas 6.1 up to 6.7 of [27] prove the following.

Lemma 9.8 *Let (X, Ξ) be a (1)-AVV of type 3 and index 1 and suppose $|\mathbb{K}| > 2$. Then the dimension of the tangent space at an arbitrary point $x \in X$ is not equal to 7.*

If $|\mathbb{K}| = 2$, then we note that only the last lemma, namely Lemma 6.7 of [27], uses the fact that the dimension of the tangent space at *each* point of (X, Ξ) is at most 7. So Lemmas 6.3 and 6.6 of [27] remain valid locally. They can be summarised as follows.

Lemma 9.9 (Lemmas 6.3 and 6.6 of [27]) *Let (X, Ξ) be a (1)-AVV of type 3 and index 1 and suppose $|\mathbb{K}| = 2$. Let $p \in X$ be arbitrary but such that $\dim T_p(X) \leq 7$.*

- (i) *Let C be a conic of (X_p, Ξ_p) and let $x \in X_p \setminus C$. Then there exists at most one member of Ξ_p containing x and disjoint from C.*
- (ii) *X_p does not contain singular planes.*

We are now going to use these two results in order to prove a lemma that will rule out ALVs of type 5 and index 2, and later ALVs of type 7 and index 3.

Lemma 9.10 *Let (X, Ξ) be a (1)-AVV of type 5 and index 2 in (a subspace of) $\mathbb{P}^{17}(\mathbb{K})$. Then each symp $X(\xi)$, $\xi \in \Xi$, contains a point $x \in X(\xi)$ such that $\dim T_x(X) > 10$.*

Proof Suppose for a contradiction that $\xi \in \Xi$ is such that $\dim T_x(X) \leq 10$, for all $x \in X(\xi)$. Let x and y be two collinear points of $X(\xi)$. If all symps on x intersect in at least a line, then the point-line geometry associated to the residue (X_x, Ξ_x) is a strong (-1)-lacunary parapolar space, contradicting Lemma 5.5, since $d = 5$. Also, Lemma 6.6 yields a symp in (X, Ξ) on y not containing x. So we have everything in place to apply Lemma 8.14, from which it follows that in (X_x, Ξ_x), all points y_* of the symp $X_x(\xi_x)$ corresponding to ξ satisfy $\dim T_{y_*}(X_x) \leq 2d - w - 1 = 7$.

Now suppose first $|\mathbb{K}| > 2$. Then Lemma 9.8 yields $\dim T_{y_*}(X_x) \leq 6$, for every point $y_* \in \xi_x$. So each point-residual of (X_x, Ξ_x) at a point of ξ_x is a $(1')$ AVV of type 1 and index 0 in $\mathbb{P}^5(\mathbb{K})$. Then Lemma 5.3 implies that it is isomorphic to the quadric Veronese variety $\mathscr{V}_2(\mathbb{K})$. Now let L_1 be an arbitrary singular line of ξ_x and let $X_x(\zeta_1)$ be a symp containing L_1, but distinct from ξ_x. Pick a point $z \in X_x(\zeta_1) \setminus L_1$ and let z_1 be the unique point on L_1 collinear to z. Pick a point $z_2 \in X_x(\xi_x)$ not collinear to z_1 and let $X_x(\zeta_2)$ be the symp containing z and z_2 (note that z_2 is not collinear to z as this would force $z \in \xi_x$). Since the point-residual in z_2 is isomorphic to $\mathscr{V}_2(\mathbb{K})$, ζ_2 and ξ_x share a unique line L_2. Then z is collinear to a unique point $z_2' \neq z_1$ on L_2, and so z, z_1, z_2' must be contained in a singular plane, contradicting the fact that there are no singular lines in the point-residual of (X_x, Ξ_x) at z_2.

Hence we have reduced the situation to the small case $|\mathbb{K}| = 2$. Let $y_* \in \xi_x$ be arbitrary and set $\Omega_{y_*} = ((X_x)_{y^*}, (\xi_x)_{y^*})$. Fix a point w in Ω_{y_*} and a conic C not containing w. By Lemma 9.9(ii) all singular lines of $\Omega_{y_*}$are pairwise disjoint. Hence we can arrange it so that, if there is a singular line on w, then it also intersects C. By Lemma 9.9(i), this implies that all points of Ω_{y_*} can be found on conics and singular lines containing w and intersecting C in exactly one point, except possibly for one conic containing w and disjoint from C. This means that the number of points of Ω_{y_*} is either 7 or 9.

Varying the point w and the conic C, we obtain that the conics and singular lines render this point set a projective plane of order 2 or an affine plane of order 3, respectively. So, back in (X_x, Ξ_x), we see that each point of X_x is either collinear to y_* (and there are exactly 14 or 18 such points, respectively), or lies on a unique symp with y_*, and there are as many such symps as there are conics in Ω_{y_*}. Hence, if there are k points and ℓ conics in Ω_{y_*}, then the number of points of X_x is equal to $1 + 2k + 8\ell$. Since $k \in \{7, 9\}$, we see that both k and ℓ are independent of $y_* \in \xi_x$. Now we bound the number of points B of $X_x \setminus \xi_x$ collinear to at least one point of ξ_x. Let ϵ be the number of singular lines in Ω_{y_*} (and note that $\ell + \epsilon = \frac{1}{6}k(k-1) \in \{7, 12\}$). Then either 0 or exactly 4ϵ points in $y_*^\perp \setminus \xi_x$ are collinear to three points of ξ_*, and all other points of $y_*^\perp \setminus \xi_*$ are collinear to only $y_*$of ξ_*. Hence there at at least $b = 15(2k - 6 - 4\epsilon) + 5(4\epsilon)$ points in B. Now it is easy to see that there are only five possible values for (k, ℓ, ϵ), and we tabulate them, together with the bound $b \leq |B|$ and $|X_x|$.

(k, ℓ, ϵ)	$\|X_x\|$	b	$b + 15$
$(7, 7, 0)$	71	90	135
$(7, 6, 1)$	63	50	95
$(9, 12, 0)$	115	150	195
$(9, 11, 1)$	107	110	155
$(9, 10, 2)$	99	79	115

Since clearly $b + 15 \leq |B| + |\xi_x| \leq |X_x|$, this table shows a contradiction and concludes the proof of the proposition. □

Proposition 9.11 *There are no abstract Lagrangian varieties of type* 5 *and index* 2.

Proof Again, we consider the point-residual (X, Ξ) of (Y, Υ) at the point $a \in Y$ (see the Standing Hypotheses 6.4), which is a $(1, 3')$-AVV of type 5 and index 2 in (a subspace of) $\mathbb{P}^{17}(\mathbb{K})$. The non-existence of such an object is proved in Lemma 9.10. □

9.3 The case $w \geq 3, (w, d) \neq (4, 8)$

By Theorem 8.15 we only need to exclude the case $w = 3$.

Theorem 9.12 *An abstract Lagrangian variety of type d and index $w = 3$ does not exist.*

Proof Referring to the Standing Hypotheses 6.4, the point-residual $(Y_a, \Upsilon_a) = (X, \Xi)$ is a $(1, 3')$-AVV of type $d \geq 6$ and index 3 in (possibly a subspace of) $\mathbb{P}^{3d+2}(\mathbb{K})$. Pick $\xi \in \partial\Xi$ and let $x \in X(\xi)$. The point-residual (X_x, Ξ_x) of (X, Ξ) at x is a (1)-AVV of type $d - 2$ and index 2 in (a subspace of) $\mathbb{P}^{2d-1}(\mathbb{K})$. Now we claim that the point $y_* \in X_x$ corresponding to the line xy in X, for any $y \in x^\perp \cap \xi \setminus \{x\}$, satisfies $\dim T_{y_*}(X_x) \leq 2d - 4$.

Indeed, first suppose that each pair of members of Ξ containing x intersects in at least a line. Then the point-line geometry related to X_x is a strong (-1)-lacunary parapolar space of constant symplectic rank 3. By Lemma 5.5 it is $\mathsf{A}_{5,2}(\mathbb{K})$ or $\mathsf{A}_{4,2}(\mathbb{K})$. Item (2) of Corollary 4.4 leads to a contradiction in case it is $\mathsf{A}_{5,2}(\mathbb{K})$ (there is no strong parapolar space with constant symplectic rank 5 having hyperbolic symps and containing $\mathsf{A}_{5,2}(\mathbb{K})$ as a line-residual (a *line-residual* being a point-residual of the point-residual) and in case it is $\mathsf{A}_{4,2}(\mathbb{K})$, then item (3) of Corollary 4.4 leads to $\mathsf{E}_{6,1}(\mathbb{K})$, which has diameter 2, also a contradiction. Hence there exist $\zeta, \zeta' \in \Xi$ with $\zeta \cap \zeta' = \{x\}$. Also, by Lemma 6.6 applied in (X_y, Ξ_y), we find a $\zeta'' \in \Xi$ containing y but not containing x. We now have everything in place to apply Lemma 8.14 and conclude that $\dim T_{y_*}(X_x) \leq 2d-4$.

First suppose that $d=6$. Then $((X_x)_{y_*}, (\Xi_x)_{y_*})$ is a (1)-AVV of type 2 and index 1 in $\mathbb{P}^7(\mathbb{K})$. Then Lemma 5.2 implies that $((X_x)_{y_*}, (\Xi_x)_{y_*})$ is either $\mathscr{S}_{1,2}(\mathbb{K})$ or $\mathscr{S}_{1,3}(\mathbb{K})$. Items (3) and (2) of Corollary 4.4 yield $(Y, \mathscr{L}) \cong \mathsf{E}_{6,1}(\mathbb{K})$, contradicting Axiom (ALV1).

Next suppose $d \geq 7$. Set $d' = d-4$. Then $((X_x)_{y_*}, (\Xi_x)_{y_*})$ is a (1)-AVV of type $d' \geq 3$ and index 1 in (a subspace of) $\mathbb{P}^{2d'+3}(\mathbb{K})$. If $d \geq 8$, we argue as in the first paragraph of the proof of Proposition 9.7: by Proposition 9.6, $((X_x)_{y_*}, (\Xi_x)_{y_*})$ is 0-lacunary. By Lemma 5.6, $d'=2$, a contradiction.

We are left with $d=7$, hence $d'=3$. Then (X_x, Ξ_x) is a (1)-AVV of type 5 and index 2 in $\mathbb{P}^{13}(\mathbb{K})$, such that the tangent spaces at the points of the symp $X_x(\xi_*)$ corresponding to ξ have dimension at most 10. Lemma 9.10 yields a contradiction and hence concludes the proof. □

This concludes the proof of Theorem 3.1.

10 Constructions and verification of the axioms

In this section, we construct the exceptional variety $\mathscr{E}_7(\mathbb{K})$ as the projective closure of the image of an affine Veronese map. To prove that this construction works, we have to show that $\mathscr{E}_7(\mathbb{K})$ is the intersection of a number of quadrics. This has been proved before, see [33]. However, we need to be slightly more explicit. In doing so, we note that the set of 133 quadrics obtained in *loc. cit.* is not minimal, and we construct a set of 129 quadrics which is minimal. Our corollaries on the exceptional variety $\mathscr{E}_6(\mathbb{K})$ are also a slightly more explicit version of the results in [32].

10.1 Construction of $\mathscr{E}_7(\mathbb{K})$ as a quadratic Zariski closure

Let $\mathbb{K}$ be any field and let $\mathbb{A}$ be a *non-degenerate quadratic alternative algebra* over $\mathbb{K}$. This means that $\mathbb{A}$ is a vector space over $\mathbb{K}$ with an alternative multiplication law (extending scalar multiplication), that is, for $a, b \in \mathbb{A}$, we have $ab \in \mathbb{A}$ and $ab^2 = (ab)b$, $a^2b = a(ab)$. Moreover, every element $a \in \mathbb{A} \setminus \mathbb{K}$ satisfies the (necessarily unique) quadratic equation $x^2 - \mathsf{t}(a)x + \mathsf{n}(a) = 0$, with $\mathsf{t}(a) \in \mathbb{K}$ the *trace* of a and $\mathsf{n}(a) \in \mathbb{K}$ the *norm*. The element $\overline{a} := \mathsf{t}(a) - a = \mathsf{n}(a)a^{-1}$ satisfies the same quadratic equation, and is sometimes called the *conjugate* of a. Setting $\overline{k} = k$ for all $k \in \mathbb{K}$, the mapping $a \mapsto \overline{a}$ is an involutive anti-automorphism of $\mathbb{A}$, called the *standard involution*. Setting $\mathsf{n}(k) = k^2$ for all $k \in \mathbb{K}$, the mapping $\mathsf{n} : \mathbb{A} \to \mathbb{K} : a \mapsto \mathsf{n}(a)$ is a quadratic form, and $\mathsf{n}(a,b) := \mathsf{n}(a+b) - \mathsf{n}(a) - \mathsf{n}(b)$ denotes its linearisation. The algebra $\mathbb{A}$ is non-degenerate if the quadratic form n is non-degenerate, i.e., for each $a \in \mathbb{A}$ with $\mathsf{n}(a) = 0$ there is a $b \in \mathbb{A}$ such that $\mathsf{n}(a,b) \neq 0$.

In case char $\mathbb{K} \neq 2$, n is non-degenerate precisely if its linearisation is non-degenerate as a bilinear form, since $\mathsf{n}(a,a) = 2\mathsf{n}(a)$. It follows from the general theory [1] that n is either *anisotropic* (that is, $\mathsf{n}(a) = 0$ if and only if $a = 0$) or *split* (that is, its null set is a hyperbolic quadric); with this definition, the trivial algebra $\mathbb{A} = \mathbb{K}$ is anisotropic and not split. We first describe the split quadratic alternative algebras. The *split octonions* $\mathbb{O}'$ over $\mathbb{K}$ are defined as follows. An element $X \in \mathbb{O}'$ and its conjugate $\overline{X}$ are defined as

$$X = \begin{pmatrix} x_0 & \begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix} \\ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} & x_7 \end{pmatrix} \text{ and } \overline{X} = \begin{pmatrix} x_7 & \begin{pmatrix} -x_4 \\ -x_5 \\ -x_6 \end{pmatrix} \\ \begin{pmatrix} -x_1 \\ -x_2 \\ -x_3 \end{pmatrix} & x_0 \end{pmatrix}.$$

where $x_i, i = 0, \ldots, 7 \in \mathbb{K}$. The x_i, $i = 0, 1, \ldots, 7$ are called the *components* of X, and the *diagonal* components of X are x_0 and x_7. Abbreviating $x_{ij\ell} = (x_i, x_j, x_\ell)$, for $(i,j,\ell) \in \{(1,2,3),(4,5,6)\}$, and denoting by $v \cdot w$ and $v \times w$ the ordinary inner product and the usual vector product of vectors $v, w \in \mathbb{K}^3$, respectively, the multiplication is, with self-explaining notation, defined by (see [36], where we use $\begin{pmatrix} \alpha & a \\ -b & \beta \end{pmatrix}$ instead of $\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}$)

$$\begin{aligned} XY &= \begin{pmatrix} x_0 & x_{456} \\ x_{123} & x_7 \end{pmatrix} \begin{pmatrix} y_0 & y_{456} \\ y_{123} & y_7 \end{pmatrix} \\ &= \begin{pmatrix} x_0y_0 + x_{456} \cdot y_{123} & x_0y_{456} + y_7x_{456} + x_{123} \times y_{123} \\ y_0x_{123} + x_7y_{123} - x_{456} \times y_{456} & x_7y_7 + x_{123} \cdot y_{456} \end{pmatrix}. \end{aligned}$$

If we restrict to x_0, x_1, x_4, x_7 (setting $x_2 = x_3 = x_5 = x_6 = 0$), then we obtain the *split quaternions* $\mathbb{H}'$ over $\mathbb{K}$. Further restriction to x_0, x_7 (so $x_1 = x_4 = 0$) yields the *split quadratic extension* $\mathbb{L}'$ of $\mathbb{K}$ (this is the Cartesian product $\mathbb{K} \times \mathbb{K}$ with componentwise addition and multiplication). These three algebras are the only split non-degenerate quadratic alternative algebras over $\mathbb{K}$, up to isomorphism (cf. [1]).

