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ORIGINAL PAPER

Local recognition of affine rank 3 graphs

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Abstract

We provide local recognition results for all infinite families of affine rank 3 graphs which are not locally a disjoint union of cliques and which are not 1-dimensional (that is, for the graphs corresponding to Liebeck's classes (A3) up to (A10) of affine rank 3 groups). We show that the local graphs alone do not characterise the global graphs by providing counterexamples. Our principal result is that the global graph is determined by the local if one adds the condition that the number of vertices adjacent to two given vertices at distance 2 from each other is a constant. Alternative conditions for specific classes are also given.

Keywords Affine rank 3 graphs · Local recognition · Clique extension

Mathematics Subject Classification 05E30 · 20E42

1 Introduction

The local graph of a graph Γ at the vertex $v \in V(\Gamma)$ is the subgraph of Γ induced by the set $\Gamma_1(v)$ of neighbours of v in Γ. By a local structure of a graph, we mean the collection of its local subgraphs, which are often assumed to be pairwise isomorphic. In a recent manuscript [19], rank 3 graphs of almost simple type related to finite groups of Lie type are characterised by their local structure (without any global assumption, that is, not even assuming the global graphs have diameter 2). In these cases, the local graphs are q-clique extensions of the point graphs of Lie incidence geometries over the finite field $\mathbb{F}_q$. It was noted in [19] that a further weakening of the assumptions to q'-cliques extension, with q' not necessarily equal to q, was not possible since there exist counterexamples for $q' = q - 1$. In the present manuscript, we precisely handle the case of graphs locally isomorphic to $(q-1)$-clique extensions of some Lie incidence geometries defined over $\mathbb{F}_q$. Such extensions turn up as local graphs when

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Table 1 Affine rank 3 graphs in notation of [3, §11.4]

Γ	Structure at infinity	Classes
$\mathsf{H}_q(2,e)$	$\mathsf{PG}(1,q)\times\mathsf{PG}(e-1,q)$	(A3), (A4), (A5)
$V\mathsf{O}^{\pm}_{2n}(q)$	$Q^{\pm}(2n-1,q)$	(A6), (A7), (A9)
$V\mathsf{J}_q(5,2)$	$\mathsf{A}_{4,2}(q)$	(A8)
$V\mathsf{D}_{5,5}(q)$	$\mathsf{D}_{5,5}(q)$	(A10)

the rank 3 graph has affine type — hence determined by a group G_0 acting on a vector space V over $\mathbb{F}_q$ with two orbits O_1 and O_2 on the 1-spaces — and one of these orbits, say O_1, has the property that, whenever two 1-spaces $\langle v\rangle$ and $\langle v'\rangle$ belong to O_1, then either all 1-spaces of the 2-space $\langle v, v'\rangle$ belong to O_1, or no other 1-space of that 2-space, except for $\langle v\rangle$ and $\langle v'\rangle$, belongs to O_1. This is precisely the case for the classes (A2), (A3), (A6), (A7), (A8), (A9), (A10) and (A11) of Table 11.4 in [3], see Table 1. However, for (A2) and (A11), the local graph is a disjoint union of $(q-1)$-cliques, and hence does not determine the global graph at all. On the other hand, the graphs in classes (A4) and (A5) are contained in the class (A3), despite the fact that they do not satisfy the above condition (which is hence sufficient, but not necessary to have a $(q-1)$-clique extension as local graph). The classes (A6) and (A9) are contained in the class (A7). Also, if $q=2$, then a $(q-1)$-clique extension is a trivial extension, and this requires very different techniques. So throughout, we will assume that the ground field has order at least 3.

Consequently, we aim to prove local characterisations of the classes (A3) up to (A10) of Table 11.4 in [3], for $q>2$, which are summarised in Table 1.

However, these graphs are not determined by their local structure alone. Indeed, we shall construct different graphs having the same local structure. Our global additional assumption will be that the number of vertices adjacent to two given vertices at distance 2 from each other is a constant. This is implied, for instance, by the assumption that the graph is strongly regular, or more relaxed, that the graph is distance-regular. Other assumptions such as assuming the diameter is 2, or setting an upper bound on the number of vertices, also work in some cases. We survey our main results, using the notation introduced in Section 2. We start with our principal result. Recall that $\Gamma_1(v)$ denotes the set of neighbours of a vertex v in the graph Γ.

Theorem A *Let Γ be a (not necessarily finite) graph with the property that at each vertex $v\in V(\Gamma)$, the graph induced on $\Gamma_1(v)$ is isomorphic to the graph induced on a neighbour of a vertex of one of the graphs $\mathsf{H}_q(2,e)$, $e\geq 2$, $V\mathsf{O}^{\pm}_{2n}(q)$, $n\geq 3$, $V\mathsf{J}_q(5,2)$ or $V\mathsf{D}_{5,5}(q)$, $q>2$ that is, one of the graphs in Table 1. If the number of vertices adjacent to two given non-adjacent vertices is a constant, then Γ is one of the mentioned graphs.*

The following consequence is immediate from Theorem A.

Corollary B *Let Γ be a graph with the property that at each vertex $v\in V(\Gamma)$, the graph induced on $\Gamma_1(v)$ is isomorphic to the graph induced on a neighbour of a vertex of one of the graphs $\mathsf{H}_q(2,e)$, $e\geq 2$, $V\mathsf{O}^{\pm}_{2n}(q)$, $n\geq 3$, $V\mathsf{J}_q(5,2)$ or $V\mathsf{D}_{5,5}(q)$ (with*

$q > 2$ in all cases). If Γ is distance-regular, then Γ is strongly regular and isomorphic to one of the mentioned graphs.

A similar result can be derived by assuming an upper bound on the number of vertices.

Theorem C *Let Γ be a graph locally isomorphic to either $\mathsf{H}_q(2, e)$, $e \geq 2$, or $V\mathsf{O}^+_{2n}(q)$, $n \geq 3$, or $V\mathsf{J}_q(5, 2)$ or $V\mathsf{D}_{5,5}(q)$ (with $q > 2$ in all cases). If Γ contains at most either q^{2e}, q^{2n}, q^{10} or q^{16} vertices, respectively, then it is isomorphic to the respective graph.*

The following result completely classifies graphs locally isomorphic to $V\mathsf{O}^\pm_{2n}(q)$, and, in particular, it improves Theorem C in the case of $V\mathsf{O}^+_{2n}(q)$.

Theorem D *Let Γ be a graph locally isomorphic to $V\mathsf{O}^\pm_{2n}(q)$, $n \geq 3$, $q > 2$. Then it is the point graph of an affine polar space obtained from $Q^\pm(2n-1, q)$ by deleting a geometric hyperplane (and there are two different isomorphism classes of graphs arising, one of diameter 2, which is $V\mathsf{O}^\pm_{2n}(q)$, and one of diameter 3).*

We already noted that Theorem A comprises all graphs of the classes (A3) up to (A10) in [3, §11.4] (see Table 1). As for the graphs in classes (A2) and (A11) of [3, §11.4], they are the point graphs of point-line geometries without larger singular subspaces than the lines. It concerns grids (generalised quadrangles with two lines on each point) and partial quadrangles. Locally they look like a bunch of disjoint cliques, which is the case for each point-line geometry without triangles (like generalised polygons of diameter at least 4 and near $2n$-gons, $n \geq 2$). Hence it is impossible to characterise these classes locally.

As for class (A1), this consists of the 1-dimensional affine rank 3 graphs (Payley graphs, Peisert graphs and van Lint–Schrijver graphs) and there is no uniform description of the local graphs, which are usually not isomorphic to a clique extension of a graph, as in the present paper. So other techniques must be used. For some of these graphs, and also for other ones, there are a number of local recognition results available in the literature, and we survey some now; see [8] for a good overview of local recognition results for graphs up to 1990, and [24] for a more recent survey.

Graphs that are locally Paley graphs with q vertices, with $q > 41$, were classified by work of Brouwer [1] and Muzychuk & Kovács [20]. For $q < 41$ Buset's thesis solves this problem [6], and Brouwer [2] explains why the case $q = 41$ also works. Buekenhout and Hubaut studied locally polar spaces [5], which includes the case $q = 2$ for the graphs $V\mathsf{O}^\pm_{2n}(q)$ of the present paper. J. Hall showed that up to isomorphisms there are three graphs which are locally Petersen graphs [16]. J. Hall and Shult studied locally cotriangular graphs [17]. The Schläfli graph is the unique connected locally Clebsch graph. The Gosset graph is the unique connected locally Schläfli graph. The graph given by the 1-skeleton of the famous 24-cell in real 4-dimensional Euclidean space is locally a cube. The only other connected locally cube graph is the complement of the 4×5 grid [7]. Gramlich, Hall and Straub complete the local recognition of commuting reflection graphs of spherical Coxeter groups arising from irreducible crystallographic root systems [14]. Cohen, Cuypers and Gramlich [9] characterise the graph on non-incident point-hyperplane pairs of a projective space (with natural adjacency) by its local structure, whereas Gramlich [13] does the same for the graph on the non-incident line-hyperline pairs of a projective space. Very recently local recognition of the point graphs of some Lie incidence geometries was studied [19]. Finally, the Payley graph on 17 vertices was shown to be the unique connected locally Möbius ladder graph by Smallwood [24]. The Payley graphs are, as pointed out above, affine rank 3 graphs in the class (A1), and the result by Smallwood shows that techniques very different from the ones in the present paper must be used to locally recognise the 1-dimensional affine rank 3 graphs.

