ON THE GENERATING RANK AND EMBEDDING RANK OF THE HEXAGONIC LIE INCIDENCE GEOMETRIES

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Abstract. Given a (thick) irreducible spherical building Ω, we establish a bound on the difference between the generating rank and the embedding rank of its long root geometry and the dimension of the corresponding Weyl module, by showing that this difference does not grow when taking certain residues of Ω (in particular the residue of a vertex corresponding to a point of the long root geometry, but also other types of vertices occur). We apply this to the finite case to obtain new results on the generating rank of mainly the exceptional long root geometries, answering an open question by Cooperstein about the generating ranks of the exceptional long root subgroup geometries. We completely settle the finite case for long root geometries of type A_n, and the case of type $\mathsf{F}_{4,4}$ over any field with characteristic distinct from 2 (which is not a long root subgroup geometry, but a hexagonic geometry).

1. Introduction and statement of the main results

The absolutely universal embedding of most "popular" Lie incidence geometries, i.e., the point-line geometries arising naturally from simple algebraic groups and their variants, are known. This knowledge allows one to treat each projective embedding of such geometry as a quotient, or in geometric terms a projection, of a unique, usually well known and natural embedding. A major exception is the class of long root geometries, which is perhaps the most important class of Lie incidence geometries in that each split algebraic group admits such geometry, and all long root geometries share a number of common intrinsic properties, turning them into a class of geometries ready-made to treat all corresponding algebraic groups simultaneously. A consequence of a group theoretic result of Völklein [36, Remark(2)] implies that also the universal embedding of the long root geometries are known, as long as their symplecta (see below for the definition) have rank at least 3, and as long as they are defined over either a perfect field in positive characteristic, or a (possibly infinite dimensional) algebraic extension of the rationals. We note that this consequence was not mentioned in the survey paper [15].

The usual geometric technique to show that a given embedding in a projective space of dimension r of a given geometry Δ is absolutely universal is to exhibit a set of $r+1$ points of Δ that (linearly) generates Δ. This method has been applied a number of times and it works well for many geometries, in particular for the exceptional geometries of type $\mathsf{E}_{6,1}$ and $\mathsf{E}_{7,7}$, see for instance Blok & Brouwer [3] and Cooperstein & Shult [16]. Hence, for those geometries, the *generating rank*, or briefly *g-rank* (that is, the smallest number of points generating the geometry) is equal to the *embedding rank*, or briefly *e-rank* (that is, the largest rank of a projective space hosting an embedding of the geometry that spans the whole projective space—the rank of a projective space is its projective dimension plus one, that is, the dimension of the underlying vector space). We will frequently use the notation $\bullet$-rank to simultaneously provide statements and reasonings for both

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the e-rank and g-rank where it is understood that once •-rank is chosen to be either the e-rank or g-rank it is fixed for the entire statement or reasoning.

However, for the long root geometries the relation between these two ranks seems to be more complex. In the classical cases, Cooperstein [13] proved that, over a finite prime field, the generating rank equals the embedding rank, but he does not say anything about other (finite) fields. The smallest case, type A_2, has been investigated by Blok & Pasini [4] and they prove that the generating rank strongly depends on the minimal number of generators of the multiplicative group of the underlying field. This is somewhat in contrast with Völklein's result mentioned above (because Völklein's result applies to fields for which this number is large, even unbounded). Indeed, it is even unknown whether the embedding rank for geometries of type $\mathsf{A}_{2,\{1,2\}}$ over fields that are not finitely generated is finite or not. This situation is particularly complicated since there might not exist a universal embedding.

The aim of the present paper is to prove some general results about both the embedding rank and generating rank of long root geometries, primarily of exceptional type, but we also handle some classical cases, by relating the respective ranks of different types of geometries. A major consequence of our investigations is that the generating rank and embedding rank of any long root geometry over a prime field (except possibly for type F_4 over $\mathbb{F}_2$) are equal to each other (Theorem C below). For finite fields other than prime fields, the generating rank is at most one more than the embedding rank if symplecta have rank at least 3, and they are equal again for type $\mathsf{A}_{n,\{1,n\}}$ (Theorem D below). We also completely settle the case of type $\mathsf{F}_{4,4}$, regardless of the underlying field in characteristic distinct from 2.

More exactly, we relate the e-rank and g-rank of a long root geometry to the e-rank and g-rank, respectively, of the long root geometry of a residual geometry in the corresponding spherical building, showing that a certain excess is non-increasing as the rank of the building increases. More precisely, and using some terminology introduced later, the *excess* is the difference between either the e-rank or g-rank of a long root geometry and the dimension of the so-called *Weyl module* associated with the longest root of the corresponding root system of the underlying split spherical building (the adjoint representation module). Concerning the exceptional cases, we have the following main result.

Theorem A. *Abbreviating the assertion "The excess of the generating rank of the long root geometry of type* X_r *over the field* $\mathbb{K}$ *is at most the excess of the long root geometry of type* Y_s *over* $\mathbb{K}$*" to "*$\mathsf{Y}_s \to \mathsf{X}_r$*", we have the following assertions:*

$$\begin{gathered}\mathsf{A}_5 \to \mathsf{E}_6,\\ \mathsf{D}_6 \to \mathsf{E}_7,\\ \mathsf{D}_5 \to \mathsf{E}_6 \to \mathsf{E}_7 \to \mathsf{E}_8,\\ \mathsf{A}_2 \to \mathsf{C}_3 \to \mathsf{F}_4.\end{gathered}$$

The same thing holds for the embedding rank.

The arrows $\mathsf{A}_2 \to \mathsf{C}_3 \to \mathsf{F}_4$ of the previous result have to be read in a specific sense, which will be explained in Subsection 6.2.2.

The same method can also be applied to the classical cases, and we obtain:

Theorem B. *With the same notation as* Theorem A*, we have the following assertions (where the geometries are defined over the field* $\mathbb{K}$*):*

$$\begin{aligned}\mathsf{A}_n \to \mathsf{A}_{n+1} \to \mathsf{D}_{n+2} \to \mathsf{D}_{n+3}, &\quad n \geq 2, \quad \mathbb{K} \text{ arbitrary},\\ \mathsf{B}_n \to \mathsf{B}_{n+1}, &\quad n \geq 2, \quad \operatorname{char} \mathbb{K} \neq 2.\end{aligned}$$

(This holds for both the embedding rank and generating rank.)

Here, also the arrow $\mathsf{B}_2 \to \mathsf{B}_3$ has to be read in a specific sense, which we explain in Subsection 7.4. The upshot of both theorems is that, if progress is made for some low-rank classical case, then this has implications on many long root geometries of higher rank, but also on (some of) the exceptional cases. In the limit, new results in the case of type $\mathsf{A}_{2,\{1,2\}}$ could imply better bounds for all other cases! As far as we know, it is the first time that this connection is made so explicit.

We now specialise to the finite case. A result of Völklein [36, Remark(2)] implies that the e-rank of the finite long root geometries that admit the universal embedding (hence with symplecta of rank at least 3; so not of type A_2 or G_2) is precisely the dimension of the Weyl module, that is, the universal embedding in case there are no symplecta of rank 2 is given by the corresponding *Weyl embedding*. We use Theorem A to show the following result (which is the analogue of the classical case).

Theorem C. *The generating rank of the long root geometry of type E_n, $n \in \{6,7,8\}$, and F_4 over a prime field (distinct from $\mathbb{F}_2$ in case of F_4) is exactly the dimension ω of the corresponding Weyl module.*

This answers an open question by Cooperstein [15, p.117]. We do not know whether the case F_4 over $\mathbb{F}_2$ is a true exception or not.

Over a general finite field, we can prove the following, as a corollary to Theorems A and B.

Theorem D. *The long root geometry of type A_n, $n \geq 2$, B_n, $n \geq 3$ and odd characteristic, D_n, $n \geq 3$, E_n, $n \in \{6,7,8\}$, and F_4 over a finite field (distinct from $\mathbb{F}_2$ in case of F_4) is generated by $\omega+1$ points, where ω is the dimension of the corresponding Weyl module. Hence the corresponding generating rank always belongs to $\{\omega, \omega+1\}$. Also, over a finite field which is not a prime field, both the generating rank and the embedding rank of the long root geometry of type A_n, $n \geq 2$, are equal to $\omega + 1 = (n+1)^2$.*

Our proof, which restricted to the classical cases is very different from the one in [13], uses the existence and construction of the so-called equator geometries, which are also long root geometries, but of lower rank. This induction process can be carried out in different ways, providing the different sufficient conditions stated in Theorem A above. It can also be applied to the classical cases and to the metasymplectic space $\mathsf{F}_{4,4}(\mathbb{K})$. However, for the latter we can use the notion of an *extended equator geometry* to determine both the g-rank and e-rank of $\mathsf{F}_{4,4}(\mathbb{K})$, $\operatorname{char}\mathbb{K} \neq 2$. This provides a complete answer to another question by Cooperstein if $\operatorname{char}\mathbb{K} \neq 2$ [15, p.120].

Theorem E. *The generating rank and embedding rank of the Lie incidence geometry of type $\mathsf{F}_{4,4}$ over an arbitrary field of characteristic distinct from 2 is 26.*

2. Definitions and notation

Henceforth let $\mathbb{K}$ be a field. We denote by $\mathbb{P}^n(\mathbb{K})$ the n-dimensional projective space over $\mathbb{K}$, where $n \geq 1$. The subspace generated by a family $\mathscr{F}$ of subsets of points is denoted by $\langle S \mid S \in \mathscr{F} \rangle$.

Point-line geometries—A *point-line geometry* Δ is a pair $(X, \mathscr{L})$, with $\mathscr{L} \subseteq 2^X$. The members of X are called the *points*, usually denoted with lower case Latin letters, those of $\mathscr{L}$ are the *lines*, usually denoted with upper case Latin letters. Since we will deal with embedded geometries, we will always assume that two points are contained in at most one line, and that lines have constant size at least 3 (this true for all Lie incidence geometries, which we will introduce in the next paragraph). Points on a common line are called *collinear*; if two points x, y are collinear we write $x \perp y$. If the joining line is unique, we denote it by $\langle x, y \rangle$. The set of points collinear to a given point x is $x^\perp$ and for $Y \subseteq X$ we define $Y^\perp = \{x \in X \mid x \perp y, \forall y \in Y\}$. Two subsets Y_1, Y_2 of X are said to be collinear, in symbols also $Y_1 \perp Y_2$, if each point of either is collinear to every point of the other.

We will frequently talk about collinear lines, for instance. The *collinearity graph* of Δ is the graph with vertices the points of Δ, adjacent when collinear.

Lie incidence geometries—The geometries of importance in the present paper are Lie incidence geometries. These are (natural) point-line geometries associated to spherical buildings (we always assume that a building is thick) or, equivalently, to simple algebraic groups and their variants like mixed groups and classical groups. A Lie incidence geometry $\Delta = (X, \mathscr{L})$ is constructed from a spherical building Ω of rank $r \geq 2$ in the following way. Let $T = \{1, 2, \ldots, r\}$ be the type set of Ω (using Bourbaki labelling [5]) and choose $J \subseteq T$. Then the *point set* X is the set of flags (or simplices) of type J; the *lines* are the sets of flags of type J completing flags of type $T \setminus \{j\}$ to a chamber, for $j \in J$. (A chamber is a flag or simplex of type T; note that different flags of type $T \setminus \{j\}$ can give rise to the same line.) If Ω has a simply laced diagram, then it is determined by its Coxeter diagram X_r and a field $\mathbb{K}$ and we denote Δ by $\mathsf{X}_{r,J}(\mathbb{K})$, and say that Δ has type $\mathsf{X}_{r,J}$. We write $\mathsf{X}_{r,j}$ if $J = \{j\}$. In this paper, we only consider subsets J consisting of one element, except in case A_r, where $J = \{1, r\}$ will play a role. If the Dynkin diagram contains a double bond, then we will only be concerned about the *split* case, that is,

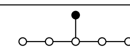
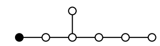
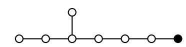
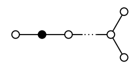
(1) for type B_n the building associated with a parabolic quadric (viewed as polar space) in $\mathbb{P}^{2n}(\mathbb{K})$ with standard equation $X_0^2 = X_{-1}X_1 + X_{-2}X_2 + \cdots + X_{-n}X_n$,
(2) for type C_n the building associated with a non-degenerate alternating form in $\mathbb{P}^{2n-1}(\mathbb{K})$,
(3) for type F_4 the building whose residue of type B_2 is precisely the case $n = 2$ in (1) above.

We also denote the associated Lie incidence geometries by $\mathsf{X}_{n,i}(\mathbb{K})$, where $\mathsf{X} \in \{\mathsf{B}, \mathsf{C}, \mathsf{F}\}$, and for appropriate n, i. If J corresponds to the set of fundamental roots not perpendicular to the longest root of the root system corresponding to the Dynkin diagram, then we say that $\mathsf{X}_{n,J}(\mathbb{K})$ is a *long root (Lie incidence) geometry*. More precisely, these are the Lie incidence geometries of split types $\mathsf{A}_{n,\{1,n\}}$, $n \geq 2$, $\mathsf{B}_{n,2}$, $n \geq 2$, $\mathsf{C}_{\mathsf{n},1}$, $n \geq 3$, $\mathsf{D}_{\mathsf{n},2}$, $n \geq 4$, $\mathsf{E}_{6,2}$, $\mathsf{E}_{7,1}$, $\mathsf{E}_{8,8}$, $\mathsf{F}_{4,1}$ and $\mathsf{G}_{2,1}$.

In spherical buildings the notion of *opposition* is an important one. Two chambers in a spherical building are *opposite* is they are at a maximal distance in the *chamber graph*, whose vertices are chambers, adjacent if they differ in one element. Two flags F, F' are opposite if for every chamber C containing F, there is a chamber C' opposite C containing F', and for every chamber C' containing F', there is a chamber C opposite C' containing F. Elements of a Lie incidence geometry which correspond to opposite vertices of the underlying building will also be called *opposite* in the Lie incidence geometry. If we want to emphasize in which geometry Δ the opposition is considered, we sometimes write Δ-*opposite*.

Embedded geometries—We are interested in embedded geometries. Let $\mathbb{K}$ be a field and let $n \geq 2$ be a natural number. Then we say that the geometry $\Delta = (X, \mathscr{L})$ is *embedded in* $\mathbb{P}^n(\mathbb{K})$ if X is a subset of the point set of $\mathbb{P}^n(\mathbb{K})$, and each member of $\mathscr{L}$ coincides (as a set of points) with a unique line of $\mathbb{P}^n(\mathbb{K})$. In the literature, this is sometimes also called a *full* embedding. Usually it is also tacitly assumed that X spans $\mathbb{P}^n(\mathbb{K})$. We can also view an embedding as a map ι from X to the point set of $\mathbb{P}^n(\mathbb{K})$. Using this point of view, an embedding ι into $\mathbb{P}^n(\mathbb{K})$ (with $\iota(X)$ generating $\mathbb{P}^n(\mathbb{K})$) is called *universal* if for each other embedding, say ι' into $\mathbb{P}^m(\mathbb{K})$, there exists an isomorphism $\theta : \mathbb{P}^m(\mathbb{K}) \to U$, with U an m-dimensional subspace of $\mathbb{P}^n(\mathbb{K})$, and a subspace W complementary to U in $\mathbb{P}^n(\mathbb{K})$, such that for each point $x \in X$, the point $\iota(x)$ is projected from W onto U onto the point $\theta(\iota'(x))$, that is, $\langle \iota(x), W \rangle \cap U = \theta(\iota'(x))$. If Δ is a long root geometry $\mathsf{X}_{n,J}(\mathbb{K})$, then there always exists an embedding that arises from the adjoint representation (we shall refer to the corresponding module briefly as the *Weyl module*, as we do not consider other representations), called briefly the *Weyl embedding* (cf. Section 4.3 of [2] (in type C_n, we consider the Veronese embedding, see below for definitions). The dimension of the Weyl module shall be denoted by $\omega(\mathsf{X}_n(\mathbb{K}))$, and it equals the number of roots of a root system of type X_n, plus the rank

n. For convenience, we tabulate this value for the different geometries appearing in the present paper.

Δ	$\omega(\Delta)$	Diagram	Δ	$\omega(\Delta)$	Diagram
$\mathsf{A}_{n,\{1,n\}}(\mathbb{K})$	n^2+2n		$\mathsf{E}_{6,2}(\mathbb{K})$	78	
$\mathsf{B}_{n,2}(\mathbb{K})$	$2n^2+n$		$\mathsf{E}_{7,1}(\mathbb{K})$	133	
$\mathsf{C}_{n,1}(\mathbb{K})$	$2n^2+n$		$\mathsf{E}_{8,8}(\mathbb{K})$	248	
$\mathsf{D}_{n,2}(\mathbb{K})$	$2n^2-n$		$\mathsf{F}_{4,1}(\mathbb{K})$	52	

Polar Spaces—Parapolar spaces were introduced to capture the (Lie incidence geometries related to the) spherical buildings of exceptional type. It is convenient to work within this framework, especially when dealing with (classes of) different Lie incidence geometries sharing some common properties. Since parapolar spaces amply contain polar spaces as subgeometries, we first provide a definition of polar spaces.

Polar spaces have been introduced by Veldkamp [35], later on included in the theory of buildings by Tits [33], and around the same time the axioms have been simplified by Buekenhout & Shult [6]. It is the latter point of view we take here.

Recall that a *subspace* of a point-line geometry $\Delta = (X, \mathscr{L})$ is a subset S of the point set X such that, if two points a, b belong to S, then all lines containing both a and b are contained in S. A subspace H is called a *geometric hyperplane* if each line of Γ intersects H nontrivially. A geometric hyperplane is *proper* if it does not coincide with X. A *singular* subspace is a subspace every two points of which are collinear. Note that the empty set and a single point are singular subspaces. A *deep point* of a subspace S is a point x such that each line containing x is contained in S.