Let V be a vector space of dimension $8 + 6\dim_K \mathbb{A}$ over $\mathbb{K}$, with either $\mathbb{A} = \{\vec{o}\}$ trivial, or $\mathbb{A} \in \{\mathbb{L}', \mathbb{H}', \mathbb{O}'\}$, or $\mathbb{A}$ a finite-dimensional quadratic alternative division algebra over $\mathbb{K}$. Below we conceive $x\overline{x}$ (where $x \mapsto \overline{x}$ denotes the standard involution) in formulae as elements of $\mathbb{K}$.

Definition 10.1 The *dual polar affine Veronese map* is defined as the map $\nu : \mathbb{K} \times \mathbb{K} \times \mathbb{K} \times \mathbb{A} \times \mathbb{A} \times \mathbb{A} \to V : (\ell_1, \ell_2, \ell_3, X_1, X_2, X_3) \mapsto$

$$\begin{gathered} (1, \ell_1, \ell_2, \ell_3, X_1, X_2, X_3, \\ X_1\overline{X}_1 - \ell_2\ell_3, X_2\overline{X}_2 - \ell_3\ell_1, X_3\overline{X}_3 - \ell_1\ell_2, \\ \ell_1\overline{X}_1 - X_2X_3, \ell_2\overline{X}_2 - X_3X_1, \ell_3\overline{X}_3 - X_1X_2, \\ \ell_1X_1\overline{X}_1 + \ell_2X_2\overline{X}_2 + \ell_3X_3\overline{X}_3 - \overline{X}_3(\overline{X}_2\overline{X}_1) - (X_1X_2)X_3 - \ell_1\ell_2\ell_3). \end{gathered}$$

If $\mathbb{A}$ is a division ring, it follows from [16] that its image $\mathscr{AV}(\mathbb{K},\mathbb{A})$ is contained in and spans $\mathbb{P}(V) \cong \mathbb{P}^{7+6d}(\mathbb{K})$, with $d = \dim_{\mathbb{K}} \mathbb{A}$. If $\mathbb{A} \in \{\{\vec{o}\}, \mathbb{L}', \mathbb{H}', \mathbb{O}'\}$, this is easy to prove:

Lemma 10.2 *If $\mathbb{A}$ is not a division ring, then the image of ν spans $\mathbb{P}(V)$.*

Proof First note that the elements of $\mathbb{A}$ with norm 0 or norm 1, respectively, generate $\mathbb{A}$ as a vector space over $\mathbb{K}$. We obtain the first $4 + 3\dim_{\mathbb{K}} \mathbb{A}$ basis vectors in the image of ν by considering the image of $(0, 0, 0, 0, 0, 0)$ and $(\ell_1, \ell_2, \ell_3, X_1, X_2, X_3)$, where we set every entry zero except $\ell_i = 1$ ($i \in \{1, 2, 3\}$) or X_i any element of $\mathbb{A} \setminus \{0\}$ with norm zero ($i \in \{1, 2, 3\}$). Then setting two of the ℓ_i's equal to 1 and all the rest zero gives us the next three basis vectors (combined with previously found basis vectors). Setting $\ell_i = 1$ and X_i varying over the norm 1 members of $\mathbb{A}$, $i \in \{1, 2, 3\}$, produces the next $3\dim_{\mathbb{K}} \mathbb{A}$ basis vectors, and finally the last basis vector is obtained from setting $\ell_1 = \ell_2 = \ell_3 = 1$ and $X_1 = X_2 = X_3 = 0$. □

In fact, $\mathscr{AV}(\mathbb{K},\mathbb{A})$ is contained in the complement of the hyperplane H_0 all points of which have 0 as their first coordinate.

In order to construct the varieties of the third row of the Freudenthal-Tits Magic Square we will need to add points to $\mathscr{AV}(\mathbb{K},\mathbb{A})$ in the hyperplane H_0. This is a kind of Zariski closure if $\mathbb{K}$ is algebraically closed, or at least infinite, and, more generally, a projective closure if $\mathbb{K}$ has at least three elements and the set contains affine lines. For our present purposes, we describe what could be called a *quadratic Zariski closure*.

Definition 10.3 Let S be a set of points of $\mathbb{P}^N(\mathbb{K})$, $2 \le N < \infty$. Then we say that S is *quadratically Zariski closed* if S is the intersection of a finite number of quadrics. The *quadratic Zariski closure* of a set T is the intersection of all quadratically Zariski closed sets that contain T, or, equivalently, the intersection of all quadrics that contain T. This is well defined since the class of quadrics is a finite dimensional vector space.

One of the aims of this section is to show the following theorem.

Theorem 10.4 *Suppose $|\mathbb{K}| > 2$. Then the quadratic Zariski closure $\mathscr{PV}(\mathbb{K},\mathbb{A})$ of $\mathscr{AV}(\mathbb{K},\mathbb{A})$ is isomorphic to*

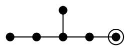
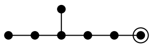
1. $\mathscr{S}_{1,1,1}(\mathbb{K})$*, if $\mathbb{A} = \{\vec{o}\}$ is trivial,*
2. $\mathscr{V}(\mathbb{K},\mathbb{A})$*, if $\mathbb{A}$ is a division ring,*
3. $\mathscr{G}_{6,3}(\mathbb{K})$*, if $\mathbb{A} \cong \mathbb{L}'$,*
4. $\mathscr{HS}_6(\mathbb{K})$*, if $\mathbb{A} \cong \mathbb{H}'$,*
5. $\mathscr{E}_7(\mathbb{K})$*, if $\mathbb{A} \cong \mathbb{O}'$.*

Remark 10.5 There are various ways to deal with the remaining case $|\mathbb{K}| = 2$. One way to incorporate it, is to consider $\mathscr{AV}(\mathbb{K},\mathbb{A})$ over a field extension of $\mathbb{F}_2$, then take its quadratic Zariski closure, and restrict the field again. The only care to be taken here is that, if $\mathbb{A}$ is the field of four elements, then the field extension should not contain $\mathbb{A}$ as a subfield.

In order to prove Theorem 10.4 we distinguish between the ovoidal ($\mathbb{A}$ division) and the hyperbolic cases (the other cases). In the ovoidal case, Theorem 10.4 follows from Lemma 3.5 of [16]. In the hyperbolic cases, the case $\mathbb{A} = \{\vec{o}\}$ is easy. The other cases will follow from the case $\mathbb{A} \cong \mathbb{O}'$. So we begin with the latter. Therefore, we introduce a second construction of $\mathscr{E}_7(\mathbb{K})$, not relying on the quadratic Zariski closure of $\mathscr{A}\mathscr{V}(\mathbb{K}, \mathbb{O}')$.

10.2 A second construction of $\mathscr{E}_7(\mathbb{K})$

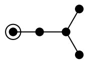
10.2.1 The Schläfli and the Gosset graph

Below we present combinatorial constructions of the Schläfli graph and Gosset graph, and also give a construction of the Gosset graph in terms of two copies of the Schläfli graph and two additional points. We explore some properties and label some of them (G1) up to (G4) for ease of further reference. We refer the reader to [2] (pages 103, 104) and mention that these graphs are the 1-skeleta of the 2_{21} polytope  and the 3_{21} polytope , respectively. Most properties we mention are direct consequences of the definition, or are standard properties of distance regular graphs. A good reference is the book [2].

The Schläfli graph. The first graph is the *Schläfli graph* $\Gamma_1 = (V_1, E_1)$, whose vertices are the points of the unique generalized quadrangle $\mathsf{GQ}(2,4)$ of order $(2,4)$, adjacent when the points are not collinear. Another, equivalent but more combinatorial description goes as follows. The 27 vertices are the pairs from the set $\{1,2,3,4,5,6\}$, together with the elements $1', 2', \ldots, 6', 1'', 2'', \ldots, 6''$. Pairs are adjacent if they intersect in precisely one element; a pair $\{i,j\}$ is adjacent to an element k' or k'' if $k \notin \{i,j\}$, two elements i' and j', or i'' and j'' are adjacent as soon as $i \neq j$ and finally, i' is adjacent to j'' if $i = j$.

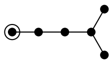
The Gosset graph. The second graph is the *Gosset graph* $\Gamma_2 = (V_2, E_2)$. Traditionally, this graph is constructed as follows. The 56 vertices are the pairs from the respective 8-sets $\{1,2,\ldots,8\}$ and $\{1',2',\ldots,8'\}$. Two pairs from the same set are adjacent if they intersect in precisely one element; two pairs $\{a,b\}$ and $\{c',d'\}$ from different sets are adjacent if $\{a,b\}$ and $\{c,d\}$ are disjoint. Consider the vertex $w = \{7',8'\}$. Identifying pairs $\{i',7'\}$ where $i' \neq 8'$ with i' and pairs $\{j',8'\}$ where $j' \neq 7'$ with j'', we see that the local graph $\Gamma_2(\{7',8'\})$ is isomorphic to the Schläfli graph Γ_1 (using the same notation with dashes and double dashes as in the previous paragraph). It is easy to see that Γ_2 is distance regular and antipodal (that is, being at maximal distance from each other is an equivalence relation among the vertices) with antipodal classes (the corresponding equivalence classes) of size 2, and has diameter 3. The unique vertex of Γ_2 at distance 3 from $w = \{7',8'\}$ is $w' = \{7,8\}$.

The Gosset graph in terms of the Schläfli graph. Let $w = \{7',8'\}$ and $w' = \{7,8\}$, as above. Let v be any vertex adjacent to w and let u' be any vertex adjacent to w'. Let v' be the antipode of v and u the antipode of u' (we will usually call antipodes *opposite vertices*) and note that u is adjacent to w (and v' to w'). Then, as Γ_2 is distance regular, has diameter 3 and is antipodal with antipodal classes of size 2, we have that $\delta(u',v) = 1$

if and only if $\delta(u, v) = 2$. Hence $\Gamma_2(u') \cap \Gamma_2(w)$ is precisely the set of vertices of $\Gamma_2(w)$ at distance 2 from u. The graph induced on $\Gamma_2(u') \cap \Gamma_2(w)$ is a cross-polytope of size 10 (the complement of five disjoint edges), also known as a *pentacross* or 5-orthoplex, with corresponding Dynkin diagram .

Identifying $\Gamma_2(w)$ with $\mathsf{GQ}(2, 4)$ as above, a pentacross is induced by the set of points collinear to but different from some other fixed point, so there are 27 such cross-polytopes in $\Gamma_2(w)$ (one for every vertex).

This implies the following description of Γ_2 in terms of Γ_1. Let $\Gamma_1' = (V_1', E_1')$ and $\Gamma_1'' = (V_1'', E_1'')$ be two disjoint copies of Γ_1 and consider two symbols ∞' and ∞''. Then the vertices of Γ_2 are the vertices of Γ_1' and Γ_1'' together with ∞' and ∞''. The vertex ∞' (resp. ∞'') is adjacent to all vertices of Γ_1' (resp. Γ_1''). Adjacency inside Γ_1' and Γ_1'' is as in Γ_1, and a vertex of Γ_1' is adjacent to the vertex of Γ_1'' if the corresponding vertices of Γ_1 are at distance 2 from one another.

Special substructures. The Gosset graph Γ_2 contains 126 cross-polytopes with 12 vertices and corresponding diagram , and no cross-polytope with 14 vertices. In terms of the first description, 56 of these are determined by an ordered pair (i, j) with $i, j \in \{1, 2, 3, 4, 5, 6, 7, 8\}$, $i \neq j$, and induced on the vertices $\{i, k\}$ and $\{j', k'\}$, $k \notin \{i, j\}$, whereas the other 70 are determined by a 4-set $\{i, j, k, \ell\} \subseteq \{1, 2, 3, 4, 5, 6, 7, 8\}$ and are induced on the vertices $\{s, t\} \subseteq \{i, j, k, \ell\}$, $s \neq t$, and $\{s', t'\} \subseteq \{1', 2', 3', 4', 5', 6', 7', 8'\} \setminus \{i', j', k', \ell'\}$. In terms of the second description, 54 are obtained by taking a pentacross in either Γ_1' (resp. Γ_1'') and adjoining ∞' (resp. ∞'') and the unique vertex of Γ_1'' (resp. Γ_1') adjacent to each point of P. The other 72 are obtained by considering a maximum clique C' in Γ_1'; then there is a unique maximum clique C'' of Γ_1'' such that $C' \cup C''$ is a cross-polytope of size 12 in Γ_2. Indeed, in terms of $\mathsf{GQ}(2, 4)$, a maximum clique of Γ_1 is induced by the set $\{p\} \cup (q^\perp \setminus p^\perp)$, for two non-collinear points p, q; so if p and q correspond to $p', q' \in V_1'$, respectively, and to $p'', q'' \in V_1''$, respectively, then if $C' = \{p'\} \cup (q'^\perp \setminus p'^\perp)$, we have $C'' = \{q''\} \cup (p''^\perp \setminus q''^\perp)$. A cross-polytope with 12 vertices in Γ_2 will be referred to as a *hexacross*, which alongside 6-orthoplex is one of its standard names. The following properties are immediate:

(G1) *The set of twelve vertices opposite the vertices of a given hexacross induces a second hexacross, called* the opposite hexacross. (So there are 63 pairs of opposite hexacrosses.)

(G2) *Every hexacross Q is* determined by *any two non-adjacent vertices $v, w \in Q$ in the sense that $Q = \{v, w\} \cup (\Gamma_2(v) \cap \Gamma_2(w))$.*

A *spread* of the Schläfli graph Γ_1 is a set of disjoint (maximal) cocliques of size 3 partitioning the vertex set. A spread of Γ_1 induces a line spread of $\mathsf{GQ}(2, 4)$ in the classical sense. There are two isomorphism classes of such spreads, but for only one of them every member has the following property when viewed in Γ_1: given two arbitrary cocliques C_1, C_2 of the spread, the set C_3 of vertices not contained in $C_1 \cup C_2$ but contained in some coclique sharing exactly two vertices with $C_1 \cup C_2$ has size 3 and is a coclique belonging to the spread. In $\mathsf{GQ}(2, 4)$, the cocliques C_1, C_2, C_3 are three disjoint lines of a subquadrangle of order $(2, 1)$. A spread with the just given property will be called a *Hermitian spread.*

A set of three disjoint lines of a subquadrangle of order $(2,1)$ in $\mathsf{GQ}(2,4)$ will be called a *regulus*. Since a pentacross of Γ_1 corresponds to the set of points of $\mathsf{GQ}(2,4)$ collinear to but different from a certain fixed point, we obtain

(G3) *each spread of* Γ_1 *has a unique member containing two vertices of any pentacross.*

We now fix a Hermitian spread $\mathscr{S}$ of Γ_1, and denote by $\mathscr{S}'$ and $\mathscr{S}''$ the copies of $\mathscr{S}$ in Γ_1' and Γ_1'', respectively. Using $\mathscr{S}$, we define a set $\mathscr{C}$ of 72 cliques of size 3 of Γ_1 covering each edge precisely once as follows. Let $\{a,b\}$ be an edge of Γ_1. There are unique and distinct cocliques $C_a, C_b \in \mathscr{S}$ containing a, b, respectively. As $\mathscr{S}$ is Hermitian, there is a unique coclique $C \in \mathscr{S}$ such that $\{C_a, C_b, C\}$ is a regulus. In $\mathsf{GQ}(2,4)$, there is a unique point c on the line C collinear to neither a nor b. The triple $\{a,b,c\}$ is a clique of Γ_1 that by definition belongs to $\mathscr{C}$. It is easy to see that $\{a,b,c\}$ is independent of the pair $\{a,b\}$ we started with. Also, Proposition 3.3 of [31] implies that

(G4) *every 6-clique of* Γ_1 *contains precisely two members of* $\mathscr{C}$*, which are moreover disjoint.*

Let $\mathscr{C}'$ and $\mathscr{C}''$ denote copies of $\mathscr{C}$ in Γ_1' and Γ_1'', respectively.