Some of the above papers on local recognition give rise to characterisations of certain groups by properties of centralisers of a conjugacy class of a certain element, see for instance [9, 13]. Other group theoretic applications of graphs related to Lie incidence geometries are for example contained in the arXiv preprint *The geometry of hyperbolic lines in polar spaces* of H. Cuypers, which was subsequently used in [11] for a characterisation of certain groups by k-transvection groups. Note that, for administrative reasons, both references by Cuypers are much older than the dates they were put on arXiv or submitted for publication to make sure they did not interfere with the PhD thesis of Ralf Gramlich.

Outline of the paper: In Section 2, we introduce affine graphs, Lie incidence geometries and polar spaces. We recall known facts about these objects that will be used in the remainder of the paper. Using these notions, we describe the four infinite families of affine rank 3 graphs, namely $\mathsf{H}_q(2,e)$, $V\mathsf{O}^{\pm}_{2n}(q)$, $V\mathsf{J}_q(5,2)$ and $V\mathsf{D}_{5,5}(q)$, a classification of which can be found in [3, Chapter 11]. In Section 3, we discuss point graphs of geometries obtained from another geometry by removing all points and lines contained in some geometric hyperplane. The existence of graphs of this type obtained via Proposition 3.1 shows that the previously mentioned rank 3 graphs are not characterised by their local structure alone. We are able to overcome this issue by introducing additional assumptions. We note that the constructed counterexamples have diameter 3 (and hence are not strongly regular) and/or have more vertices. The rest of the paper is dedicated to proving that one can characterise graphs $\mathsf{H}_q(2,e)$, $V\mathsf{O}^{\pm}_{2n}(q)$, $V\mathsf{J}_q(5,2)$ and $V\mathsf{D}_{5,5}(q)$ via their local structure under the additional assumption that the number of neighbours shared by two non-adjacent vertices at distance 2 is constant or, alternatively, that the number of vertices is appropriately bounded.

In Section 4, we initiate a study of graphs for which the local graph at every vertex is a clique extension of a graph with certain properties (see Lemma 4.2). For such graphs, we introduce the notion of a graph geometry. We then specialise to the case when the local graphs are clique extension of point graphs of certain geometries. We introduce affine singular planes and define parallelism for the graph geometry. Finally, we are able to prove that in this context, all local geometries are isomorphic (see Proposition 4.17).

In Section 5, we study the case when local geometries are isomorphic to the point graphs of the quadrics $Q^+(2n-1,q)$, $n \geq 2$, and $Q^-(2n-1,q)$, $n \geq 3$. We show that the corresponding graph geometry is an affine polar space in the sense of Cohen & Shult [10]. The result of their paper allows us to characterise the graph $V\mathsf{O}^{\pm}_{2n}(q)$. In the following sections we deal with the remaining families of affine rank 3 graphs.

In Section 6, given a graph Γ having the appropriate local structure, we study the structure of the graph induced by the common neighbours of two vertices at mutual distance 2. We call this subgraph the mu-graph. We are able to show that the set of points of a local geometry induced by the mu-graph is in fact a complement of a geometric hyperplane of a symp in that local geometry (see Proposition 6.4 for more details). This crucial observation lets us count the number of vertices in the mu-graph, and for each type of local structure, we obtain two possible parameters, which we denote by μ and μ' (see Eq. 6.6 for these values). Using counting arguments and applying either of our additional assumptions, we are able to prove that all mu-graphs of the given graph Γ have the same size given by μ (which is equal to the μ parameter of the affine rank 3 strongly regular graph with the same local structure as Γ).

Finally, in Section 7, we are able to describe adjacency of vertices $x, z \in \Gamma_2(y)$ for a given vertex y entirely in terms of the geometry of the set of their common neighbours $x^{\perp} \cap y^{\perp} \cap z^{\perp}$ viewed as a set of points of the local geometry at y, as explained in Proposition 7.1. This allows us to characterise the graphs $\mathsf{H}_q(2, e)$, $V\mathsf{J}_q(5, 2)$ and $V\mathsf{D}_{5,5}(q)$ under either of the two additional assumptions in Theorem 7.2 and Theorem 7.3.

Section 8 contains a group theoretic application along the lines of [9].

2 Preliminaries

In this section we set notation, introduce the necessary background in geometry and recall the main known properties of some Lie incidence geometries that will be used in our arguments. We restrict ourselves to the finite cases and examples, since our characterisation will only work in the locally finite case.

2.1 Affine (rank 3) graphs

A graph $\Gamma = (V(\Gamma), E(\Gamma))$ is strongly regular with parameters (v, k, λ, μ) if $|V(\Gamma)| = v$, the valency (or degree) of each vertex is k, each edge is in precisely λ triangles, and every two non-adjacent vertices share precisely μ neighbours. We use standard notation $\Gamma_j(v)$ for the set of vertices at distance j from a given vertex $v \in V(\Gamma)$. For a general graph Γ, we call the graph induced on the common neighbours of two vertices at mutual distance 2 the *mu-graph (of these vertices)*.

An *affine expansion graph*, or briefly, an *affine graph*, is a graph Γ whose vertex set is the set of points of an affine space, and where two vertices are adjacent if the joining line intersects the projective space at infinity in a point that belongs to a given subset S of points at infinity. An equivalent description runs as follows. The vertices are the vectors of a given vector space, adjacent if their difference belongs to the union of a given set of 1-spaces (hence it is some Cayley graph for an elementary abelian group). In our case, the subset S will always be an embedded geometry. This will enable us in Section 3 to describe the corresponding graph in yet another way, only using abstract Lie incidence geometries, which we introduce in the next paragraph.

2.2 Grassmannians, Lie incidence geometries

In the present paper, a *point-line geometry* $\Delta = (X, \mathscr{L})$ is a pair consisting of a set X of *points*, and a set $\mathscr{L}$ of subsets of X, called *lines*.

Let $\Delta = (X, \mathscr{L})$ be a point-line geometry. Points x and y on a common line are called *collinear*, denoted $x \perp y$ (sometimes $x \perp_\Delta y$ for clarity). For a point x we denote by $x^\perp$ or $x^{\perp_\Delta}$ the set of points collinear to x (this includes x itself if, and only if, there exists some line on x). A *subspace* of Δ is a set of points intersecting each line in none, exactly one, or all of its points. A *(geometric) hyperplane* is a subspace intersecting each line nontrivially. A *singular subspace* is a subspace all points of which are mutually collinear.

A point-line geometry is a *partial linear space* if every line contains at least three points, and there is a unique line, sometimes denoted xy, through every pair $\{x, y\}$ of distinct collinear points (however, we will also use the notation $\langle x, y \rangle$). A point-line geometry is degenerate if there is some point collinear to each other point.

A *gamma space* is a point-line geometry $\Delta = (X, \mathscr{L})$ such that for each line $L \in \mathscr{L}$ and each point $p \in X$, either no, or exactly one, or all points of L are collinear to p.

The standard example of a point-line geometry is the one arising from a projective space $\mathsf{PG}(V)$, constructed from a vector space V with dimension $d + 1$ at least 3 by taking for points the 1-spaces of V and the lines can be identified with the 2-spaces of V in the natural way (each 2-space is regarded to be the collection of its 1-spaces). We denote that point-line geometry often by $\mathsf{A}_{d,1}(q)$ or $\mathsf{PG}(d, q)$, where V is defined over the finite field $\mathbb{F}_q$ (remember we restrict to finite graphs and geometries). Its (projective) dimension is d. More generally, given a vector space V over $\mathbb{F}_q$ of dimension $d + 1 \geq 3$, and a natural number j, with $1 \leq j < n$, we can define a geometry $\mathsf{A}_{d,j}(q)$ as follows. The point set X is the set of j-spaces of V. This set can be seen as a set of points of the geometry $\mathsf{A}_{n,1}(q)$, with $n + 1 = \binom{d+1}{j}$ via the Plücker coordinates. The lines of $\mathsf{A}_{d,j}(q)$ are then the lines of $\mathsf{A}_{n,1}(q)$ completely contained in X. Alternatively, a line is the set of j-spaces of V containing a given $(j-1)$-space and contained in a given $(j+1)$-space (containing the $(j-1)$-space). The geometry $\mathsf{A}_{d,j}(q)$ is called a *j-Grassmannian*. For $j = 2$, it is sometimes referred to as a *line-Grassmannian of type d*. Note that $\mathsf{A}_{d,j}(q)$ is canonically isomorphic to $\mathsf{A}_{d,d-j}(q)$ by duality.

A projective space (plane) from which a hyperplane is removed is called an *affine space (plane)*.