We can now define polar spaces.

Definition 2.1. A point-line geometry $\Delta = (X, \mathscr{L})$ is called a *polar space* if the following hold.

(PS1) Every line contains at least three points.
(PS2) No point is collinear to all other points.
(PS3) Every nested sequence of singular subspaces is finite.
(PS4) For any point x and any line L, either one or all points on L are collinear to x.

Some basic properties—Let Δ be a polar space. Then every one of its singular subspaces is a projective space, and its dimension can hence be defined as the dimension of the projective space. There exists an integer $r \geq 2$ such that each nested sequence of singular subspaces has length $r+1$. We call r the *rank* of Δ. Consequently, the maximal singular subspaces of Δ have dimension $r-1$. Note that axiom (PS3) implies that the rank is finite, which is not strictly necessary for polar spaces. As we will only consider polar spaces of finite rank, we preferred to include this axiom.

Parapolar spaces—Let $\Delta = (X, \mathscr{L})$ be a point-line geometry. A subspace S of Δ is called *convex* if, for any pair of points $\{p, q\} \subseteq S$, every point incident with a line occurring in a shortest path between p and q is contained in S. Also, Δ is called *connected* if its incidence graph is connected.

Definition 2.2. A *parapolar space* is a point-line geometry $\Delta = (X, \mathscr{L})$ such that:

(PPS1) Δ is connected and, for each line L and each point $p \notin L$, p is collinear to either none, one or all of the points of L and there exists a pair $(p, L) \in X \times \mathscr{L}$ with $p \notin L$ such that p is collinear to no point of L.

(PPS2) For every pair of non-collinear points p and q in $\mathscr{P}$, one of the following holds:
   (a) the convex closure of $\{p, q\}$ is a polar space, called a *symplecton*; we say that p and q are *symplectic* and denote $p \perp\!\!\!\perp q$;

(b) $p^\perp \cap q^\perp$ is a single point called the *centre*; we say p is *special to* q, denoted $p\bowtie q$. The centre of the special pair $\{p,q\}$ is denoted $\mathfrak{c}(p,q)$;

(c) $p^\perp \cap q^\perp = \emptyset$.

(PPS3) Every line is contained in at least one symplecton,

For $p \in X$, denote by $p^{\perp\!\!\!\perp} = \{q \in X \mid p \perp\!\!\!\perp q\}$ and let $p^{\bowtie} = \{q \in X \mid p \bowtie q\}$.

Some basic properties—Let $\Delta = (X, \mathscr{L})$ be a parapolar space. First of all, note that it is never a polar space by (PPS1) and (PS4). In general, a singular subspace of a parapolar space should not necessarily be projective; however, in the Lie incidence geometries that we will consider all singular spaces are projective spaces. Therefore, a *plane* always means a singular subspace of projective dimension 2. This enables us to define the *residue at a point* $x \in X$ in the usual way: it is the geometry $\mathrm{Res}_\Delta(x)$ with point set the set of lines through x and line set the set of (full) planar line pencils with vertex x. It corresponds to the building-theoretic notion of the residue at x. Likewise, the residue at any other singular subspace can be defined, as long as the rank of the symps is at least two more than the dimension of the subspace. Objects that are opposite in the residue at a point x will be briefly called *locally opposite at* x.

If there are no special pairs in Δ, we say that Δ is *strong*. Symplecta are also briefly called *symps*. We denote symps with greek letters like ξ and ζ. By Kasikova & Shult [24], all Lie incidence geometries with symps of rank at least 3 that we will encounter admit a universal embedding.

A *para* is a proper convex subspace of Δ, whose points and lines form a parapolar space themselves. The set of symps of a para is a subset of the set of symps of Δ. Paras are rather rare in long root geometries; they are classified in [26]. Restricted to the exceptional types, Main Result 1 of [26] says that all paras of $\mathsf{E}_{7,1}(\mathbb{K})$ are isomorphic to $\mathsf{E}_{6,1}(\mathbb{K})$, and all paras of $\mathsf{E}_{6,2}(\mathbb{K})$ are isomorphic to $\mathsf{D}_{5,5}(\mathbb{K})$; the geometry $\mathsf{E}_{8,8}(\mathbb{K})$ does not contain paras, just like any geometry of type $\mathsf{F}_{4,1}$ or $\mathsf{F}_{4,4}$.

Hexagonic geometries—In the present paper we are mainly concerned with a particular class of parapolar spaces, some specific Lie incidence geometries of exceptional type, known as the (split) exceptional hexagonic geometries. They are the (exceptional) long root geometries of type $\mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8, \mathsf{F}_4$, but also the Lie incidence geometry $\mathsf{F}_{4,4}(\mathbb{K})$, which very much behaves like a long root geometry, but is not. They have in common the following properties, which are the defining axioms for abstract hexagonic geometries and can be found in [30, Chapter 17]:

(H1) If x is a point and ξ is a symplecton, with $x \notin \xi$, then $x^\perp \cap \xi$ is not exactly one point.

(H2) If a plane π and a line L meet at a point p, then either

(a) every line of π containing p lies in a common symplecton with L, or

(b) exactly one such line incident with p and π has this property.

(H3) If (p, L) is an incident point-line pair, then there exists a second line N such that $L \cap N = \{p\}$ and no symplecton contains $L \cup N$, i.e., $x \bowtie y$ for each $x \in L \setminus \{p\}$ and $y \in N \setminus \{p\}$.

By definition, all symps have rank at least 3. The diameter of the collinearity graph of a hexagonic geometry is 3 [25, Theorem 39]. Points p, q at distance 3 are opposite and we denote this by $p \leftrightarrow q$.

The hexagonic Lie incidence geometries that we will be considering are $\mathsf{B}_{n,2}(\mathbb{K})$, $n \geq 3$, $\mathsf{D}_{n,2}(\mathbb{K})$, $n \geq 4$, $\mathsf{E}_{6,2}(\mathbb{K})$, $\mathsf{E}_{7,1}(\mathbb{K})$, $\mathsf{E}_{8,8}(\mathbb{K})$, $\mathsf{F}_{4,1}(\mathbb{K})$ and $\mathsf{F}_{4,4}(\mathbb{K})$ and for the purposes of the present paper, we also call the parapolar space $\mathsf{A}_{n,\{1,n\}}(\mathbb{K})$, $n \geq 3$, hexagonic. All these, except for $\mathsf{F}_{4,4}(\mathbb{K})$, are long root geometries. We will also work with $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$, $\mathsf{B}_{2,2}(\mathbb{K})$ and $\mathsf{C}_{n,1}(\mathbb{K})$, but these are not parapolar spaces and are hexagonic in a broader sense, namely, in the sense of *root filtration spaces* [10, 11]. Without going into details, we mention that all hexagonic geometries are root filtration spaces, but the latter is more general. In the present paper, we shall use the notion *exceptional hexagonic geometries* to refer to the Lie incidence geometries $\mathsf{E}_{6,2}(\mathbb{K})$, $\mathsf{E}_{7,1}(\mathbb{K})$, $\mathsf{E}_{8,8}(\mathbb{K})$, $\mathsf{F}_{4,1}(\mathbb{K})$ and $\mathsf{F}_{4,4}(\mathbb{K})$.

Generating rank—Let $\Delta = (X, \mathscr{L})$ be a point-line geometry. Let $S \subseteq X$. Since obviously the intersection of an arbitrary family of subspaces of Δ is again a subspace, and since X itself is a subspace, the intersection of all subspaces containing S is well defined and is a subspace again, which we denote by $\langle S \rangle$. A subset S is said to generate Δ if $\langle S \rangle = X$. The generating rank $\rho_{\mathsf{g}}(\Delta)$ of Δ is the minimal cardinality of a generating set. For a long root geometry $\mathsf{X}_{n,J}(\mathbb{K})$, we write $\rho_{\mathsf{g}}(\mathsf{X}_n(\mathbb{K})) = \rho_{\mathsf{g}}(\mathsf{X}_{n,J}(\mathbb{K}))$. We sometimes abbreviate 'generating rank' to *g-rank*.

Embedding rank—Let $\Delta = (X, \mathscr{L})$ be a point-line geometry. If Δ does not admit any embedding into some finite dimensional projective space, then we say that its embedding rank is 0. Otherwise, the embedding rank $\rho_{\mathsf{e}}(\Delta)$ is equal to

$$1 + \sup\{n \in \mathbb{N} \mid \Delta \text{ is embedded in } \mathbb{P}^n(\mathbb{K}), \text{ for some field } \mathbb{K}, \text{ with } \langle X \rangle = \mathbb{P}^n(\mathbb{K})\}.$$

If Δ admits a universal embedding in $\mathbb{P}^n(\mathbb{K})$, then the embedding rank is equal to $n+1$. We sometimes abbreviate 'embedding rank' to *e-rank*. Note that the e-rank of Δ is always at most the g-rank of Δ. For $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$, we need a more restrictive notion of the embedding rank:

Segre embedding rank of $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$—Let Δ be the Lie incidence geometry $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$. This geometry is a subgeometry of all hexagonic parapolar spaces, in particular those with symps of rank at least 3, and so, in particular of those that admit a universal embedding. That universal embedding admits a projection onto the Weyl embedding, by definition of universality. This Weyl embedding, however, contains the Weyl embedding of $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$. Hence, we are only interested in those embeddings of $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$ that admit a projection onto the Weyl embedding of $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$. One plus the corresponding supremum of such (projective) dimensions will be called the *Segre embedding rank of* $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$ and denoted by $\rho^{\circ}_{\mathsf{e}}(\mathsf{A}_2(\mathbb{K}))$. To motivate this name, we note that the Weyl embedding of $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$ arises from intersecting the Segre variety corresponding to the product geometry $\mathbb{P}^2(\mathbb{K}) \times \mathbb{P}^2(\mathbb{K})$ with a generic hyperplane. Indeed, that Segre geometry is given, after introducing homogeneous coordinates in $\mathbb{P}^2(\mathbb{K})$ and $\mathbb{P}^8(\mathbb{K})$, by the image of the map

$$\mathbb{P}^2(\mathbb{K}) \times \mathbb{P}^2(\mathbb{K}) \to \mathbb{P}^8(\mathbb{K}) : (a, b, c; x, y, z) \mapsto (ax, ay, az; bx, by, bz; cx, cy, cz),$$

and we can choose the hyperplane such that it induces the equality $ax + by + cz = 0$. This provides the Weyl embedding of $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$.

Veronese embedding rank and Veronese generating rank—In order for our procedure to make sense for the arrows in Theorem A involving type C_3, and to be uniform across all types, we need to consider a different type of embedding and generation of symplectic polar spaces, but also of projective spaces. Let $\Delta = (X, \mathscr{L})$ be a point-line geometry. A *Veronese subspace* V is a set of points such that each line not entirely contained in V intersects V in at most two points. Any (ordinary) subspace is a Veronese subspace, but the converse is not true: consider two collinear points. The intersection of an arbitrary family of Veronese subspaces is again a Veronese subspace, and X itself is a Veronese subspace, hence we can again consider the Veronese subspace V spanned by a subset $S \subseteq X$; we say that V is *Veronese generated by* S. The minimal cardinality of such a set S Veronese generating X is called the *Veronese generating rank* and denoted by $\rho^*_{\mathsf{g}}(\Delta)$.

We say that $\Delta = (X, \mathscr{L})$ is *Veronese embedded in* $\mathbb{P}^n(\mathbb{K})$ if X is a subset of the point set of $\mathbb{P}^n(\mathbb{K})$ (generating it), and each member of $\mathscr{L}$ is a nondegenerate conic in some plane of $\mathbb{P}^n(\mathbb{K})$. The *Veronese embedding rank* $\rho^*_{\mathsf{e}}(\Delta)$ is 0 if there does not exist any Veronese embedding of Δ in a finite dimensional projective space; otherwise it is the supremum of all natural numbers n for which Δ is embedded in $\mathbb{P}^{n-1}(\mathbb{K})$, for some field $\mathbb{K}$. Given a Veronese embedding $\epsilon : X \subseteq \mathbb{P}^n(\mathbb{K})$ of Δ, then the *ϵ-relative Veronese embedding rank* is the supremum of all natural numbers m for which Δ admits an embedding in $\mathbb{P}^{m-1}(\mathbb{K})$ that projects onto ϵ.

Again we abbreviate 'Veronese generating rank' and 'Veronese embedding rank' to *Veronese g-rank* and *Veronese e-rank*, respectively.

Structure of the paper—In Section 3 we gather some known results on the g-rank and e-rank of a number of Lie incidence geometries. In particular, we focus on long root geometries of types A_n and D_n over finite fields, in particular over (finite) prime fields. In Section 4 we discuss various properties of long root geometries, starting with general properties in Section 4.1, before focussing on $E_{6,2}$ in Section 4.2 and $E_{7,1}$ in Section 4.3. We define the equator geometries related to paras. In Section 5 we prove Theorem A for the long root geometries of types $\mathsf{E}_6, \mathsf{E}_7$ and E_8. In Section 4.4 we explain the role of geometric hyperplanes (since these are essential to our arguments). In Sections 5.1 and 5.2 we define certain subspaces and prove these are hyperplanes. We show that these are designed to allow us to inductively compute the g-rank and e-rank, and we conclude the proof of Theorem A in the case E. In Section 6 we discuss the cases of geometries of type $\mathsf{F}_{4,1}$ and $\mathsf{F}_{4,4}$ in detail. We prove Theorem E and the remainder of Theorem A. The proof of Theorem B is very similar and we only sketch the proof, leaving the details to the reader, in Section 7. The proof of Theorem C (being well known for the classical cases [13]) is given in Section 5.3 for the case of type E, and in Section 6.3 for type F_4. Finally, Theorem D is proved in Section 8, where we also provide a geometric proof of Völklein's result (restricted to finite fields) using the statements of the present paper, for types D_n, $n \geq 4$, $\mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$ and F_4 (the latter in characteristic distinct from 2).

3. Generation and embeddings of some Lie incidence geometries

3.1. **Projective spaces, polar spaces, strong parapolar spaces.** In the table below we list the g-rank and e-rank for several Lie incidence geometries, strong if they are parapolar spaces. Since in all cases $\rho_g(\Delta) = \rho_e(\Delta)$ we write $\rho(\Delta)$ in the table. The table includes the so-called minuscule embeddings of geometries of type $\mathsf{E}_{6,1}$ and $\mathsf{E}_{7,7}$. The results there follow from the existence of embeddings in the given (vector) dimension (see for instance [1] and [12]), the fact that the g-rank is exactly equal to that dimension (see [3] or [16, Corollary 7.5]), and the existence of the absolutely universal embedding (see [24]). The results we mention in this section are also surveyed in [15].

Fact 3.1. *The following is known for g-rank and e-rank, where $\mathbb{K}$ is an arbitrary field:*

Δ	$\rho(\Delta)$	References	Δ	$\rho(\Delta)$	References
$\mathsf{A}_{n,k}(\mathbb{K})$	$\binom{n+1}{k}$	[3, 16]	$\mathsf{E}_{6,1}(\mathbb{K})$	27	[1, 3, 16, 24]
$\mathsf{D}_{n,1}(\mathbb{K})$	$2n$	[3, 16]	$\mathsf{E}_{7,7}(\mathbb{K})$	56	[3, 12, 16, 24]
$\mathsf{D}_{\mathsf{n},\mathsf{n}}(\mathbb{K})$	2^{n-1}	[3, 16, 37]			

In particular, the g-rank and e-rank of $\mathsf{A}_{5,3}(\mathbb{K})$, $\mathsf{D}_{6,6}(\mathbb{K})$ and $\mathsf{E}_{7,7}(\mathbb{K})$ are equal to $3.2^{r-3}+8$, where r is the rank of the corresponding building.

All geometries mentioned in the table of Fact 3.1 are generated by the set of points contained in one given apartment, as is proved in [3, 16]. The constructions and decompositions of apartments given in Section 7 of [34] imply immediately the following facts.

Fact 3.2. *The Lie incidence geometry $\mathsf{D}_{6,6}(\mathbb{K})$ is generated by two opposite 5-spaces and the set of points collinear to a plane in each of these 5-spaces.*

Proof. This follows from the third to last diagram of Section 7.2 in [34]. One can also (easily) prove this directly using the associated polar space. □

Proposition 3.3. *The Lie incidence geometries $\mathsf{A}_{5,3}(\mathbb{K})$, $\mathsf{D}_{6,6}(\mathbb{K})$ and $\mathsf{E}_{7,7}(\mathbb{K})$ are generated by two opposite symps and the set of points collinear to maximal singular subspaces in both these symps.*

Proof. This follows from the fourth diagram of Section 7.2 in [34] for type $\mathsf{E}_{7,7}$. The other types are similar. □

3.2. **Some classical hexagonic geometries over finite fields and prime fields.** In order to prove Theorems C and D, we collect some known results about the e-rank and the g-rank of long root geometries of types A_n and D_n over finite fields, in particular over (finite) prime fields.

Fact 3.4 (Theorem 4.1 of [13]). *The $\bullet$-rank of $\mathsf{A}_{n,\{1,n\}}(\mathbb{K})$, for $\mathbb{K}$ a finite prime field, is n^2+n.*

In fact, Cooperstein [13] only proves the above for finite fields. However, the proof of this, and of the next fact, also works for $\mathbb{K}=\mathbb{Q}$ and $n=2$.