10.2.2 Some quadratic forms

Let V be a 56-dimensional vector space over $\mathbb{K}$ the basis vectors of which are labeled by the vertices of the Gosset graph Γ_2. We define for each hexacross of Γ_2, and for each pair of opposite hexacrosses, a quadratic form, determined up to a non-zero scalar. Later on, we will use precisely these quadratic forms to describe $\mathscr{E}_7(\mathbb{K})$.

We use coordinates relative to the standard basis of V, denoting the variable related to the basis vector corresponding to the vertex v of Γ_2 by X_v. The set of all quadratic forms will (only) depend on Γ_2, the vertex ∞' of Γ_2 and the spread $\mathscr{S}'$ of V_1'. We will refer to the first two classes of quadratic forms below as the *short quadratic forms belonging to* $(\Gamma_2, \infty', \mathscr{S}')$, and to those of the last two classes as the *long quadratic forms belonging to* $(\Gamma_2, \infty', \mathscr{S}')$. Hence there are four classes in total.

- *Let Q be a hexacross defined by a vertex $v'' \in \Gamma_1''$, that is, $Q = (\Gamma_2(v'') \cap V_1') \cup \{\infty', v''\}$.* By the above Property (G3), there are exactly two vertices i, j of $\Gamma_2(v'') \cap V_1'$ belonging to a common member of $\mathscr{S}'$. Let P be the partition of $(\Gamma_2(v'') \cap V_1') \setminus \{i,j\}$ in pairs of non-adjacent vertices. We define the quadratic form

$$\beta_Q : V \to \mathbb{K} : (X_v)_{v \in V_2} \mapsto -X_i X_j + X_{\infty'} X_{v''} + \sum_{\{k,\ell\} \in P} X_k X_\ell.$$

  Similarly, one defines 27 quadratic forms using a hexacross defined by a vertex of Γ_1' and ∞''.

- *Let Q be a hexacross consisting of the union of a 6-clique W' of Γ_1' and a 6-clique W'' of Γ_1''.*
  By Property (G4), there are unique 3-cliques $C_1, C_2 \in \mathscr{C}$ with $C_1 \cup C_2 = W'$. For

each $w' \in W'$, let $w'' \in W''$ denote the unique vertex of W'' not adjacent to w'. Then we define the quadratic form

$$\beta_Q : V \to \mathbb{K} : (X_v)_{v\in V_2} \mapsto \sum_{w'\in C_1} X_{w'}X_{w''} - \sum_{w'\in C_2} X_{w'}X_{w''}.$$

- *Let (Q', Q'') be a pair of opposite hexacrosses with $\infty' \in Q'$ and $\infty'' \in Q''$.* Then Q' and Q'' have a unique vertex v' and v'' in Γ_1'' and Γ_1', respectively. For each $w' \in Q'$, let $w'' \in Q''$ denote the unique vertex of Γ_2 opposite w'. Then we define the quadratic form

$$\beta_{Q',Q''} : V \to \mathbb{K} : (X_v)_{v\in V_2} \mapsto -X_{\infty'}X_{\infty''} - X_{v'}X_{v''} + \sum_{w'\in Q'\setminus\{\infty',v'\}} X_{w'}X_{w''}.$$

- *Let (Q', Q'') be a pair of opposite hexacrosses with $\infty' \notin Q'$ and $\infty'' \notin Q''$.* Set $W' = Q' \cap V_1'$ and $W'' = Q'' \cap V_1'$. For each $w \in W' \cup W''$, let w_* be the vertex of Γ_2 opposite w. Then we define the quadratic form

$$\beta_{Q',Q''} : V \to \mathbb{K} : (X_v)_{v\in V_2} \mapsto \sum_{w'\in W'} X_{w'}X_{w'_*} - \sum_{w''\in W''} X_{w''}X_{w''_*}.$$

We now have the following theorem, which we prove in the following section.

Theorem 10.6 *The variety $\mathscr{E}_7(\mathbb{K})$ is isomorphic to the intersection of the respective null sets in $\mathbb{P}(V)$ of the 126 quadratic forms β_Q, for Q ranging over the set of hexacrosses of Γ_2, and the 63 quadratic forms $\beta_{Q',Q''}$, with $\{Q', Q''\}$ ranging over the set of pairs of opposite hexacrosses of Γ_2.*

The previous theorem can be improved in that we do not need all 126+63=189 quadratic forms, but only 126+3=129, see Corollary 10.32.

10.3 Proof that the second construction works

We show Theorem 10.6 in a sequence of lemmas. For the rest of this subsection we denote by $\mathfrak{E}$ the intersection of the respective null sets in V or in $\mathbb{P}(V)$ of the 126 quadratic forms β_Q, for Q ranging over the set of hexacrosses of Γ_2, and the 63 quadratic forms $\beta_{Q',Q''}$, with $\{Q', Q''\}$ ranging over all pairs of opposite hexacrosses of Γ_2. Recall that the standard basis of V is $(e_v)_{v\in V_2}$.

We say that two points of $\mathfrak{E}$ are *collinear* if the line joining them entirely belongs to $\mathfrak{E}$. Recall that E_2 is the set of edges of Γ_2.

Lemma 10.7 *For each $v \in V_2$, the point $p_v := \mathbb{K}e_v$ belongs to $\mathfrak{E}$. For each pair of vertices $v, w \in V_2$, the line $\langle p_v, p_w\rangle$ entirely belongs to $\mathfrak{E}$ if and only if $\{v, w\} \in E_2$. Also, if a point p with coordinates $(x_v)_{v\in V_2}$ belongs to $\mathfrak{E}$ and is collinear to p_w, for some $w \in V_2$, then $x_v = 0$ for all v not adjacent to w in Γ_2.*

Proof The first assertion follows from the fact that no quadratic form β_Q or $\beta_{Q,Q'}$ contains the square of a variable. The second assertion follows from the fact that v and w are non-adjacent vertices of Γ_2 if and only if X_vX_w occurs in at least one of the said quadratic forms without other occurrences of X_v or X_w in it. The same observation shows the third assertion. □

Lemma 10.8 *For each $\varphi \in \mathrm{Aut}(\Gamma_2)$ there exist $\epsilon_v \in \{+1, -1\}$, $v \in V_2$, such that the linear transformation Φ of V defined by $e_v \mapsto \epsilon_v e_{\varphi(v)}$ preserves $\mathfrak{E}$.*

Proof First suppose that φ fixes ∞' (and hence also ∞''). If φ stabilizes the spread $\mathscr{S}'$, then clearly, there is nothing to prove (choose all ϵ_v equal to 1). If φ does not stabilize $\mathscr{S}'$, then it suffices to consider the case where $\mathscr{S}'^{\varphi}$ has three members in common with $\mathscr{S}'$. Indeed, the graph with vertices the Hermitian spreads of $\mathsf{GQ}(2,4)$, adjacent when intersecting in three lines (so, a regulus), is the collinearity graph of the symplectic generalized quadrangle of order 3 (this can be deduced from the description of maximal subgroups of $U_4(2) \cong S_4(3)$ on page 26 of the Atlas of Finite Simple Groups [11]), and is hence connected. Now, possibly by composing with an automorphism of Γ_2 preserving ∞' and preserving the spread $\mathscr{S}'$, we may assume that φ fixes all points of the members in $\mathscr{S}' \cap \mathscr{S}'^{\varphi}$. Now we define $\epsilon_v = -1$ if v is adjacent to ∞' and v belongs to a member of $\mathscr{S}' \cap \mathscr{S}'^{\varphi}$, or if v is adjacent to ∞'' and v belongs to a member of $\mathscr{S}'' \cap \mathscr{S}''^{\varphi}$. In all other cases $\epsilon_v = 1$. One verifies that the corresponding linear transformation Φ preserves all quadratic forms β_Q and $\beta_{Q',Q''}$, up to a constant in $\{1, -1\}$.

Now suppose that φ does not fix ∞'. By connectivity, we may without loss of generality assume that $w' := \infty'^{\varphi} \in V_1'$. Set $w'' := \infty''^{\varphi}$ and note that w'' is adjacent to ∞'' and opposite w'. Composing with an appropriate automorphism of Γ_2 fixing ∞', we may assume that φ interchanges ∞' with w' and pointwise fixes $(\Gamma_2(\infty') \cap \Gamma_2(w')) \cup (\Gamma_2(\infty'') \cap \Gamma_2(w''))$. It maps a vertex u in the pentacross $\Gamma_2(\infty') \setminus (\Gamma_2(w') \cup \{w'\})$ to the opposite u^* of the unique vertex of $\Gamma_2(\infty') \setminus (\Gamma_2(w') \cup \{w'\})$ not adjacent to u. The vertex u^* is also the unique vertex of the hexacross containing w' and u not adjacent to ∞'. Also, u^* is mapped to u. We define $\epsilon_v = -1$ if either $v \in \{w', \infty''\}$, or $v \in \Gamma_2(\infty') \setminus \Gamma_2(w')$ and v does not belong to the same spread element of $\mathscr{S}'$ that contains w', or if $v \in V_2'' \setminus \{w''\}$ and v belongs to the same spread element of $\mathscr{S}''$ as w''. One verifies that the corresponding Φ preserves all quadratic forms β_Q and $\beta_{Q',Q''}$ up to a constant in $\{1, -1\}$. The lemma is proved. □

Our next aim is to show that each pair of points of $\mathfrak{E}$ is equivalent to a pair of points from the standard basis, see Proposition 10.17. Therefore we introduce linear mappings $\sigma_Q(a)$ of V, with $a \in \mathbb{K}$, and Q a hexacross of Γ_2. In fact, these correspond to certain central elations, also called central collineations, or long root elations, of the building $\mathsf{E}_7(\mathbb{K})$, see [4]. We need the following observation, the verification of which we leave to the reader.

Lemma 10.9 *Let Q_1 be a hexacross containing 6-cliques of Γ_1' and Γ_1''. Let Q_2 be the opposite hexacross. Then*

(*i*) *For each vertex $v_1 \in Q_1$, the opposite vertex $v_2 \in Q_2$ is adjacent to a unique vertex $v_1^* \in Q_1$, namely to the unique vertex of Q_1 non-adjacent to v_1.*

(*ii*) *The mapping* $v_1 \mapsto v_1^*$ *defined in* (*i*) *permutes the four members of* $\mathscr{C}'$ *and* $\mathscr{C}''$ *contained in* Q_1 *(cf.* Property (G4)*).*

We are ready to define the central elations. By Lemma 10.8, it suffices to do this for hexacrosses not containing ∞' or ∞''.

Definition 10.10 Let W_1' be a 6-clique of Γ_1' which, together with the 6-clique $W_1'' \subseteq V_1''$, forms a hexacross denoted Q_1. Let $W_2'' \subseteq V_1''$ be the set of vertices of Γ_2 opposite the vertices of W_1', and let $W_2' \subseteq V_1'$ be the set of vertices of Γ_2 opposite the vertices of W_1'', and denote $Q_2 = W_2' \cup W_2''$. By Property (G1), Q_2 is a hexacross. Let $W_1' = C_1' \cup D_1'$ and $W_1'' = C_1'' \cup D_1''$, with $C_1', D_1' \in \mathscr{C}'$ and $C_1'', D_1'' \in \mathscr{C}''$. According to Lemma 10.9(*ii*) we may assume that the vertex opposite an arbitrary vertex of C_1' is adjacent to a vertex of C_1''.

We define the linear mapping $\sigma_{Q_1}(a)$ of V, with $a \in \mathbb{K}$ arbitrary, by its action on the basis vectors as follows. For $v \in Q_1$, we denote by v^o its opposite in Γ_2 (which belongs to Q_2), and by v^* the unique vertex of Q_1 adjacent to v^o (using (i) of Lemma 10.9).

$$\sigma_{Q_1}(a) : V \to V : \begin{cases} e_{v^o} & \mapsto & e_{v^o} + a e_{v^*}, & \text{for } v \in C_1' \cup D_1'' \\ e_{v^o} & \mapsto & e_{v^o} - a e_{v^*}, & \text{for } v \in D_1' \cup C_1'' \\ e_v & \mapsto & e_v & \text{for all } v \in V_2 \setminus Q_2. \end{cases}$$

In terms of the coordinates, $\sigma_{Q_1}(a)$ transforms $(X_v)_{v \in V_2}$ into $(X_v')_{v \in V_2}$ as follows

$$\begin{cases} X_{v^*}' & = & X_{v^*} - a X_{v^o} & \text{for } v \in C_1' \cup D_1'' \\ X_{v^*}' & = & X_{v^*} + a X_{v^o} & \text{for } v \in D_1' \cup C_1'' \\ X_v' & = & X_v & \text{for all } v \in V_2 \setminus Q_2. \end{cases}$$

Now let Q be a hexacross containing ∞'. We fix a hexacross Q_1 not containing ∞' and a linear map Φ obtained as in Lemma 10.8 from an automorphism of Γ_2 mapping Q_1 onto Q (there are two choices, say Φ and Φ', and their product is minus the identity). Then we define $\sigma_Q(a)$ as the conjugate $\sigma_{Q_1}(a)^{\Phi}$. Choosing Φ' instead of Φ yields $\sigma_{Q_1}(a)^{\Phi'} = \sigma_{Q_1}(-a)^{\Phi}$. Conjugation is $\Phi \sigma_{Q_1}(a) \Phi^{-1}$ or $\Phi^{-1} \sigma_{Q_1}(a) \Phi$, which will not bother us because we will only use these maps for transitivity properties (and these are independent of the choice made). Likewise, a different choice of Q_1 produces the same group.

Lemma 10.11 *Let* Q *be a hexacross of* Γ_2*,* Q' *its opposite and let* w *be a vertex of* Q*. Then, for all* $a \in \mathbb{K}$*,* $\sigma_Q(a)$ *fixes* $\pm e_v$ *for every* $v \in V_2 \setminus Q'$*, in particular, for each* $v \in \Gamma_2(w) \setminus \{w_*\}$*, with* w_* *the unique vertex in* Q' *collinear to* w*.*

Proof This follows immediately from the definition of $\sigma_Q(a)$. □

Lemma 10.12 *Let* Q_1 *be any hexacross disjoint from* $\{\infty', \infty''\}$*. Then, for each* $a \in \mathbb{K}$*, the mapping* $\sigma_{Q_1}(a)$ *maps each quadratic form* β_Q *and* $\beta_{Q,Q'}$*, to a linear combination of such quadratic forms. Also,* $\sigma_{Q_1}(a)$ *maps* $\mathfrak{E}$ *bijectively to itself.*

Proof We have to calculate the image of each quadratic form β_Q and $\beta_{Q,Q'}$. This is an elementary exercise, which we shall perform in the most elaborate case (most quadratic forms remain the same), namely the case $Q = Q_1$. We use the notation of Definition 10.10. For each vertex $v \in W_1'$, the vertex v^o is opposite v; the latter is adjacent to v^*, which belongs to C_1''. Let $v_* = (v^*)^o$. A generic term of β_{Q_1} is, up to ±1, given by $X_v X_{v^*}$. The latter is transformed by $\sigma_{Q_1}(a)$ to

$$(X_v \pm aX_{v_*})(X_{v^*} \mp aX_{v^o}) = X_v X_{v^*} \mp a(X_v X_{v^o} - X_{v^*}X_{v_*}) - a^2 X_{v_*}X_{v^o}.$$

Now $X_{v_*}X_{v^o}$ is a generic term of β_{Q_2}, and $V_v X_{v^o} - X_{v^*}X_{v_*}$ is a generic pair of terms of β_{Q_1,Q_2}. It then follows from Lemma 10.9(ii) (to get the signs in the image of β_{Q_1} right) that the image of β_{Q_1} under $\sigma_{Q_1}(a)$ is equal to $\beta_{Q_1} \pm a\beta_{Q_1,Q_2} \pm a^2\beta_{Q_2}$ (where the two sign symbols are not coupled).