Other geometries arise from forms, in particular from quadratic forms in a vector space V over $\mathbb{F}_q$. A quadratic form is non-degenerate if the radical of the associated bilinear form has trivial intersection with the null set of the quadratic form. The 1- and 2-spaces of V contained in that null set are called the *singular points* and *lines*, respectively, in $\mathsf{PG}(V)$. The dimension of a largest subspace of V contained in the null set is called the *Witt index* of the quadratic form, and such a subspace, seen as subspace of $\mathsf{PG}(V)$, is often called a *generator*. The singular points of a non-degenerate quadratic form constitute a non-degenerate quadric, and it is well-known that there are precisely three possibilities (still assuming V is finite). If $\dim V = 2n + 1$ is odd, then there is a projectively unique non-degenerate quadric, called a *parabolic quadric*. Its Witt index is n. If $\dim V = 2n$ is even, then there are two projectively different non-degenerate quadrics. One is the *hyperbolic* quadric, and has Witt index n; the other one is an *elliptic* quadric and has Witt index $n-1$. We use the standard notation $Q(2n, q)$, $Q^+(2n-1, q)$ and $Q^-(2n-1, q)$, respectively, for those quadrics.

The point-line geometry defined by the singular points and lines of $Q^+(2n-1, q)$ is also sometimes denoted by $\mathsf{D}_{n,1}(q)$, which emphasises the rank more clearly. The *half spin geometry* $\mathsf{D}_{n,n}(q)$ *of rank n* has as point set one of the natural systems of generators of $Q^+(2n-1, q)$, and a line is the set of generators containing a given (singular) subspace of codimension 2 in each generator. We will be especially interested in the half spin geometry $\mathsf{D}_{5,5}(q)$ of rank 5.

At last, we will also need the so-called *minuscule geometry* of type E_6, denoted by $\mathsf{E}_{6,1}(q)$, and arising from the exceptional finite group of Lie type of type E_6 over the field $\mathbb{F}_q$. A construction can be found in Chapter 4 of [3]. All geometries we introduced so far are standard geometries related to finite groups of Lie type which are called *Lie incidence geometries*.

In the next paragraph, we will mention some well-known properties of the geometries we just introduced.

2.3 Trivia

The point-line geometries defined by the quadrics $Q(2n, q)$, $Q^+(2n-1, q)$ and $Q^-(2n+1, q)$ are *polar spaces of rank n*, that is, point-line geometries satisfying the so-called *one-or-all axiom:*

each point is collinear to exactly one, or all points of a given line,

and the dimension of each maximal singular subspace (which automatically is a projective space) is equal to $n-1$. Moreover, they are non-degenerate geometries in the above sense.

Finite polar spaces of rank at least 3 are traditionally subdivided into three classes: the *orthogonal* ones arising from quadratic forms as described above, the *Hermitian* ones arising from Hermitian forms, and the *symplectic* ones arising from alternating forms. We will not need the latter two classes, except for some elementary property mentioned in the proof of Proposition 3.1.

There is also a notion of *affine polar space*, which basically comes down to a polar space where a geometric hyperplane is removed. We will revisit this in Section 5.

Let $\Delta = (X, \mathscr{L})$ be a point-line geometry such that all singular subspaces are projective spaces or affine spaces, and each point is contained in at least one singular subspace of dimension at least 2. Pick $x \in X$. Then the *point residual*, or the $\mathsf{Res}_\Delta(x)$ *of* Δ at x, is the pointline geometry with point set the set of lines of Δ through x, and a line of $\mathsf{Res}_\Delta(x)$ is the set of members of $\mathscr{L}$ containing x and lying in a given singular subspace of Δ of dimension 2 (a projective or affine plane) containing x. Defining the product of two geometries $\Delta_1 = (X_1, \mathscr{L}_1)$ and $\Delta_2 = (X_2, \mathscr{L}_2)$ as the geometry with point set $X_1 \times X_2$ and set of lines $\{\{x_1\} \times L_2 \mid x_1 \in X_1, L_2 \in \mathscr{L}_2\} \cup \{\{L_1 \times \{x_2\} \mid$

Table 2 Some geometries and their point residuals

Geometry	Point residual
$\mathsf{A}_{d,1}(q)$	$\mathsf{A}_{d-1,1}(q)$
$\mathsf{A}_{d+1,2}(q)$	$\mathsf{A}_{1,1}(q) \times \mathsf{A}_{\mathsf{d}-1,1}(q)$
$\mathsf{D}_{5,5}(q)$	$\mathsf{A}_{4,2}(q)$
$\mathsf{E}_{6,1}(q)$	$\mathsf{D}_{5,5}(q)$
$Q(2n, q)$	$Q(2n-2, q)$
$Q^{\pm}(2n+1, q)$,	$Q^{\pm}(2n-1, q)$

Table 3 Parameters of some infinite classes of rank 3 graphs

	Γ	$\|V(\Gamma)\|$	k	λ	μ
(A3)	$\mathsf{H}_q(2, e)$	q^{2e}	$(q+1)(q^e-1)$	q^e+q^2-q-2	q^2+q
(A7)	$V\mathsf{O}^{\pm}_{2n}(q)$	q^{2n}	$(q^n \mp 1)(q^{n-1} \pm 1)$	$q^{2n-2} \pm q^n \mp q^{n-1} - 2$	$q^{2n-2} \pm q^{n-1}$
(A8)	$V\mathsf{J}_q(5, 2)$	q^{10}	$(q^2-1)(q^5+1)$	$q^5+q^4-q^2-2$	q^4+q^2
(A10)	$V\mathsf{D}_{5,5}(q)$	q^{16}	$(q^8-1)(q^3+1)$	$q^8+q^6-q^3-2$	q^6+q^3

$L_1 \in \mathscr{L}_1, x_2 \in X_2\}$, we have Table 2, where in the last row the superscripts $\pm$ match each other.

The *point graph* of a point-line geometry $\Delta = (X, \mathscr{L})$ is the graph on X, where two points are adjacent if they are collinear. The point graphs of $\mathsf{A}_{d,2}(q)$, $\mathsf{D}_{5,5}(q)$, $\mathsf{E}_{6,1}(q)$ and every non-degenerate polar space are strongly regular. Actually, the point graph of $\mathsf{A}_{d,2}(q)$ is the q-Johnson graph $\mathsf{J}_q(d+1, 2)$. Since the geometries $Q^{\pm}(2n-1, q)$ and $\mathsf{A}_{4,2}(q)$ live in some projective space, they define an affine expansion graph, which, following [3], we denote by $V\mathsf{O}^{\pm}_{2n}(q)$ and $V\mathsf{J}_q(5, 2)$, respectively. As explained in Section 3.3.3 of [3], also $\mathsf{D}_{5,5}(q)$ is naturally embedded in some projective space (of dimension 16 over $\mathbb{F}_q$) and gives rise to the affine expansion graph $V\mathsf{D}_{5,5}(q)$. All these affine graphs are strongly regular.

If a geometry Δ is embedded in projective space, that is, if the point and line set are a subset of the point and line set, respectively, of some projective space, then obviously the same is true for the point residuals. Hence, the geometry $\mathsf{A}_{1,1}(q) \times \mathsf{A}_{\mathsf{d}-1,1}(q)$, which we refer to as a *Segre geometry of type* $(1, d-1)$, is also embedded in projective space, and this is known as the *Segre variety* $\mathscr{S}_{1,d-1}(q)$ in $\mathsf{A}_{2d-1,1}(q)$. The corresponding affine expansion graph is the bilinear forms graph $\mathsf{H}_q(2, d)$, see §3.4.1 of [3]. It is a strongly regular graph.

In Table 3, we list the parameters of the strongly regular graphs defined so far, according to their labels in Table 1 and Table 11.4 of [3] of the infinite classes of affine rank 3 graphs.

Now let $\Omega = (X, \mathscr{L})$ be one of the geometries $\mathsf{A}_{d+1,2}(q)$, $\mathsf{A}_{1,1}(q) \times \mathsf{A}_{\mathsf{d}-1,1}(q)$, $\mathsf{D}_{5,5}(q)$, $\mathsf{E}_{6,1}(q)$, or a polar space of rank at least 3. Call a subspace *convex* if the corresponding set of vertices in the point graph is convex (in the usual sense, that is, every shortest path between two given vertices of a convex set is completely contained in the set). Then in Ω, every pair of points x, y at mutual distance 2 in the point graph, is contained in a unique convex subspace isomorphic to a polar space (if Ω is a polar space, then this unique subspace is the whole polar space itself). Such a subspace is called a *symplecton*, or *symp* for short. Every line of Ω is also contained in such a symp. Geometries with these properties are called *strong parapolar spaces* in the literature, see the later chapters of [23] for more background. In the following table we list the isomorphism class of symplecta of the various types of geometries, different from polar spaces.

Geometry	Symplecta
$\mathsf{A}_{1,1}(q) \times \mathsf{A}_{d-1,1}(q)$	$\mathsf{D}_{2,1}(q)$
$\mathsf{A}_{d+1,2}(q)$	$\mathsf{D}_{3,1}(q)$
$\mathsf{D}_{5,5}(q)$	$\mathsf{D}_{4,1}(q)$
$\mathsf{E}_{6,1}(q)$	$\mathsf{D}_{5,1}(q)$

A geometric hyperplane of a polar space is called *degenerate* if the induced point-line geometry in that hyperplane is degenerate. That is equivalent with the geometric hyperplane coinciding with $x^\perp$, for some point x. A geometric hyperplane H of Ω will be called *degenerate* if it intersects each symp not contained in H in a degenerate hyperplane. We have the following classification.