Fact 3.5. *The $\bullet$-rank of $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$, for $\mathbb{K}$ a finite field, but not a prime field, is 9. The Segre embedding rank of $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$ for a finite field $\mathbb{K}$ is equal to 8.*

Proof. It follows from Section 2 of [31] that the g-rank is at least 9, that the e-rank is equal to 9 and that the Segre e-rank is equal to 8. The rest follows straight from Theorem 1.1 of [4]. □

Remark 3.6. If $\mathbb{K}$ is not finitely generated, for example when $\mathbb{K}$ is the algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$, then $\rho_g(\mathsf{A}_{n,\{1,n\}}(\mathbb{K}))$ is infinite as shown by Cardinali, Giuzzi and Pasini [7]. On the other hand $\rho_e(\mathsf{A}_{n,\{1,n\}}(\overline{\mathbb{F}}_p)) \in \{(n+1)^2-1,(n+1)^2\}$, as also shown in [7]. However, it will follow from Proposition 8.1 that $\rho_e(\mathsf{A}_{n,\{1,n\}}(\overline{\mathbb{F}}_p)) = (n+1)^2$.

Fact 3.7 (Theorem 5.1 of [13]). *The $\bullet$-rank of $\mathsf{D}_{n,2}(\mathbb{K})$, for $\mathbb{K}$ a finite prime field, is $2n^2-n$.*

4. Properties of the long root geometries of exceptional type

4.1. **General properties.** We start by listing a number of general properties which we will use later on. Note that the references we use may use a labelling convention different from our Bourbaki labelling.

Fact 4.1 (Proposition 2 of [9]). *In the Lie incidence geometries $\mathsf{B}_{3,3}(\mathbb{K})$, $\mathsf{A}_{5,3}(\mathbb{K})$, $\mathsf{D}_{6,6}(\mathbb{K})$ and $\mathsf{E}_{7,7}(\mathbb{K})$, given a point p and a symp ξ, p is collinear to at least one point of ξ. If $p \notin \xi$ is collinear to at least a line of ξ, then p is collinear to a maximal singular subspace of ξ. If $p \notin \xi$ is collinear to precisely a point x of ξ, then p is at distance 3 (in the collinearity graph) of all points of ξ not collinear to x.*

Fact 4.2. *In the Lie incidence geometries $\mathsf{B}_{3,3}(\mathbb{K})$, $\mathsf{A}_{5,3}(\mathbb{K})$, $\mathsf{D}_{6,6}(\mathbb{K})$ and $\mathsf{E}_{7,7}(\mathbb{K})$, collinearity is an isomorphism between the point sets of two symps if and only if these symps are opposite.*

Proof. This follows from Theorem 3.28 and Proposition 3.29 of [33]. □

Fact 4.3. *Let Δ be a hexagonic Lie incidence geometry or one of $\mathsf{B}_{3,3}(\mathbb{K})$, $\mathsf{A}_{5,3}(\mathbb{K})$, $\mathsf{D}_{6,6}(\mathbb{K})$ and $\mathsf{E}_{7,7}(\mathbb{K})$. If p and q are opposite points of Δ, and L is any line containing q, then L contains a unique point at distance 2 from p.*

Proof. This is condition (F) for root filtration spaces, see [10]. For the geometries $\mathsf{B}_{3,3}(\mathbb{K})$, $\mathsf{A}_{5,3}(\mathbb{K})$, $\mathsf{D}_{6,6}(\mathbb{K})$ and $\mathsf{E}_{7,7}(\mathbb{K})$, this follows from Theorem 17.1.2(2) in [30]. □

In several ways, $\mathsf{F}_{4,1}(\mathbb{K})$ or $\mathsf{F}_{4,4}(\mathbb{K})$ behave slightly different compared to the other exceptional hexagonic geometries.

Fact 4.4 (Theorem 2 of [9]). *Let Δ be an exceptional hexagonic geometry. If a point p of Δ is collinear to a point of a symp ξ, and $p \notin \xi$, then $p^\perp \cap \xi$ is either a line, or a maximal singular subspace of ξ. The latter possibility does not occur in $\mathsf{F}_{4,1}(\mathbb{K})$ and in $\mathsf{F}_{4,4}(\mathbb{K})$.*

Fact 4.5 ([8]). *Let Δ be $\mathsf{F}_{4,1}(\mathbb{K})$ or $\mathsf{F}_{4,4}(\mathbb{K})$. If two symps of Δ share a line, their intersection is a plane.*

Fact 4.6 (Lemma 6 of [29]). *Let Δ be a hexagonic Lie incidence geometry. If a point x of Δ is collinear to a unique line L of a symp ξ, then all points of ξ collinear to L, but not on L, are symplectic to x, whereas all points of $\xi \setminus L^\perp$ are special to x.*

Lemma 4.7. *Let Δ be a hexagonic Lie incidence geometry. Each point x of Δ is symplectic to at least one point of each symp ξ. Moreover, being symplectic is an isomorphism between the point sets of two symps if and only if these symps are opposite. In particular, if ξ contains a point opposite x, then ξ contains a unique point symplectic to x. Also, if Δ is $\mathsf{F}_{4,1}(\mathbb{K})$ or $\mathsf{F}_{4,4}(\mathbb{K})$ and $x^\perp \cap \xi = \emptyset$, then ξ contains a unique point symplectic to x.*

Proof. Let x be a point not contained in a symp ξ. Assume for a contradiction that no point of ξ is collinear or symplectic to x. Hence by Fact 4.3, there exists a point $y \in \xi$ special to x. Set $r = \mathfrak{c}(x, y)$. By (H1), r is collinear to some line $L \subseteq \xi$ and since all points of L are now at distance 2 from x, (H2) implies that some point of L is symplectic to x. This shows the first assertion. The second assertion follows from Theorem 3.28 and Proposition 3.29 of [33]. The third assertion follows from the second, as there exists a symp ξ' through x opposite ξ (this follows from the definition of opposition in [33, §2.39]).

For the last assertion, suppose for a contradiction that $x^\perp \cap \xi = \emptyset$ and x is symplectic to two points y and z of ξ. If $y \perp z$, then y is collinear to a line $L \ni z$ of the symp $\xi(x, z)$ determined by x and z by Fact 4.4. Since y and x are symplectic, Fact 4.6 implies that x is collinear to L and hence to z, a contradiction. So $y \perp\!\!\!\perp z$. Let u be a point of $y^\perp \cap z^\perp \subseteq \xi$. Again by Fact 4.4, u is collinear to lines L and M of the respective symps $\xi(x, y)$ and $\xi(x, z)$. Let y' be the unique point on L collinear to x, likewise, z' the unique point on M collinear to x. If $y' = z'$ then this point is contained in ξ, contradicting $x^\perp \cap \xi = \emptyset$. So $y' \neq z'$ and hence x and u are symplectic, a contradiction to Fact 4.6 as x is not collinear to L. ☐

Lemma 4.8. *Let Δ be a hexagonic Lie incidence geometry. If a point p of Δ is symplectic to a unique point x of a symp ξ, then all points of ξ collinear to x, but distinct from x, are special to p, whereas all points of $\xi \setminus p^\perp$ are opposite p. In particular, $p^{\perp\!\!\!\perp} \cap q^\perp = \emptyset$ for opposite points p, q.*

Proof. Note that p cannot be collinear with any point ξ since otherwise there would be more than one point of ξ symplectic to p. Let t be a point of $\xi \cap x^\perp \setminus \{x\}$. By the foregoing, t is either special to or opposite p. Therefore, t is not contained in the symp $\xi(p, x)$ and hence, by (H1), t is collinear to a line M of $\xi(p, x)$. Since p is collinear with a point on M we find that p and t are at distance 2 and hence special. Note that $p \bowtie t$ is collinear to x. Observe that the last statement of the lemma can be deduced from this argument.

Next, let y be a point in $\xi \setminus x^\perp$ and suppose for a contradiction that y is special to p, and let $r = \mathfrak{c}(p, y)$ (note that $r \notin \xi$ because $r \perp p$). By (H1), r is collinear with at least a line L of ξ. Note that r is not collinear to x for otherwise $r \in x^\perp \cap y^\perp \subseteq \xi$, whereas we deduced above $r \notin \xi$. So x is collinear to a unique point t of $L \setminus \{y\}$. By the above, p and t are special and $r = \mathfrak{c}(p, t)$. However, in the previous paragraph, we noted that r is collinear to x, contradicting the above. ☐

From Lemma 2(v) of [11] we immediately obtain

Lemma 4.9. *Let Δ be a hexagonic Lie incidence geometry and let x_0, x_1, x_2 and x_3 be four points of Δ. If $x_0 \perp x_1 \perp x_2 \perp x_3$, with $x_0 \bowtie x_2$ and $x_1 \bowtie x_3$, then $x_0 \leftrightarrow x_3$.*

Lastly, we will use the following.

Lemma 4.10. *If in a hexagonic Lie incidence geometry a point $p \in X$ is special to all points of a line $L \in \mathscr{L}$, then there exists a unique line M collinear to p such that M and L are ξ-opposite lines in a symp ξ.*

Proof. Select $x, y \in L$, $x \neq y$, and set $c = \mathfrak{c}(p, x)$. Then $\{c, y\}$ is not special as otherwise p would be opposite y by Lemma 4.9. Also, by Condition (H2) of hexagonic geometry, c is not collinear to y. Hence we may consider $\xi := \xi(c, y)$. Then, by Fact 4.4, p is collinear to a line M of ξ, obviously with $c \in M$. No point of L is collinear to M as such a point would be automatically symplectic to p, contrary to our assumption that all points of L are special to p. Hence L and M are ξ-opposite. Moreover M is unique, otherwise at least two points of L are symplectic to p. The lemma is proved. $\square$

4.2. **The long root geometry of type E_6.** Since a point of the long root geometry $\mathsf{E}_{6,2}(\mathbb{K})$ is given by a 5-space of the Lie incidence geometry $\mathsf{E}_{6,1}(\mathbb{K})$, we first state facts about the latter.

Fact 4.11. *The Lie incidence geometry $\mathsf{E}_{6,1}(\mathbb{K})$ is a strong parapolar space of diameter 2, which is self-dual, that is, the geometry $(\Xi, \mathscr{M})$, where Ξ is the set of symps of Δ and a typical member of $\mathscr{M}$ consists of all symps of $\mathsf{E}_{6,1}(\mathbb{K})$ containing a given maximal singular 4-space, is isomorphic to $\mathsf{E}_{6,1}(\mathbb{K})$. In particular, two symps of $\mathsf{E}_{6,1}(\mathbb{K})$ either intersect in a unique point or in a 4-space. Given a point p and a symp ξ with $p \notin \xi$, the intersection $p^\perp \cap \xi$ is either empty or a singular space of dimension 4 corresponding to a flag of type $\{2, 6\}$ of the underlying spherical building of type E_6.*

Proof. The first statement is 3.7 of [32]; the second follows from Section 3.3 of [32]. The rest is an immediate consequence of these two statements. $\square$

A singular space of dimension 4 that corresponds to a flag of type $\{2, 6\}$ of the underlying building will be referred to as a $4'$-space. It is obviously always contained in a (unique, maximal) 5-space. If $p^\perp \cap \xi = \emptyset$ (which means that p and ξ are opposite), we have:

Fact 4.12. *Given a point p and a symp ξ in $\mathsf{E}_{6,1}(\mathbb{K})$ with $p^\perp \cap \xi = \emptyset$, each symp through p meets ξ in a unique point and this correspondence induces an isomorphism between the dual of the point residue at p and ξ. In particular, each line L containing p contains a unique point p_L with $p_L^\perp \cap \xi$ a $4'$-space V_L (and hence $\langle p_L, V_L \rangle$ is a 5-space), and for each 5-space U containing p, there is a unique $4'$-space V_U in U which is in a symp together with a unique 4-space V_U' of ξ.*

Proof. This follows from Theorem 3.28 and Proposition 3.29 of [33]. $\square$

Proposition 4.13. *Given a symp ξ of $\mathsf{E}_{6,1}(\mathbb{K})$, the set of points $\{p \mid p^\perp \cap \xi \neq \emptyset\}$ is a geometric hyperplane of $\mathsf{E}_{6,1}(\mathbb{K})$.*

Proof. This is exactly (5.3.1) of Section 5.3 in [17]. $\square$

Since the 5-spaces of $\mathsf{E}_{6,1}(\mathbb{K})$ are the points of $\mathsf{E}_{6,2}(\mathbb{K})$, their relation with respect to each other and to points and symps is also relevant for us.

Fact 4.14 (Tits [32])**.** *Two 5-spaces of $\mathsf{E}_{6,1}(\mathbb{K})$ meet in at most a plane; for a point p and a 5-space U, either p and U are incident (so $p \in U$) or $p^\perp \cap U$ is a unique point or a 3-space. Dually, for a symp ξ and a 5-space U, either ξ and U are incident (so $U \cap \xi$ is a $4'$-space) or $\xi \cap U$ is a line or a $4'$-space.*

The set of points of $\mathsf{E}_{6,2}(\mathbb{K})$ on a line corresponds to the set of 5-spaces incident with a plane of $\mathsf{E}_{6,1}(\mathbb{K})$. Using the diagram, one sees that the maximal singular subspaces of $\mathsf{E}_{6,2}(\mathbb{K})$ are 4-spaces, and that the symps of $\mathsf{E}_{6,2}(\mathbb{K})$ are of type D_4 (they correspond to a flag of type $\{1, 6\}$). There are two types of paras in $\mathsf{E}_{6,2}(\mathbb{K})$, corresponding to a residue related to a node of type 1 and to a residue related to a node of type 6, respectively. Both carry the structure of a $\mathsf{D}_{5,5}(\mathbb{K})$ geometry. We refer to the first type as a *para of point-type* and to the latter as a *para of symp-type*. We list the possibilities for the mutual relations between these paras.

Lemma 4.15. *Let Π_1 and Π_2 be distinct paras of $\mathsf{E}_{6,2}(\mathbb{K})$. If Π_1 and Π_2 have the same type, then they intersect each other either in the empty subspace, or in a 4-space; if Π_1 and Π_2 have different types, then they either meet in a symp, in a unique point, or are disjoint. In the latter case, no point of Π_1 is collinear to a point of Π_2, and every point of Π_1 (resp. Π_2) is contained in a unique para Π with a unique 4-space of Π_2 (resp. Π_1), and Π has the same type as Π_2 (resp. Π_1).*

Proof. Suppose first that Π_1 and Π_2 have the same type. By duality, we may assume that both have point-type. Let p_1 and p_2 be the corresponding (distinct) points of $\mathsf{E}_{6,1}(\mathbb{K})$. Obviously, p_1 and p_2 are either on a unique line L, or there is no line joining them. Since the points in $\Pi_1 \cap \Pi_2$ correspond to the 5-spaces of $\mathsf{E}_{6,1}(\mathbb{K})$ containing both p_1, p_2, it follows that in the first case, $\Pi_1 \cap \Pi_2$ is a 4-space (corresponding to the residue of L), and in the latter case, $\Pi_1 \cap \Pi_2$ is empty.

Next, suppose that Π_1 and Π_2 have different types. By symmetry, we may assume that Π_1 corresponds to a point p_1 of $\mathsf{E}_{6,1}(\mathbb{K})$ and Π_2 to a symp ξ_2. Again using Fact 4.11, either $p_1 \in \xi_2$ or $p_1^\perp \cap \xi_2$ is either a $4'$-space or the empty set. In the first case, $\Pi_1 \cap \Pi_2$ is a symp, since it corresponds to the set of 5-spaces of $\mathsf{E}_{6,1}(\mathbb{K})$ incident with both p_1 and ξ_2, i.e., to a flag of type $\{1, 6\}$. In the second case, $\langle p_1, p_1^\perp \cap \xi_2 \rangle$ is the unique 5-space incident with both p_1 and ξ_2 and hence $\Pi_1 \cap \Pi_2$ is a unique point. Finally, in the last case, $p_1^\perp \cap \xi_2$ is empty so every 5-space containing p_1 is disjoint from ξ_2, leading to $\Pi_1 \cap \Pi_2 = \emptyset$.

We continue with the final case. In that case, no point of Π_1 is collinear to a point of Π_2, as this would correspond to a plane π in $\mathsf{E}_{6,1}(\mathbb{K})$ which is contained in a 5-space incident with p_1 and in a 5-space incident with ξ_2, implying that π shares a line with $p_1^\perp \cap \xi_2$, which is empty however. Now, a point in Π_1 corresponds to a 5-space U containing p_1, and according to Fact 4.12, U contains a unique $4'$-space V_U contained in a symp ξ_U together with a unique 4-space V'_U of ξ_2. Moreover, each point of V_U is contained in a unique 5-space with a $4'$-space of ξ_2, and each such 5-space is incident with both ξ_U and ξ_2. Hence ξ_U corresponds to a para (of symp-type, i.e., same type as Π_2) which meets Π_1 in a point (corresponding to U) and Π_2 in a 4-space (corresponding to the 5-spaces incident with ξ_U and ξ_2). By duality, we may interchange the roles of Π_1 and Π_2. The statement follows. $\square$

Disjoint paras of different types are opposite, as they correspond to opposite elements of E_6.

Let Π_1 and Π_2 be two opposite paras of $\mathsf{E}_{6,2}(\mathbb{K})$, where Π_1 corresponds to a point p_1 and Π_2 to a symp ξ_2 of $\mathsf{E}_{6,1}(\mathbb{K})$.

Definition 4.16. Given opposite paras Π_1, Π_2 of $\mathsf{E}_{6,2}(\mathbb{K})$, the set $E(\Pi_1, \Pi_2)$ of points x of $\mathsf{E}_{6,2}(\mathbb{K})$ with the property that $x^\perp \cap \Pi_1$ and $x^\perp \cap \Pi_2$ are maximal 3-spaces in Π_1 and Π_2, respectively, equipped with the lines of $\mathsf{E}_{6,2}(\mathbb{K})$ fully contained in it, is called the *equator geometry* $E(\Pi_1, \Pi_2)$ with *poles* Π_1 and Π_2.

