Desarguesian Finite Generalized Quadrangles are Classical or Dual Classical

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Abstract. Let $\mathcal{S} = (P, B, I)$ be a finite generalized quadrangle of order (s, t), $s > 1$, $t > 1$. Given a flag (p, L) of $\mathcal{S}$, a (p, L)-collineation is a collineation θ of $\mathcal{S}$ which fixes each point on L and each line through p. For any line N incident with p, $N \neq L$, and any point u incident with L, $u \neq p$, the group $G(p, L)$ of all (p, L)-collineations acts semiregularly on the lines M concurrent with N, p not incident with M, and on the points w collinear with u, w not incident with L. If the group $G(p, L)$ is transitive on the lines M, or equivalently, on the points w, then we say that $\mathcal{S}$ is (p, L)-transitive. We prove that the finite generalized quadrangle $\mathcal{S}$ is (p, L)-transitive for all flags (p, L) if and only if $\mathcal{S}$ is classical or dual classical. Further, for any flag (p, L), we introduce the notion of (p, L)-desarguesian generalized quadrangle, a purely geometrical concept, and we prove that the finite generalized quadrangle $\mathcal{S}$ is (p, L)-desarguesian if and only if it is (p, L)-transitive.

1. Introduction

Let $\mathcal{S} = (P, B, I)$ be a finite generalized quadrangle (GQ) of order (s, t), $s > 1$, $t > 1$; that is, $\mathcal{S}$ is an incidence structure of points and lines, with a symmetric point-line incidence relation satisfying (i) each point is incident with $1 + t$ lines ($t \geq 1$) and two distinct points are incident with at most one line, (ii) each line is incident with $1 + s$ points ($s \geq 1$) and two distinct lines are incident with at most one point, and (iii) if x is a point and L is a line not incident with x, then there is a unique pair $(y, M) \in P \times B$ for which $xI MIyIL$. Results on generalized quadrangles may be found in the monograph (Payne and Thas 1984).

Given a flag $(p, L) \in P \times B$ of $\mathcal{S}$, a *(p, L)-collineation* is a collineation θ of $\mathcal{S}$ which fixes each point on L and each line through p. By 2.4.1 of (Payne and Thas 1984) θ acts semiregularly on the set $P \backslash p^{\perp}$ and also on the set $B \backslash L^{\perp}$ (i.e., θ is an elation about p and an elation about L in the terminology of (Payne and Thas 1984)). In particular it follows that the group $G(p, L)$ of (p, L)-collineations has order dividing st. For let N be a line incident with p, $N \neq L$, and u a point incident with L, $u \neq p$. Then $G(p, L)$ acts semiregularly on the lines M concurrent with N, p not incident with M. It follows that $|G(p, L)| = st$ iff $G(p, L)$ is transitive on the set of st lines meeting N at points different from p iff $G(p, L)$ is transitive on the set of points collinear with u but not incident with L.

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If these equivalent conditions hold, we say $\mathcal{S}$ is *(p, L)-transitive*. Note that $\mathcal{S}$ is (p, L)-transitive iff (the point-line dual of) $\mathcal{S}$ is (L, p)-transitive. Further, one can check that each finite classical or dual classical GQ is (p, L)-transitive for all flags (p, L).

This note has two main results. The first is:

THEOREM 1. The GQ $\mathcal{S}$ is (p, L)-transitive for all flags (p, L) iff $\mathcal{S}$ is classical or dual classical.