Lemma 2.1 *A geometric hyperplane of* $\mathsf{A}_{d+1,2}(q)$, $d \geq 3$, $\mathsf{D}_{5,5}(q)$ *or* $\mathsf{E}_{6,1}(q)$ *is degenerate if, and only if, it consists of the points collinear to some point of a subgeometry isomorphic to* $\mathsf{A}_{d-1,2}(q)$, $\mathsf{A}_{4,1}(q)$ *or* $\mathsf{D}_{5,1}(q)$, *respectively. For the case* $\mathsf{A}_{4,2}(q)$, *the subgeometry* $\mathsf{A}_{2,2}(q)$ *is assumed to be a maximal singular subspace (of dimension* 2*).*

Proof For $\mathsf{A}_{d+1,2}(q)$, this follows from the classification of its hyperplanes in [22]. For the other two cases, this follows from the discussion in Sections 3.1 and 3.2, respectively, of [12]. □

3 The graphs in question

Let Ω be a (finite) point-line geometry isomorphic to one of the following: a polar space of rank at least 3, a line Grassmannian of a projective space, a half spin geometry $\mathsf{D}_{5,5}(q)$ or the famous minuscule geometry $\mathsf{E}_{6,1}(q)$. Let H be a geometric hyperplane in Ω. Let Δ be the geometry obtained from Ω by removing all points and lines contained in H from Ω. At each point p of Δ, the point residual as defined above is isomorphic to either a polar space again, or a Segre geometry, or $\mathsf{A}_{4,2}(q)$, or $\mathsf{D}_{5,5}(q)$, respectively, because it coincides with the point residual at p in Ω.

We now wonder when (the point graph of) Δ has diameter 2. This has a nice and clean answer.

Proposition 3.1 *With the above set-up, the diameter of* Δ *is equal to* 2 *if, and only if, the hyperplane* H *is degenerate and, in the case when* Ω *is a polar space, it is an orthogonal polar space.*

Proof Clearly, the diameter of Δ is equal to 2 if, and only if, for every pair of points x and y at distance 2 in Ω, the common perp $x^\perp \cap y^\perp$ is not contained in H. Such common perp is a (non-degenerate) polar space of the same type as the symp ξ containing x and y, but one rank lower. If $H \cap \xi$ is non-degenerate, then its perp contains at least two points, and so we find x and y with $x^\perp \cap y^\perp \subseteq H \cap \xi$.

Hence, if H intersects at least one symp in a non-degenerate hyperplane, then the diameter of Δ is at least 3. Hence H is degenerate and is as in Lemma 2.1.

Now let Ω be a polar space. A necessary condition for the diameter of Δ to be equal to 2 is that H is the perp of some point p of Ω. Then, for an arbitrary point $x \notin H$, $(x^\perp \cap H)^\perp$ contains exactly 2 points if Ω is orthogonal, $\sqrt{q}+1$ points if Ω is unitary, and $q+1$ points if Ω is symplectic. In any case $p \in (x^\perp \cap H)^\perp$, so the diameter of Δ is equal to 2 exactly when Ω is orthogonal. □

Proposition 3.2 *With the above set-up, suppose Δ has diameter 2. Then the point graph Γ of Δ is strongly regular, except if Ω is of parabolic type. Moreover, if Γ is strongly regular, then it is an affine rank 3 graph.*

Proof Suppose Ω is a parabolic quadric and let $H = p^\perp$, with p a point of Ω. Let x be a point off H. Then $x^\perp \cap H$ is a parabolic quadric Q_x. Let, for each $\epsilon \in \{+,-\}$, Q_x^ϵ be a non-degenerate geometric hyperplane of Q_x of type ϵ, that is, Q_x^ϵ is an elliptic or hyperbolic quadric depending on whether ϵ is $-$ or $+$. Let $y_\epsilon \perp p$ be a point of Ω with the property that the line py_ϵ does not intersect Q_x^ϵ. Then y_ϵ and Q_x^ϵ generate a parabolic quadric in $p^\perp$, which coincides with $z_\epsilon^\perp \cap p^\perp$, for a unique point z_ϵ of Ω not collinear to p. Since the number of points of an elliptic quadric is different from the number of points of a hyperbolic quadric in the same projective space, we see that

$$|(x^\perp \cap z_-^\perp) \setminus p^\perp| \neq |(x^\perp \cap z_+^\perp) \setminus p^\perp|,$$

showing that Γ is not strongly regular, since the number of common neighbours of two non-adjacent vertices is not constant (i.e., the corresponding μ is not well-defined).

Now let Ω be either an elliptic of hyperbolic polar space $Q^\pm(2n+1,q)$, $n \geq 3$, or isomorphic to the line Grassmannian $\mathsf{A}_{d+1,2}(q)$, $d \geq 2$, or the half spin geometry $\mathsf{D}_{5,5}(q)$, or the exceptional minuscule geometry $\mathsf{E}_{6,1}(q)$. Let H be a degenerate hyperplane, more exactly, due to Lemma 2.1, H is the set of points of Ω collinear to a given point p, to some point of a given subgeometry Π isomorphic to $\mathsf{A}_{d-1,2}(q)$, to some point of a given maximal singular subspace π of dimension 4, or to some point of a given symp ξ, respectively. Then $\{p\}$, π and ξ are the sets D of *deep points* of H, that is, points whose perp is entirely contained in H.

Now consider the natural embedding of Ω in

$$\mathsf{PG}(2n+1,q), \mathsf{PG}(\binom{d+2}{2}-1,q), \mathsf{PG}(15,q) \text{ or } \mathsf{PG}(26,q),$$

respectively, briefly and in general denoted by $\mathsf{PG}(N,q)$. There is a unique hyperplane Υ of $\mathsf{PG}(N,q)$ such that Υ intersects Ω in H. We claim that every subspace Σ of $\mathsf{PG}(N,q)$ of dimension $1+\dim\langle D\rangle$ containing D without being contained in Υ intersects Ω, and hence Δ, in a unique point. Due to a counting argument, it suffices to prove that no such subspace intersects Ω in at least two points (indeed, there are exactly $q^{2n}, q^{2d}, q^{10}, q^{16}$ points of Ω opposite D, or equivalently, not contained in H, and the dimensions of the affine spaces obtained from $\mathsf{PG}(N, q)$ by factoring out $\langle D\rangle$ and deleting $H/\langle D\rangle$ is equal to $(2n+1)-1$, $\binom{d+2}{2}-\binom{d}{2}-1$, $15-5$ and $26-10$, respectively). Note that the claim is obvious for polar spaces by properties of the tangent spaces. We now treat the three other cases.

Suppose Σ contains two points x, y of Δ. Then x and y are not collinear as otherwise the line xy intersects D, hence contains a deep point, and so $x, y \in H$, a contradiction. Hence there exists a unique symp ζ containing x and y. The point $xy \cap \langle D\rangle$ lies in ζ (this is obvious if D is a singular subspace; in the other cases this follows from Axiom (MM2) in [21]), and so the line xy contains at least three points of Ω and hence is contained in Ω, a contradiction.

Hence, Δ is represented in an affine space, which we denote by $\mathsf{AG}(N', q)$, with $N' = 2n, 2d, 10, 16$, respectively. Now, the projection of H from $\langle D\rangle$ onto a subspace of $\langle H\rangle$ complementary to $\langle D\rangle$ is a polar space isomorphic to $Q^{\epsilon}(2n-1, q)$, the Segre geometry $\mathsf{A}_{1,1}(q) \times \mathsf{A}_{d-1,1}(q)$, the line Grassmannian $\mathsf{A}_{4,2}(q)$, or the half spin geometry $\mathsf{D}_{5,5}(q)$, respectively, which can be thought of as embedded in the projective space at infinity of $\mathsf{AG}(N', q)$. It is easy to see that an affine line L of $\mathsf{AG}(N', q)$ belongs to Δ if and only if the line L contains as point at infinity a point of the above mentioned projection of H.

In view of the definitions of $V\mathsf{O}^{\pm}_{2n}(q)$, $\mathsf{H}_q(2, d)$, $V\mathsf{J}_q(5, 2)$ and $V\mathsf{D}_{5,5}(q)$, the assertion follows. $\square$

Hence we obtain the affine rank 3 graphs $V\mathsf{O}^{\pm}_{2n}(q)$, $\mathsf{H}_q(2, d)$, $V\mathsf{J}_q(5, 2)$ and $V\mathsf{D}_{5,5}(q)$. It is these graphs that we will characterise. Note that these graphs are not characterised by their local graphs alone. Counterexamples arise in Proposition 3.1, when a non-degenerate hyperplane H is removed from appropriate geometries.