In $\mathsf{E}_{6,1}(\mathbb{K})$, a 5-space U corresponds to a point of $E(\Pi_1, \Pi_2)$ if and only if $p_1^\perp \cap U$ is 3-dimensional and $U \cap \xi_2$ is a line, as is easily verified.

The definition hints at a bijection between the maximal 3-spaces of Π_1 and the points of $E(\Pi_1, \Pi_2)$. Indeed, consider any maximal 3-space W in Π_1. Then W is contained in a unique 4-space V_W of $\mathsf{E}_{6,2}(\mathbb{K})$, which corresponds to a 4-space V'_W of $\mathsf{E}_{6,1}(\mathbb{K})$ containing p_1. By Fact 4.12, V'_W contains a unique 3-space U_W, which is collinear to a unique line L_W in ξ_2. The 5-space $\langle U_W, L_W \rangle$ gives us a point of $E(\Pi_1, \Pi_2)$, as it is contained in a 4-space incident with Π_2 too, namely the one corresponding to the line L_W of ξ_2. By duality, the points of $E(\Pi_1, \Pi_2)$ are also in bijection with the maximal 3-spaces of $E(\Pi_1, \Pi_2)$. Another verification using the correspondence with $\mathsf{E}_{6,1}(\mathbb{K})$ shows that two points of $E(\Pi_1, \Pi_2)$ which are on a line L of Δ, correspond to 3-spaces of Π_1 which share a line L', moreover, each point on L corresponds to a 3-space of Π_1 containing L' and hence L is a full line of $E(\Pi_1, \Pi_2)$. More precisely, this shows:

Fact 4.17. *Given opposite paras Π_1, Π_2 in $\mathsf{E}_{6,2}(\mathbb{K})$, the equator geometry $E(\Pi_1, \Pi_2)$ is a $\mathsf{D}_{5,2}(\mathbb{K})$ geometry and collinearity induces the natural isomorphism between Π_i and $E(\Pi_1, \Pi_2)$, $i = 1, 2$. A point of Π_1 is hence collinear with a subgeometry of $E(\Pi_1, \Pi_2)$ isomorphic to $\mathsf{A}_{4,2}(\mathbb{K})$.*

4.3. **The long root geometry of type E_7.** The points of the long root geometry $\mathsf{E}_{7,1}(\mathbb{K})$ can be identified with the symps of the Lie incidence geometry $\mathsf{E}_{7,7}(\mathbb{K})$, which is a strong parapolar space of diameter 3 and hence more manageable than the long root geometry (which also has diameter 3 but is non-strong). By Main Result 1 of [26], the paras of $\mathsf{E}_{7,1}(\mathbb{K})$ correspond to the points of $\mathsf{E}_{7,7}(\mathbb{K})$ and are isomorphic to $\mathsf{E}_{6,1}(\mathbb{K})$.

Since points, lines and symps of $\mathsf{E}_{7,7}(\mathbb{K})$ correspond to paras, symps and points, respectively, of $\mathsf{E}_{7,1}(\mathbb{K})$, we deduce the following possibilities for the mutual position of paras in $\mathsf{E}_{7,1}(\mathbb{K})$:

Proposition 4.18. *Two paras in Δ either are disjoint (in which case they are opposite), or meet in exactly one point, or meet exactly in a symp.*

Two opposite paras Π_1 and Π_2 define an *equator geometry* $E(\Pi_1, \Pi_2)$ as follows (see [21])

Definition 4.19. Given opposite paras Π_1, Π_2 of $\mathsf{E}_{7,1}(\mathbb{K})$, the set $E(\Pi_1, \Pi_2)$ of points x of $\mathsf{E}_{7,1}(\mathbb{K})$ with the property that $x^\perp \cap \Pi_1$ and $x^\perp \cap \Pi_2$ are 5-spaces in Π_1 and Π_2, respectively, equipped with the lines of $\mathsf{E}_{7,1}(\mathbb{K})$ fully contained in it, is called the *equator geometry* $E(\Pi_1, \Pi_2)$ with *poles* Π_1 and Π_2.

It is shown in Lemma 6.7 of [21] that the poles of the equator geometry are unique. It is noted in Section 6 of the same paper that $E(\Pi_1, \Pi_2)$ is isomorphic to $\mathsf{E}_{6,2}(\mathbb{K})$.

We also have the following property. In the proof, a $5'$-space of $\mathsf{E}_{7,7}(\mathbb{K})$ corresponds to a flag in the corresponding building of type $\{1, 2\}$.

Proposition 4.20. *If Π_1 and Π_2 are two opposite paras of $\mathsf{E}_{7,1}(\mathbb{K})$, then every 6-space intersecting $\Pi_1 \cup \Pi_2$ in a 5-space contains a unique point of $E(\Pi_1, \Pi_2)$.*

Proof. Translated to $\mathsf{E}_{7,7}(\mathbb{K})$, we are given two opposite points p, q (points at distance 3) and a maximal singular subspace W of dimension 6 containing p. We have to find a symp ξ intersecting W in a $5'$-space and such that q is collinear to a $5'$-space of ξ. By Fact 4.3, each line of W through p contains a unique point at distance 2 from q, and this yields a $5'$-space $U \subseteq W$ of points symplectic to q. Since U corresponds to a flag of type $\{1, 2\}$ of the underlying spherical building, it is contained in a unique symp ξ. Since q is symplectic to all points of a $5'$-space of ξ, Fact 4.1 implies that either q is collinear to a point of U, contradicting p and q being opposite, or q is collinear to a unique $5'$-space of ξ, which concludes the proof of the proposition. □

4.4. **Geometric hyperplanes.** Our technique to prove Theorem A uses geometric hyperplanes of the long root geometries in question. Essential in the arguments will be the fact that the complement of these hyperplanes is a connected geometry, which is Theorem 2.2 in [23].

We will need the following lemma by Hall and Shult [22, Lemma 3.1(2)].

Lemma 4.21. *No polar space is the union of two (proper) geometric hyperplanes.*

Proposition 4.22 (Kasikova [23])**.** *The complement of any geometric hyperplane H of any hexagonic Lie incidence geometry $\Delta = (X, \mathscr{L})$ with no rank 2 symplecta is connected.*

This will be applied in two well-known ways (we include a proof for completeness):

Lemma 4.23. *Let $\Delta = (X, \mathscr{L})$ be a hexagonic geometry with no rank 2 symplecta and let $H \subseteq X$ be a geometric hyperplane (which may also coincide with X).*

(*i*) *If Δ admits an embedding in a projective space of dimension d, then H spans a subspace of dimension at least $d-1$.*
(*ii*) *If the generating rank of H is r, then the generating rank of Δ is at most $r+1$.*

Proof. If $H = X$ the statements are trivially true, so suppose $H \subsetneq X$. Let x be a point of $X \setminus H$. Since each line of Δ through x intersects H in a point, all points of $X \setminus H$ collinear to x are generated by x and H. By connectivity (see Proposition 4.22), all points of $X \setminus H$ are generated by x and H, showing the two assertions. □

We will also need the connectivity of the complement of a geometric hyperplane in the point residues of the long root geometries of type $\mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$. This has been proved by Shult [28, Lemma 5.2].

Proposition 4.24 (Shult [28])**.** *The complement of any geometric hyperplane H of a Lie incidence geometry of type $\mathsf{A}_{5,3}, \mathsf{D}_{6,6}, \mathsf{E}_{7,7}$ is connected.*

5. Geometries of type $\mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$

Recall that
$$\rho_\bullet(\mathsf{X}_r(\mathbb{K})) = \omega(\mathsf{X}_r(\mathbb{K})) + \epsilon_\bullet(\mathsf{X}_r(\mathbb{K}))$$
for the $\bullet$-rank of the long root geometry of type X_r over the field $\mathbb{K}$, where $\omega(\mathsf{X}_r(\mathbb{K}))$ is the dimension of the corresponding Weyl module, and $\epsilon_\bullet(\mathsf{X}_r(\mathbb{K}))$ is the *excess*.

5.1. Bounds by point-equator geometries.

Let $(\mathsf{X}_r, \mathsf{Y}_{r-1}) \in \{(\mathsf{E}_6, \mathsf{A}_5), (\mathsf{E}_7, \mathsf{D}_6), (\mathsf{E}_8, \mathsf{E}_7)\}$. Our principal aim is to show

Theorem 5.1. $\epsilon_\bullet(\mathsf{X}_r(\mathbb{K})) \le \epsilon_\bullet(\mathsf{Y}_{r-1}(\mathbb{K}))$.

We will do this by showing the following slightly more explicit form.

Theorem 5.2. $\rho_\bullet(\mathsf{X}_r(\mathbb{K})) \le 3.2^{r-3} + 19 + \rho_\bullet(\mathsf{Y}_{r-1}(\mathbb{K}))$.

To show that Theorem 5.1 really follows from Theorem 5.2, we notice that, for all $r \in \{6,7,8\}$,
$$\omega(\mathsf{X}_r(\mathbb{K})) = 3.2^{r-3} + 19 + \omega(\mathsf{Y}_{r-1}(\mathbb{K}))$$
(use the explicit value of the dimension of the Weyl module, which is 35, 66 and 133 for types A_5, D_6 and E_7, respectively). Hence Theorem 5.2 yields
$$\omega(\mathsf{X}_r(\mathbb{K})) + \epsilon(\mathsf{X}_r(\mathbb{K})) = \rho(\mathsf{X}_r(\mathbb{K})) \le -1 + \omega(\mathsf{X}_r(\mathbb{K})) + 1 + \epsilon(\mathsf{Y}_{r-1}(\mathbb{K})),$$
which proves Theorem 5.1.

Theorem A then follows from Theorem 5.1 for the cases $\mathsf{A}_5 \to \mathsf{E}_6$, $\mathsf{D}_6 \to \mathsf{E}_7$, $\mathsf{E}_7 \to \mathsf{E}_8$.

Remark 5.3. (1) It is not by coincidence that the number $3.2^{r-3} + 18$ is 2 more than the double of the e-rank and g-rank of the Lie incidence geometries mentioned in Fact 3.1, as will become apparent in the proof.
(2) There is also a closed formula for $\omega(\mathsf{Y}_{r-1})$, which reads $2^{2r-12} + 27.2^{r-6} + r + 1$. But we will not need this.

In order to prove Theorem 5.1, we use Lemmas 4.4 and 4.23. Throughout we denote by $\Delta = (X, \mathscr{L})$ the long root geometry of type X_r, $r = 6,7,8$. We will establish a geometric hyperplane $H \subseteq X$ of Δ. Let p and q be two opposite points of Δ and define H as the subspace of Δ generated by $p^{\perp\!\!\perp}$ and $q^{\perp\!\!\perp}$, that is,
$$H := \langle \{x \in X \mid x \perp\!\!\perp p \text{ or } x \perp\!\!\perp q\} \rangle = \langle p^{\perp\!\!\perp} \cup q^{\perp\!\!\perp} \rangle.$$
We first prove a bound on the g-rank and e-rank of H, and then we show that H is really a geometric hyperplane of Δ. We begin with a lemma. Recall that $E(p,q) = \{x \in X \mid p \perp\!\!\perp x \perp\!\!\perp q\} = p^{\perp\!\!\perp} \cap q^{\perp\!\!\perp}$.

Proposition 5.4. *The subspace H of Δ is generated by $p^\perp$, $q^\perp$ and $E(p,q)$.*

Proof. Set $H' = \langle p^\perp, q^\perp, E(p,q)\rangle$. We show $H = H'$. By their definitions, $E(p,q) \subseteq H$. Let L be any line containing p. By (PPS3), there is a symp ξ containing L. Since the points of ξ not collinear to p are symplectic to p, and $p^{\perp\!\!\!\perp} \cap \xi$ generates ξ, we see that $p^\perp \subseteq H$. Likewise $q^\perp \subseteq H$, and we conclude $H' \subseteq H$.

Now let ξ be an arbitrary symp containing p. Since $\langle p^\perp \cap \xi\rangle$ is a geometric hyperplane of ξ and $\xi \cap q^{\perp\!\!\!\perp}$ is nonempty by Lemma 4.7 and belongs to $E(p,q)$ by the last statement of Fact 4.8, we deduce that $\xi \subseteq H'$. Hence each point $x \perp\!\!\!\perp p$ is contained in H'. Similarly, every point $y \perp\!\!\!\perp q$ belongs to H'. This yields $H \subseteq H'$. $\square$

Lemma 5.5. *The $\bullet$-rank of H is at most $3.2^{r-3} + 18 + \rho_\bullet(\mathsf{Y}_{r-1}(\mathbb{K}))$.*

Proof. We know that $p^\perp$ is a cone with vertex p and with basis $\mathsf{A}_{5,3}(\mathbb{K})$, $\mathsf{D}_{6,6}(\mathbb{K})$, and $\mathsf{E}_{7,7}(\mathbb{K})$, for $r = 6,7,8$, respectively. Fact 3.1 implies that both the g-rank and e-rank of $p^\perp$, as a point-line geometry, are bounded above by $3.2^{r-4} + 9$ (in fact they are easily seen to be equal to it). Likewise for $q^\perp$. Now, since $E(p,q)$ is isomorphic to the long root geometry of type Y_{r-1} over $\mathbb{K}$, the assertion follows from Proposition 5.4. $\square$

Now we embark on the proof that H is a geometric hyperplane of Δ. Throughout, let $L \in \mathscr{L}$ be arbitrary. In a series of lemmas, we will show that $L \cap H \neq \emptyset$. We start, though, with showing that H is proper. Note that this is not necessary for the proof of Theorem A (as $H = X$ would even give a stronger upper bound), but it is good to know.

Lemma 5.6. *The subspace H of Δ does not coincide with X.*

Proof. By letting the Lie algebra $\mathfrak{e}_7$ act in its adjoint representation on $\mathfrak{e}_8$, one deduces that in the Weyl embedding of Δ, the equator geometry $E(p,q)$ is the Weyl embedding of the long root geometry of type Y_{r-1}. Now the subspace of the ambient projective space generated by H has (projective) dimension at most $3.2^{r-3} + 17 + \omega(\mathsf{Y}_{r-1})$, which is equal to $(\omega(\mathsf{X}_r) - 1) - 1$, as one can compute in the three cases $r = 6,7,8$. The lemma follows. $\square$

Recall that a *deep point* of a hyperplane of a Lie incidence geometry is a point for which every line containing this point is fully contained in the hyperplane.

We will call a line *rebellious* if it has empty intersection with H.

Lemma 5.7. *Each line with a point in $E(p,q)$ belongs to H, that is, each point of $E(p,q)$ is a deep point of H. Hence a line contained in a symp ξ with $\xi \cap E(p,q) \neq \emptyset$ is not rebellious.*

Proof. Let $x \in E(p,q)$ be arbitrary. In $\mathrm{Res}_\Delta(x)$, the lines through x contained in $\xi(p,x) \cup \xi(q,x)$ form the union of two opposite symps ζ_1 and ζ_2. By Proposition 3.3, $\mathrm{Res}_\Delta(x)$ is generated by ζ_1 and ζ_2 and the set S of points collinear to maximal singular subspaces in both these symps. Let S' be the set of points s' of Δ such that the line xs' is a point of S and take $s' \in S'$. Then s' is collinear to a maximal singular subspace of $\xi(p,x)$, and also to one of $\xi(q,x)$. This implies that $p \perp\!\!\!\perp s' \perp\!\!\!\perp q$, and so $s' \in E(p,q)$. We conclude that $x^\perp = \langle \zeta_1, \zeta_2, S'\rangle \subseteq \langle p^\perp, q^\perp, E(p,q)\rangle = H$ (the latter equality by Proposition 5.4), from which the first assertion follows. The second follows immediately from the fact that the union of the set of lines through a certain point (the perp of that point) in a polar space is a geometric hyperplane. $\square$

Remark 5.8. The next results and their proofs, until Proposition 5.13, only use Lemma 5.7 and the fact that Δ is a hexagonic geometry with no rank 2 symps. In particular, they also hold in $\mathsf{F}_{4,1}(\mathbb{K})$. We will need this in Proposition 6.4.

Lemma 5.9. *If all points of a line L are special to either p or q, that is, $L \subseteq p^{\bowtie}$ or $L \subseteq q^{\bowtie}$, then L is not rebellious.*

Proof. Without loss of generality we may assume that all points of L are special to p. By Lemma 4.10, there is a line M collinear to p and contained in a symp ξ together with L, and M and L are ξ-opposite. By Fact 4.6, all points of ξ collinear to M are symplectic to p and hence contained in H. By Lemma 4.7, there is at least one point $x \in \xi$ symplectic to q. If $x \in M$, then Lemma 4.8 contradicts p and q being opposite. If $M \not\ni x \perp M$, then $x \in p^{\perp\!\!\!\perp} \cap q^{\perp\!\!\!\perp} = E(p,q)$ and the result follows from Lemma 5.7. Finally, if $x \notin M^{\perp}$, then $M^{\perp}$ and x generate a geometric hyperplane T of ξ contained in H, proving the assertion as L will meet T in at least a point. $\square$

Recall that, according to Lemma 4.7, if a point x and a symp ξ are such that ξ contains a point opposite x, then ξ contains a unique point symplectic to x. We will use this a couple of times.

Lemma 5.10. *Suppose L is a rebellious line. Then L contains a point opposite p and q. Consequently, if ξ is a symp containing L, then ξ contains unique points p' and q' symplectic to p and q, respectively, and $p' \neq q'$.*

Proof. We show that L contains a point opposite p. Firstly, since $p^{\perp} \subseteq H$ by Proposition 5.4 and since $p^{\perp\!\!\!\perp} \subseteq H$ by definition, L contains no points collinear or symplectic to p. By Lemma 5.9, not all points of L are special to p. Therefore, all points of L but one are opposite p (and the unique remaining one is special to p by the last statement of Lemma 4.8). The same goes for q.