Let $V = (p_1, L_1, \dots, p_4, L_4)$ be an ordered quadrilateral of $\mathcal{S}$. So $p_1, \dots, p_4$ are four distinct points, $L_1, \dots, L_4$ are four distinct lines, and $p_i I L_i I p_{i+1}$, where subscripts are taken modulo 4. The quadrilateral V is said to be *opposite* a flag (p, L) provided p is not incident with L_i and p_i is not incident with L, $i = 1, 2, 3, 4$. So let V be a quadrilateral opposite a flag (p, L), and let θ be a nonidentity (p, L)-collineation. Then V^θ is also a quadrilateral opposite (p, L) and V, V^θ are in perspective from the flag (p, L) according to the following definition: Two (ordered) quadrilaterals $V = (p_1, L_1, \dots, L_4)$ and $V' = (p_1', L_1', \dots, L_4')$ which are opposite a flag (p, L) are said to be *in perspective from* (p, L) provided p_i, p_i' are collinear with a common point on L, and L_i, L_i' are concurrent with a common line through p, $i = 1, 2, 3, 4$. Note that this definition is also self-dual.

The GQ $\mathcal{S}$, with flag (p, L), is said to be *(p, L)-desarguesian* provided the following condition holds: For any quadrilateral $V = (p_1, L_1, \dots, L_4)$ opposite (p, L) and any flag (p_1', L_1') satisfying (*i*) p_1' is not on L and p is not on L_1', (*ii*) $p_1 \perp r_1 \perp p_1'$ for some r_1 on L, (*iii*) $L_1 \perp M_1 \perp L_1'$ for some M_1 through p, there is a quadrilateral $V' = (p_1', L_1', \dots, L_4')$ opposite (p, L) and in perspective with V from (p, L).

Note that $\mathcal{S}$ is (p, L)-desarguesian iff (the point-line dual of) $\mathcal{S}$ is (L, p)-desarguesian.

It is easy to see that if $\mathcal{S}$ is (p, L)-transitive, then $\mathcal{S}$ is (p, L)-desarguesian. The converse yields our second main result.

THEOREM 2. For a given flag (p, L), $\mathcal{S}$ is (p, L)-transitive iff $\mathcal{S}$ is (p, L)-desarguesian.

There is an immediate corollary.

COROLLARY 1. The GQ $\mathcal{S}$ is (p, L)-desarguesian for all flags (p, L) iff $\mathcal{S}$ is classical or dual classical.

2. (p, L)-Transitivity

Let $\mathcal{S} = (P, B, I)$ be a finite GQ of order (s, t), as in Section 1. This section is devoted to a proof of Theorem 1, and for that we need the concept of property (H). For distinct points x, y, recall that the *closure* of the pair (x, y) is $cl(x, y) = \{z \in P : z^\perp \cap \{x, y\}^{\perp\perp} \neq \emptyset\}$. Then a point p has *property* (H) provided that, whenever $T = \{x, y, z\}$ is a triad of points contained in $p^\perp$ and $x \in cl(y, z)$, then $y \in cl(x, z)$. By 5.6.2 of (Payne and Thas 1984) we see that property (H) is a fundamental property for points (or lines) in the classical or dual classical GQ.

LEMMA 1. Suppose $\mathcal{S}$ is (p, L)-transitive for fixed p and all lines L through p. Then p has property (H).

Proof. Let $T = \{x, y, z\}$ be a triad in $p^{\perp}$ for which $x \in cl(y, z)$ say $wIpx$ with $w \in \{y, z\}^{\perp\perp}$. By hypothesis there is a (p, pz)-collineation θ mapping w to x. Then $y^{\theta}Ipy$ and $\theta : \{w, y, z\}^{\perp\perp} \to \{x, y^{\theta}, z\}^{\perp\perp}$, forcing $y \in cl(x, z)$. Hence $x \in cl(y, z)$ implies $y \in cl(x, z)$.