In the case of $V\mathsf{O}^{\pm}_{2n}(q)$, the non-degenerate hyperplane of $Q^{\pm}(2n+1, q)$ is given by a parabolic subquadric $Q^{\pm}(2n, q)$. An elementary count reveals that removing such a hyperplane results in a graph with $q^n(q^n \pm 1)$ vertices. For $V\mathsf{O}^{+}_{2n}(q)$, we obtain a counterexample with strictly more vertices, while for $V\mathsf{O}^{-}_{2n}(q)$, the counterexample has fewer vertices.

In the case of $\mathsf{A}_{d+1,2}(q)$, $d \geq 2$, a non-degenerate hyperplane is given by the lines of $\mathsf{PG}(d+1, q)$ which are totally isotropic with respect to an alternating form with the codimension $2r$ of the radical at least 4. An elementary count reveals that the removal of such a hyperplane leaves a graph with

$$q^{2d} + q^{2d-2} + \cdots + q^{2d-2r+2}$$

vertices. For $r = 1$ we get the number of vertices of $\mathsf{H}_q(2, d)$. For any $r > 1$, we get a graph with strictly more vertices.

In the case of $\mathsf{D}_{5,5}(q)$, there is only one type of non-degenerate hyperplane, and it is of size $q^4\frac{q^6-1}{q-1} - \frac{(q^3+1)(q^4-1)}{q-1}$, whose removal results in a graph with the same local graph as $V\mathsf{J}_q(5, 2)$ with $q^{10} + q^7$ vertices.

In the case of $\mathsf{E}_{6,1}(q)$, the non-degenerate hyperplane is either of so-called type F_4 and has size $\frac{(q^{12}-1)(q^4+1)}{q-1}$ whose removal results in a graph with $q^{16} + q^{12} + q^8$

vertices, or, by Section 3.2 of [12], a hyperplane of size $q^8\frac{q^8-1}{q-1} + q(q+1)(q^2+1)(q^3+1)(q^4+1)+1$ whose removal results in a graph with $q^{16}+q^{12}$ vertices, both having the same local graph as $V\mathsf{D}_{5,5}(q)$.

Therefore, we see that all previously constructed counterexamples, except for $V\mathsf{O}^-_{2n}(q)$, have more vertices than their affine rank 3 strongly regular counterparts. This leads us to introduce the alternative additional assumptions for these graphs that

$$\begin{cases} |V(\Gamma)| \le q^{2n} \text{ if the local polar space of } \Gamma is Q^+(2n-1,q), \\ |V(\Gamma)| \le q^{2e} \text{ if the local parapolar space is } \mathsf{A}_{1,1}(q)\times \mathsf{A}_{e-1,1}(q), \\ |V(\Gamma)| \le q^{16} \text{ if the local parapolar space is } \mathsf{A}_{4,2}(q), \text{ and} \\ |V(\Gamma)| \le q^{10} \text{ if the local parapolar space is } \mathsf{D}_{5,5}(q), \end{cases} \tag{3.3}$$

which also looks pretty uniform. According to Theorem D, it is possible to relax the upper bound for the case of $Q^+(2n-1,q)$. It seems plausible that the other bounds in (3.3) could also be relaxed according to the above counting arguments for the sizes of derived counterexamples. However, our methods do not allow for this at the moment.

Hence we give ourselves a graph Γ that has a specific local structure, aiming at the local structures of the above graphs $V\mathsf{O}^{\pm}_{2n}(q)$, $n\ge 3$, $\mathsf{H}_q(2,d)$, $d\ge 2$, $V\mathsf{J}_q(5,2)$ and $V\mathsf{D}_{5,5}(q)$. But for the time being, we would like to start off more generally to allow for more graphs to be included in possible future work.

Hence we first derive some general properties, and then specialise and treat the different cases separately.

4 General properties

4.1 General set-up

This section parallels [19]. However, in the present paper we deal with $(q-1)$-clique extensions of the point graphs of geometries with $q+1$ points per line, whereas [19] deals with q-clique extensions of such graphs. Therefore, we will generalise things slightly in such a way that it is applicable to both situations (and potentially additional ones).

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a graph. We will write $x\sim y$ to mean that vertices $x, y\in V(\Gamma)$ are adjacent (i.e., $\{x,y\}\in E(\Gamma)$). For $x\in V(\Gamma)$ a vertex and $S\subseteq V(\Gamma)$ an arbitrary subset of vertices we define the *perp*:

$$x^\perp := \{y\in V(\Gamma) \,|\, x=y \text{ or } x\sim y\} \quad \text{and} \quad S^\perp := \bigcap_{x\in S} x^\perp.$$

We make some easy observations about the perp.

$$\begin{aligned} x\in y^\perp &\iff y\in x^\perp \\ y\in (x^\perp)^\perp &\iff x^\perp \subseteq y^\perp \end{aligned} \tag{4.1}$$

Let c be a positive integer. The *c-clique extension* of Γ is the graph $c\Gamma$ with vertex set $\{t_i(x) : 1 \le i \le c, x \in V(\Gamma)\}$ and two vertices $t_i(x)$ and $t_j(y)$ with $1 \le i, j \le c$ and $x, y \in V(\Gamma)$ adjacent if either $x \sim y$, or $x = y$ and $i \ne j$. This construction is a particular instance of a *lexicographic product* of graphs. In fact, using notation from Chapter 10 of [18], $c\Gamma$ is the lexicographic product $\Gamma \circ K_c$ of Γ with K_c, the complete graph on c vertices. Subsets of vertices of $c\Gamma$ of the form $\{t_i(x)\colon 1 \le i \le c\}$ with $x \in V(\Gamma)$ are called *rays*. Hence, two vertices of $c\Gamma$ are adjacent if and only if they are in the same ray or lie in rays corresponding to adjacent vertices of Γ.

Lemma 4.2 *(Lemma 3.1, [19]) Let Γ be a graph and c a positive integer. Suppose that for every $x \in V(\Gamma)$, it holds that*

$$(x^{\perp})^{\perp} = \{x\}. \tag{4.3}$$

Then in $c\Gamma$ we have the following for all $i \in \{1, \dots, c\}$ and $x \in V(\Gamma)$.

$$(t_i(x)^{\perp})^{\perp} = \{t_1(x), \dots, t_c(x)\}$$

Proof Apply observations from Eq. 4.1. □

When assumption Eq. 4.3 holds, we call the ray $\{t_i(x)\colon 1 \le i \le c\}$ *reconstructable*. The number c will be referred to as the *height*.

4.2 From local graphs to geometries

Let Γ be a graph and $p \in V(\Gamma)$. The *local graph* of Γ at p is the subgraph of Γ induced by the (open) neighbourhood of p in Γ. We will denote both the set of neighbours of p in Γ and the local graph of Γ at p by $\Gamma(p)$.

Let now Γ be connected graph such that at each $p \in V(\Gamma)$ the local graph at p is of the form $c_p\Gamma_p$, where c_p is a positive integer and Γ_p is a graph such that each of its vertices has the property described by Eq. 4.3 (note that both c_p and Γ_p are allowed to depend on the particular vertex p of Γ).

We will denote by $\perp_p$ the previously defined $\perp$ observed in the local graph $\Gamma(p)$. Neighbours of x in $\Gamma(p)$ are just neighbours of x in Γ that are also adjacent to p, hence we have that

$$x^{\perp_p} = (x^{\perp} \cap p^{\perp}) \setminus \{p\}.$$

By assumption, each local graph $\Gamma(p)$ is a clique extension $c_p\Gamma_p$ with reconstructable rays (since we assumed Γ_p satisfies Eq.4.3). By Lemma 4.2, the ray of a vertex x in $\Gamma(p)$ is given by $(x^{\perp_p})^{\perp_p}$.

Note then by Eq. 4.1, we have that

$$\begin{aligned}(x^{\perp_p})^{\perp_p} &= \{y \in p^{\perp} \setminus \{p\} : x^{\perp_p} \subseteq y^{\perp_p}\} = \\ &= \{y \in p^{\perp} \setminus \{p\} : (x^{\perp} \cap p^{\perp}) \subseteq (y^{\perp} \cap p^{\perp})\} = \\ &= (x^{\perp} \cap p^{\perp})^{\perp} \setminus \{p\}.\end{aligned}$$

We can now define an *extended ray* for two adjacent vertices x and p of Γ by setting

$$R[x, p] := (x^{\perp} \cap p^{\perp})^{\perp}. \tag{4.4}$$

It is important to note that the extended ray $R[x, y]$ is symmetric in x and y and moreover, it is determined by any two of its distinct members.

Using this, we can now generalise Lemma 3.2 of [19].

Lemma 4.5 *The heights of local graphs at any two distinct vertices of* Γ *coincide.*

Proof Let $x, y \in V(\Gamma)$ be distinct, adjacent vertices of Γ. Then they define the same extended ray $R[x, y]$. By definition of the extended ray, we get

$$c_x + 1 = |(x^{\perp_y})^{\perp_y} \cup \{y\}| = |R[x, y]| = |(y^{\perp_x})^{\perp_x} \cup \{x\}| = c_y + 1,$$

which proves the lemma. □

Our next step is to introduce a certain geometry related to Γ. Define a point-line geometry $\Delta = \Delta(\Gamma)$ with

- the point set given by $V(\Gamma)$, and
- the line set $\mathscr{R}$ consisting of the extended rays $R[x, y]$ with $x, y \in V(\Gamma), x \neq y, x \sim y$.