Now let ξ be a symp containing L. Since L contains a point opposite p, ξ has a unique point p' symplectic to p by Lemma 4.7; likewise $\xi \cap q^{\perp\!\!\!\perp}$ is a unique point q'. If $p' = q'$ then this point belongs to $\xi \cap E(p,q)$, and Lemma 5.7 implies that L is not rebellious, a contradiction. So $p' \neq q'$. $\square$

Lemma 5.11. *Suppose a line L is contained in a symp ξ which has a unique point p' symplectic to p and a unique point q' symplectic to q, with $p' \neq q'$. Let r be the unique point of $\xi(p,p')$ symplectic to q. Then $p' \perp\!\!\!\perp r$ if and only if $p' \perp\!\!\!\perp q'$. Moreover, if $p' \perp\!\!\!\perp q'$, then L is not rebellious and every point $a \in (p^{\bowtie} \cap \xi) \setminus H$ is contained in a line $M \subseteq \xi$ which intersects H in a point opposite p.*

Proof. Note that $p' \neq q'$ implies that $p' \neq r$. Suppose that $q' \perp p' \perp\!\!\!\perp r$. Then by Lemma 4.8, since $q \perp\!\!\!\perp q' \perp p'$, we see that $q \bowtie p'$. Since $p' \perp\!\!\!\perp r$, Lemma 4.8 yields $q \leftrightarrow p'$ and hence $p' \perp\!\!\!\perp r$ implies $p' \perp\!\!\!\perp q'$. Likewise, $p' \perp\!\!\!\perp q'$ implies $p' \perp\!\!\!\perp r$.

Now, every line in $p'^{\perp} \cap \xi$ not through p' only contains points special to p and hence contains at least one point of H (by Lemma 5.9). It follows that each plane in $p'^{\perp}$ containing p' either contains a unique line through p' in H, or is contained in H. Hence we may assume that $H_p := H \cap (p'^{\perp} \cap \xi)$ is a geometric hyperplane of $p'^{\perp} \cap \xi$ containing p' (if H_p would coincide with the whole of $p'^{\perp} \cap \xi$, then $L \cap H$ is nontrivial and the lemma is proved; in fact in this case $L \subseteq H$ because $q' \in H \setminus H_p$). Since $q' \notin H_p$, we see that H_p and q' generate a hyperplane of ξ, and so L has a point in common with $\langle H_p, q' \rangle \subseteq H$. This is the first assertion.

If $\xi \subseteq H$, then the second assertion is trivial. If not, then every line M in ξ through a and not contained in $p'^{\perp}$ intersects H in unique point (as we showed above that $\xi \cap H$ is a geometric hyperplane of ξ) which is automatically opposite p. $\square$

The remaining problem is when a line is not contained in a symp satisfying the assumptions of Lemma 5.11. So, a rebellious line contains points opposite both p and q, and for every symp containing L, the unique points p' and q' symplectic to p and q, respectively, are collinear.

Lemma 5.12. *If there exists a rebellious line, then there is one, say L, with the properties that it contains unique points $t, u \in L$ with $t \neq u$, $t \bowtie p$ and $u \bowtie q$ and such that there exists a line $M \ni t$ which intersects H in a point opposite p.*

Proof. Let ξ be any symp containing some rebellious line. Define $p', q' \in \xi$ as before (symplectic to p, q, respectively). By Lemma 5.11, $p' \perp q'$. Define H_p and H_q as in the proof of Lemma 5.11; so $H_p = H \cap p'^{\perp} \cap \xi$ and $H_q = H \cap q'^{\perp} \cap \xi$. If these do not coincide, then they generate at least a geometric hyperplane of ξ and no line of ξ is rebellious. Hence $H_p = H_q = \{x \in \xi \mid x \perp \langle p', q'\rangle\}$ (as H_p and H_q are geometric hyperplanes of $p'^{\perp}$ and $q'^{\perp}$, respectively). Note that, since ξ contains a rebellious line, we have $H \cap \xi = H_p = H_q$.

By Fact 4.2 applied in the residue of p', each line K contained in H_p and containing p' is coplanar with a unique line K^α in $\xi(p, p')$, which itself is coplanar with $(p'q')^\alpha$. All such lines K^α hence constitute a geometric hyperplane of $p'^{\perp} \cap \xi(p, p')$. It is easy to see that we can select a line T through p' in $\xi(p, p')$ not belonging to that geometric hyperplane and not collinear to $r \perp p'$, where $q \perp\!\!\!\perp r \in \xi(p, p')$. Then T is collinear to a unique line T' through p' in ξ, which does not belong to H_p. Pick any $t \in T' \setminus \{p'\}$, then there exists a line $L \ni t$ such that the unique point $u \in L$ collinear to q' does not belong to H_q, and clearly $t \neq u$. Since t is not collinear to q', we see that $L \cap H_p = \emptyset$ and hence L is rebellious.

Take a point p^* on $T \setminus \{p'\}$ not collinear to p and let U be the line tp^*. Let ζ be a symp containing U and locally opposite $\xi(p, p')$ at p^* (and note $p^* \perp\!\!\!\perp p$). Since the unique point of ζ symplectic to p is p^*, and p^* is symplectic to r (by the choice of T), the unique point q^* of ζ symplectic to q is symplectic to p^* by Lemma 5.11. The second assertion of the same lemma yields a line $M \subseteq \zeta$ through t containing a point $z \in H \cap p^{\leftrightarrow}$. $\square$

We now show that rebellious lines cannot exist.

Proposition 5.13. *There do not exist rebellious lines, that is, H is a geometric hyperplane of Δ.*

Proof. Suppose for a contradiction that there exists a rebellious line L. By Lemma 5.12, we may assume that it contains unique distinct points $t \bowtie p$ and $u \bowtie q$, and that there exists a line $M \ni t$ containing a point $z \in H$ opposite p. Consider the subspace $W := t^{\perp}$. The points not opposite p in W form a geometric hyperplane G; by Lemma 5.10, H intersects that geometric hyperplane in a geometric hyperplane thereof. But also z belongs to H. It suffices to show that L has nonempty intersection with $J := \langle G \cap H, z\rangle \subseteq H$. Since W is a cone with vertex t, the subspace $\langle t, J\rangle$ just consist of the points on a line connecting t with a point of J. But $\langle t, J\rangle = \langle t, z, G \cap H\rangle = \langle z, G\rangle$, and, by Proposition 4.24, the latter coincides with W. Hence J contains a point of every line of W through t. In particular $L \cap J \neq \emptyset$. $\square$

Now combining Lemmas 4.23, 5.5 and Proposition 5.13 yields Theorem 5.2, and hence also Theorem 5.1. This shows the arrows $\mathsf{A}_5 \to \mathsf{E}_6$, $\mathsf{D}_6 \to \mathsf{E}_7$ and $\mathsf{E}_7 \to \mathsf{E}_8$ of Theorem A.

5.2. **Bounds by para-equator geometries.** In this subsection we show the arrows $\mathsf{D}_5 \to \mathsf{E}_6$, $\mathsf{E}_6 \to \mathsf{E}_7$ of Theorem A. Although there are similarities, there are also differences between the cases, so we treat these two arrows separately.

5.2.1. *The case* $\mathsf{E}_{6,2}(\mathbb{K})$ *from* $\mathsf{D}_{5,2}(\mathbb{K})$. In this section, $\Delta = (X, \mathscr{L})$ is a long root geometry of type E_6 over the field $\mathbb{K}$. With previous notation, we show:

Theorem 5.14. $\epsilon_\bullet(\mathsf{E}_6(\mathbb{K})) \leq \epsilon_\bullet(\mathsf{D}_5(\mathbb{K}))$.

This is a consequence of the following theorem.

Theorem 5.15. $\rho_\bullet(\mathsf{E}_6(\mathbb{K})) \leq 33 + \rho_\bullet(\mathsf{D}_5(\mathbb{K}))$.

Indeed, $\omega(\mathsf{E}_6(\mathbb{K})) - \omega(\mathsf{D}_5(\mathbb{K})) = 78 - 45 = 33$. Hence

$$\epsilon_\bullet(\mathsf{E}_6(\mathbb{K})) = \rho_\bullet(\mathsf{E}_6(\mathbb{K})) - \omega(\mathsf{E}_6(\mathbb{K})) \leq 33 + \rho_\bullet(\mathsf{D}_5(\mathbb{K})) - \omega(\mathsf{D}_5(\mathbb{K})) - 33,$$

thus $\epsilon_\bullet(\mathsf{E}_6(\mathbb{K})) \leq \epsilon_\bullet(\mathsf{D}_5(\mathbb{K}))$.

The method to show Theorem 5.15 is the same as in the previous section: we exhibit a geometric hyperplane of Δ, determine a bound on its e-rank and its g-rank, and use Lemma 4.23. So we start by introducing the geometric hyperplane H.

Let Π_1 and Π_2 be two opposite paras of Δ, where Π_1 corresponds to a point p_1 and Π_2 to a symp ξ_2 of $\mathsf{E}_{6,1}(\mathbb{K})$. Let H be the subspace of $\mathsf{E}_{6,2}(\mathbb{K})$ generated by $\Pi_1, \Pi_2, E(\Pi_1, \Pi_2)$, see Definition 4.16. Then we already have the following result.

Lemma 5.16. *The $\bullet$-rank of H is at most $32 + \rho_\bullet(\mathsf{D}_5(\mathbb{K}))$.*

Proof. This follows immediately from Fact 3.1. □

Our next aim is to show that H is a geometric hyperplane of $\mathsf{E}_{6,2}(\mathbb{K})$. To that end, we first show that all points of $\Pi_1 \cup \Pi_2$ are deep points of H.

Lemma 5.17. *For any point $x \in \Pi_1 \cup \Pi_2$, its perp $x^\perp$ is contained in H.*

Proof. We may suppose that $x \in \Pi_1$. The residue of x in Π_1 is isomorphic to $\mathsf{A}_{4,2}(\mathbb{K})$. Moreover, we know that $x^\perp \cap E(\Pi_1, \Pi_2)$ is also isomorphic to $\mathsf{A}_{4,2}(\mathbb{K})$ by Fact 4.17. On the other hand, in $\mathsf{E}_{6,2}(\mathbb{K})$, the residue of x is isomorphic to $\mathsf{A}_{5,3}(\mathbb{K})$. Now $x^\perp \cap \Pi_1$ and $x^\perp \cap E(\Pi_1, \Pi_2)$ are clearly disjoint. Since two disjoint $\mathsf{A}_{4,2}(\mathbb{K})$ geometries generate $\mathsf{A}_{5,3}(\mathbb{K})$, as can easily be checked, it follows that $x^\perp$ is generated by $x^\perp \cap \Pi_1$ and $x^\perp \cap E(\Pi_1, \Pi_2)$ and hence $x^\perp \subset H$. □

Next, we show that H contains certain paras which intersect $\Pi_1 \cup E(\Pi_1, \Pi_2) \cup \Pi_2$. Let $\Delta^* = \mathsf{E}_{6,1}(\mathbb{K})$.

Lemma 5.18. *For each symp Σ of Π_1, there is a unique para Π_Σ containing Σ and meeting $E(\Pi_1, \Pi_2)$ in the symp Σ' corresponding to Σ. Moreover, Π_Σ is of symp-type, and the corresponding symp contains p_1. Finally, the para Π_Σ is generated by Σ and Σ' and as such contained in H.*

Proof. Recall that Π_1 corresponds to the point p_1 of Δ^* and note that Σ corresponds to a flag $\{p_1, \xi_\Sigma\}$, where ξ_Σ is a symp of Δ^* containing p_1. Then ξ_Σ corresponds to the unique para Π_Σ of Δ distinct from Π_1 and containing Σ. Since p_1 is opposite ξ_2, the symps ξ_2 and ξ_Σ intersect in a point x. Every 5-space U incident with $\{x, \xi_\Sigma\}$ intersects ξ_2 in a line and shares a 3-space with $p_1^\perp$. Hence the symp Σ' of Δ corresponding to $\{x, \xi_\Sigma\}$ entirely belongs to $E(\Pi_1, \Pi_2)$. This shows the first two assertions. Since $\mathsf{D}_{5,5}(\mathbb{K})$ is generated by two disjoint symps, such as Σ and Σ', also the final statement follows. □

We need one other type of para.

Lemma 5.19. *For each point z of Π_2, there is a unique para Π_z containing z and meeting Π_1 in the unique 4-space of Π_1 containing the points symplectic to z. Moreover, Π_z is of point-type, and the corresponding point p_z is collinear to p_1 and to a $4'$-space of ξ_2 (which, together with p_z generates the 5-space U_z corresponding to z). Finally, Π_z is contained in H.*

Proof. We argue in Δ^*. Since U_z shares a $4'$-space with ξ_2, and p_1 is opposite ξ_2, there is a unique point p_z in U_z collinear to p_1. Recalling that symplectic points of Δ correspond to 5-spaces intersecting in a point, we now see that the set $z^{\perp\!\!\!\perp} \cap \Pi_1$ of points of Π_1 symplectic to z corresponds to the set of 5-spaces of Δ^* containing the line $\langle p_1, p_z \rangle$. Since the intersection of all these 5-spaces is exactly p_z, the unique para Π_z we are looking for corresponds to p_z and all assertions except the last one follow.

For the last assertion, we translate the situation to the polar space $\mathsf{D}_{5,1}(\mathbb{K})$ corresponding to the para Π_z, where the points of Π_z correspond to 4-spaces; the other maximal subspaces will be called $4'$-spaces (they correspond to 4-dimensional subspaces of Π_z). Then z corresponds to a 4-space W_z and the 4-dimensional subspace $z^{\perp\!\!\!\perp} \cap \Pi_1$ corresponds to a $4'$-space V_z. Since $z^\perp \subseteq H$ and $s^\perp \subseteq H$ for any point $s \in z^{\perp\!\!\!\perp} \cap \Pi_1$ by Lemma 5.17, H induces in $\mathsf{D}_{5,1}(\mathbb{K})$ a set of 4-spaces containing all 4-spaces intersecting W_z in a plane, or intersecting V_z in a line. Any other 4-space W of $\mathsf{D}_{5,1}(\mathbb{K})$ intersects W_z in a point x. Select an arbitrary plane $\pi \subseteq W$ with $x \in \pi$. Then $\langle \pi, \pi^\perp \cap V_z \rangle$ and $\langle \pi, \pi^\perp \cap W_z \rangle$ are two distinct 4-spaces induced by H containing π; hence also W is induced by H (since H is a subspace). $\square$

Definition 5.20. By the previous two lemmas, we may introduce the following paras:

- For each symp Σ of Π_1, the para Π_Σ of symp-type meeting Π_1 in Σ and meeting $E(\Pi_1, \Pi_2)$ in the symp Σ' corresponding to Σ.
- For each point $z \in \Pi_2$, the para Π_z of point-type meeting Π_2 in $\{z\}$ and Π_1 in $z^{\perp\!\!\!\perp} \cap \Pi_1$.

Using these paras, we can show that H is a possibly nonproper geometric hyperplane of Δ. In fact, analogously to Lemma 5.6, one shows that H is proper, but since we do not strictly need this, we only state:

Proposition 5.21. *H is a (possibly nonproper) geometric hyperplane of Δ.*

Proof. We argue in Δ^*. Set H^* the set of 5-spaces corresponding to points of H. Recall that the lines of Δ correspond to the planes of Δ^* (and the points on the line are the 5-spaces through that plane of course). Recall also that p_1 is the point corresponding to Π_1. Let π be an arbitrary plane in Δ^*. We have to show that some 5-space through π is contained in H^*. Let $S(\pi)$ be the set of 5-spaces containing π. By Lemma 5.17, we may assume that no member of $S(\pi)$ contains a plane collinear to p_1.

Then p_1 is collinear to a unique point r_W of W, for each $W \in S(\pi)$. Suppose first that $r_W \in \pi$ (that is, all points r_W coincide; denote this common point by r). Select two points $s, t \in \pi$ not on one line with r. Set $\xi := \xi(p_1, s)$ and $W = \langle t, t^\perp \cap \xi \rangle$. Note that, since $r, s \in t^\perp \cap \xi$, the space W is 5-dimensional, and it contains π. So $W \in S(\pi)$ and $\{r\} \subsetneq p_1^\perp \cap W$, contradicting our assumption. Hence we may assume that the points r_W, $W \in S(\pi)$, do not belong to π and are hence all distinct. We claim that they are exactly the points of a line L. Indeed, firstly, they are pairwise collinear for otherwise the unique symp determined by two noncollinear ones among them contains both p_1 and π, contradicting $p_1^\perp \cap \pi = \emptyset$. Since $U := \langle \pi \cup \{r_W \mid W \in S(\pi)\} \rangle$ is a singular subspace sharing, for each $W \in S(\pi)$, the 3-space $\langle r_W, \pi \rangle$ with W, we see that U is 4-dimensional. Since $p_1^\perp \cap U$ is a subspace disjoint from π and containing all r_W, $W \in S(\pi)$, we conclude that $\langle r_W \mid W \in S(\pi) \rangle$ is a line L. Since each point of L is collinear to π and there is a unique 5-space through a singular 3-space, $L = \{r_W \mid W \in S(\pi)\}$ indeed. Now, Proposition 4.13 yields a point $r_W \in L$, for some $W \in S(\pi)$, collinear to some point, and hence some $4'$-space, of ξ_2. As r_W is also collinear to p_1, Lemma 5.19 implies that the point of Δ corresponding to W belongs to the para Π_z of point-type, with z the point of Δ corresponding to the 5-space of Δ^* generated by r_W and $r_W^\perp \cap \xi_2$. $\square$

Now combining Lemmas 4.23, 5.16 and Proposition 5.21 yields Theorem 5.15, and hence also Theorem 5.14.

This shows the arrow $\mathsf{D}_5 \to \mathsf{E}_6$ of Theorem A.