A *panel* of $\mathcal{S}$ is an ordered triple (L, p, M), where L and M are distinct lines incident with the point p, or an ordered triple (x, L, y) where x and y are distinct points incident with the line L. For a given panel (x, L, y), let $H(x, L, y)$ be the group of all collineations of $\mathcal{S}$ which are both $(x, L)-$ and $(L, y)-$collineations. For each line M through x, $M \neq L$, $H(x, L, y)$ acts semiregularly on the points of M different from x. Hence $|H(x, L, y)|$ divides s. We say the panel (x, L, y) is *Moufang* provided $|H(x, L, y)| = s$. For a panel (L, p, M) the group $H(L, p, M)$ is defined in a dual manner. And (L, p, M) is *Moufang* provided $|H(L, \mathrm{p}, M)| = t$. If every panel of $\mathcal{S}$ is Moufang, then $\mathcal{S}$ is said to be a *Moufang* GQ. By a theorem of P. Fong and G.M. Seitz [1974] (cf. J. Tits [1976]), a Moufang GQ must be classical or dual classical. The main result of (Thas, Payne and Van Maldeghem 1991) is that if every panel of $\mathcal{S}$ of one type (say of the form (x, L, y)) is Moufang, then every panel is Moufang.

LEMMA 2. If $\mathcal{S}$ is (p, L)-transitive for every flag (p, L), then $\mathcal{S}$ is Moufang, and hence classical or dual classical.

Proof. Suppose $\mathcal{S}$ is (p, L)-transitive for all flags (p, L). So all points (and dually, all lines) have property (H). By 5.6.2 of (Payne and Thas 1984) we have the following:

(i) Each point is regular, or
(ii) each hyperbolic line $\{x, y\}^{\perp\perp}$, $x \not\perp y$, has just two points, or
(iii) $\mathcal{S} \cong H(4, s)$.

And dually,

(i)′ Each line is regular, or
(ii)′ for each pair of nonconcurrent lines L, M, $\{L, M\}^{\perp\perp} = \{L, M\}$, or
(iii)′ $\mathcal{S} \cong H^*(4, t)$, the point-line dual of $H(4, s)$.

We may assume $\mathcal{S} \not\cong H(4, s)$ and $\mathcal{S} \not\cong H^*(4, t)$. Suppose all lines are regular, so $s \leq t$. If each point is regular, then $s = t$ and $\mathcal{S} \cong W(2^e)$ (cf. 5.2.1 of (Payne and Thas 1984)). So we may assume each hyperbolic line has two points. Let p, p' be distinct points on a line L; so (p, L, p') is a panel for which we shall show $\mathcal{S}$ is (p, L, p')-transitive. Let q, q' be any points $(\neq p)$ on a line N $(\neq L)$ through p. We show there is a $\theta \in H(p, L, p')$ mapping q to q'. Let N' be any line $(\neq L)$ through p', and M, M' the lines through q, q' meeting N'. Let x be any point of L, $p \neq x \neq p'$. By the regularity of lines there is a line K through x meeting M at a point y and M' at a point y'. Then the (p, L)-collineation θ mapping M to M' maps q to q', y to y', and fixes K and N'. Now let N'' be any line through p', $N' \neq N'' \neq L$. Define lines M'' and T by $yIM'' \perp N''$ and $pIT \perp M''$. By regularity of the pair (M'', L), the line through y' meeting T also meets N''. As θ fixes T and maps y to y', clearly θ fixes N''. So θ fixes all lines through p'. (Actually, this shows θ is an ordinary symmetry about L, i.e., it fixes all lines meeting L.) As $\mathcal{S}$ is (p, L, p')-transitive for all panels (p, L, p') of this type, $\mathcal{S}$ is classical or dual classical.

By duality, the only remaining case has all hyperbolic lines of size 2 and all spans $\{L, M\}^{\perp\perp}$, $L \not\perp M$, of size two. Also, by duality we may assume $s \leq t$.