We will call this the *graph geometry* of Γ.

Lemma 4.6 *The graph geometry* Δ *of a graph* Γ *is a partial linear space.*

Proof Distinct collinear points x and y of Δ determine adjacent vertices x and y in Γ and a unique extended ray $R[x, y]$. □

We now generalise Lemma 3.3 of [19].

Lemma 4.7 *The graph geometry* Δ *is a gamma space.*

Proof Suppose that $L \in \mathscr{R}$ is a line of Δ and $p \in V(\Gamma)$ is an arbitrary point. We need to show that p is collinear to 0, 1 or all points of L. Hence, it suffices to prove that if p is collinear with two distinct points $x, y \in L$ and p does not lie on L then p is collinear with all points of L.

Because x and y are distinct and collinear in Δ, they correspond to distinct adjacent vertices of Γ inducing a unique extended ray $R[x, y]$.

Recall that (with $\perp$ observed in Γ)

$$R[x, y] = (x^{\perp} \cap y^{\perp})^{\perp}.$$

As p is adjacent to both x and y, it lies in $x^{\perp} \cap y^{\perp}$. Consequently, by definition, p is adjacent with every element of $L = R[x, y]$ (as it is assumed not to lie on L). Equivalently, p is collinear with all points on L in the graph geometry Δ. □

Corollary 3.4 from [19] does not generalise at this point, as we did not make an assumption on the local graphs being point graphs of geometries. In the context of the present paper, the counterpart of that corollary will be Proposition 4.9.

4.3 When the local graphs are also related to geometries

We now specialise our set-up further. Let $q \geq 3$ be a positive integer. Let $\mathscr{G}$ denote a class of point-line geometries $G = (X_G, \mathscr{L}_G)$ satisfying the following:

1. all lines in $\mathscr{L}_G$ contain $q + 1$ points;
2. the residual geometries $\mathsf{Res}_G(x)$ and $\mathsf{Res}_G(y)$ are isomorphic for all points $x, y \in X_G$;
3. for all $G_1, G_2 \in \mathscr{G}$, the geometries G_1 and G_2 are isomorphic if, and only if, $\mathsf{Res}_{G_1}(x_1)$ and $\mathsf{Res}_{G_2}(x_2)$ are isomorphic, for some $x_1 \in X_{G_1}, x_2 \in X_{G_2}$;
4. every singular subspace of G is a projective space;
5. for all lines $L \in \mathscr{L}_G$, we have that $(L^{\perp_G})^{\perp_G} = L$;
6. for points $x, y, z \in \mathscr{P}_G$, with $x^\perp \cap y^\perp \neq \varnothing$ and x not collinear to y, we have $x^\perp \cap y^\perp = x^\perp \cap z^\perp$ if and only if $y = z$.

Let Γ be a possibly infinite connected graph such that for each $p \in V(\Gamma)$, the local graph $\Gamma(p)$ is $(q-1)\Gamma_p$ i.e., a $(q-1)$-clique extension of a graph Γ_p, where Γ_p is the point graph of a geometry belonging to a given class $\mathscr{G}$ satisfying the above conditions. Since the vertices of the graph Γ will end up being points of a geometry, we will often call a vertex of Γ a *point of* Γ.

Notice that, since the perps of $x \in V(\Gamma_p)$ as a vertex of Γ_p and a point in the corresponding geometry (with Γ_p being its point graph) coincide, the assumption 5 above precisely implies that Γ_p satisfies the condition given by Eq. 4.3 at every vertex. In particular, the rays of $\Gamma(p) = (q-1)\Gamma_p$ are reconstructable, for all $p \in V(\Gamma)$. An extended ray in Γ will be referred to as a *singular affine line*.

With this set-up, Lemma 4.6 specialises to the following result.

Lemma 4.8 *Two adjacent vertices of* Γ *are contained in a unique singular affine line, which has size* q*. Also, for arbitrary vertices* $x, y \in V(\Gamma)$*, we have the equivalence*

$$x = y \iff x^\perp \subseteq y^\perp \iff x^\perp = y^\perp.$$

Proof The first assertion is Lemma 4.6. We now show the second assertion. Suppose $x^\perp \subseteq y^\perp$ and $x \neq y$. Let $z \in \Gamma_1(x)$ be arbitrary but distinct from y. Then clearly $y \in \{x, z\}^{\perp\perp}$, so x, y, z are contained in a singular affine line L. By the first assertion and the fact that z was arbitrary, we now conclude $\Gamma_1(x) = L \setminus \{x\}$, implying Γ_x is trivial, a contradiction. Hence

$$x^\perp \subseteq y^\perp \implies x = y \implies x^\perp = y^\perp \implies x^\perp \subseteq y^\perp,$$

which concludes the proof. □

We now define singular affine planes. This is similar to Lemma 3.3 and Corollary 3.4 in [19].

Let x, y, z be three pairwise adjacent vertices not contained in a common singular affine line. Then we denote $\langle x, y, z\rangle := \{x, y, z\}^{\perp\perp}$. We have the following property.

Proposition 4.9 *Let x, y, z be three pairwise adjacent vertices of Γ not contained in a common singular affine line. Then the set $\langle x, y, z\rangle$ endowed with all singular affine lines contained in it, is an affine plane.*

Proof Consider the local graph $\Gamma_1(x)$ at x. The sets $\langle x, y\rangle \setminus \{x\}$ and $\langle x, z\rangle \setminus \{x\}$ are rays in $\Gamma_1(x)$. By the very definition of $(q-1)$-clique extension, either $\langle x, y\rangle \cup \langle x, z\rangle$ is a clique, or no vertex of $\langle x, y\rangle \setminus \{x\}$ is adjacent to any vertex of $\langle x, z\rangle \setminus \{z\}$. Since the latter is excluded by the assumption $y \perp z$, the former occurs.

Now assumption 4.3 implies that $\langle x, y, z\rangle$ coincides with the union of $\{x\}$ with $q+1$ rays corresponding to the $q+1$ points of the appropriate line of Γ_x. It follows that $\langle x, y, z\rangle$ is a q^2-clique. Consider two arbitrary points u, v in $\langle x, y, z\rangle$. Since $\{x, y, z\}^\perp \subseteq \{u, v\}^\perp$, we have $\{u, v\}^{\perp\perp} \subseteq \{x, y, z\}^{\perp\perp}$. This implies that $\langle x, y, z\rangle$ endowed with the singular affine lines it contains is a $2-(q^2, q, 1)$ design, that is, an affine plane. □

This immediately implies the following consequence, which we record for further reference.

Corollary 4.10 *Let x, y, z be three pairwise adjacent vertices in Γ. Then x is collinear to each point of $\langle y, z\rangle$.* □

Definition 4.11 For a vertex $x \in V(\Gamma)$, we will from now on consider Γ_x as the point-line geometry whose points are the affine singular lines through x and whose lines are singular affine planes containing x. We call Γ_x the *local geometry at x*.

Now let $x \in V(\Gamma)$ be arbitrary. Let U be a singular subspace of Γ_x. Then the union of all the points of the singular affine lines through x contained in U forms a clique C_U of size q^i, where $i-1$ is the projective dimension of U, viewed as singular subspace of $Q^\epsilon(2n-1, q)$. Since, in $Q^\epsilon(2n-1, q)$ we have $U^{\perp\perp} = U$, we again see that C_U can be defined starting from any point of C_U and hence that every singular line sharing at least two points with it is entirely contained in it. However, we cannot conclude directly from this that C_U, endowed with the singular affine lines it contains, is an affine space. We need to do some intermediate steps.

First of all, we prove this for $i = 3$.

Lemma 4.12 *Let $x \in V(\Gamma)$ be arbitrary. Let U be a singular plane of Γ_x. Then, with the above notation, C_U, endowed with the singular affine lines contained in it, is an affine space of dimension* 3.

Proof The previous discussion implies that $C_U = \{x, y, z, u\}^{\perp\perp}$, with x, y, z, u four vertices of C_U not contained in a singular affine plane. If t, v, w are three vertices of C_U not contained in a common singular affine line, then this implies, similarly as before, that $\langle t, v, w\rangle \subseteq C_U$. It now easily follows that C_U, endowed with all singular affine lines it contains, is a planar space of size q^3 whose planes are all affine of size q^2. By results of Buekenhout [4, Théorème p.368] and Hall [15, Theorem 4.2], this implies that C_U is a 3-dimensional (singular) affine plane. □

Lemma 4.13 *Let C be a q^i-clique in Γ with $C^{\perp\perp} = C$ and $i \in \{4, \ldots, n\}$. Call two singular affine lines in C parallel if they are disjoint and contained in a common singular affine plane. Then parallelism is an equivalence relation.*

Proof Clearly parallelism is a reflexive and symmetric relation. We ought to prove that it is transitive. Let L_1, L_2, L_3 be three lines in C with L_1 parallel to L_2 and L_2 parallel to L_3. Selecting points $x_1 \in L_1, x_3 \in L_3$ and distinct $x_2, y_2 \in L_2$, we see that, since L_1, L_2 and L_2, L_3 are contained in respective common singular affine planes α and β, the set $\langle x_1, x_2, y_2, x_3\rangle$ is, due to Lemma 4.12, a singular affine 3-space. Hence L_1 and L_3 are parallel. □

Using a result of Hall [15, Theorem 4.2] again, this now immediately implies the following.