5.2.2. *The case* $\mathsf{E}_{7,1}(\mathbb{K})$ *from* $\mathsf{E}_{6,2}(\mathbb{K})$. Let $\Delta = (X, \mathscr{L})$ be the long root geometry $\mathsf{E}_{7,1}(\mathbb{K})$. We show the arrow $\mathsf{E}_6 \to \mathsf{E}_7$ of Theorem A, see the theorem below (using the same notation as before). Although this case is somewhat similar to the previous case, the details of the arguments are quite different, so we provide an explicit proof.

Theorem 5.22. $\epsilon_\bullet(\mathsf{E}_7(\mathbb{K})) \leq \epsilon_\bullet(\mathsf{E}_6(\mathbb{K}))$.

As before this is a consequence of the following theorem.

Theorem 5.23. $\rho_\bullet(\mathsf{E}_7(\mathbb{K})) \leq 55 + \rho_\bullet(\mathsf{E}_6(\mathbb{K}))$.

Indeed, $\omega(\mathsf{E}_7(\mathbb{K})) - \omega(\mathsf{E}_6(\mathbb{K})) = 133 - 78 = 55$. Hence

$$\epsilon_\bullet(\mathsf{E}_7(\mathbb{K})) = \rho_\bullet(\mathsf{E}_7(\mathbb{K})) - \omega(\mathsf{E}_7(\mathbb{K})) \leq 55 + \rho_\bullet(\mathsf{E}_6(\mathbb{K})) - \omega(\mathsf{E}_6(\mathbb{K})) - 55$$

thus $\epsilon_\bullet(\mathsf{E}_7(\mathbb{K})) \leq \epsilon_\bullet(\mathsf{E}_6(\mathbb{K}))$.

Again, the method to show Theorem 5.23 is the same as in Section 5.1: we exhibit a geometric hyperplane of Δ, determine a bound on its $\bullet$-rank, and use Lemma 4.23. So we start by introducing the geometric hyperplane H.

Select two opposite paras Π_1 and Π_2 in Δ and recall that these are isomorphic to $\mathsf{E}_{6,1}(\mathbb{K})$. Denote by H the subspace of Δ generated by Π_1, Π_2 and $E(\Pi_1, \Pi_2)$, cf. Definition 4.19.

Lemma 5.24. *The $\bullet$-rank of H is at most $54 + \rho_\bullet(\mathsf{E}_6(\mathbb{K}))$.*

Proof. Both Π_1 and Π_2 are isomorphic to $\mathsf{E}_{6,1}(\mathbb{K})$ whose $\bullet$-rank is 27. □

Now we show that H is a geometric hyperplane of Δ. As before we do not insist on the fact $H \neq X$.

Lemma 5.25. *If x is a point of $E(\Pi_1, \Pi_2)$, then $x^\perp \subseteq H$, that is, the points of $E(\Pi_1, \Pi_2)$ are deep points of H.*

Proof. The residue $\mathrm{Res}_\Delta(x)$ is a geometry isomorphic to $\mathsf{D}_{6,6}(\mathbb{K})$. The lines joining x to a point of Π_1 and Π_2 correspond to opposite 5-spaces W_1 and W_2, respectively, in $\mathrm{Res}_\Delta(x)$. Fact 3.2 implies that $\mathrm{Res}_\Delta(x)$ is generated by W_1, W_2 and the set S of points collinear to planes in both W_1 and W_2. Similarly as in the proof of Lemma 5.7, it follows that the lines xs' corresponding to points of S in $\mathrm{Res}_\Delta(x)$ are contained in $E(\Pi_1, \Pi_2)$, leading to $x^\perp \subseteq H$. □

Lemma 5.26. *If p is a point of Δ collinear to at least a plane of $\Pi_1 \cup \Pi_2$, then $p \in H$.*

Proof. We may assume that p is collinear to some plane π of Π_1. Select any symp ξ in Π_1 containing π (since Π_1 is a para, ξ is also a symp of Δ). By Fact 4.4, $p^\perp \cap \xi$ is a 4-dimensional subspace U containing π. Let V be the unique 5-space in Π_1 containing U and let W be the unique 6-space in Δ containing U. Then W contains both V and p (as otherwise a standard argument shows that the symp through two non-collinear points of $W \cup V \cup \{p\}$ contains a subspace of projective dimension at least 5, a contradiction). But W has a unique point x in $E(\Pi_1, \Pi_2)$ by virtue of Proposition 4.20. Hence $p \in \langle x, V \rangle \subseteq H$. □

Lemma 5.27. *Every para Π sharing a symp ξ with Π_1 intersects H in at least a hyperplane of Π; if moreover Π contains a point of Π_2, then $\Pi \subseteq H$.*

Proof. The set $H \cap \Pi$ contains all points of ξ and all points close to ξ (collinear to a 4-space of ξ) by Lemma 5.26. The first assertion now follows from Proposition 4.13. A point of Π_2 is never collinear with any point of Π_1, so the second assertion now also follows from Proposition 4.13. □

We now translate the situation to $\Delta^* := \mathsf{E}_{7,7}(\mathbb{K})$, where it is somewhat easier to argue. Note that points, lines, symps and paras of $\Delta = \mathsf{E}_{7,1}(\mathbb{K})$ correspond to symps, maximal 5-spaces, lines and points, respectively, of Δ^*. Moreover, paras of Δ intersecting in symps correspond to collinear points of Δ^*, paras in Δ intersecting in just a point correspond to symplectic points of Δ^*. Denote by p_1, p_2 the points of Δ^* corresponding to Π_1, Π_2. Then the set of paras in Δ intersecting Π_1 in a symp and intersecting Π_2 in a point, corresponds to the set $p_1^\perp \cap p_2^{\perp\!\!\!\perp}$ of points of Δ^* collinear to

p_1 and symplectic to p_2. Also, the set of paras of Δ intersecting Π_1 in a symp corresponds to $p_1^\perp$ in Δ^*. From this discussion and Lemma 5.27 follows:

Lemma 5.28. *Let L be a line of Δ and let U be the corresponding maximal singular 5-space in Δ^*. Then L contains a point of H as soon as either $p_1^\perp \cap U$ is nonempty, or some symp of Δ^* contains U and a point of $p_1^\perp \cap p_2^{\perp\!\!\perp}$.*

Proof. Suppose first that U is such that $p_1^\perp \cap U$ is a point a. Then a corresponds to a para Π_a in Δ sharing a symp ξ with Π_1, and U corresponds to a line L in Π_a. By Lemma 5.27, L has at least a point contained in H. Next, suppose U is such that it is contained in a symp ζ of Δ meeting $p_1^\perp \cap p_2^{\perp\!\!\perp}$ in a point b. Then b corresponds to a para Π_b meeting Π_1 in a symp and Π_2 in a point and is hence contained in H by Lemma 5.27. The symp ζ corresponds to a point x in Π_b and U corresponds to a line containing x. Since $x \in H$, the statement follows. $\square$

We are now ready to show that:

Proposition 5.29. *$H \cap X$ is a geometric hyperplane of Δ.*

Proof. As above, we argue in Δ^*. Let U be any maximal 5-space of Δ^* and suppose $p_1^\perp \cap U = \emptyset$. Let $S(U)$ be the set of symps of Δ^* containing U. By Fact 4.1, for each symp $\xi \in S(U)$, there exists at least one point $p_\xi \in \xi$ collinear to p_1. Select two distinct members ξ, ζ of $S(U)$ and suppose for a contradiction that p_ξ and p_ζ are not collinear. Since they are collinear to at least a common 3-space of U, they are symplectic and the symp ξ_1 containing them also contains p_1. But then $p_1^\perp \cap U$ is at least a 2-space, contradicting our hypothesis. Hence p_ξ and p_ζ are collinear.
Now note that p_ξ is collinear to a 4-space of U and hence a 5-space of ζ, i.e., a maximal singular subspace of ζ, implying that $p_1^\perp \cap \zeta = \{p_\zeta\}$. Similarly, p_ξ is unique. Now let ξ, ζ, υ be three distinct symps containing U. We claim that $p_\xi, p_\zeta, p_\upsilon$ are contained in a common line L. Suppose not, then the span is a plane π all points of which are collinear to p_1. The convexity of ξ implies that every point of $p_\zeta^\perp \cap U$ is collinear to p_ξ; hence $V := p_\xi^\perp \cap U = p_\zeta^\perp \cap U$ and likewise $V = p_\upsilon^\perp \cap U$. So V and π are contained in a singular subspace, which has dimension at most 6 in Δ^*. Since $\dim V = 4$, it follows that $\pi \cap V \neq \emptyset$, contradicting our assumption that $p_1^\perp \cap U = \emptyset$. The claim is proved.
So $L \subseteq p_1^\perp$. Now there is some point $x \in L$ contained in $p_2^{\perp\!\!\perp}$. Due to Lemma 5.28, it suffices to show that there is a symp containing $U \cup \{x\}$. Notice that the previous paragraph yields $V \subseteq L^\perp$. Pick $y \in U \setminus V$. Since U is a maximal singular subspace, it follows that $y \notin x^\perp$. The symp defined by x and y contains U and x and hence the proposition is proved. $\square$

Now combining Lemmas 4.23, 5.24 and Proposition 5.29 yields Theorem 5.23, and hence also Theorem 5.22. This shows the arrow $\mathsf{E}_6 \to \mathsf{E}_7$ of Theorem A.

5.3. **Proof of Theorem C for type E.** By Fact 3.7, the excess $\epsilon_\bullet(\mathsf{D}_5(\mathbb{K}))$ for a finite prime field $\mathbb{K}$, is equal to 0. Indeed, the number of roots of a root system of type D_5 is equal to 40; hence the Weyl module has dimension $40 + 5 = 45 = 2n^2 - n$ for $n = 5$. Now Theorem A, in particular the arrows $\mathsf{D}_5 \to \mathsf{E}_6 \to \mathsf{E}_7 \to \mathsf{E}_8$, implies that the excesses $\epsilon_\bullet(\mathsf{E}_i(\mathbb{K}))$, $i = 6, 7, 8$, are 0, too.

6. Geometries of type $\mathsf{F}_{4,1}$ and $\mathsf{F}_{4,4}$

6.1. **The embedding rank and generating rank of $\mathsf{F}_{4,4}(\mathbb{K})$.** The e-rank and g-rank of $\mathsf{F}_{4,4}(\mathbb{K})$ can be completely determined for all fields $\mathbb{K}$ not of characteristic 2.

Theorem 6.1. *Let $\mathbb{K}$ be any field not of characteristic 2. Then both the embedding rank and generating rank of $\mathsf{F}_{4,4}(\mathbb{K})$ is 26.*

Proof. Since the standard embedding of $\mathsf{F}_{4,4}(\mathbb{K})$ happens in a projective space of dimension 25, it suffices to show that $\mathsf{F}_{4,4}(\mathbb{K})$ is generated by 26 points.

By [20], each pair of opposite points p, q is contained in a unique so-called *extended equator geometry*, which can be described as follows. Each symplectic pair of points x, y is contained in a unique symp $\xi(x, y)$, which is isomorphic to a symplectic polar space of rank 3. The set $\{x, y\}^{\perp\perp} =: L(x, y)$ contains x and y, and is called a *hyperbolic line*. In the standard embedding of the polar space in 5-dimensional projective space over $\mathbb{K}$, it is an ordinary, though non-isotropic, line. Now, the points symplectic to both p and q, together with p and q generate by hyperbolic lines a polar space $\widehat{E}(p, q)$ isomorphic to $\mathsf{B}_{4,1}(\mathbb{K})$ (whose lines are thus hyperbolic lines), This is the extended equator geometry defined by p and q. The set $\widehat{T}(p, q)$ of all points collinear to a maximal singular subspace of $\widehat{E}(p, q)$, together with all lines it contains, is a geometry isomorphic to the dual polar space $\mathsf{B}_{4,4}(\mathbb{K})$ (called the *tropic circle geometry of p and q* in [20]). Now the set of points of $\mathsf{F}_{4,4}(\mathbb{K})$ contained in some line joining a point of $\widehat{E}(p, q)$ with a point of $\widehat{T}(p, q)$, constitutes a geometric hyperplane $\widehat{H}(p, q)$ of $\mathsf{F}_{4,4}(\mathbb{K})$ by Lemma 5.37(iv) of [21].

Now let T be a minimal generating set of $\widehat{T}(p, q)$, and E a minimal generating set of $\widehat{E}(p, q)$ (as a polar space, hence with respect to hyperbolic lines). By Theorem 5.3 of [18], we have $|T| = 16$. Since $\widehat{E}(p, q)$ is a parabolic polar space, we have $|E| = 9$. Proposition 5.3.1 of [20] implies that, for any pair of symplectic points $x, y \in \widehat{E}(p, q)$, the set $\{x, y\}^{\perp} \cap \xi(x, y)$ is contained in $\langle T\rangle$. Now $\xi(x, y)$ is isomorphic to $\mathsf{C}_{3,1}(\mathbb{K})$, a symplectic polar space of rank 3 over a field of characteristic different from 2. Since x and $\{x, y\}^{\perp}$ generates a singular hyperplane of $\xi(x, y)$, we see that $T \cup \{x, y\}$ generates $\xi(x, y)$ and hence also $L(x, y)$. It follows that $T \cup E$ generates $\widehat{H}(p, q)$. Hence, by Lemma 4.23(ii), the g-rank of $\mathsf{F}_{4,4}(\mathbb{K})$ is at most $|T| + |E| + 1 = 16 + 9 + 1$, which is 26. As noted in the beginning of this proof, this implies that the g-rank is exactly 26, as is the e-rank. $\square$

This proves Theorem E.

Remark 6.2. The proof of the previous theorem also works for perfect fields of characteristic 2 not of size 2. In this case one obtains that bot the embedding and generating rank of $\mathsf{F}_{4,1}(\mathbb{K})$ equals 52.

6.2. **The generating rank and embedding rank of $\mathsf{F}_{4,1}(\mathbb{K})$.** Let $\Delta = (X, \mathscr{L})$ be the Lie incidence geometry $\mathsf{F}_{4,1}(\mathbb{K})$. Recall that the Segre e-rank of $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$ is denoted by $\rho^{\circ}_{\mathsf{e}}(\mathsf{A}_2(\mathbb{K}))$. In this section we want to prove:

Theorem 6.3. *If $\mathbb{K}$ is a field with characteristic distinct from 2 and size at least 5, then $\rho_e(\mathsf{F}_4(\mathbb{K})) \leq 44 + \rho^{\circ}_{\mathsf{e}}(\mathsf{A}_2(\mathbb{K}))$ and also $\rho_{\mathsf{g}}(\mathsf{F}_4(\mathbb{K})) \leq 44 + \rho_{\mathsf{g}}(\mathsf{A}_2(\mathbb{K}))$.*

6.2.1. *The equator geometry for* $\mathsf{F}_{4,1}(\mathbb{K})$. We consider two opposite points p and q and define $H = \langle p^{\perp\!\!\perp} \cup q^{\perp\!\!\perp}\rangle$. Recall that $E(p, q) = p^{\perp\!\!\perp} \cap q^{\perp\!\!\perp}$. We start by showing that H is a geometric hyperplane.

Proposition 6.4. *The subspace H is a geometric hyperplane of Δ.*

Proof. By Remark 5.8, we only have to show that each point of $E(p, q)$ is a deep point of H (not using the fact that H is a geometric hyperplane).

So let L be a line containing a point x of $E(p, q)$ and suppose L is not contained in $\xi(p, x) \cup \xi(q, x)$. In the polar space $\mathsf{C}_{3,1}(\mathbb{K})$ corresponding to $\mathrm{Res}_{\Delta}(x)$, the symps $\xi(p, x)$ and $\xi(q, x)$ correspond to two opposite points a and b. The line L corresponds to a plane π neither containing a nor b. Then a point $c \in \pi \cap a^{\perp} \cap b^{\perp}$ corresponds to a symp ξ through L intersecting both $\xi(p, x)$ and $\xi(q, x)$ in (different) planes, say α_p and α_q, respectively. Set $L_p = \alpha_p \cap p^{\perp}$ and $L_q = \alpha_q \cap q^{\perp}$ and note that $\xi = \langle L_p, L_q, L_p^\perp \cap L_q^\perp\rangle$. Then every point of $L_p^\perp \cap L_q^\perp$ is symplectic to both p and q and hence belongs to $E(p,q)$. Since also L_p and L_q are contained in H, it follows that ξ is entirely contained in H. Since $L \subseteq \xi$, Lemma 5.7 follows for the current Δ and H. $\square$

A proof similar to that of Proposition 5.4 yields:

Proposition 6.5. *The subspace H of Δ is generated by $p^\perp$, $q^\perp$ and $E(p,q)$.*

We now first concentrate on the equator geometry $E(p,q)$. We will equip this point set with "lines" with the help of the following lemma. Recall that the symps of Δ are polar spaces isomorphic to a parabolic quadric $\mathsf{B}_{3,1}(\mathbb{K})$ in $\mathbb{P}^6(\mathbb{K})$.