Another quadratic form which is not mapped onto itself is β_Q for Q the hexacross determined by ∞' and, using the notation of Definition 10.10, the vertex $v^* \in W_1''$, with $v \in W_1'$ arbitrary (cf. Property (G2)). One calculates that $\sigma_{Q_1}(a)$ maps β_Q to $\beta_Q \pm a\beta_{Q'}$, with Q' the hexacross determined by ∞' and v^o (and the sign depends on the inclusion of v in either C_1' or D_1').

The other cases are left to the reader. Since $\sigma_{Q_1}(-a)$ is obviously the inverse of $\sigma_{Q_1}(a)$, both map $\mathfrak{E}$ bijectively to itself. The second assertion follows and the lemma is proved. □

We also note the following.

Lemma 10.13 *For each hexacross Q and each point $p \in \mathfrak{E}$, the set $\{p^{\sigma_Q(a)} \mid a \in \mathbb{K}\}$ is an affine line completely contained in $\mathfrak{E}$.*

Proof This follows from the fact that, in the definition of $\sigma_Q(a)$, the parameter a appears linearly (so that $\{p^{\sigma_Q(a)} \mid a \in \mathbb{K}\}$ is an affine line), and from Lemma 10.12 (so that $\{p^{\sigma_Q(a)} \mid a \in \mathbb{K}\} \subseteq \mathfrak{E}$). □

Lemma 10.14 *A vector $p \in V$ with coordinates $(x_v)_{v \in V_2}$, where for some $w \in V_2$, we have $x_w \neq 0$ and $x_u = 0$ for all u adjacent to w, belongs to $\mathfrak{E}$ if and only if $p \in e_w\mathbb{K}$.*

Proof By Lemma 10.8 we may assume $w = \infty'$. Then it is easy to see that the coordinates of p belong to the null set of β_Q, with $\infty' \in Q$ and $v'' \in Q \cap V_1''$, if and only if $x_{v''} = 0$. Now considering the quadratic form $\beta_{Q,Q'}$, with $\infty' \in Q$ and Q' opposite Q, we see that $x_{\infty''} = 0$. □

Definition 10.15 Define the group $G \leq \mathsf{GL}(V)$ as the group generated by all $\sigma_Q(a)$, Q a hexacross and $a \in \mathbb{K}$, and all Φ obtained from Lemma 10.8. Note that G acts as an automorphism group on $\mathfrak{E}$, by Lemma 10.12.

Lemma 10.16 *Let $p \in \mathfrak{E}$ have coordinates $(x_v)_{v \in V_2}$, where for some $w \in V_2$, we have $x_w \neq 0$. Then there exists $g \in G$ such that $g(p) \in e_w\mathbb{K}$ and $g(e_{w^o}) = e_{w^o}$, with $w^o \in V_2$ opposite w.*

Proof Let $v \in V_2$ be any vertex adjacent to w and let $w^o \in V_2$ be opposite w. Then w^o and v are at distance 2 from one another and hence define a unique hexacross Q. One of the maps $\sigma_Q(\pm x_v/x_w)$ maps p to a vector with zero v-coordinate, while all other u-coordinates, with $u \in V_2$ equal or adjacent to w, stay the same by Lemma 10.11. This map also fixes e_{w^o}. Doing this for all vertices v adjacent to w produces an element $g \in G$ and a vector $q = g(p)$ in $\mathfrak{E}$ with non-zero w-coordinate and all v-coordinates zero, for v adjacent to w. Moreover $g(e_{w^o}) = e_{w^o}$. By Lemma 10.14, $q \in e_w\mathbb{K}$ and the lemma is proved. $\square$

The following proposition basically says that G acts distance-transitively on $\mathfrak{E}$.

Proposition 10.17 *For every pair of points $p, q \in \mathfrak{E}$ there exists $g \in G$ such that both $g(p)$ and $g(q)$ are multiples of standard basis vectors.*

Proof By Lemma 10.16 we already may assume that $p = e_w\mathbb{K}$, for some $w \in V_2$. Set $q = (x_v)_{v \in V_2}$. We consider three cases.

- *Assume that $x_{w^o} \neq 0$, where w^o is opposite w in Γ_2.*
  This case follows immediately from Lemma 10.16 with the roles of w and w^o interchanged.

- *Assume that $x_{w^o} = 0$, but $x_v \neq 0$ for some vertex v at distance 2 from w.*
  Let $u \in \Gamma_2(v)$ be arbitrary, but distinct from w^o. Let $v^o \in V_2$ be opposite v and denote by Q_v the hexacross determined by u and v^o. Then $w^o \notin Q_v$ since w^o is not adjacent to v^o (as this would imply $u = w^o$, contrary to our assumptions). This now implies that $\sigma_{Q_v}(\pm x_u/x_v)$ fixes w, and, as before in the proof of Lemma 10.16, for one choice of the sign, maps q to a point with zero u-coordinate. Varying u, and using Lemma 10.11, we thus produce a member $g \in G$ fixing p and mapping q to a point with zero u-coordinate, for all $u \in \Gamma_2(v)$, but non-zero v-coordinate. Then $g(q) \in e_v\mathbb{K}$ by Lemma 10.14.

- *Assume that $x_v = 0$, for all $v \in V_2$ not equal or adjacent to w.*
  In this case, there exists $v \in V_2$ adjacent to w for which $x_v \neq 0$ (otherwise $p = q$ and the assertion is trivial). Let v^o and w^o be as above and take any $u \in \Gamma_2(v) \cap \Gamma_2(w)$. Then, as in the previous case, the unique hexacross determined by u and v^o does not contain w^o. The rest of the proof applies verbatim.

The proof of the proposition is complete. $\square$

Corollary 10.18 *Let $w \in V_2$, denote by w^0 its opposite, and suppose $q \in \mathfrak{E}$ has coordinates $(x_v)_{v \in V_2}$. Then q is collinear to $e_w\mathbb{K}$ if and only if $x_v = 0$ for all $v \in V_2 \setminus (\Gamma_2(w) \cup \{w\})$; q is at distance 2 from $e_w\mathbb{K}$ if and only if $x_{w^o} = 0$ and $x_v \neq 0$ for some $v \in V_2 \setminus (\Gamma_2(w) \cup \{w\})$; and finally q is at distance 3 from $e_w\mathbb{K}$ if and only if $x_{w^o} \neq 0$.*

Proof We use the case distinction of the proof of Proposition 10.17: In all three cases, we considered a vertex $v \in V_2$ such that $x_v \neq 0$ and obtained an automorphism $g \in G$ such that $g(q) \in e_v\mathbb{K}$, and hence p and q are at the same distance from each other as v and w, which is distance 3, 2 or 1, respectively. Since this exhausts all cases (but the trivial one $p = q$), the lemma follows. □

Now let $\mathfrak{L}$ be the set of projective lines contained in $\mathfrak{E}$ (viewed as a set of points of $\mathbb{P}(V)$).

Proposition 10.19 *The point-line geometry $\Delta = (\mathfrak{E}, \mathfrak{L})$ is isomorphic to the parapolar space $\mathsf{E}_{7,7}(\mathbb{K})$.*

Proof We first show that Δ is a parapolar space with all symps isomorphic to $\mathsf{D}_{6,1}(\mathbb{K})$.

Note that Corollary 10.18 implies that the distance between $e_v\mathbb{K}$ and $e_w\mathbb{K}$ in Δ is the same as the distance between v and w in Γ_2.

Proposition 10.17 now ensures that Δ has diameter 3, hence is connected. Now consider two points $p, q \in \mathfrak{E}$ at distance 2. By Proposition 10.17, we may assume that $p = e_v\mathbb{K}$ and $q = e_w\mathbb{K}$, for two vertices v, w of Γ_2 at distance 2. Let Q be the unique hexacross determined by v and w. Let U be the subspace of $\mathbb{P}(V)$ generated by all e_u, $u \in Q$. Let Ω be the null set of the quadratic form β_Q restricted to U. Then Ω is a hyperbolic polar space isomorphic to $\mathsf{D}_{6,1}(\mathbb{K})$ containing p and q as non-collinear points. Hence Ω is contained in the convex subspace closure $S(p, q)$ of p and q. Note that $\Omega \subseteq \mathfrak{E}$ since every point of U is in the null set of every quadratic form β_{Q_*}, with Q_* a hexacross distinct from Q, and every quadratic form β_{Q_*,Q'_*}, now for every pair of opposite hexacrosses Q_*, Q'_*. If we can show that $p^\perp \cap q^\perp \subseteq \Omega$, then, since p and q can be seen as arbitrary non-collinear points of Ω, it follows that $\Omega = S(p, q)$. So suppose $r \in p^\perp \cap q^\perp$. Then by the definition of a hexacross and Corollary 10.18, we conclude $r \in U$ and hence $r \in \Omega$. So we have shown that $\Omega = S(p, q)$.

Lemma 10.17 implies that every member of $\mathfrak{L}$ is contained in the convex subspace closure of two points at distance 2. Since clearly no such subspace contains all points of $\mathfrak{E}$, we have shown that Δ is a parapolar space all symps of which are isomorphic to $\mathsf{D}_{6,1}(\mathbb{K})$.

Consider a clique C of Γ_2 of size 5. By Lemma 10.7, the subspace $W = \langle e_v\mathbb{K} \mid v \in C\rangle$ is a singular subspace of Δ. Notice that C is contained in exactly two maximal cliques of Γ_2, one of size 6 (say, C_1), and one of size 7 (say, C_2). Let $p \in \mathfrak{E}$ be a point collinear to all points of W. Then Corollary 10.18 implies that p is contained in one of $\langle e_v \mid v \in C_i\rangle$, $i = 1, 2$. This implies that W is contained in exactly two maximal singular subspaces and Corollary 4.4(3) concludes the proof of the proposition. □

Proposition 6.7(H) completes, together with Proposition 10.19, the proof of Theorem 10.6.

10.4 Proof that the first construction works: equivalence of the two constructions

We now prove Theorem 10.4 for the case $\mathbb{A} = \mathbb{O}'$. This will be done by establishing the equivalence with the second construction. More exactly, let $\mathfrak{E}^*$ be the quadratic Zariski closure of $\mathscr{A}\mathscr{V}(\mathbb{K}, \mathbb{O}')$. Then we show in this subsection that $\mathfrak{E}^*$ is projectively equivalent to $\mathfrak{E}$. In order to do so, we need to establish a basis of the target vector space V of the dual polar affine Veronese map ν defined before, and relate this basis to the Gosset graph, two opposite vertices in it and a spread in the neighbourhood of these vertices, as above.

Construction 10.20 Let V be as in the definition of the dual polar affine Veronese map. We view V as a 56-dimensional vector space over $\mathbb{K}$ consisting of the direct sum $\mathbb{K}^4 \oplus \mathbb{O}'^3 \oplus \mathbb{K}^3 \oplus \mathbb{O}'^3 \oplus \mathbb{K}$. In the components in $\mathbb{K}$ we choose the standard basis and introduce the following notation. The basis vector related to the i-th coordinates, $i = 1, 2, 3, 4, 29, 30, 31, 56$ will be denoted by e_∞, e_1, e_2, e_3, f_1, f_2, f_3, f_∞, respectively. In each $\mathbb{O}'$-component, we choose the standard basis of the corresponding split octonions, numbered $0, 1, \ldots, 7$ as the subscripts in the definition of X in the beginning of Section 10.1. The basis vectors of V related to the i-th coordinates, $i = 5, 6, \ldots, 12, 13, \ldots, 28$, will be denoted by $e_{1,0}$, $e_{1,1}, \ldots$, $e_{1,7}, e_{2,0}$, $\ldots$, $e_{3,7}$, respectively (and we conceive the first subscript as belonging to $\mathbb{Z}/3\mathbb{Z}$, as we also do with the subscripts of $e_1, e_2, \ldots, f_3$). Likewise, the basis vectors of V related to the i-th coordinates, $i = 32, 33, \ldots, 40, 41, \ldots, 55$, will be denoted by $f_{1,0}$, $f_{1,1}, \ldots$, $f_{1,7}, f_{2,0}$, $\ldots$, $f_{3,7}$. Let, for $i \in \{0, 1, \ldots, 7\}$, $a_i \in \mathbb{O}'$ be the split octonion $X = (x_0, x_1, \ldots, x_7)$ with $x_i = 1$ and $x_j = 0$, $j \in \{0, 1, \ldots, 7\} \setminus \{i\}$ using the notation of the beginning of Section 10.1.

We define a graph Γ with as set of vertices the (standard) basis vectors of V and with adjacency, denoted $\sim$, as follows. Define the involutive permutation ι of $\{0, 1, \ldots, 7\}$ as $(0, 7)$,$(1, 4)$, $(2, 5)$, $(3, 6) \in \iota$. Further, for all $j, j', k \in \mathbb{Z}/3\mathbb{Z}$ and $i, i' \in \{0, 1, \ldots, 7\}$, define

1. $e_j \sim e_\infty \sim e_{j,i}$
2. $f_j \sim f_\infty \sim f_{j,i}$
3. $f_j \sim e_k \sim e_{j,i}$ if $k \neq j$; $e_k \sim f_{j,i}$ if $k = j$;
4. $e_j \sim f_k \sim f_{j,i}$ if $k \neq j$; $f_k \sim e_{j,i}$ if $k = j$;
5. $e_{j,i} \sim e_{j+1,i'}$, $j \in \mathbb{Z}/3\mathbb{Z}$, if $a_i a_{i'} = 0$;
6. $f_{j,i} \sim f_{j-1,i'}$, $j \in \mathbb{Z}/3\mathbb{Z}$, if $a_i a_{i'} = 0$;
7. $e_{j,i} \sim e_{j,i'}$ if $(i, i') \notin \iota$ and $i \neq i'$;
8. $f_{j,i} \sim f_{j,i'}$ if $(i, i') \notin \iota$ and $i \neq i'$;
9. $e_{j,i} \sim f_{j',i'}$ if $(j, i) \neq (j', i^*)$ and $e_{j,i} \not\sim e_{j',i^*}$, with $i^* = i'$ if $i \in \{0, 7\}$ and $i^* = \iota(i')$ otherwise.

There are no further adjacencies.

Remark 10.21 The mapping ι can also be defined as $\iota(i) = i^*$ if $(a_i + a_{i^*})^2 = a_0 + a_7$.

Lemma 10.22 *The graph Γ is isomorphic to the Gosset graph.*

Proof This is just an explicit check, which can be done by the reader. A useful tool for the computations involved is the following multiplication table (elements of left column times elements of upper row).