Proposition 4.14 *Let C be a q^i-clique in Γ with $C^{\perp\perp} = C$ and $4 \le i \le n$. Then C, endowed with all singular affine lines it contains, is an affine space of dimension i.* □

A particular case is the following, which is easily derived from Lemma 4.12.

Corollary 4.15 *Four pairwise adjacent points not in a common singular affine plane, are contained in a unique common singular affine subspace of dimension* 3.

Finally we have the following characterisation:

Lemma 4.16 *For three arbitrary vertices x, y, z we have that $\emptyset \neq x^\perp \cap y^\perp = x^\perp \cap z^\perp$ if, and only if, x, y, z are contained in a common singular line.*

Proof If x, y, z are contained in a common singular affine line, then in the local graph at x, the vertices y and z are contained in the same ray, implying that they are adjacent to the same vertices in the local graph at x except for y and z themselves. Hence $x^\perp \cap y^\perp = x^\perp \cap z^\perp$ and the "if"-part is proved.

Now we proceed with the "only if"-part. Let $u \in x^\perp \cap y^\perp = x^\perp \cap z^\perp$ be arbitrary.

Case 1 *Suppose x and y are not adjacent.*

We look at the local geometry Γ_u at u. Then x, y and z lie on respective singular affine lines through u, which are points of Γ_u. Note that $\langle u, y\rangle^\perp \cap \langle u, x\rangle^\perp = \langle u, z\rangle^\perp \cap \langle u, x\rangle^\perp$ (in Γ_u), which is only possible if $\langle u, y\rangle = \langle u, z\rangle$ by assumption 6 of Section 4.3.

Now we can choose a new point $v \in x^\perp \cap y^\perp$, which is adjacent (hence distinct from) to u. By the same argument, y and z now must lie on the same singular affine line through v, but this is a contradiction as no point $(\langle u, z\rangle \setminus \{u\}) \ni v$ is adjacent to x.

Case 2 *Suppose x and y are adjacent.*

As in this case $x \in x^\perp \cap y^\perp$, this implies that z is adjacent to both x and y as well. If the corresponding singular affine lines through x are different, then they induce different perps in Γ_x by assumption 6 of Section 4.3, a contradiction. □

Now we can show the following important result.

Proposition 4.17 *All local geometries are mutually isomorphic.*

Proof Let x and y be two adjacent vertices of Γ. Then for each singular affine line L_x through x but not through y, and contained in $x^\perp \cap y^\perp$, there exists a unique singular affine line L_y through y lying in the plane $\langle x, y, z\rangle$, with $z \in L_x \setminus \{x\}$, disjoint from L_x; it is the parallel to L_x in that affine plane. The mapping $\theta : \Gamma_x \to \Gamma_y$ sending $L_x \mapsto L_y$, where $\theta(\langle x, y\rangle) = \langle x, y\rangle$ by definition, is clearly an isomorphism between the point residuals of Γ_x and Γ_y at $\langle x, y\rangle$. By the assumption that the point residuals of members of $\mathscr{C}$ determine the geometry, we conclude $\Gamma_x \cong \Gamma_y$. By connectivity, the proposition follows. □

The class $\mathscr{G}$ of geometries to which we wish to apply the results of this section consists of all finite elliptic and hyperbolic polar spaces of rank at least 3, Segre geometries of type $(1, e)$, $e \geq 1$, line Grassmannians of type 4, and half spin geometries of rank 5. Hence, we may from now on assume that all local geometries are isomorphic. Since all geometries in $\mathscr{G}$ are parapolar spaces, we will sometimes call the local geometry the *local parapolar space*.

In the next section, we deal with the (elliptic and hyperbolic) polar spaces as local parapolar spaces, and then we continue with the other geometries.

5 The local geometries are polar spaces

In this case we show that the graph geometry $\Delta = (V(\Gamma), \mathscr{R})$ of Γ is an affine polar space in the sense of Cohen & Shult [10], that is, a polar space from which a geometric hyperplane has been removed. Cohen & Shult axiomatised these structures (see Section 3.1 of [10]).

First we need a lemma.

Lemma 5.1 *Let $\Omega = (X, \mathscr{L})$ be the point-line geometry associated to $Q^+(2n+1, q)$, $n \geq 1$, or to $Q^-(2n+1, q)$, $n \geq 2$. Let H be a non-degenerate hyperplane of Ω, that is a hyperplane with the structure of $Q(2n, q)$. Let ϵ be $+$ or $-$ according to whether Ω is $Q^+(2n+1, q)$ or $Q^-(2n+1, q)$. Then, for each point $x \in X \setminus H$, there exists points u, v, w such that*

$$\begin{cases} |(x^\perp \cap u^\perp) \setminus H| = 0, \\ |(x^\perp \cap v^\perp) \setminus H| = q^{2n-2}, \\ |(x^\perp \cap w^\perp) \setminus H| = q^{2n-2} + \epsilon q^{n-1}. \end{cases}$$

Proof Let $\epsilon \in \{+, -\}$. Since H is not degenerate, $(x^\perp \cap H)^\perp$ consists of x and a unique other point u. Hence $|(x^\perp \cap u^\perp) \setminus H| = 0$.

Now let $H_1 \subseteq H$ be a degenerate hyperplane of H, say $H_1 = y^\perp \cap H$. Then $H_1^\perp = xy \cup uy$, with u as above. Each point v on $uy \setminus \{u, y\}$ is non-collinear to x and hence $|(x^\perp \cap v^\perp) \setminus H| = |Q^\epsilon(2n-1, q)| - (1 + q|Q^\epsilon(2n-3, q)|) = q^{2n-2}$, where $Q^+(-1, q)$ represents the empty set.

Now for any other point $w \in X \setminus (H \cup x^\perp \cup u^\perp)$ we have $x^\perp \cap w^\perp \cap H$ is non-degenerate in H and hence $|(x^\perp \cap w^\perp) \setminus H| = |Q^\epsilon(2n-1, q)| - |Q(2n-2, q)| = q^{2n-2} + \epsilon q^{n-1}$. □

An *axiomatic affine polar space* is a connected gamma space equipped with a nonempty collection Π of subspaces, called affine planes (motivated by Axiom (APS2) below), such that:

(APS1) any two collinear points are on a unique line, and any three pairwise collinear points not on a line are contained in a unique member of Π;
(APS2) for each $\pi \in \Pi$, the incidence system $(\pi, \mathscr{L}(\pi))$ is an affine plane;
(APS3) if $p \in P$ and $\pi \in \Pi$, then $p^\perp \cap \pi$ is either empty, is the set of points on a line or coincides with the set of all points in π;
(APS4) $x^\perp \subseteq y^\perp$ implies $x = y$ for any two points x and y.

Then Cohen & Shult prove that axiomatic affine polar spaces and affine polar spaces of rank at least 3 are the same things, see Corollary 4.2 of [10].

Theorem 5.2 *Let $q > 2$. If all local geometries of Γ are polar spaces isomorphic to $Q^+(2n-1, q)$, $n \geq 2$, or to $Q^-(2n-1, q)$, $n \geq 3$, then Γ is isomorphic to the corresponding graph $V\mathsf{O}^{\pm}_{2n}(q)$.*

Proof Consider the graph geometry $\Delta = (V(\Gamma), \mathscr{L})$. We verify the axioms for this geometry given above, taking as set Π the family of singular affine planes. Then Axioms (APS1) and (APS2) follow from Lemma 4.8 and Proposition 4.9. Axiom (APS4) follows from Lemma 4.8. By Corollary 4.10, to prove Axiom (APS3), it suffices to show that $|p^\perp \cap \pi| \neq 1$. Suppose $x \in p^\perp \cap \pi$. Then $x^\perp$ contains both p and π. Since Γ_x is a polar space, there exists at least one singular affine plane α through $\langle p, x\rangle$ intersecting π in a singular affine line L. Then $p^\perp \cap \pi$ contains L and the verification of the axioms is complete.

Hence Δ is an affine polar space, obtained from a polar space Ω by deleting a geometric hyperplane H. Since a polar space of rank at least 3 is determined by it point residuals, and the point residual is in our case isomorphic to the local geometry at any vertex, that is, $Q^{\pm}(2n-1, q)$, we see that Ω is isomorphic to $Q^{\pm}(2n+1, q)$. By the assumption that $|x^\perp \cap y^\perp|$ is constant, for $y \in \Gamma_2(x)$, Lemma 5.1 implies that the removed hyperplane is degenerate. Hence we obtain $V\mathsf{O}^{\pm}_{2n}(q)$ by Proposition 3.2. □

In view of the construction in Section 3, the first paragraph of the previous proof shows Theorem D.