Lemma 6.6. *Let $\xi = \xi(x,y)$ be a symp with x,y a symplectic pair of points of $E(p,q)$. Then $L := p^\perp \cap \xi$ and $M := q^\perp \cap \xi$ are lines, which are opposite in ξ, and $L^\perp \cap M^\perp$ is a conic $C \subseteq E(p,q)$ in the ambient projective space of ξ. Moreover, the set of symps $\xi(p,c)$ with $c \in C$ coincides with the set of all symps of Δ through the plane $\langle p, L\rangle$.*

Proof. Since p is, by definition of $E(p,q)$, symplectic to the two points x and y of ξ, it follows from Fact 4.5 that $p^\perp \cap \xi$ is non-empty. Fact 4.4 then implies that $p^\perp \cap \xi$ is a line, say L. Likewise, $q^\perp \cap \xi$ is a line M. We claim that L and M are opposite in ξ. Since p and q have distance 3, it is clear that L and M are disjoint. Let r be any point of L and suppose for a contradiction that r is collinear to M. Then r and q are symplectic (see also Fact 4.6), which however implies that p and q are not opposite (cf. Fact 4.9), a contradiction. We conclude that L and M are opposite lines in ξ. Hence $C := L^\perp \cap M^\perp$ (which is contained in ξ by convexity of ξ) is a conic in the ambient 6-dimensional projective space of ξ as a polar space isomorphic to $\mathsf{B}_{3,1}(\mathbb{K})$. Note that the points of $L^\perp \cap M^\perp$ are symplectic to both p and q by Fact 4.6 and hence $C \subseteq E(p,q)$ indeed. Moreover, for any point $c \in C$, $\xi(p,c)$ contains the plane $\langle p, L\rangle$. To prove the last statement, suppose ξ' is a symp containing the plane $\langle p, L\rangle$. Then $\xi' \cap \xi$ is a plane π by Fact 4.5. The plane π contains L and hence contains a unique point z collinear to M, so $z \in C$ and $\xi' = \xi(p,z)$. $\square$

We now declare two points x,y of $E(p,q)$ collinear if they are symplectic in Δ, and the joining "line" is given by $L^\perp \cap M^\perp$, where $L = p^\perp \cap \xi(x,y)$ and $M = q^\perp \cap \xi(x,y)$. Noting that a point x of $E(p,q)$ corresponds to a symp containing p (namely $(\xi(p,x))$ and that, by the above lemma, the just defined "lines" correspond to the symps containing a plane of Δ through p; we see that $E(p,q)$ equipped with the new lines has the structure of the symplectic polar space $\mathsf{C}_{3,1}(\mathbb{K})$.

Since the Weyl embedding of $\mathsf{F}_{4,1}(\mathbb{K})$ induces the Weyl embedding of $\mathsf{C}_{3,1}(\mathbb{K})$, and the latter induces the Weyl embedding of its planes (as in the proof of Lemma 5.6, these assertions follow by considering the adjoint action of the corresponding Lie subalgebras on the appropriate Lie algebra), and since the universal embedding of Δ projects onto the Weyl embedding of Δ, we see that the "planes" of $E(p,q)$ span a subspace of dimension at least 5. By Theorem 2.3 of [27], these planes correspond to ordinary Veronese surfaces in projective 5-space. Therefore we may consider the Veronese e-rank of $E(p,q)$. Likewise, we are only interested in the Veronese g-rank of $E(p,q)$.

6.2.2. *The Veronese generating rank and Veronese embedding rank of $\mathsf{C}_{3,1}(\mathbb{K})$.* We now determine the Veronese e-rank $\rho^*_{\mathsf{e}}(\mathsf{C}_{3,1}(\mathbb{K}))$ and the Veronese g-rank $\rho^*_{\mathsf{g}}(\mathsf{C}_{3,1}(\mathbb{K}))$ of $\mathsf{C}_{3,1}(\mathbb{K})$ in terms of the Segre e-rank $\rho^\circ_{\mathsf{e}}(\mathsf{A}_2(\mathbb{K}))$ and the g-rank $\rho_{\mathsf{g}}(\mathsf{A}_2(\mathbb{K}))$, respectively, of $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$, where $\mathbb{K}$ is a field whose characteristic is not equal to 2.

Let $\Delta' = (X', \mathscr{L}')$ be the symplectic polar space $\mathsf{C}_{3,1}(\mathbb{K})$. We choose two opposite (that is, disjoint) planes π_1 and π_2 in Δ'. Consider the following set P of points of Δ':

$$P = \{x \in X' \mid \exists L \in \mathscr{L}' : x \in L \text{ and } \pi_i \cap L \neq \emptyset, i = 1,2\}.$$

In this section we forget the notation $E(p,q)$; in particular the letters p and q do not refer to the points introduced in the previous subsection. Recall that a hyperbolic line of Δ' is an ordinary line of the 5-dimensional projective space $\mathbb{P}^5(\mathbb{K})$ in which Δ' naturally embeds.

Lemma 6.7. *The set P, endowed with all lines and hyperbolic lines of Δ' fully contained in P, is isomorphic to a Klein quadric, that is, an irreducible hyperbolic quadric of rank* 3.

Proof. This follows from the fact that P is the union of all planes of Δ' intersecting π_1 in a point x and π_2 in a line L, the mapping $p \mapsto L$ being a duality, using Proposition 5.2 of [34]. $\square$

Lemma 6.8. *Let p,q be two noncollinear points of the generalized quadrangle $\mathsf{B}_{2,1}(\mathbb{K})$. Then $\{p,q\}^{\perp\perp} = \{p,q\}$.*

Proof. Since the characteristic of $\mathbb{K}$ is not equal to 2, the relation $\perp$ is induced by a non-degenerate polarity ρ in the ambient projective space $\mathbb{P}^4(\mathbb{K})$ of $\mathsf{B}_{2,1}(\mathbb{K})$. Let Z be the point set on $\mathsf{B}_{2,1}(\mathbb{K})$ in $\mathbb{P}^4(\mathbb{K})$. Then $\{p,q\}^{\perp\perp} = Z \cap \langle p,q\rangle^{\rho\rho} = Z \cap \langle p,q\rangle = \{p,q\}$. $\square$

Lemma 6.9. *Each singular line contained in P intersects $\pi_1 \cup \pi_2$ nontrivially.*

Proof. Assume for a contradiction that some line L disjoint from $\pi_1 \cup \pi_2$ is contained in P. Then each point x of L is contained in a unique line L_x intersecting π_i in a point $t_{x,i}$, $i = 1,2$. If $t_{x,1} = t_{y,1}$ for two distinct points $x, y \in L$, then L is contained in the plane spanned by $t_{x,1}, t_{x,2}, t_{y,2}$ and hence intersects π_2 nontrivially, a contradiction. Note that in $\mathbb{P}^5(\mathbb{K})$, the point $t_{x,1}$ is the projection of x from π_2 onto π_1. Hence $\{t_{x,1} \mid x \in L\}$ is the projection of L from π_2 onto π_1 and is hence the point set of a line L_1. Likewise, $\{t_{x,2} \mid x \in L\}$ is a line L_2.

Set $p_1 = L_2^\perp \cap \pi_1$ and $p_2 = L_1^\perp \cap \pi_2$. Assume that $p_1 \in L_1$ and let α be the plane spanned by L_2 and p_1. Let $x \in L$ be such that $p_1 = t_{x,1}$ (note $x \in \alpha$) and pick $y_1, y_2 \in L \setminus \{x\}$. Then $y_1 \perp t_{y_1,2}$ and inside the plane $\langle y_1, x, t_{y_1,2}\rangle = \langle L, t_{y_1,2}\rangle$ we see that $y_2 \perp t_{y_1,2}$, implying $t_{y_1,2} = t_{y_2,2}$, so by the above $y_1 = y_2$, contradicting the fact that L contains at least three points. Hence $p_1 \notin L_1$ and likewise $p_2 \notin L_2$. Hence $p_1 \not\perp p_2$, so $\{p_1,p_2\}^\perp$ is isomorphic to $\mathsf{C}_{2,1}(\mathbb{K})$, i.e., the dual of $\mathsf{B}_{2,1}(\mathbb{K})$. But L intersects every line which intersects both L_1 and L_2, contradicting Lemma 6.8. $\square$

Corollary 6.10. *A singular plane α of Δ' disjoint from $\pi_1 \cup \pi_2$ intersects P in a (possibly empty) non-degenerate conic.*

Proof. Considering the situation in $\mathbb{P}^5(\mathbb{K})$, the intersection $\alpha \cap P$ is given by a quadratic equation in the coordinates, hence is a possibly degenerate conic. If it is degenerate, then by possibly considering the situation over a quadratic extension, we may assume that $\alpha \cap P$ contains a (singular) line, which contradicts Lemma 6.9. $\square$

Lemma 6.11. *Let $(\Gamma, \sim)$ be the graph with vertex set $X' \setminus P$, where two vertices x_1, x_2 are adjacent if they are collinear in Δ' and contained in a common singular plane α disjoint from $\pi_1 \cup \pi_2$ which intersects P nontrivially. Then Γ is connected.*

Proof. First we claim that any singular line L disjoint from $\pi_1 \cup \pi_2$ is contained in at least one singular plane disjoint from $\pi_1 \cup \pi_2$ and having nonempty intersection with P. Indeed, if $L \cap P \neq \emptyset$, then it suffices to select a plane of Δ' through L distinct from $\langle L, L^\perp \cap \pi_1\rangle$ and $\langle L, L^\perp \cap \pi_2\rangle$. Now suppose $L \cap P = \emptyset$. It is easy to select a line $L_1 \subseteq \pi_1$ such that $L^\perp \cap L_1 = \emptyset$ and that $p_2 := L_1^\perp \cap \pi_2 \neq L^\perp \cap \pi_2$. Let α be the plane spanned by L and put $y := L^\perp \cap \langle L_1, p_2\rangle$. Note that $y \in P$ and that our assumptions on L_1 and p_2 imply that $y \notin \pi_1 \cup \pi_2 \cup L$ (so α is really a plane). Assume for a contradiction that α contains some point p_1 of π_1. Then $p_1 = L^\perp \cap \pi_1$ and $y \perp \langle p_1, L_1\rangle$. The latter is, by assumption on L_1, the whole of π_1, forcing $y \in \pi_1$, a contradiction. Next, assume for a contradiction that α contains some point x_2 of π_2. Then $p_2 \perp x_2 \perp y$ and the plane $\langle x_2, p_2, y\rangle$ intersects π_1 in some point z_1 (the latter is $L_1 \cap \langle p_2, y\rangle$). Hence the point $L \cap \langle x_2, y\rangle$ belongs to P, a contradiction. The claim is proved.

Let x_1, x_2 be two distinct points of $X' \setminus P$. Suppose first that $x_1 \perp x_2$ in Δ'. If $L := \langle x_1, x_2\rangle$ is disjoint from $\pi_1 \cup \pi_2$, then $x_1 \sim x_2$ by the first paragraph. So assume that L intersects $\pi_1 \cup \pi_2$. It is easy to see that we can find a plane α containing L so that $\alpha \setminus L$ is disjoint from $\pi_1 \cup \pi_2$. Pick $x \in \alpha \setminus (L \cup P)$. Note that this is possible by Corollary 6.10 (including any line of α disjoint from $\pi_1 \cup \pi_2$ in a plane disjoint from $\pi_1 \cup \pi_2$). Then both $\langle x_1, x\rangle$ and $\langle x_2, x\rangle$ are disjoint from $\pi_1 \cup \pi_2$ and hence $x_1 \sim x \sim x_2$ by the previous paragraph.

At last suppose that x_1 is not collinear to x_2. Let α_1 be any plane through x_1 disjoint from $\pi_1 \cup \pi_2$ and not disjoint from P (this exists by the first paragraph). Let α_2' be the plane generated by x_2 and $L_2 := x_2^{\perp} \cap \alpha_1$. Then $L_2 \cap P$ has size at most 2 (since $P \cap \alpha_1$ is a non-degenerate conic). Hence we can pick $x_3 \in L_2 \setminus P$. By the previous cases, both x_1 and x_2 are in the same connected component of Γ as x_3. $\square$

Lemma 6.12. *Let $(\Gamma', \sim)$ be the graph with vertex set $X' \setminus P$, where two vertices x_1, x_2 are adjacent if they are collinear in Δ' and the joining line $\langle x_1, x_2\rangle$ intersects P in exactly two points. Then Γ' is connected.*

Proof. If $|\mathbb{K}| \geq 5$, then this follows straight from Lemma 6.11. So we may suppose that $|\mathbb{K}| = 3$. We coordinatize $\mathbb{P}^5(\mathbb{F}_3)$ such that the underlying alternating form is given by

$$\big((x_1, \cdots, x_6)(y_1, \cdots, y_6)\big) \mapsto x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3 + x_5y_6 - x_6y_5.$$

Then we may define π_1 as the plane with equations $X_2 = X_4 = X_6 = 0$ and π_2 as the plane with equations $X_1 = X_3 = X_5 = 0$. One easily calculates that a point with coordinates $(x_1, x_2, x_3, x_4, x_5, x_6)$ belongs to P if and only if $x_1x_2 + x_3x_4 + x_5x_6 = 0$ (which indeed represents an irreducible hyperbolic quadric). Now, two points $(x_1, x_2, x_3, x_4, x_5, x_6)$ and $(y_1, y_2, y_3, y_4, y_5, y_6)$ belong to $X' \setminus P$ and are adjacent in Γ' if and only if

$$\begin{cases} x_1x_2 + x_3x_4 + x_5x_6 \neq 0, \\ y_1y_2 + y_3y_4 + y_5y_6 \neq 0, \\ x_1y_2 + x_3y_4 + x_5y_6 = x_2y_1 + x_4y_3 + x_6y_5, \\ (x_1 + y_1)(x_2 + y_2) + (x_3 + y_3)(x_4 + y_4) + (x_5 + y_5)(x_6 + y_6) = 0, \\ (x_1 - y_1)(x_2 - y_2) + (x_3 - y_3)(x_4 - y_4) + (x_5 - y_5)(x_6 - y_6) = 0. \end{cases}$$

Define the *weight* of a point as the number of nonzero coordinates of each of its coordinate tuples. Now let $x_1, x_2, x_3, x_4, x_5, x_6$ be six arbitrary but nonzero elements of $\mathbb{F}_3$ (hence each of them is 1 or -1). Using the above conditions, we see that $(x_1, x_2, 0, 0, 0, 0) \sim (0, 0, x_1, -x_2, 0, 0)$. Permuting the coordinates in blocks of two in the obvious way, this implies that all points of weight 2 of Γ' belong to the same connected component, say C.

Up to permuting coordinates, a generic weight 3 vertex of Γ' is given by $(x_1, x_2, x_3, 0, 0, 0)$, and one calculates that this is adjacent to $(0, 0, 0, 0, x_1, -x_2)$. Hence also all weight 3 points in $X' \setminus P$ belong to C. Likewise, a generic weight 4 vertex has coordinates either $(x_1, x_2, x_3, x_4, 0, 0)$ with $x_1x_2 + x_3x_4 \neq 0$, or $(x_1, x_2, x_3, 0, x_5, 0)$. The former is adjacent to $(0, 0, 0, 0, x_1, x_2)$, and the latter is adjacent to $(0, 0, x_3, x_1x_2x_3, -x_5, -x_1x_2x_5)$ (note that $x_3^2 = x_5^2 = 1$). Hence all weight 4 elements belong C. A generic weight 5 vertex is a point with coordinates $(x_1, x_2, x_3, x_4, x_5, 0)$, with $x_1x_2 + x_3x_4 \neq 0$ (and note that this implies $x_1x_2 - x_3x_4 = 0$). This point is now adjacent to $(x_1, -x_2, -x_3, x_4, 0, 0)$, showing that all weight 5 vertices belong to C. Finally, for the vertex with coordinates $(x_1, x_2, x_3, x_4, x_5, x_6)$ we may, without loss of generality, assume that $x_1x_2 = x_3x_4 = -x_5x_6$. It follows that this vertex is adjacent to $(x_1, x_2, -x_3, -x_4, 0, 0)$.

We have shown that all vertices belong to C, and the assertion follows. $\square$

Define the following geometry $\Delta'' = (X'', \mathscr{L}'')$. The set X'' is the set of lines of Δ' intersecting both π_1 and π_2 nontrivially. A typical member of $\mathscr{L}''$ is the pencil determined by (p, α), where $p \in \pi_i$ and α is a plane of Δ' containing p and intersecting π_{3-i} in a line, $i \in \{1, 2\}$.

Lemma 6.13. *The geometry Δ'' is isomorphic to $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$.*

Proof. Let $\{p, L\}$ be a flag of π_1, where $p \in \pi_1$ is a point and $L \subseteq \pi_1$ a line containing p. Obviously, the mapping $\{p, L\} \mapsto \langle p, L^\perp \cap \pi_2\rangle$ is a bijection between the set of flags of π_1 and X''. Also, for each line L of π_1, that bijection maps the set of flags $\{\{p, L\} \mid p \in L\}$ onto the line pencil determined by $(L^\perp \cap \pi_2, L)$, which belongs to $\mathscr{L}''$, and, for each point $p \in \pi_1$, it maps the set of flags $\{\{p, L\} \mid p \in L \in \mathscr{L}', L \subseteq \pi_1\}$ onto the pencil determined by $(p, p^\perp \cap \pi_2)$, which belongs to $\mathscr{L}''$. One checks that this correspondence is bijective onto $\mathscr{L}''$. $\square$

The next lemma holds for all fields with at least 3 elements.

Lemma 6.14. *Let $\mathbb{K}$ be an arbitrary field with at least three elements. Then the Veronese generating rank and the Veronese embedding rank of $\mathbb{P}^2(\mathbb{K})$ are both equal to 6.*

Proof. Clearly a line and a point do not Veronese generate $\mathbb{P}^2(\mathbb{K})$. Since five arbitrary points no four on a line determine a (possibly degenerate) conic of $\mathbb{P}^2(\mathbb{K})$, and every such conic intersects each line that it does not contain in at most two points, the Veronese g-rank of $\mathbb{P}^2(\mathbb{K})$ is at least 6. Let p_1, p_2, p_3 be a triangle in $\mathbb{P}^2(\mathbb{K})$ (that is, they are not contained in a common line). Select a point q_i in $\langle p_j, p_k\rangle \setminus \{p_j, p_k\}$, with $\{1, 2, 3\} = \{i, j, k\}$. Then $\{p_1, p_2, p_3, q_1, q_2, q_3\}$ generates the three lines $\langle p_1, p_2\rangle$, $\langle p_2, p_3\rangle$, $\langle p_1, p_3\rangle$. Since $|\mathbb{K}| > 2$, every point is contained in a line intersecting the union of these three lines in three distinct points. Whence the assertion concerning the Veronese g-rank. The assertion concerning the Veronese e-rank follows straight from Theorem 2.3 of [27]. $\square$

Let $\epsilon' : X' \subseteq \mathbb{P}^{20}$ be the Veronese embedding of Δ' obtained from the Veronese map on the underlying projective space $\mathbb{P}^5(\mathbb{K})$. Let ϵ be the restriction of ϵ' to P. We note that ϵ, and hence every embedding that projects onto it, satisfies the following easy to verify statements.