$\cdot$	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7
a_0	a_0	0	0	0	a_4	a_5	a_6	0
a_1	a_1	0	a_6	$-a_5$	a_7	0	0	0
a_2	a_2	$-a_6$	0	a_4	0	a_7	0	0
a_3	a_3	a_5	$-a_4$	0	0	0	a_7	0
a_4	0	a_0	0	0	0	$-a_3$	a_2	a_4
a_5	0	0	a_0	0	a_3	0	$-a_1$	a_5
a_6	0	0	0	a_0	$-a_2$	a_1	0	a_6
a_7	0	a_1	a_2	a_3	0	0	0	a_7

□

Construction 10.23 Construction 10.20 implies the following construction of $\mathsf{GQ}(2,4)$ on the 27 points e_j and $e_{j,i}$, $j \in \{1,2,3\}$, $i \in \{0,1,\ldots,7\}$. There are three types of lines:

- $e_1e_2e_3$ is a line;
- $e_je_{j,i}e_{j,\iota(i)}$ is a line for all $j \in \{1,2,3\}$ and all $i \in \{0,1,\ldots,7\}$;
- $e_{1,i_1}e_{2,i_2}e_{3,i_3}$ is a line if $0 \notin \{a_{i_1}a_{i_2}, a_{i_2}a_{i_3}, a_{i_3}a_{i_1}\}$ (in fact, two of these non-zero implies the third is non-zero).

We now define the following spread $\mathscr{S}$ in this $\mathsf{GQ}(2,4)$:

$$\begin{array}{lll} e_1e_{1,0}e_{1,7}, & e_{1,1}e_{3,2}e_{2,3}, & e_{1,4}e_{2,5}e_{3,6}, \\ e_2e_{2,0}e_{2,7}, & e_{2,1}e_{1,2}e_{3,3}, & e_{2,4}e_{3,5}e_{1,6}, \\ e_3e_{3,0}e_{3,7}, & e_{3,1}e_{2,2}e_{1,3}, & e_{3,4}e_{1,5}e_{2,6}. \end{array}$$

Conceiving the above arrangement of the spread lines as a 3×3 matrix, the reguli of the spread correspond to the rows, the columns, and terms which are the product of 3 entries occurring in the expansion of the determinant, e.g. via Sarrus' rule.

Definition 10.24 We now define some quadratic forms on V. We use the generic coordinates

$$(x, \ell_1, \ell_2, \ell_3, X_1, X_2, X_3, k_1, k_2, k_3, Y_1, Y_2, Y_3, y)$$

of a vector in V, where $x, y, \ell_1, \ell_2, \ell_3, k_1, k_2, k_3 \in \mathbb{K}$ and $X_1, X_2, X_3, Y_1, Y_2, Y_3 \in \mathbb{O}'$. The twelve quadratic forms in the second and third column below which seemingly have values in $\mathbb{O}'$ should be read componentwise so that each of them stands for eight forms with values in $\mathbb{K}$.

Consider the following list (L) of 102 quadratic forms (with abbreviations for further use):

$$\begin{array}{lll}
\varphi_{x,1} = xk_1 + \ell_2\ell_3 - X_1\overline{X}_1 & \varphi_{x,23} = xY_1 + X_2X_3 - \ell_1\overline{X}_1 & \varphi_{23} = k_2\overline{X}_1 + \ell_3Y_1 + X_2\overline{Y}_3 \\
\varphi_{x,2} = xk_2 + \ell_3\ell_1 - X_2\overline{X}_2 & \varphi_{x,31} = xY_2 + X_3X_1 - \ell_2\overline{X}_2 & \varphi_{32} = k_3\overline{X}_1 + \ell_2Y_1 + \overline{Y}_2X_3 \\
\varphi_{x,3} = xk_3 + \ell_1\ell_2 - X_3\overline{X}_3 & \varphi_{x,12} = xY_3 + X_1X_2 - \ell_3\overline{X}_3 & \varphi_{31} = k_3\overline{X}_2 + \ell_1Y_2 + X_3\overline{Y}_1 \\
\varphi_{y,1} = y\ell_1 + k_2k_3 - Y_1\overline{Y}_1 & \varphi_{y,32} = yX_1 + Y_3Y_2 - k_1\overline{Y}_1 & \varphi_{13} = k_1\overline{X}_2 + \ell_3Y_2 + \overline{Y}_3X_1 \\
\varphi_{y,2} = y\ell_2 + k_3k_1 - Y_2\overline{Y}_2 & \varphi_{y,13} = yX_2 + Y_1Y_3 - k_2\overline{Y}_2 & \varphi_{12} = k_1\overline{X}_3 + \ell_2Y_3 + X_1\overline{Y}_2 \\
\varphi_{y,3} = y\ell_3 + k_1k_2 - Y_3\overline{Y}_3 & \varphi_{y,21} = yX_3 + Y_2Y_1 - k_3\overline{Y}_3 & \varphi_{21} = k_2\overline{X}_3 + \ell_1Y_3 + \overline{Y}_1X_2
\end{array}$$

and the following list (M) of 3 quadratic forms:

$$\begin{aligned}
\psi_1 &= xy + \ell_1 k_1 - \ell_2 k_2 - \ell_3 k_3 - X_1 Y_1 - \overline{Y}_1 \overline{X}_1 \\
\psi_2 &= xy + \ell_2 k_2 - \ell_3 k_3 - \ell_1 k_1 - X_2 Y_2 - \overline{Y}_2 \overline{X}_2 \\
\psi_3 &= xy + \ell_3 k_3 - \ell_1 k_1 - \ell_2 k_2 - X_3 Y_3 - \overline{Y}_3 \overline{X}_3
\end{aligned}$$

Lemma 10.25 *The* 102 *quadratic forms of the list* (L) *are exactly the short quadratic forms belonging to* $(\Gamma, e_\infty, \mathscr{S})$ *with the property that the corresponding hexacross contains one of* e_∞*,* e_1*,* e_2*,* e_3*,* f_∞*,* f_1*,* f_2 *or* f_3*. The other* 24 *short quadratic forms belonging to* $(\Gamma, e_\infty, \mathscr{S})$ *are the following (using the same subscripts for the coordinate as for the corresponding basis vector, though omitting the comma):*

$$\begin{array}{c}
x_{10}y_{11} + x_{11}y_{17} + x_{36}y_{35} - x_{21}y_{20} - x_{27}y_{21} - x_{35}y_{36} \\
x_{20}y_{21} + x_{21}y_{27} + x_{16}y_{15} - x_{31}y_{30} - x_{37}y_{31} - x_{15}y_{16} \\
x_{30}y_{31} + x_{31}y_{37} + x_{26}y_{25} - x_{11}y_{10} - x_{17}y_{11} - x_{25}y_{26} \\
x_{14}y_{10} + x_{17}y_{14} + x_{32}y_{33} - x_{20}y_{24} - x_{24}y_{27} - x_{33}y_{32} \\
x_{24}y_{20} + x_{27}y_{24} + x_{12}y_{13} - x_{30}y_{34} - x_{34}y_{37} - x_{13}y_{12} \\
x_{34}y_{30} + x_{37}y_{34} + x_{22}y_{23} - x_{10}y_{14} - x_{14}y_{17} - x_{23}y_{22}
\end{array}$$

$$\begin{array}{c}
x_{10}y_{12} + x_{12}y_{17} + x_{34}y_{36} - x_{22}y_{20} - x_{27}y_{22} - x_{36}y_{34} \\
x_{20}y_{22} + x_{22}y_{27} + x_{14}y_{16} - x_{32}y_{30} - x_{37}y_{32} - x_{16}y_{14} \\
x_{30}y_{32} + x_{32}y_{37} + x_{24}y_{26} - x_{12}y_{10} - x_{17}y_{12} - x_{26}y_{24} \\
x_{15}y_{10} + x_{17}y_{15} + x_{33}y_{31} - x_{20}y_{25} - x_{25}y_{27} - x_{31}y_{33} \\
x_{25}y_{20} + x_{27}y_{25} + x_{13}y_{11} - x_{30}y_{35} - x_{35}y_{37} - x_{11}y_{13} \\
x_{35}y_{30} + x_{37}y_{35} + x_{23}y_{21} - x_{10}y_{15} - x_{15}y_{17} - x_{21}y_{23}
\end{array}$$

$$\begin{array}{c}
x_{10}y_{13} + x_{13}y_{17} + x_{35}y_{34} - x_{23}y_{20} - x_{27}y_{23} - x_{34}y_{35} \\
x_{20}y_{23} + x_{23}y_{27} + x_{15}y_{14} - x_{33}y_{30} - x_{37}y_{33} - x_{14}y_{15} \\
x_{30}y_{33} + x_{33}y_{37} + x_{25}y_{24} - x_{13}y_{10} - x_{17}y_{13} - x_{24}y_{25} \\
x_{16}y_{10} + x_{17}y_{16} + x_{31}y_{32} - x_{20}y_{26} - x_{26}y_{27} - x_{32}y_{31} \\
x_{26}y_{20} + x_{27}y_{26} + x_{11}y_{12} - x_{30}y_{36} - x_{36}y_{37} - x_{12}y_{11} \\
x_{36}y_{30} + x_{37}y_{36} + x_{21}y_{22} - x_{10}y_{16} - x_{16}y_{17} - x_{22}y_{21}
\end{array}$$

$$\begin{array}{c}
x_{11}y_{15} + x_{21}y_{25} + x_{31}y_{35} - x_{15}y_{11} - x_{25}y_{21} - x_{35}y_{31} \\
x_{11}y_{16} + x_{21}y_{26} + x_{31}y_{36} - x_{16}y_{11} - x_{26}y_{21} - x_{36}y_{31} \\
x_{12}y_{14} + x_{22}y_{24} + x_{32}y_{34} - x_{14}y_{12} - x_{24}y_{22} - x_{34}y_{32} \\
x_{12}y_{16} + x_{22}y_{26} + x_{32}y_{36} - x_{16}y_{12} - x_{26}y_{22} - x_{36}y_{32} \\
x_{13}y_{14} + x_{23}y_{24} + x_{33}y_{34} - x_{14}y_{13} - x_{24}y_{23} - x_{34}y_{33} \\
x_{13}y_{15} + x_{23}y_{25} + x_{33}y_{35} - x_{15}y_{13} - x_{25}y_{23} - x_{35}y_{33}
\end{array}$$

Proof This is a straightforward verification using Construction 10.20 and the definition of the spread $\mathscr{S}$ above. □

Lemma 10.26 *The image* $\mathscr{A}\mathscr{V}(\mathbb{K}, \mathbb{A})$ *of the dual polar affine Veronese map is contained in the common null set of the short quadratic forms belonging to* $(\Gamma, e_\infty, \mathscr{S})$*.*

Proof This is easy for the quadratic forms in the list (L). As an example, take the set of eight quadratic forms determined by $k_2\overline{X}_3 + \ell_1 Y_3 + \overline{Y}_1 X_2$. Substitute (see the explicit form of ν)

$$\begin{cases} k_2 &= X_2\overline{X}_2 - \ell_3\ell_1, \\ \overline{Y}_1 &= \ell_1 X_1 - \overline{X}_3\overline{X}_2, \\ Y_3 &= \ell_3\overline{X}_3 - X_1 X_2. \end{cases}$$

Then we obtain $k_2\overline{X}_3 + \ell_1 Y_3 + \overline{Y}_1 X_2 = (X_2\overline{X}_2)\overline{X}_3 - (\overline{X}_3\overline{X}_2)X_2 = 0$, since $\overline{X}_2$ belongs to the quaternion subalgebra generated by X_2 and X_3, and hence associativity holds (also use that $\overline{X}_2 X_2 = X_2\overline{X}_2$ belongs to $\mathbb{K}$ and hence commutes with everything).

For the other forms given in Lemma 10.25, an explicit calculation with $\mathbb{K}$-coordinates must be performed. In fact, it suffices to only check two of these calculations because of the obvious symmetry $x_{1j} \mapsto x_{2j} \mapsto x_{3j} \mapsto x_{1j}$, and the same for the y_{ij}, $i \in \{1,2,3\}$, $j \in \{0,1,\ldots,7\}$, and the less obvious symmetry $x_{i0} \leftrightarrow x_{i7}$, $x_{i1} \leftrightarrow -x_{i4}$, $x_{i2} \leftrightarrow -x_{i5}$, $x_{i3} \leftrightarrow -x_{i6}$, and the same for the y_{ij}, $i \in \{1,2,3\}$, $j \in \{0,1,\ldots,7\}$. The latter symmetry is due to the automorphism of $\mathbb{O}'$ obtained by composing the standard involution with the ordinary transpose (in the sense of matrices). Under these two symmetries, the first eighteen forms given in Lemma 10.25 are equivalent (up to sign) and the last six are equivalent. In order to check the first form we calculate

$$\begin{cases} y_{11} &= x_{21}x_{30} - x_{25}x_{36} + x_{26}x_{35} + x_{27}x_{31}, \\ y_{17} &= x_{21}x_{34} + x_{22}x_{35} + x_{23}x_{36} + x_{27}x_{37}, \\ y_{20} &= x_{30}x_{10} + x_{34}x_{11} + x_{35}x_{12} + x_{36}x_{13}, \\ y_{21} &= x_{31}x_{10} - x_{35}x_{16} + x_{36}x_{15} + x_{37}x_{11}, \\ y_{35} &= x_{10}x_{25} - x_{11}x_{23} + x_{13}x_{21} + x_{15}x_{27}, \\ y_{36} &= x_{10}x_{26} + x_{11}x_{22} - x_{12}x_{21} + x_{16}x_{27}. \end{cases}$$

Substituting these values for y_{ij}, for the given i,j, in $x_{10}y_{11} + x_{11}y_{17} + x_{36}y_{35} - x_{21}y_{20} - x_{27}y_{21} - x_{35}y_{36}$ gives identically zero. Similarly for one of the last six forms given in Lemma 10.25. □

We now concentrate on the long quadratic forms. Recall the definition of "diagonal components" in Section 10.1.

Lemma 10.27 *All* 3 *quadratic forms of the list* (M) *are long quadratic forms belonging to* $(\Gamma, e_\infty, \mathscr{S})$. *Moreover, also the diagonal components of the quadratic forms*

$$\begin{aligned} \psi_{11} &= xy - \ell_1 k_1 + Y_1 X_1 - \overline{Y}_1\overline{X}_1 - X_2 Y_2 - Y_3 X_3, \\ \psi_{22} &= xy - \ell_2 k_2 + Y_2 X_2 - \overline{Y}_2\overline{X}_2 - X_3 Y_3 - Y_1 X_1, \\ \psi_{33} &= xy - \ell_3 k_3 + Y_3 X_3 - \overline{Y}_3\overline{X}_3 - X_1 Y_1 - Y_2 X_2, \end{aligned}$$

are long quadratic forms belonging to $(\Gamma, e_\infty, \mathscr{S})$.

Proof Straightforward from Construction 10.20. □

Lemma 10.28 *The image* $\mathscr{AV}(\mathbb{K}, \mathbb{A})$ *of the dual polar affine Veronese map is contained in the common null set of the long quadratic forms belonging to* $(\Gamma, e_\infty, \mathscr{S})$ *of the list* (M).

Proof Easy verification using the explicit form of ν. $\square$

Lemma 10.29 *The following are identities in the above set of quadratic forms:*

(1) $x\psi_2 = x\psi_1 - 2\ell_1\varphi_{x,1} + 2\ell_2\varphi_{x,2} + X_1\varphi_{x,23} + \overline{\varphi}_{x,23}\overline{X}_1 - X_2\varphi_{x,31} - \overline{\varphi}_{x,31}\overline{X}_2$.