Fix a point p, and let $L_0, \ldots, L_t$ be the lines through p. Each (p, L_i)−collineation is an elation about p. If $G = \langle \theta : \theta$ is a (p, L_i)-collineation for some L_i, $0 \leq i \leq t\rangle$, then by 8.2.4(iii) of (Payne and Thas 1984) G is a group of elations about p. Now let L_{j_0}, L_{j_1} be any two of the lines through p, x_0 on L_{j_0}, x_1 on L_{j_1}, $x_0 \not\perp x_1$. Let N_0, M_0 be two lines ($\neq L_{j_0}$) through x_0 and N_1, M_1 the two lines through x_1 with $N_0 \perp N_1$, $M_0 \perp M_1$. Say N_0IqIN_1, M_0IrIM_1. Let θ_0 be the (p, L_{j_0})-collineation mapping N_1 to M_1. Clearly θ_0 also maps N_0 to M_0. Similarly, θ_1 is the (p, L_{j_1})-collineation mapping N_0 to M_0, and hence mapping N_1 to M_1. Then both $\theta_0, \theta_1 \in G$, and they both map q to r. So $\theta = \theta_0 = \theta_1$ is the unique element of G mapping q to r. Hence $\theta \in H(L_{j_0}, p, L_{j_1})$. It follows that $|H(L, p, L')| = t$ for each panel of the form (L, p, L'). By (Thas, Payne and Van Maldeghem 1991) the dual result holds; so $\mathcal{S}$ is classical or dual-classical. (However, by 9.5.1 of (Payne and Thas 1984) $\mathcal{S}$ cannot exist.)

3. (p, L)-Desarguesian Implies (p, L)-Transitive

To prove Theorem 2 it must be shown that, if $\mathcal{S}$ is (p, L)-desarguesian, then it is (p, L)-transitive. So suppose $\mathcal{S}$ is (p, L)-desarguesian for a given flag (p, L). Let L' be an arbitrary line through p, $L' \neq L$. Suppose $MIp'IL'$, $M'Ip''IL'$, $p' \neq p \neq p'' \neq p'$, $M \neq L' \neq M'$. To show that $\mathcal{S}$ is (p, L)-transitive, it will suffice to construct a (p, L)-collineation θ mapping M to M'. Since any GQ with $t = 2$ and $s > 1$ is classical, hence (p, L)-transitive, we may assume that $t > 2$.

The set of all points incident with a line N will be denoted by N^*. The idea of the proof is to define a bijection θ from the pointset $(P \backslash p'^{\perp}) \cup L^* \cup M^*$ onto the pointset $(P \setminus p''^{\perp}) \cup L^* \cup M'^*$ which preserves collinearity, and then to extend θ to an automorphism α of $\mathcal{S}$ for which α is a (p, L)-collineation mapping M to M'.

The definition of θ proceeds according to six cases determined by the type of point involved.

Type (*i*) For $x \in L^*$, $x^\theta = x$.

Type (*ii*) For $x = p'$, $x^\theta = p''$.

Type (*iii*) For $p' \neq xIM$, put $x^\theta = x'$, where $x'IM'$ and $x \perp r \perp x'$ for some point r, $r \in L^*$. So $\theta : M^* \to M'^*$.

Type (*iv*) Suppose y is any point of P not on any line of $\{L, M\}^{\perp}$. Let (N, x) be the flag defined by $yINIxIM$, and define the line L'' by $pIL'' \perp N$. If $x' = x^\theta$, N' is the line defined by $x'IN' \perp L''$. Then $y^\theta = y'$ where $y'IN'$ and $y \perp r \perp y'$ for some point r on L. By hypothesis N cannot meet L; so $L'' \neq L$, forcing y' to be on no line of $\{M', L\}^{\perp}$. Hence a counting argument shows that θ is a bijection from the set of points of type (*iv*), i.e., on no line of $\{L, M\}^{\perp}$ to the set of points on no line of $\{M', L\}^{\perp}$, which is the set of points of type $(iv)^\theta$. Note that each point of N^* other than x is of type (*iv*). The points of N^* are mapped bijectively to the points of N'^*, and each point of N' other than x' is of type $(iv)^\theta$. Note that if $y \perp p$, then $L'' = yp$ and y^θ is the unique point for which $x'Ix'y^\theta Iy^\theta Iyp$. So $\theta : \{yp\}^* \to \{yp\}^*$.