(a) Every projective plane contained in P spans a 5-space and hence defines an ordinary quadric Veronese surface (use Theorem 2.3 of [27] again);
(b) The span of two disjoint planes of P intersects P in the union of those planes.

Lemma 6.15. *Consider P as a subgeometry of Δ' with induced line set. Then the Veronese generating rank of P is at most $12 + \rho_{\mathsf{g}}(\mathsf{A}_2(\mathbb{K}))$. Also, the ϵ-relative Veronese embedding rank of P is at most $12 + \rho^\circ_{\mathsf{e}}(\mathsf{A}_2(\mathbb{K}))$.*

Proof. We first prove the assertion about the Veronese g-rank of P. Let G be a minimum generating set of points for Δ''. For each $g \in G$, we select an arbitrary point p_g on the corresponding line L_g of Δ', but not belonging to $\pi_1 \cup \pi_2$. We claim that $G^* := \pi_1 \cup \pi_2 \cup \{p_g \mid g \in G\}$ generates P, which then proves the assertion using Lemmas 6.13 and 6.14. Indeed, it suffices to show that, if $g_1, g_2 \in G$ and g_1 is collinear to g_2 in Δ'', then each point of the singular plane $\alpha := \langle L_{g_1}, L_{g_2}\rangle$ of Δ' is (Veronese) generated by G^*. Now α intersects $\pi_1 \cup \pi_2$ in the union of a point and a line, say $\{x\} \cup K$. Then clearly $\{x, p_{g_1}, p_{g_2}\} \cup K$ is a Veronese generating set of α (because it contains the triangle $\{x, p_{g_1}, p_{g_2}\}$ together with an additional point on each side, namely the intersection of that side with K). The claim follows.

Next we consider the ϵ-relative Veronese e-rank. So we assume that P is embedded in some projective space $\mathbb{P}$ such that each of its lines is a plane conic, and P projects into the usual Veronese (Weyl) embedding of $\mathsf{C}_{3,1}(\mathbb{K})$ obtained from the ordinary Veronese embedding of the ambient projective space $\mathbb{P}^5(\mathbb{K})$. By (a), the planes contained in P correspond to ordinary Veronese surfaces. The subspace spanned by $\pi_1 \cup \pi_2$ in $\mathbb{P}$ is strictly contained in the one generated by P (as this is the case in the Weyl embedding). So we can project $P \setminus (\pi_1 \cup \pi_2)$ from $W := \langle \pi_1, \pi_2 \rangle$ (generation is now in $\mathbb{P}$) onto some complementary subspace U of $\mathbb{P}$. Let α be a singular plane in P intersecting π_1 in some point p_1 and π_2 in some line L_2. By (b), the projection of α in $\mathbb{P}$ from $\langle p_1, L_2 \rangle$ is either a point or a (full) line. If it were a point, then some 4-space of $\langle \alpha \rangle$ would intersect α in just $\{p_1\} \cup L_2$, a contradiction. It follows that the projection of $P \setminus (\pi_1 \cup \pi_2)$ from W onto U is isomorphic to an embedded $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$. Moreover, in the Weyl embedding, the same procedure yields the Weyl embedding, hence the dimension of the subspace generated by the image of the projection of $P \setminus (\pi_1 \cup \pi_2)$ from W onto U is at most $\rho^\circ_{\mathsf{e}}(\mathsf{A}_2(\mathbb{K})) - 1$. Since $\dim\langle \pi_1 \cup \pi_2 \rangle \leq 11$, the lemma is proved. $\square$

We are now ready to prove the main result of this subsection.

Proposition 6.16. *Let $\mathbb{K}$ be a field with characteristic distinct from 2. Then $\rho^*_{\mathsf{g}}(\mathsf{C}_{3,1}(\mathbb{K})) \leq 13 + \rho_{\mathsf{g}}(\mathsf{A}_2(\mathbb{K}))$ and $\rho^*_{\mathsf{e}}(\mathsf{C}_{3,1}(\mathbb{K})) \leq 13 + \rho^\circ_{\mathsf{e}}(\mathsf{A}_2(\mathbb{K}))$.*

Proof. First we consider the Veronese g-rank. It suffices to prove that, for an arbitrary point $x \in X' \setminus P$, the set $P \cup \{x\}$ is a Veronese generating set of Δ'. Clearly, all points on each line L through x that intersects P in two points are Veronese generated by $P \cup \{x\}$. Now the assertion follows from Lemma 6.12.

Concerning the Veronese e-rank, we have to show that, if Δ' is Veronese embedded in the projective space $\mathbb{P}$, then X' is contained in the subspace of $\mathbb{P}$ generated by P and one additional point $x \in X' \setminus P$. Suppose α is a plane disjoint from $\pi_1 \cup \pi_2$, containing x and meeting P non-trivially, i.e., in a non-degenerate conic by Corollary 6.10. Then, since every conic in α generates a hyperplane in the corresponding ambient projective 5-space of the Veronese surface, and that hyperplane does not contain any other points than those of the conic, we see that all points of α are contained in the (projective) subspace of $\mathbb{P}$ spanned by P and x. Now again Lemma 6.11 completes the proof. $\square$

6.3. **Conclusion.** In this subsection, we let p and q again be two opposite points of $\Delta \cong \mathsf{F}_{4,1}(\mathbb{K})$, and H is again $\langle p^{\perp\!\!\perp} \cup q^{\perp\!\!\perp} \rangle$ as in Subsection 6.2.1. We can now complete the proof of Theorem 6.3. First we notice that this theorem follows from Lemmas 4.23 and Proposition 6.4 as soon as we show the following lemma:

Lemma 6.17. *The generating rank of H is at most $30 + \rho^*_{\mathsf{g}}(\mathsf{C}_{3,1}(\mathbb{K})) \leq 43 + \rho_{\mathsf{g}}(\mathsf{A}_2(\mathbb{K}))$, and the embedding rank is at most $30 + \rho^*_{\mathsf{e}}(\mathsf{C}_{3,1}(\mathbb{K})) \leq 43 + \rho^\circ_{\mathsf{e}}(\mathsf{A}_2(\mathbb{K}))$.*

Proof. We start by noting that $\mathrm{Res}_\Delta(p)$ is isomorphic to $\mathsf{C}_{3,3}(\mathbb{K})$. Such a geometry has e-rank and g-rank equal to $\binom{6}{3} - \binom{6}{1} = 14$, by [14] and [19]. Hence the e-rank of $p^\perp \cup q^\perp$ is at most 30. The second assertion of Proposition 6.16 implies that the e-rank of H is at most $30 + 13 + \rho^\circ_{\mathsf{e}}(\mathsf{A}_2(\mathbb{K}))$.

Now consider a set T of 30 points generating $p^\perp \cup q^\perp$ and a set E Veronese generating $E(p,q) = p^{\perp\!\!\perp} \cap q^{\perp\!\!\perp}$. Let C be a line of $E(p,q)$, the latter viewed as a symplectic polar space. Let $\xi(C)$ be the symplecton containing C. Then, as explained in Lemma 6.6 and just after it, $C = L^\perp \cap M^\perp$, with $L = p^\perp \cap \xi(C)$ and $M = q^\perp \cap \xi(C)$. Hence if E contains at least three points of C, then the whole of C is generated by $T \cup E$. It follows that, if E is a Veronese generating set of $E(p,q)$, then $T \cup E$ generates H. The first assertion now follows from the first assertion of Proposition 6.16. $\square$

Since $\omega(\mathsf{F}_4(\mathbb{K})) - \omega(\mathsf{C}_3(\mathbb{K})) = 52 - 21 = 31$ and $\omega(\mathsf{C}_3(\mathbb{K})) - \omega(\mathsf{A}_2(\mathbb{K})) = 21 - 8 = 13$, the arrows $\mathsf{A}_2 \to \mathsf{C}_3 \to \mathsf{F}_4$ of Theorem A follow, as before, from Lemma 4.23, Propositions 6.4 and 6.16, and Lemma 6.17. Moreover, Fact 3.4 implies that the g-rank of $\mathsf{F}_{4,1}(p)$ is equal to 52, for any prime p. This shows Theorem C for type F_4.

7. The classical cases $\mathsf{A}_n, \mathsf{B}_n$ and D_{n+1}, $n \geq 2$

The structure of the proof of Theorem B is exactly the same as that of Theorem A. Hence we are not going to repeat it here. We content ourselves with only mentioning the various geometric hyperplanes H, based on the various equator geometries. We consider all cases separately.

7.1. **Case $\mathsf{A}_{n-1} \to \mathsf{A}_n$, $n \geq 3$.** Recall that $\mathsf{A}_{n,\{1,n\}}(\mathbb{K})$ is the geometry with point set the incident point-hyperplane pairs of the projective space $\mathbb{P}^n(\mathbb{K})$, where lines are given by the incident line-hyperplane pairs and incident point-subhyperplane pairs, with natural incidence (a *subhyperplane* is a subspace of codimension 2, that is, a hyperplane of a hyperplane).

Let $\Delta = (X, \mathscr{L})$ be isomorphic to $\mathsf{A}_{n,\{1,n\}}(\mathbb{K})$. Pick a non-incident point-hyperplane pair (p, W) in the underlying projective space $\mathbb{P}^n(\mathbb{K})$. Define H to be the subspace of Δ generated by all points $(x, W) \in X$, $x \in W$, all points $(p, U) \in X$, $p \in U$ and all points $(x, U) \in X$, $x \in W$ and $p \in U$. One easily checks that H indeed generates a hyperplane. Moreover, the singular subspaces $\{(x, W) \in X \mid x \in W\}$ and $\{(p, U) \in X \mid p \in U\}$ have $\bullet$-rank n, whereas the subspace $\{(x, U) \in X \mid x \in W \text{ and } p \in U\}$ is isomorphic to $\mathsf{A}_{n-1}(\mathbb{K})$. The arrow now follows from the numerical equality $\omega(\mathsf{A}_n(\mathbb{K})) = (n+1)^2 - 1 = ((n^2-1) + n + n) + 1 = \omega(\mathsf{A}_{n-1}(\mathbb{K})) + 2n + 1$.

Remark 7.1. This arrow implies that the $\bullet$-rank of the geometry $\mathsf{A}_{n,\{1,n\}}(\mathbb{K})$, for $\mathbb{K}$ finite but not prime, is equal to $\omega(\mathsf{A}_n(\mathbb{K})) + 1$. Indeed, this is true for $n = 2$ by Lemma 3.5, and it follows from Proposition 8.1 below for $n \geq 3$.

Remark 7.2. This arrow can also be interpreted as an arrow in the class of Segre embeddings, with the same proof. Hence, as a consequence, the Weyl embedding of $\mathsf{A}_{n,\{1,n\}}(\mathbb{K})$, for $\mathbb{K}$ a finite field, is relatively universal.

7.2. **Case $\mathsf{A}_{n-1} \to \mathsf{D}_n$, $n \geq 3$.** Recall that $\mathsf{D}_{n,2}(\mathbb{K})$ is the geometry with point set the set of lines of a hyperbolic polar space Γ of rank n and lines the planar line pencils.

Let $\Delta = (X, \mathscr{L})$ be isomorphic to $\mathsf{D}_{n,2}(\mathbb{K})$. Pick two disjoint (opposite) maximal singular subspaces W_1, W_2 in the underlying polar space $\Gamma = (Y, \mathscr{M})$; hence $X = \mathscr{M}$. Define H to be the subspace of Δ generated by all points $M \in \mathscr{M}$ either contained in W_1 or W_2, or intersecting both W_1 and W_2 nontrivially. It is routine to check that H indeed generates a hyperplane of Δ (use the fact that every point of Γ is contained in a line intersecting both W_1 and W_2 nontrivially). Clearly the subspace on the set $\{M \in \mathscr{M} \mid M \subseteq W_i\}$, $i = 1, 2$, is isomorphic to $\mathsf{A}_{n-1,2}(\mathbb{K})$, whereas the subspace on the set $\{M \in \mathscr{M} \mid M \cap W_i \neq \emptyset, i = 1, 2\}$ is clearly isomorphic to the long root geometry $\mathsf{A}_{n-1,\{1,n-1\}}(\mathbb{K})$. Now, by Fact 3.1, the g-rank and e-rank of $\mathsf{A}_{n-1,2}(\mathbb{K})$ are both equal to $\frac{n(n-1)}{2}$. The arrow then follows from the numerical equality $\omega(\mathsf{D}_n(\mathbb{K})) = 2n^2 - n = \omega(\mathsf{A}_{n-1}(\mathbb{K})) + n(n-1) + 1$.

7.3. **Case $\mathsf{D}_{n-1} \to \mathsf{D}_n$, $n \geq 4$.** Let $\Delta = (X, \mathscr{L})$ again be isomorphic to $\mathsf{D}_{n,2}(\mathbb{K})$. This time, we pick two non-collinear (opposite) points p_1, p_2 in the underlying polar space $\Gamma = (Y, \mathscr{M})$. Define H to be the subspace of Δ generated by all points $M \in \mathscr{M}$ either incident with p_1 or with p_2, or contained in $p_1^\perp \cap p_2^\perp$. It is again routine to check that H indeed generates a hyperplane of Δ (use the fact that every singular plane of Γ intersects $p_1^\perp \cap p_2^\perp$ nontrivially). Clearly the subspace on the set $\{M \in \mathscr{M} \mid p_i \in M\}$, $i = 1, 2$, is isomorphic to $\mathsf{D}_{n-1,1}(\mathbb{K})$, whereas the subspace on the set $\{M \in \mathscr{M} \mid M \subseteq p_1^\perp \cap p_2^\perp\}$ is isomorphic to the long root geometry $\mathsf{D}_{n-1,2}(\mathbb{K})$. Now, by Fact 3.1, the g-rank and e-rank of $\mathsf{D}_{n-1,1}(\mathbb{K})$ are equal to $2(n-1)$. The arrow then follows from the numerical equality $\omega(\mathsf{D}_n(\mathbb{K})) = 2n^2-n = (2(n-1)^2-(n-1))+2(n-1)+2(n-1)+1 = \omega(\mathsf{D}_{n-1,2}(\mathbb{K}))+4n-3$.

7.4. **Case $\mathsf{B}_n \to \mathsf{B}_{n+1}$, $n \geq 2$.** This case is completely similar to the previous arrow. Note that, for the case $n = 2$, we have to use $\rho^*_g(\mathsf{B}_{2,2}(\mathbb{K}))$ and $\rho^*_{\mathsf{e}}(\mathsf{B}_{2,2}(\mathbb{K}))$, which are the Veronese g-rank and Veronese e-rank, respectively, as defined in Section 2.

8. Proof of Theorem D

Putting Theorems A and B together, we see that the excess in the g-rank of $\mathsf{A}_{2,\{1,2\}}(\mathbb{K})$ is at least the excess in g-rank of all long root geometries mentioned in the statement of Theorem D. Since for a finite field, this excess is 0 in the prime case, and 1 otherwise, the first part of Theorem D follows. We now show the last part. This will follow from the next result.

Proposition 8.1. *If $\mathbb{K}$ is a field with* $\mathrm{Aut}(\mathbb{K}) \neq 1$, *then* $\rho_e(\mathsf{A}_{n,\{1,n\}}(\mathbb{K})) \geq (n+1)^2$.

Proof. Let $\theta \in \mathrm{Aut}(\mathbb{K})$. Consider the following map from $\mathsf{A}_{n,\{1,n\}}(\mathbb{K})$ to $\mathbb{P}^{n^2+2n}(\mathbb{K})$. We label the coordinates in the latter with $(x_{ij})_{1\leq i\leq n, 1\leq j\leq n}$. We also denote a point of $\mathsf{A}_{n,\{1,n\}}(\mathbb{K})$ by the coordinates of a point-hyperplane flag in $\mathbb{P}^n(\mathbb{K})$, that is, with a pair $((x_i)_{1\leq i\leq n}, (a_j)_{1\leq j\leq n})$, all elements in $\mathbb{K}$, and $\sum_{i=1}^n a_i x_i = 0$:

$$((x_i)_{1\leq i\leq n}, (a_j)_{1\leq j\leq n}) \mapsto (x_i a_j^\theta)_{1\leq i\leq n, 1\leq j\leq n}.$$

If $\theta = 1$, this induces the ordinary Weyl embedding. If θ is nontrivial, one shows, exactly as in Section 2 of [31], that this induces an embedding spanning $\mathbb{P}^{n^2+2n}(\mathbb{K})$. $\square$

Now the second part of Theorem D follows from the first arrow of Theorem B and Fact 3.5.

Finally, we can prove the finite case of Völklein's result in a purely geometric way.

Proposition 8.2. *The embedding rank of any finite long root subgroup geometry of type D_n, $n \geq 4$, $\mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$ and F_4 (the latter in characteristic distinct from 2) is exactly equal to the dimension of the Weyl module.*

Proof. This follows from Fact 3.5, the fact that the stated geometries admit the universal embedding by [24], Theorem 6.3, Remark 7.2 and the arrows $\mathsf{A}_{n-1} \to \mathsf{D}_n$ and $\mathsf{D}_5 \to \mathsf{E}_6 \to \mathsf{E}_7 \to \mathsf{E}_8$. $\square$

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