(2) $\psi_1 X_2 = x\varphi_{y,13} + \ell_1\overline{\varphi}_{13} + k_2\overline{\varphi}_{x,31} - \ell_3\overline{\varphi}_{31} - Y_1\varphi_{x,12} - \overline{X}_1\varphi_{21}$.

(3) $x\psi_{33} = x\psi_1 - \ell_1\varphi_{x,1} + \ell_2\varphi_{x,2} + \overline{\varphi}_{x,23}\overline{X}_1 + \varphi_{x,12}X_3 - \overline{\varphi}_{x,12}\overline{X}_3 - \varphi_{x,31}X_2$.

Proof This is a straightforward check, using the following well known properties of the associator $(a\ b\ c) = a(bc) - (ab)c$ and commutator $[a, b] = ab - ba$. Let σ be an arbitrary permutation of $\{1, 2, 3\}$ or of $\{1, 2\}$, respectively. Let θ_i, $i = 1, 2, 3$, be either the identity or the standard involution of $\mathbb{O}'$. Let ϵ be the sign of σ, if $\theta_1\theta_2\theta_3$ or $\theta_1\theta_2$ is the identity, and minus that sign otherwise. Then

$$\left(x_{\sigma(1)}^{\theta_1}\ x_{\sigma(2)}^{\theta_2}\ x_{\sigma(3)}^{\theta_3}\right) = \epsilon(x_1\ x_2\ x_3), \text{ and } \left[x_{\sigma(1)}^{\theta_1}\ x_{\sigma(2)}^{\theta_2}\right] = \epsilon(x_1\ x_2),$$

for all $x_1, x_2, x_3 \in \mathbb{O}'$. $\square$

Before we go on, we need the following transitivity properties of the Gosset graph Γ_2.

Lemma 10.30 *Let $\Gamma_2 = (V_2, E_2)$ be the Gosset graph and let D, E be two hexacrosses. Let D' and E' be the respective opposite hexacrosses. Then*

(*i*) *the stabilizer of $D \cup D'$ in* $\mathrm{Aut}(\Gamma_2)$ *acts transitively on $V_2 \setminus (D \cup D')$, and*

(*ii*) *the common stabilizer of $D \cup D'$ and $E \cup E'$ in* $\mathrm{Aut}(\Gamma_2)$ *acts transitively on the set of vertices $(D \cup D') \cap (E \cup E')$.*

Proof (*i*) It is easy to check that every vertex of $V_2 \setminus (D \cup D')$ is adjacent to a unique maximal clique of D. Also, the stabilizer of D in $\mathrm{Aut}(\Gamma_2)$ is transitive on the maximal cliques of D that are properly contained in a maximal clique of Γ_2, since this stabilizer acts on D as the Weyl group of type D_6. Finally, D' is automatically stabilized if D is stabilized.

(*ii*) One verifies that $(D \cup D') \cap (E \cup E')$ is either the disjoint union of four edges, or the disjoint union of two 6-cliques. In the former case, $D \cap E$ is an edge $e \in E$. We can map any edge e' of $(D \cup D') \cap (E \cup E')$ to e. The stabilizer of e is the Weyl group of type $\mathsf{A}_1 \times \mathsf{D}_5$, which acts transitively on the pairs (v, C), where $v \in e \subseteq C$, with C a hexacross. Hence we choose the map which maps e' to e in such a way that it maps some member of $\{D, D', E, E'\}$ that contains e' to D. Then, since E is the unique hexacross of Γ_2 intersecting D in e, the map preserves $\{D \cup D', E \cup E'\}$. Suppose now that $(D \cup D') \cap (E \cup E')$ is the union of two 6-cliques. Then arguing in the Weyl group of type $\mathsf{A}_5 \times \mathsf{A}_1$ corresponding to the stabilizer of such a 6-clique, the result follows similarly as before. $\square$

Lemma 10.31 *The common null set of the short quadratic forms belonging to $(\Gamma, e_\infty, \mathscr{S})$ and the long quadratic forms in the list* (M) *is exactly the variety $\mathscr{E}_7(\mathbb{K})$. In other words, every point in the common null set of the short quadratic forms belonging to $(\Gamma, e_\infty, \mathscr{S})$ and the long quadratic forms in the list* (M), *is also in the null set of every other long quadratic form belonging to $(\Gamma, e_\infty, \mathscr{S})$. In particular, $\mathscr{A}\mathscr{V}(\mathbb{K}, \mathbb{A})$ is a subset of $\mathscr{E}_7(\mathbb{K})$.*

Proof Let $p = (x, \ell_1, \ell_2, \ldots, Y_3, y)$ be an arbitrary point of $\mathbb{P}(V)$ in the common null set of all short quadratic forms belonging to $(\Gamma, e_\infty, \mathscr{S})$. Let $\{Q, Q'\}$ be an arbitrary pair of opposite hexacrosses. We claim that, if some non-zero coordinate of p corresponds to a vertex outside $Q \cup Q'$, then p is in the null set of the long quadratic form $\beta_{Q,Q'}$. Indeed, by Lemmas 10.8 and 10.30(i), we may assume that $\beta_{Q,Q'}$ is ψ_1, and $X_2 \neq 0$. Then it follows from Lemma 10.29(2) that $\psi_1 X_2$ vanishes at p, and hence ψ_1 does. The claim is proved.

Now let $p = (x, \ell_1, \ell_2, \ldots, Y_3, y)$ be an arbitrary point of $\mathbb{P}(V)$ in the common null set of all short quadratic forms belonging to $(\Gamma, e_\infty, \mathscr{S})$ and the long quadratic forms in the list (M). Let $\{Q, Q'\}$ be an arbitrary pair of opposite hexacrosses so that $\beta_{Q',Q} \notin \{\pm\psi_1, \pm\psi_2, \pm\psi_3\}$. We claim that, if some non-zero coordinate of p corresponds to a vertex v of $Q \cup Q'$, then p is in the null set of the long quadratic form $\beta_{Q,Q'}$. Indeed, in this case, at least one of ψ_1, ψ_2, ψ_3 contains v, say, without loss of generality, ψ_1. By Lemmas 10.8 and 10.30(ii), there is a linear map θ preserving $\mathscr{E}_7(\mathbb{K})$, interchanging the coordinates, up to sign, and thus inducing an automorphism of Γ_2 mapping v to ∞, stabilizing ψ_1 and mapping $\beta_{Q,Q'}$ to ψ_2 (if $\beta_{Q,Q'}$ and ψ_1 share exactly four terms) or to a diagonal component of ψ_{33} (if $\beta_{Q,Q'}$ and ψ_1 share exactly six terms). Now Lemma 10.29(1) and (3) imply that $\theta(p)$ is in the null set of ψ_2 or ψ_{33}, respectively, and hence p is in the null set of $\beta_{Q,Q'}$, proving the claim. Now the lemma follows from Lemmas 10.26 and 10.28. $\square$

This already has the following consequence, which is an improvement of Theorem 10.6.

Corollary 10.32 *The variety $\mathscr{E}_7(\mathbb{K})$ is the intersection of 129 quadrics, namely, those corresponding to the short quadratic forms belonging to $(\Gamma, e_\infty, \mathscr{S})$, together with the three long quadratic forms in the list (M). No quadric can be deleted, that is, the intersection of each proper subset of these 129 quadrics contains points not contained in $\mathscr{E}_7(\mathbb{K})$.*

Proof We only need to show the last assertion. Note first that every product $X_v X_w$ of distinct variables, with v and w vertices of Γ_2 at distance 2, is contained in exactly one of the 126 short quadratic forms belonging to $(\Gamma, e_\infty, \mathscr{S})$, and not in any of the long quadratic forms. Hence the line of $\mathbb{P}(V)$ joining the base points corresponding to v and w entirely belongs to each of the said 129 quadrics except for exactly one (short). Similarly, every quadratic form in the list (M) contains a product $X_v X_w$, with v and w opposite vertices of Γ_2, which does not appear in any other of the 129 quadratic forms. $\square$

Proposition 10.33 *Assuming $|\mathbb{K}| > 2$, we have $\mathscr{PV}(\mathbb{K}, \mathbb{O}') = \mathscr{E}_7(\mathbb{K})$.*

Proof Since $\mathscr{E}_7(\mathbb{K})$ is quadratically Zariski closed, Lemma 10.31 implies that $\mathscr{PV}(\mathbb{K}, \mathbb{O}')$ is contained in $\mathscr{E}_7(\mathbb{K})$, where the latter is defined as the common null set of all short and long quadratic forms belonging to $(\Gamma, e_\infty, \mathscr{S})$.

Now let $p = (x, \ell_1, \ell_2, \ldots, Y_3, y)$ be an arbitrary point of $\mathbb{P}(V)$ belonging to $\mathscr{E}_7(\mathbb{K})$. Suppose first $x \neq 0$, in which case we may assume $x = 1$. Then p is in the null sets of $\varphi_{x,i}$, $i = 1, 2, 3$, $\varphi_{x,ij}$, $ij \in \{23, 31, 12\}$ and ψ_1 determines the coordinates $k_1, k_2, \ldots, Y_3, y$ unambiguously, showing p belongs to $\mathscr{AV}(\mathbb{K}, \mathbb{O}')$.

Now suppose $x = 0$ and $(\ell_1, \ell_2, \ell_3, X_1, X_2, X_3) \neq (0, 0, 0, 0, 0, 0)$. Then we select a hexacross Q containing e_∞ and such that the vertex $v \in V_2$ corresponding to an arbitrary non-zero coordinate in $(\ell_1, \ell_2, \ell_3, X_1, X_2, X_3)$ has no neighbours in Q besides ∞. Then by Lemma 10.13, the set $\{p^{\sigma_Q(a)} \mid a \in \mathbb{K}\}$ is an affine line contained in $\mathscr{E}_7(\mathbb{K})$, and by the definition of $\sigma_Q(a)$, the first coordinate of $p^{\sigma_Q(a)}$ is non-zero if $a \neq 0$. So p belongs to a line entirely contained in $\mathscr{E}_7(\mathbb{K})$ and intersecting $\mathscr{AV}(\mathbb{K}, \mathbb{O}')$ in an affine line. It follows that $p \in \mathscr{PV}(\mathbb{K}, \mathbb{O}')$.

Now suppose $(x, \ell_1, \ldots, X_3) = (0, \ldots, 0)$ and $(k_1, k_2, k_3, Y_1, Y_2, Y_3) \neq (0, 0, 0, 0, 0, 0)$. Then we select an arbitrary vertex w adjacent to e_∞ and also adjacent to the vertex v corresponding to an arbitrary non-zero coordinate in $(k_1, \ldots, Y_3)$. The argument of the previous paragraph with now w in place of e_∞ shows that p is contained in a projective line contained in $\mathscr{E}_7(\mathbb{K})$ intersecting $\mathscr{PV}(\mathbb{K}, \mathbb{O}')$ in at least an affine line. Hence also $p \in \mathscr{PV}(\mathbb{K}, \mathbb{O}')$.

It remains to show that the point $p = (0, 0, \ldots, 0, 1)$ belongs to $\mathscr{PV}(\mathbb{K}, \mathbb{O}')$. This follows from the fact $(0, \ldots, 0, 1, a)$ belongs to $\mathscr{E}_7(\mathbb{K})$, for all $a \in \mathbb{K}$, and hence to $\mathscr{PV}(\mathbb{K}, \mathbb{O}')$.

The proposition is proved. □

The following corollary concludes the proof of Theorem 10.4.

Corollary 10.34 *Assuming $|\mathbb{K}| > 2$, we have $\mathscr{PV}(\mathbb{K}, \mathbb{L}') \cong \mathscr{G}_{6,3}(\mathbb{K})$ and $\mathscr{PV}(\mathbb{K}, \mathbb{H}') \cong \mathscr{HS}_6(\mathbb{K})$.*

Proof Set

$$Q_1 = \{e_{1,2}, e_{1,6}, e_{2,2}, e_{2,6}, e_{3,2}, e_{3,6}, f_{1,2}, f_{1,6}, f_{2,2}, f_{2,6}, f_{3,2}, f_{3,6}\}$$

and

$$Q_2 = \{e_{1,3}, e_{1,5}, e_{2,3}, e_{2,5}, e_{3,3}, e_{3,5}, f_{1,3}, f_{1,5}, f_{2,3}, f_{2,5}, f_{3,3}, f_{3,5}\}.$$

Then Q_1 and Q_2 are opposite hexacrosses. They determine unique symps ξ_1 and ξ_2, respectively. According to Section 4.4 of [31], the set of points of $\mathscr{E}_7(\mathbb{K})$ collinear to respective maximal singular subspaces of ξ_1 and ξ_2 is the point set $\mathscr{X}$ of a subgeometry isomorphic to $\mathsf{D}_{6,6}(\mathbb{K})$. Now, each base point corresponding to a vertex of Γ_2 not in $Q_1 \cup Q_2$ belongs to $\mathscr{X}$; these generate a subspace U of dimension 31 of $\mathbb{P}(V)$. By Proposition 6.7(H), $U \cap \mathscr{E}_7(\mathbb{K})$ contains $\mathscr{HS}_6(\mathbb{K})$.

We claim that $U \cap \mathscr{E}_7(\mathbb{K}) \equiv \mathscr{HS}_6(\mathbb{K})$. Indeed, suppose $p \in U \cap \mathscr{E}_7(\mathbb{K})$ does not belong to $\mathscr{HS}_6(\mathbb{K})$. Then without loss of generality, we may assume that p is collinear to a unique point $p_1 \in \xi_1$. Since the coordinates of p corresponding to the vertices of Q_2 are 0, it follows from Corollary 10.18 that p is at distance 2 from every point $e_{i,j}\mathbb{K}$, with $e_{i,j} \in Q_1$. Hence p_1 is collinear to every such point, a contradiction.

Now a point $p \in V$ belongs to U if and only if its coordinates corresponding to the vertices of $Q_1 \cup Q_2$ are 0. These coordinates correspond precisely to the components of $\mathbb{O}'$ corresponding to x_2, x_3, x_5 and x_6. Hence if the first coordinate of p is 1, this is precisely if p belongs to the image of the dual polar affine Veronese map restricted to the quaternion subalgebra $\mathbb{H}'$ of $\mathbb{O}'$ obtained by putting $x_2 = x_3 = x_5 = x_6 = 0$ in the matrix form of an arbitrary octonion. Consequently, $\mathscr{AV}(\mathbb{K}, \mathbb{H}')$, and hence $\mathscr{PV}(\mathbb{K}, \mathbb{H}')$, is contained in U.

We now claim that $U \cap \mathscr{E}_7(\mathbb{K}) \equiv \mathscr{PV}(\mathbb{K},\mathbb{H}')$. It suffices to show that $U \cap \mathscr{E}_7(\mathbb{K}) \subseteq \mathscr{PV}(\mathbb{K},\mathbb{H}')$. Now, $\mathscr{AV}(\mathbb{K},\mathbb{H}')$ is precisely the set of points of $\mathscr{HS}_6(\mathbb{K})$ opposite the point $(0,\ldots,0,1)$ (as follows from Corollary 10.18). Since every affine line of $\mathscr{AV}(\mathbb{K},\mathbb{H}')$ is contained in a line of $\mathscr{HS}_6(\mathbb{K})$, the quadratic Zariski closure of $\mathscr{AV}(\mathbb{K},\mathbb{H}')$ is precisely $\mathscr{HS}_6(\mathbb{K})$.

Hence we have shown that $\mathscr{HS}_6(\mathbb{K}) \equiv U \cap \mathscr{E}_7(\mathbb{K}) \equiv \mathscr{PV}(\mathbb{K},\mathbb{H}')$.