Let y_1, y_2 be distinct collinear points of type (*iv*). We shall prove that $y_1^\theta \perp y_2^\theta$. By the preceding comments we may assume that $p \notin (y_1y_2)^*$ and $y_1y_2 \not\perp M$. Let $MIx_i \perp y_i$, $i = 1, 2$. Then

$$V = (y_2, y_2x_2, x_2, x_2x_1, x_1, x_1y_1, y_1, y_1y_2)$$

is a quadrilateral opposite (p, L). Since $\mathcal{S}$ is (p, L)-desarguesian, there is a quadrilateral $V' = (y_2^\theta, y_2^\theta x_2^\theta, \ldots)$ opposite (p, L) and in perspective with V from (p, L). Clearly $V' = (y_2^\theta, y_2^\theta x_2^\theta, x_2^\theta, x_2^\theta x_1^\theta, x_1^\theta, x_1^\theta y_1^\theta, y_1^\theta, ?)$, forcing the last line to be $y_1^\theta y_2^\theta$. Moreover, as in the earlier two cases, y_1y_2 and $y_1^\theta y_2^\theta$ must be concurrent with the same line through p. So θ yields a bijection from the set of points of type (*iv*) to the set of points of type $(iv)^\theta$ which preserves collinearity. A counting argument shows that it must also preserve noncollinearity.

Type (*v*) Let uIL', $p \neq u \neq p'$. If $y \perp u$ with y of type (*iv*), then let u^θ be the point of L' collinear with y^θ. Note that since $t > 1$ there are such points y. We now show that u^θ is independent of the choice of y. So let y_1 be of type (*iv*) with $u \perp y_1$ and $y \neq y_1$. If y_1Iuy, then $y^\theta \perp y_1^\theta$ and $L' \perp y^\theta y_1^\theta$ by case (*iv*). Now assume that $y_1 \notin (uy)^*$. Suppose $L'Iu' \perp y_1^\theta$ with $u' \neq u^\theta$. If y_2Iuy and y_3Iuy_1, with y_2 and y_3 of type (*iv*), then necessarily $y_2^\theta \not\perp y_3^\theta$. (By the last line of case (*iv*).) In $((yu)^* \cup (y_1u))^* \setminus \{u\}$ there are at least s points of type (*iv*). Since points y_2^θ and y_3^θ, respectively, on $u^\theta y^\theta$ and $u'y_1^\theta$ are never collinear (so the point of $y_1^\theta u'$ collinear with y_2^θ must not be the image of a type (*iv*) point of y_1u), it follows that $((yu)^* \cup (y_1u)^*) \setminus \{u\}$ has at least s points on lines of $\{L, M\}^\perp$, hence exactly s points on lines of $\{L, M\}^\perp$. Since $t > 2$, there is a line Y through u for which all points of $Y^* \setminus \{u\}$ are of type (*iv*). Let $y_4 \in Y^* \setminus \{u\}$. Then by the preceding argument, $y_4^\theta \perp u^\theta$ and $y_4^\theta \perp u'$. Hence $u^\theta = u'$, a contradiction. Consequently u^θ is independent of the choice of y.

Type (*vi*) Let z be on a line of $\{L, M\}^\perp$, $z \notin M^* \cup L^* \cup L'^*$. Let $L'Iu \perp z$, $LIr \perp z$, and let R' be defined by $rIR' \perp M'$. Then z^θ is defined by $R'Iz^\theta \perp u^\theta$. So the $s^2 - s$ points of type (*vi*) are mapped bijectively to the $s^2 - s$ points (of type $(vi)^\theta$) on a line of $\{L, M'\}^\perp$ but not on M' or L or L'.