The assertion about $\mathscr{G}_{6,3}(\mathbb{K})$ follows similarly, now relying on the fact that $\mathscr{G}_{6,3}(\mathbb{K})$ arises as the set of points of $\mathscr{HS}_6(\mathbb{K})$ collinear to respective planes of two respective opposite singular subspaces of projective dimension 5. The canonical choice for the latter (to make the identification with $\mathbb{L}'$ as above with $\mathbb{H}'$) are the subspaces generated by the points corresponding to the vertices $e_{1,1}, e_{2,1}, e_{3,1}, f_{1,1}, f_{2,1}, f_{3,1}$, and $e_{1,4}, e_{2,4}, e_{3,4}, f_{1,4}, f_{2,4}, f_{3,4}$, respectively. The details are left to the reader. $\square$

The same technique as in the previous proof can be used to show the following construction results.

Corollary 10.35 *Let V be the 32-dimensional vector space over $\mathbb{K}$ consisting of the direct sum $\mathbb{K}^4 \oplus \mathbb{H}'^3 \oplus \mathbb{K}^3 \oplus \mathbb{H}'^3 \oplus \mathbb{K}$. We label the standard basis and coordinates as in* Construction 10.20 *restricting the standard basis of the split octonions $\mathbb{O}'$ to those with subscripts $0,1,4,7$ so as to obtain the split quaternions $\mathbb{H}'$. Then the intersection of the null sets in $\mathbb{P}(V)$ of the following sixty-three quadratic forms is the point set of the half spin variety $\mathscr{HS}_6(\mathbb{K})$:*

$$\begin{array}{lll}
xk_1+\ell_2\ell_3-X_1\overline{X}_1, & xY_1+X_2X_3-\ell_1\overline{X}_1, & k_2\overline{X}_1+\ell_3Y_1+X_2\overline{Y}_3,\\
xk_2+\ell_3\ell_1-X_2\overline{X}_2, & xY_2+X_3X_1-\ell_2\overline{X}_2, & k_3\overline{X}_1+\ell_2Y_1+\overline{Y}_2X_3,\\
xk_3+\ell_1\ell_2-X_3\overline{X}_3, & xY_3+X_1X_2-\ell_3\overline{X}_3, & k_3\overline{X}_2+\ell_1Y_2+X_3\overline{Y}_1,\\
y\ell_1+k_2k_3-Y_1\overline{Y}_1, & yX_1+Y_3Y_2-k_1\overline{Y}_1, & k_1\overline{X}_2+\ell_3Y_2+\overline{Y}_3X_1,\\
y\ell_2+k_3k_1-Y_2\overline{Y}_2, & yX_2+Y_1Y_3-k_2\overline{Y}_2, & k_1\overline{X}_3+\ell_2Y_3+X_1\overline{Y}_2,\\
y\ell_3+k_1k_2-Y_3\overline{Y}_3, & yX_3+Y_2Y_1-k_3\overline{Y}_3, & k_2\overline{X}_3+\ell_1Y_3+\overline{Y}_1X_2,
\end{array}$$

$$\begin{array}{ll}
x_{10}y_{11}+x_{11}y_{17}-x_{21}y_{20}-x_{27}y_{21}, & x_{20}y_{21}+x_{21}y_{27}-x_{31}y_{30}-x_{37}y_{31},\\
x_{30}y_{31}+x_{31}y_{37}-x_{11}y_{10}-x_{17}y_{11}, & x_{14}y_{10}+x_{17}y_{14}-x_{20}y_{24}-x_{24}y_{27},\\
x_{24}y_{20}+x_{27}y_{24}-x_{30}y_{34}-x_{34}y_{37}, & x_{34}y_{30}+x_{37}y_{34}-x_{10}y_{14}-x_{14}y_{17},
\end{array}$$

and

$$\begin{array}{l}
xy+\ell_1k_1-\ell_2k_2-\ell_3k_3-X_1Y_1-\overline{Y}_1\overline{X}_1,\\
xy+\ell_2k_2-\ell_3k_3-\ell_1k_1-X_2Y_2-\overline{Y}_2\overline{X}_2,\\
xy+\ell_3k_3-\ell_1k_1-\ell_2k_2-X_3Y_3-\overline{Y}_3\overline{X}_3.
\end{array}$$

Also, no quadratic form can be deleted, that is, the intersection of each proper subset of the set of null sets of these sixty-three quadratic forms contains points not contained in $\mathscr{HS}_6(\mathbb{K})$.

Corollary 10.36 *Let V be the 20-dimensional vector space over $\mathbb{K}$ consisting of the direct sum $\mathbb{K}^4 \oplus \mathbb{L}'^3 \oplus \mathbb{K}^3 \oplus \mathbb{L}'^3 \oplus \mathbb{K}$. We label the standard basis and coordinates as in* Construction 10.20 *restricting the standard basis of the split octonions $\mathbb{O}'$ to those with* *subscripts* 0 *and* 7 *so as to obtain the split quadratic extension* $\mathbb{L}'$. *Then the intersection of the null sets in* $\mathbb{P}(V)$ *of the following thirty-three quadratic forms is the point set of the plane Grassmannian* $\mathscr{G}_{6,3}(\mathbb{K})$*:*

$$\begin{array}{lll}
xk_1 + \ell_2\ell_3 - X_1\overline{X}_1, & xY_1 + X_2X_3 - \ell_1\overline{X}_1, & k_2\overline{X}_1 + \ell_3Y_1 + X_2\overline{Y}_3, \\
xk_2 + \ell_3\ell_1 - X_2\overline{X}_2, & xY_2 + X_3X_1 - \ell_2\overline{X}_2, & k_3\overline{X}_1 + \ell_2Y_1 + \overline{Y}_2X_3, \\
xk_3 + \ell_1\ell_2 - X_3\overline{X}_3, & xY_3 + X_1X_2 - \ell_3\overline{X}_3, & k_3\overline{X}_2 + \ell_1Y_2 + X_3\overline{Y}_1, \\
y\ell_1 + k_2k_3 - Y_1\overline{Y}_1, & yX_1 + Y_3Y_2 - k_1\overline{Y}_1, & k_1\overline{X}_2 + \ell_3Y_2 + \overline{Y}_3X_1, \\
y\ell_2 + k_3k_1 - Y_2\overline{Y}_2, & yX_2 + Y_1Y_3 - k_2\overline{Y}_2, & k_1\overline{X}_3 + \ell_2Y_3 + X_1\overline{Y}_2, \\
y\ell_3 + k_1k_2 - Y_3\overline{Y}_3, & yX_3 + Y_2Y_1 - k_3\overline{Y}_3, & k_2\overline{X}_3 + \ell_1Y_3 + \overline{Y}_1X_2,
\end{array}$$

and

$$\begin{array}{l}
xy + \ell_1k_1 - \ell_2k_2 - \ell_3k_3 - X_1Y_1 - \overline{Y}_1\overline{X}_1, \\
xy + \ell_2k_2 - \ell_3k_3 - \ell_1k_1 - X_2Y_2 - \overline{Y}_2\overline{X}_2, \\
xy + \ell_3k_3 - \ell_1k_1 - \ell_2k_2 - X_3Y_3 - \overline{Y}_3\overline{X}_3.
\end{array}$$

Also, no quadratic form can be deleted, that is, the intersection of each proper subset of the set of null sets of these thirty-three quadratic forms contains points not contained in $\mathscr{G}_{6,3}(\mathbb{K})$.

We can now verify the axioms (ALV1), (ALV2) and (ALV3) for the varieties $\mathscr{G}_{6,3}(\mathbb{K})$, $\mathscr{HS}_6(\mathbb{K})$ and $\mathscr{E}_7(\mathbb{K})$. We leave the straightforward case of the Segre variety $\mathscr{S}_{1,1,1}(\mathbb{K})$ to the reader.

Theorem 10.37 *Let* Y *be the point set of* $\mathscr{G}_{6,3}(\mathbb{K})$, $\mathscr{HS}_6(\mathbb{K})$, *or* $\mathscr{E}_7(\mathbb{K})$. *Let* Υ *be the set of all subspaces that are generated by some symp of the respective varieties. Then* (Y, Υ) *is an abstract Lagrangian variety of type* 2, 4, 8, *respectively, and index* 1, 2, 4, *respectively.*

Proof We show the assertion for $\mathscr{E}_7(\mathbb{K})$. The other cases follow by restriction, as in Corollaries 10.36 and 10.35.

We begin by noting that the group G introduced in Definition 10.15 is the little projective group of the corresponding building of type E_7. Hence G acts as a group with a natural BN-pair on $\mathscr{E}_7(\mathbb{K})$.

We first claim that (Y, Υ) is an abstract variety. Indeed, let S be any symp of $\mathscr{E}_7(\mathbb{K})$. By the mentioned transitivity of G we may assume that S contains the points corresponding to the vertices e_∞ and f_1. The proof of Proposition 10.19 implies that $\langle S\rangle$ is generated by the points corresponding to the hexacross determined by e_∞ and f_1, and S is given by restricting the null set of $\varphi_{x,1}$ to $\langle S\rangle$. The latter clearly does not contain any other point of $\mathscr{E}_7(\mathbb{K})$. The claim is proved.

Now (ALV1) follows from Lemma 10.7 and Proposition 10.17.

In order to show (ALV2), we note that the transitivity properties of G imply that any pair of symps can be simultaneously mapped into the standard apartment (given by the Gosset graph). Since the vertices of the Gosset graph label the standard basic vectors of V, and the said symps correspond to the hexacrosses, Axiom (ALV2) holds.

Finally, (ALV3) follows directly from Lemma 10.7 and the transitivity of the group G on the point set of $\mathscr{E}_7(\mathbb{K})$. ☐

10.5 The ovoidal case: intersection of quadrics

Just like Theorem 10.4 also holds for the ovoidal case, Theorem 10.6 also has an analogue for the ovoidal case. In the ovoidal case, the list (L) and one quadratic form from the list (M) suffice. Explicitly:

Theorem 10.38 *Let $\mathbb{A}$ be a finite-dimensional alternative quadratic division algebra over $\mathbb{K}$ and set $d = \dim_{\mathbb{K}} \mathbb{A}$. Let V be the $(6d+8)$-dimensional vector space over $\mathbb{K}$ consisting of the direct sum $\mathbb{K}^4 \oplus \mathbb{A}^3 \oplus \mathbb{K}^3 \oplus \mathbb{A}^3 \oplus \mathbb{K}$. We label the coordinates according to the generic point $(x, \ell_1, \ell_2, \ell_3, X_1, X_2, X_3, k_1, k_2, k_3, Y_1, Y_2, Y_3, y)$. Then the intersection of the null sets in $\mathbb{P}(V)$ of the following $12d + 7$ quadratic forms, abbreviated as in* Definition 10.24, *is the point set of the dual polar Veronese variety $\mathscr{V}(\mathbb{K}, \mathbb{A})$:*

$$\begin{array}{lll}
\varphi_{x,1} = xk_1 + \ell_2\ell_3 - X_1\overline{X}_1, & \varphi_{x,23} = xY_1 + X_2X_3 - \ell_1\overline{X}_1, & \varphi_{23} = k_2\overline{X}_1 + \ell_3Y_1 + X_2\overline{Y}_3, \\
\varphi_{x,2} = xk_2 + \ell_3\ell_1 - X_2\overline{X}_2, & \varphi_{x,31} = xY_2 + X_3X_1 - \ell_2\overline{X}_2, & \varphi_{32} = k_3\overline{X}_1 + \ell_2Y_1 + \overline{Y}_2X_3, \\
\varphi_{x,3} = xk_3 + \ell_1\ell_2 - X_3\overline{X}_3, & \varphi_{x,12} = xY_3 + X_1X_2 - \ell_3\overline{X}_3, & \varphi_{31} = k_3\overline{X}_2 + \ell_1Y_2 + X_3\overline{Y}_1, \\
\varphi_{y,1} = y\ell_1 + k_2k_3 - Y_1\overline{Y}_1, & \varphi_{y,32} = yX_1 + Y_3Y_2 - k_1\overline{Y}_1, & \varphi_{13} = k_1\overline{X}_2 + \ell_3Y_2 + \overline{Y}_3X_1, \\
\varphi_{y,2} = y\ell_2 + k_3k_1 - Y_2\overline{Y}_2, & \varphi_{y,13} = yX_2 + Y_1Y_3 - k_2\overline{Y}_2, & \varphi_{12} = k_1\overline{X}_3 + \ell_2Y_3 + X_1\overline{Y}_2, \\
\varphi_{y,3} = y\ell_3 + k_1k_2 - Y_3\overline{Y}_3, & \varphi_{y,21} = yX_3 + Y_2Y_1 - k_3\overline{Y}_3, & \varphi_{21} = k_2\overline{X}_3 + \ell_1Y_3 + \overline{Y}_1X_2
\end{array}$$

and $\psi_1 = xy + \ell_1k_1 - \ell_2k_2 - \ell_3k_3 - X_1Y_1 - \overline{Y}_1\overline{X}_1$.
Also, no quadratic form can be deleted, that is, the intersection of each proper subset of the set of null sets of these $12d+7$ quadratic forms contains points not contained in $\mathscr{V}(\mathbb{K}, \mathbb{A})$.

Proof The quadratic Zariski closure of the image of the affine dual polar Veronese map has been explicitly calculated in [16]. In our notation and coordinates, the variety $\mathscr{V}(\mathbb{K}, \mathbb{A})$ consists of the following points, divided into eight types (and we use the same numbering as in Section 3 of [16], but the points have undergone a mild coordinate change):

Type VIII: These points are exactly the points in the image of the affine dual polar Veronese map.

Type VII: For each 5-tuple $(Y_1, X_2, X_3, k_2, k_3) \in \mathbb{A}^3 \times \mathbb{K}^2$, the point

$$(0, 1, X_3\overline{X}_3, X_2\overline{X}_2, \overline{X}_3\overline{X}_2, X_2, X_3, k_2X_2\overline{X}_2 + k_3X_3\overline{X}_3 + \overline{Y}_1(X_2X_3) + (\overline{X}_3\overline{X}_2)Y_1, k_2, k_3,$$
$$Y_1, -k_3\overline{X}_2 - X_3\overline{Y}_1, -k_2\overline{X}_3 - \overline{Y}_1X_2, Y_1\overline{Y}_1 - k_2k_3).$$

Type VI: For each 4-tuple $(X_1, Y_2; k_1, k_3) \in \mathbb{A}^2 \times \mathbb{K}^2$, the point

$$(0, 0, 1, X_1\overline{X}_1, X_1, 0, 0, k_1, k_3X_1\overline{X}_1, k_3, -k_3\overline{X}_1, Y_2, -X_1\overline{Y}_2, k_1k_3 - Y_2\overline{Y}_2).$$

Type IV: For each triple $(Y_3; k_1, k_2) \in \mathbb{A} \times \mathbb{K}^2$, the point

$$(0, 0, 0, 1, 0, 0, 0, k_1, k_2, 0, 0, 0, Y_3, Y_3\overline{Y}_3 - k_1k_2).$$

Type V: For each triple $(Y_2, Y_3; y) \in \mathbb{A}^2 \times \mathbb{K}$, the point

$$(0, 0, 0, 0, 0, 0, 0, 1, Y_3\overline{Y}_3, Y_2\overline{Y}_2, \overline{Y}_2\overline{Y}_3, Y_2, Y_3, y).$$

Type III: For each pair $(Y_1; y) \in \mathbb{A} \times \mathbb{K}$, the point $(0,0,0,0,0,0,0,0,1,Y_1\overline{Y}_1,Y_1,0,0,y)$.