Now we prove that the restriction of θ to $P \setminus p'^\perp$ preserves collinearity. If $y_1 \perp y_2$ with y_1, y_2 of type (*iv*), then we already have proved that $y_1^\theta \perp {y_2}^\theta$. So let $y \perp z$, with y, $z \in P \setminus p'^\perp$, y of type (*iv*), z on a line of $\{L, M\}^\perp$. If zIL, clearly $z^\theta \perp y^\theta$. Suppose $z \notin L$, but yz meets L' at a point u. Let $LIr \perp z$, and let R be any line through r with $L \neq R \neq rz$. If $RIm \perp u$, clearly m is of type (*iv*). If $y^\theta Iu^\theta m^\theta$, then $y \perp m$, a contradiction. Hence y^θ is not on the line through u^θ and concurrent with $R' = rm^\theta$. It follows that y^θ is a point of the line through u^θ and concurrent with rz^θ (which meets M'), i.e., $y^\theta Iu^\theta z^\theta$. Hence $y^\theta \perp z^\theta$ and y^θ, z^θ are collinear with the common point u^θ of L'.

Let z_1, z_2 be distinct elements of type (*vi*), with $z_1 \perp z_2$ and z_1, z_2 collinear with a common point u of L'. Assume that $z_1^\theta \not\perp z_2^\theta$. We have $z_1^\theta \perp u^\theta \perp z_2^\theta$. Let $z_1 \perp r_1IL$ and $u^\theta z_2^\theta Iy_2' \perp r_1$. Then y_2' is not on $r_1z_1^\theta$, so is not on a line of $\{M', L\}^\perp$, so $y_2' = y_2^\theta$ for a y_2 of type (*iv*). Analogously, on $u^\theta z_1^\theta$ there is a point y_1^θ ($y_1^\theta \perp r_2$, $z_2 \perp r_2IL$) with y_1 of type (*iv*). Since y_1 and y_2 are on $uz_1 = uz_2$ and $y_1 \perp y_2$, we have $y_1^\theta \perp y_2^\theta$, a contradiction. Hence $z_1^\theta \perp z_2^\theta$, and z_1^θ, z_2^θ are collinear with the common point u^θ of L'.

Let y be of type (*iv*), z of type (*vi*), $y \perp z$, and assume that $L'Iu \perp y$ with $u \not\perp z$. Let $L'Im \perp z$. First assume that all points of $(mz)^* \setminus \{m\}$ are of type (*vi*). Then z is the only point of yz which is of type (*vi*), while the s other points are of type (*iv*). The images of the points of $(yz)^* \setminus \{z\}$ are incident with a common line V. Let z' be the remaining point of V. Since m is collinear with no point of $(yz)^* \setminus \{z\}$, m^θ is collinear with no point of $V^* \setminus \{z'\}$. Hence $m^\theta \perp z'$. Let $LIr \perp z$. Since $(z')^{\theta^{-1}}$ is of type (*vi*), we have $rz' = rz^\theta$. And since $m^\theta \perp z^\theta$, necessarily $z' = z^\theta$. Consequently $y^\theta \perp z^\theta$. Now assume that in $(mz)^*$ there is at least one point y_1 of type (*iv*). Let $yzIy_2 \perp p$. Then y_2 is of type (*iv*). Further, let $y_1IR \perp M$ and $RIy_3 \perp y_2$. Then y_3 is of type (*iv*). Since $\mathcal{S}$ is (p, L)-desarguesian, there is a quadrilateral $V' = (y_2^\theta, y_2^\theta y_3^\theta, \ldots)$ in perspective with $V = (y_2, y_2y_3, y_3, y_3y_1, y_1, y_1z, z, zy_2)$ from (p, L). Consequently $y_2^\theta \perp z^\theta$. Hence we may assume that $y \neq y_2$ and $y^\theta \not\perp z^\theta$. Since $y_2^\theta \perp z^\theta$, the line $y^\theta y_2^\theta$ is not concurrent with $m^\theta z^\theta$. Let $m^\theta \perp z''Iy^\theta y_2^\theta$. Then z'' is of type $(vi)^\theta$, for otherwise there arises the triangle $m(z'')^{\theta^{-1}}z$. No point v of type $(iv)^\theta$ of $(y^\theta y_2^\theta)^* \setminus \{z''\}$ is collinear with a point w of type $(iv)^\theta$ of $(m^\theta z^\theta)^* \setminus \{m^\theta\}$, for then $v^{\theta^{-1}}$ on yy_2 would be collinear with $w^{\theta^{-1}}$ on mz. On $(yz)^* \cup (mz)^*$ there are at least s points of type (*iv*). Hence on $((y^\theta y_1^\theta)^* \setminus \{z''\}) \cup ((m^\theta z^\theta)^* \setminus \{m^\theta\})$ there are at least s points of type $(iv)^\theta$, and hence also at least s points of type $(vi)^\theta$. Consequently in $(yz)^* \cup (mz)^*$ there are exactly $s + 1$ points on lines of $\{L, M\}^\perp$. Let T be a line through z with $mz \neq T \neq yz$ and $T \not\perp L$. (Since $t > 2$ the line T exists.) All points of $T^* \setminus \{z\}$ are of type (*iv*). By the preceding arguments $(T^*)^\theta$ is the pointset of a line through z^θ. Let $y^\theta \perp y_4^\theta \in T^{*\theta}$. Then yy_4z is a triangle, giving a contradiction. We conclude that $y^\theta \perp z^\theta$.

Finally, let z_1, z_2 be distinct collinear points on lines of $\{L, M\}^\perp$ but not in $p'^\perp$. If z_1IL, then clearly $z_1^\theta \perp z_2^\theta$. So we may assume that $z_1, z_2 \notin L^*$. If on z_1z_2 there are at least two points y_1, y_2 of type (*iv*), then by preceding cases $z_i^\theta \perp y_i^\theta$, $i = 1, 2$, $y_1^\theta \perp y_2^\theta$. Hence $z_1^\theta \perp z_2^\theta$. If, on the other hand, z_1z_2 has at least s points on lines of $\{L, M\}^\perp$, then by (the point-line dual of) 1.3.4(iv) of (Payne and Thas 1984), all points of z_1z_2 are on lines of $\{L, M\}^\perp$, implying $z_1z_2 \perp L'$. Again by a preceding case $z_1^\theta \perp z_2^\theta$.

Hence θ induces a bijection of $P \setminus p'^\perp$ onto $P \setminus p''^\perp$, and collinear points are mapped onto collinear points. A counting argument shows that noncollinear points of $P \setminus p'^\perp$ are mapped to noncollinear points of $P \setminus p''^\perp$.

Let T be a line of $\mathcal{S}$ not incident with p', with $L' \not\perp T \not\perp M$, and let $TIq \perp p'$. All points of T different from q are mapped by θ onto the s points of a line T' not collinear with p''. Let $T'Iq' \perp p''$. Suppose T_1 $(\neq T)$ is a second line through q and not incident with p'. Assume that the corresponding line T_1' is incident with q_1', $q_1' \perp p''$. Since no point of $(T')^* \setminus \{q'\}$ is collinear with any point of $(T_1')^* \setminus \{q_1'\}$, clearly T' meets T_1' at $q_1' = q'$. Put $q^\theta = q'$. Now let q, q_1 be on a line through p', with $q \neq q_1 \neq p' \neq q$. If $q^\theta \not\perp q_1^\theta$, then let n' be a point for which $q^\theta \perp n' \perp q_1^\theta$ and $n' \not\perp p''$. So $q \perp (n')^{\theta^{-1}} \perp q_1$ with $(n')^{\theta^{-1}} \not\perp p'$, a contradiction. Consequently $q^\theta \perp q_1^\theta$. Now θ is defined on all points of $\mathcal{S}$, and is the restriction to P of a uniquely defined collineation α of $\mathcal{S}$. Clearly the restriction of α to $(P \setminus p'^\perp) \cup L'^* \cup M^*$ is θ. The collineation α is a (p, L)-collineation mapping (p', M) to (p'', M'). We conclude that $\mathcal{S}$ is (p, L)-transitive.

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