Type II: For each $y \in \mathbb{K}$, the point $(0,0,0,0,0,0,0,0,0,1,0,0,0,y)$.

Type I: The point $(0,0,0,0,0,0,0,0,0,0,0,0,0,1)$.

One easily checks that all the points just mentioned are in the null set of all the quadratic forms mentioned in the statement.

Conversely, let the point p with coordinates $(x,\ell_1,\ell_2,\ell_3,X_1,X_2,X_3,k_1,k_2,k_3,Y_1,Y_2,Y_3,y)$ be a point in the common null set of all the said quadratic forms.

(VIII) Suppose $x \neq 0$. Then we set $x = 1$. The quadratic forms $\varphi_{x,i}$, $i = 1,2,3,23,31,12$, and ψ_1 determine k_1,k_2,k_3,Y_1,Y_2,Y_3 and y uniquely, given $\ell_1,\ell_2,\ell_3,X_1,X_2,X_3$ and show that p belongs to the image of the affine dual polar Veronese map. Hence p is of Type VIII.

(VII) Suppose now $x = 0$ and $\ell_1 \neq 0$, so we may assume $\ell_1 = 1$. Then $\varphi_{x,2}$, $\varphi_{x,3}$, $\varphi_{y,1}$, $\varphi_{x,23}$, φ_{31}, φ_{21} and ψ_1 uniquely determine $\ell_3,\ell_2,y,X_1,Y_2,Y_3$ and k_1, respectively, in terms of Y_1,X_2,X_3,k_2,k_3. Precisely: $\ell_3 = X_2\overline{X}_2$, $\ell_2 = X_3\overline{X}_3$, $y = Y_1\overline{Y}_1 - k_2k_3$, $X_1 = \overline{X}_3\overline{X}_2$, $Y_2 = -k_3\overline{X}_2 - X_3\overline{Y}_1$, $Y_3 = -k_2\overline{X}_3 - \overline{Y}_1X_2$ and

$$k_1 = \ell_2k_2 + \ell_3k_3 + X_1Y_1 + \overline{Y}_1\overline{X}_1 = k_2X_3\overline{X}_3 + k_3X_2\overline{X}_2 + (\overline{X}_3\overline{X}_2)Y_1 + \overline{Y}_1(X_2X_3),$$

respectively, which exactly yields a point of Type VII.

(VI) Suppose $x = \ell_1 = 0$, and assume $\ell_2 = 1$. Similarly as above, $\varphi_{x,1},\varphi_{x,2},\varphi_{x,3},\varphi_{y,2},\varphi_{32},\varphi_{12}$ and ψ_1 uniquely yield ℓ_3,X_2,X_3,y,Y_1,Y_3 and k_2, respectively. More precisely, $\ell_3 = X_1\overline{X}_1$, $X_2 = 0 = X_3$, $y = Y_2\overline{Y}_2 - k_1k_3$, $Y_1 = -k_3\overline{X}_1$, $Y_3 = -X_1\overline{Y}_2$ and

$$k_2 = -\ell_3k_3 - X_1Y_1 - \overline{Y}_1\overline{X}_1 = -k_3X_1\overline{X}_1 + k_3X_1\overline{X}_1 + k_3X_1\overline{X}_1 = k_3X_1\overline{X}_1,$$

respectively, which exactly gives rise to a point of Type VI.

(IV) Suppose $x = \ell_1 = \ell_2 = 0$, and assume $\ell_3 = 1$. Then $\varphi_{x,i}$, $i = 1,2,3$, yields $X_1 = X_2 = X_3 = 0$, and ψ_1,φ_{23} and φ_{13} yield $k_3 = 0$, $Y_1 = 0$ and $Y_2 = 0$, respectively. Finally, $\varphi_{y,3}$ yields $y = Y_3\overline{Y}_3 - k_1k_2$ and p belongs to Type IV.

(V) Suppose $x = \ell_1 = \ell_2 = \ell_3 = 0$, and assume $k_1 = 1$. Then again $\varphi_{x,i}$, $i = 1,2,3$, yields $X_1 = X_2 = X_3 = 0$. Also, $\varphi_{y,2}$, $\varphi_{y,3}$ and $\varphi_{y,32}$ yield $k_3 = Y_2\overline{Y}_2$, $k_2 = Y_3\overline{Y}_3$ and $Y_1 = \overline{Y}_2\overline{Y}_3$, respectively. We obtain a point of Type V.

(III) Suppose $x = \ell_1 = \ell_2 = \ell_3 = k_1 = 0$, and assume $k_2 = 1$. As before, we deduce $X_1 = X_2 = X_3 = 0$ and $\phi_{y,i}$, $i = 2,3$, yields $Y_2 = Y_3 = 0$. Then φ_{y_1} yields $k_3 = Y_1\overline{Y}_1$ and we have a point of Type III.

(I-II) Suppose $x = \ell_1 = \ell_2 = \ell_3 = k_1 = k_2 = 0$. Then, similarly as above, we deduce $X_1 = X_2 = X_3 = Y_1 = Y_2 = Y_3 = 0$ and we clearly have a point of Type II (if $k_3 \neq 0$) or Type I (if $k_3 = 0$).

In order to show that the list of quadratic forms is minimal, we note that every quadratic form of the list contains a term whose factors are only together in one term in that unique quadratic form. For instance, xY_3 only appears in $\varphi_{x,12}$ (in other words, a point with all coordinates 0, except x and Y_3, is automatically in the null set of all other quadratic forms). If we would delete one of the d quadratic forms bundled together in $\varphi_{x,12}$ from the list, then the point with all coordinates 0 except $x = 1$ and the corresponding coordinate of Y_3 equal to 1 would belong to the intersection of the remaining null sets, but not to $\mathscr{V}(\mathbb{K}, \mathbb{A})$.

This completes the proof of the theorem. □

We now verify the axioms of an abstract Lagrangian variety for the Veronese representation of a dual polar space of rank 3 related to an alternative quadratic division algebra.

Theorem 10.39 *Let Y be the Veronese representation $\mathscr{V}(\mathbb{K}, \mathbb{A})$ in $\mathbb{P}^{6d+7}(\mathbb{K})$ of the dual polar space $\mathsf{C}_{3,3}(\mathbb{K}, \mathbb{A})$, where $\mathbb{A}$ is a quadratic alternative division algebra over $\mathbb{K}$ with $\dim_{\mathbb{K}} \mathbb{A} = d$. Let Υ be the set of all subspaces of $\mathbb{P}^{6d+7}(\mathbb{K})$ that are generated by the symps of $\mathsf{C}_{3,3}(\mathbb{K}, \mathbb{A})$ (as a parapolar space) in this representation. Then (Y, Υ) is an abstract Lagrangian variety of type d and index 0.*

Proof It is noted right after Lemma 6.1 in [16] that $\mathscr{V}(\mathbb{K}, \mathbb{A})$ admits the full automorphism group of the corresponding (dual) polar space. By Lemma 6.2 of [16] collinearity in $\mathscr{V}(\mathbb{K}, \mathbb{A})$ coincides with collinearity in $\mathsf{C}_{3,3}(\mathbb{K}, \mathbb{A})$.

We first claim that (Y, Υ) is an abstract variety, that is, the subspace generated by any symp S intersects $\mathscr{V}(\mathbb{K}, \mathbb{A})$ precisely in S. Indeed, by the mentioned transitivity, we may assume that S contains the points $(1, 0, \ldots, 0)$ and $(0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0)$. Then the null set of $\varphi_{x,1}$ restricted to the subspace with equations $\ell_1 = k_2 = k_3 = y = X_2 = X_3 = Y_1 = Y_2 = Y_3 = 0$ is S, and $\langle S \rangle$ clearly does not contain any other point of $\mathscr{V}(\mathbb{K}, \mathbb{A})$.

By Lemma 5.6 of [16] and the transitivity of $\mathrm{Aut}\mathscr{V}(\mathbb{K}, \mathbb{A})$ on pairs of points at mutual distance 3, we have $T_x \cap T_y = \emptyset$ when $\delta(x, y) = 3$, which implies that (ALV1) holds. This now immediately implies that $\dim T_x \leq 3d + 3$, for all x, that is (ALV3) holds.

We finally verify (ALV2). Since $\mathrm{Aut}\mathscr{V}(\mathbb{K}, \mathbb{A})$ acts as a permutation group of (permutation) rank 4 on the set of symps, it suffices to check the axiom for only three specific cases, one where the two symps intersect in a line and two where the two symps are disjoint. The former situation is given by the two quadratic forms $\varphi_{x,1}$ and $\varphi_{x,2}$ (and the corresponding host spaces indeed intersect exactly in a line) and the latter by $\varphi_{x,1}$ and $\varphi_{y,1}$ or $\varphi_{y,21}$ (and the corresponding host spaces are clearly disjoint).

This completes the proof of the theorem. □

10.6 Application to the varieties of the second row of the FTMS

Denote by W the 27-dimensional subspace of V generated by the e_i and the $e_{i,j}$, $i = 1, 2, 3$, $j \in \{0, 1, \ldots, 7\}$. If follows from Corollary 10.18 that W intersects $\mathscr{E}_7(\mathbb{K})$ in the Cartan variety $\mathscr{E}_6(\mathbb{K})$. Then we obtain the following elegant constructions of $\mathscr{E}_6(\mathbb{K})$. Note that it is known that the latter can be described as the intersection of 27 quadrics, which are even explicitly given in [7]. Here, we provide a combinatorial way to "remember" the equations, and a compact algebraic way to write them down. Both follow from our construction of $\mathscr{E}_7(\mathbb{K})$ above by restricting to $\mathbb{P}(W)$.

Corollary 10.40 *Let Γ_1 be the Schäfli graph and let $\mathscr{S}_1$ be a Hermitian spread of Γ_1. Let a basis of W be indexed by the vertices of Γ_1, say $(e_v)_{v\in V_1}$. For each set of vertices $\{v_{-5},\ldots,v_{-1},v_1,\ldots,v_5\}$ of a pentacross D, with v_i not adjacent to v_{-i}, $i\in\{1,\ldots,5\}$, and where we have chosen the indices so that $\{v_{-1},v_1\}$ belongs to a member of $\mathscr{S}_1$, we define the quadratic form φ_D, in coordinates $X_{-1}X_1-X_{-2}X_2-X_{-3}X_3-X_{-4}X_4-X_{-5}X_5$, where X_i is the coordinate corresponding to the basis vector e_{v_i}, $i\in\{-5,\ldots,-1,1,\ldots,5\}$. Then $\mathscr{E}_6(\mathbb{K})$ is the common null set of the quadratic forms φ_D, for D ranging over all pentacrosses of Γ_1.*

Proof With the notation of Subsection 10.2.2, this follows from restricting the quadratic forms belonging to $(\Gamma_2,\infty,\mathscr{S}')$ to W. □

The second consequence also holds in the ovoidal case, so we state it as such. We denote by $\mathscr{V}_2(\mathbb{K},\mathbb{A})$ the usual Veronese representation of the projective plane $\mathbb{P}^2(\mathbb{A})$, for $\mathbb{A}$ a quadratic alternative division algebra over $\mathbb{K}$.

Corollary 10.41 *Let $\mathbb{A}$ be a finite dimensional quadratic alternative algebra over $\mathbb{K}$. Set $d=\dim_{\mathbb{K}}\mathbb{A}$. Identify $\mathbb{K}^{3d+3}$ with $\mathbb{K}\times\mathbb{K}\times\mathbb{K}\times\mathbb{A}\times\mathbb{A}\times\mathbb{A}$. Then the set of points of $\mathbb{P}^{3d+2}(\mathbb{K})$ with generic coordinates $(x_1,x_2,x_3,X_1,X_2,X_3)$, $x_i\in\mathbb{K}$, $X_i\in\mathbb{A}$, $i=1,2,3$, satisfying each of the quadratic equations $X_i\overline{X}_i=x_{i+1}x_{i+2}$ and $x_i\overline{X}_i=X_{i+1}X_{i+2}$, for all $i\in\{1,2,3\}$ mod 3, is the point set of the Segre variety $\mathscr{S}_{2,2}(\mathbb{K})$ if $\mathbb{A}\cong\mathbb{L}'$, the line Grassmannian variety $\mathscr{G}_{6,2}(\mathbb{K})$ if $\mathbb{A}\cong\mathbb{H}'$, the Cartan variety $\mathscr{E}_6(\mathbb{K})$ if $\mathbb{A}\cong\mathbb{O}'$ and the Veronese variety $\mathscr{V}_2(\mathbb{K},\mathbb{A})$ if $\mathbb{A}$ is a division algebra.*

Proof The proof for the hyperbolic case is similar to the proof of Corollary 10.40, now using the explicit forms of the quadratic forms containing the coordinate x in List (L), possibly restricted to the appropriate subspace as in the proof of Corollary 10.34. The ovoidal case follows similarly from Theorem 10.38. □

Corollary 10.42 *Let $|\mathbb{K}|>2$. Then the quadratic Zariski closure of the image of the affine Veronese map $\mu:\mathbb{A}\times\mathbb{A}\to W:(X_2,X_3)\mapsto(1,X_2\overline{X}_2,X_3\overline{X}_3,\overline{X}_2X_3,\overline{X_3},X_2)$ is $\mathscr{S}_{2,2}(\mathbb{K})$ if $\mathbb{A}\cong\mathbb{L}'$, it is $\mathscr{G}_{6,2}(\mathbb{K})$ if $\mathbb{A}\cong\mathbb{H}'$, it is $\mathscr{E}_6(\mathbb{K})$ if $\mathbb{A}\cong\mathbb{O}'$, and it is $\mathscr{V}_2(\mathbb{K},\mathbb{A})$ if $\mathbb{A}$ is a division algebra.*

Proof Clearly, every point in the image of μ satisfies the quadratic equations given in Lemma 10.41. A direct computation shows that a point belongs to the quadratic Zariski closure of the image of μ and not to the image of μ if and only if it can be written as $(0,X_2\overline{X}_2,X_3\overline{X}_3,\overline{X}_2X_3,0,0)$, which also satisfies the said quadratic equations. Also, it is easy to check that a point $(1,y_2,y_3,Y_1,Y_2,Y_3)$ satisfies the equations of Lemma 10.41 if and only if it can be written as $(1,X_2\overline{X}_2,X_3\overline{X}_3,\overline{X}_2X_3,\overline{X_3},X_2)$. Now the corollary follows. □

Remark 10.43 It is easy to show that, if $\mathbb{A}$ is associative, then the quadratic Zariski closure of the image of μ coincides with the image of the *projective Veronese map* $\overline{\mu}$: $\mathbb{A} \times \mathbb{A} \times \mathbb{A} \to W : (X_1, X_2, X_3) \mapsto (X_1\overline{X}_1, X_2\overline{X}_2, X_3\overline{X}_3, \overline{X}_2X_3, \overline{X_3}X_1, \overline{X}_1X_2)$. We leave the straightforward proof to the reader.

Remark 10.44 Corollaries 10.41 and 10.42 also hold for infinite dimensional quadratic alternative division algebras $\mathbb{A}$ over $\mathbb{K}$, in which case $\mathbb{A}$ is an inseparable field extension of $\mathbb{K}$ where $\text{char}\mathbb{K} = 2$.

Acknowledgment. The research leading to the first part of this paper was carried out in Auckland while the first and third author were trapped in their accommodation by lock down due to COVID-19, still graciously enjoying the help and hospitality of the second author, who is funded by New Zealand Marsden Fund grant MFP-UOA-2122.

Our gratitude also goes to the referee for their willingness and courage to read in detail through this long paper.

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