Remark 5.3 If in Theorem 5.2 we assume that Γ has diameter 2 and drop the assumption that $|x^\perp \cap y^\perp|$ is a constant for vertices x, y at mutual distance 2, then the same conclusion holds since, with the notation of the proof of Theorem 5.2, if H is non-degenerate, the diameter is 3 by Proposition 3.1.

6 The structure of the mu-graph

Henceforth we assume that the local parapolar spaces of Γ are all isomorphic to either $\mathsf{A}_{1,1}(q) \times \mathsf{A}_{\mathsf{d}-1,1}(q)$, with $d \geq 3$, $\mathsf{A}_{4,2}(q)$ or $\mathsf{D}_{5,5}(q)$. The case $\mathsf{A}_{1,1}(q) \times \mathsf{A}_{\mathsf{d}-1,1}(q)$ for $d = 2$ is the case $Q^+(3, q)$ handled in the previous section.

In this section, we are interested in the structure of the subgraph of Γ induced by $x^\perp \cap y^\perp$ with $x, y \in V(\Gamma)$ at mutual distance 2, called the *mu-graph*. We first need a general lemma for polar spaces and a lemma for the other parapolar spaces in $\mathscr{G}$.

Lemma 6.1 *Let $\Omega = (X, \mathscr{L})$ be a polar space of rank at least 2. Suppose $Y \subseteq X$ is nonempty with the property that, for each $y \in Y$, the set $y^\perp \setminus Y$ is a geometric hyperplane of $y^\perp$. Then $H := X \setminus Y$ is a geometric hyperplane of Ω.*

Proof Note that, for each $y \in Y$, because $y \notin y^\perp \setminus Y$, every line through y has exactly one point not in Y. We now show that H is a subspace of Ω. Suppose $x, y \in H$ are collinear but some $z \in L = xy$ does not lie in H i.e., $z \in Y$. However, L is a line that violates our assumptions on elements of Y.

Let L be a line of Ω and assume that $L \cap H = \emptyset$ i.e., $L \subseteq Y$. Then pick any $y \in L$. It follows that $L \subseteq y^\perp$. Hence, L is a line of the subspace $y^\perp$ and must intersect the geometric hyperplane $y^\perp \setminus Y$ of $y^\perp$, but that is a contradiction with $L \subseteq Y$.

Hence, $L \cap H \neq \emptyset$ and H is a geometric hyperplane of Ω. □

Lemma 6.2 *Let $\Omega = (X, \mathscr{L})$ be a parapolar space isomorphic to either $\mathsf{A}_{1,1}(q) \times \mathsf{A}_{\mathsf{d}-1,1}(q)$ with $d \geq 3$, $\mathsf{A}_{4,2}(q)$ or $\mathsf{D}_{5,5}(q)$. Suppose $Y \subseteq X$ is nonempty with the property that, for each $y \in Y$, the set $y^\perp \cap Y$ is the complement of a geometric hyperplane of $y^\perp \cap \xi_y$, for some symplecton ξ_y. Then Y is contained in a symplecton ξ_Y and $H := \xi_Y \setminus Y$ is a geometric hyperplane of ξ_Y.*

Proof We first handle the case of $\mathsf{A}_{1,1}(q) \times \mathsf{A}_{\mathsf{d}-1,1}(q)$. Pick $y \in Y$ arbitrarily. Then there is a unique line L of ξ_y containing y and contained in a $(d-1)$-dimensional singular subspace, say U_y. Then each point z of L is on a unique line L_z of Ω which is itself a maximal singular subspace, and the union of such L_z is precisely ξ_y. Now, there is a unique point $u \in L$ not contained in Y. Hence, for all $z \in L \setminus \{u\}$, there is a unique point $u_z \in L_z$ not contained in Y. It is straightforward to see that either all points u_z are collinear, say contained in the line $M \subseteq \xi_y$, or the points u_z form an *ovoid*, which is a non-degenerate hyperplane of ξ_y. In both cases, suppose there exists a point $y' \in Y \setminus \xi_y$. By possibly replacing y' with a point in Y on the unique line through y' which is a maximal singular subspace of Ω, and the same for y, we may assume that y' is contained in U_y. But then $L \subseteq y'^\perp$, contradicting the hypotheses on Y. The assertion follows.

Now we may assume that Ω is isomorphic to $\mathsf{A}_{4,2}(q)$ or $\mathsf{D}_{5,5}(q)$.

Select $y \in Y$ arbitrarily and let $z \in y^\perp \cap Y$. Then $z \in \xi_y$. Now $y^\perp \cap z^\perp \cap Y$ contains at least two planes through the line yz not contained in a singular 3-space of ξ_y. Hence $\xi_z = \xi_y$. Now let $u \in Y \cap \xi_y$ be arbitrary. If u is collinear to at least one point of $(y^\perp \cup z^\perp) \cap Y$, then $\xi_y = \xi_u$, by the foregoing. If u is not collinear to any point colllinear to some point of $y^\perp \cap Y$, then clearly $\{u\} \cup (\xi_y \setminus Y) = u^\perp \cap \xi_y$. But $\xi_u \cap \xi_y$ contains a plane, and so there are points of Y in $u^\perp \cap \xi_y$ distinct from u, a contradiction.

Hence for each point $u \in Y \cap \xi_y$, we have $\xi_u = \xi_y$. Lemma 6.1 then implies that $Y \cap \xi_y$ is the complement of a geometric hyperplane of ξ_y. Suppose for a contradiction that for some $y^* \in Y$, we have $\xi_y \neq \xi_{y^*}$. If $\xi_y \setminus Y$ is a non-degenerate hyperplane, then ${y^*}^\perp \cap \xi_y \cap Y$ spans a maximal singular subspace distinct from $\xi_y \cap \xi_{y^*}$, hence contains points of $Y \setminus \xi_{y^*}$, contradicting the fact that all points of Y collinear to y^* are contained in ξ_{y^*}, by definition of ξ_{y^*}.

Hence we may assume that both $\xi_y \setminus Y$ and $\xi_{y^*} \setminus Y$ are degenerate hyperplanes in the respective symps, say $\xi_y \setminus Y := p^\perp \cap \xi_y$ and $\xi_{y^*} \setminus Y := p^{*\perp} \cap \xi_{y^*}$. Suppose first that ξ_y and ξ_{y^*} intersect in a maximal singular subspace W. Select a hyperplane U of W containing neither p nor p^*. Pick $w \in \xi_y \setminus W$ collinear to each point of U. Then $w \in Y$ and w is collinear to each point of a maximal singular subspace V^* of ξ_{y^*} containing U. Each point of $V^* \setminus U$ violates the fact that $w^\perp \cap Y \subseteq \xi_{y^*}$. So, we may assume that ξ_y and ξ_{y^*} are disjoint. Then Δ is isomorphic to $\mathsf{D}_{5,5}(q)$ and the map ρ that sends a point $a \in \xi_y$ to the 3-space $a^\perp \cap \xi_{y^*}$ is an isomorphism from ξ_y to the half-spin geometry defined by the 3-spaces of ξ_{y^*} that are not maximal in Δ. Now each point $v \in \xi_{y^*} \cap Y$ not in $\rho(p)$ is contained in a 3-space $U_v \subseteq \xi_{y^*}$ disjoint from $\rho(p)$. Then U_v is the image under ρ of some point $b \in \xi_y$ nor collinear to p, hence belonging to Y, contradicting $\xi_b = \xi_y$ and $v \in Y \setminus (\xi_y) \cap b^\perp$.

Hence $Y \subseteq \xi_y$. Setting $\xi_y = \xi_Y$, the assertions follow. □

We now apply these results in our situation.

Lemma 6.3 *Let $x, y \in V(\Gamma)$ be at distance 2 from each other in Γ. Then the set of singular affine lines through x containing at least one point of $y^\perp$ (and then precisely one point) is the complement of a geometric hyperplane of a symplecton ξ_x of the local parapolar space Γ_x at x.*

Proof Let $z \in x^\perp \cap y^\perp$ be arbitrary. We look at the local graph $\Gamma(z)$ of Γ at z.

Let Z be the set of points of the local parapolar space Γ_z induced by the intersection $x^\perp \cap y^\perp$. Then Z is contained in a unique symp ξ_z of Γ_z, and hence also in a unique symp ξ_x of Γ_x, since the cone in Γ_z with vertex $\langle x, z\rangle$ and base Z has the structure of a quadric, and hence is contained in a symp in any local parapolar space Γ_u for $u \in \langle x, z\rangle$.

Define Y to be the set of affine singular lines through x that intersect Z in at least one point. Note, however, if an affine singular line through x intersected Z in more than one point, then, by Corollary 4.10, x would be collinear to y, a contradiction.

We take an affine singular plane π through x and z in ξ_x. Note that π contains a unique line L_π through z in $y^\perp$. There is a unique line M_π in π parallel to L_π and containing x; it is disjoint from Y. This corresponds to the unique point of the line of the local parapolar space Γ_x defined by π that is not contained in Y.

Let π' be an affine singular plane through x such that the corresponding line $L_{\pi'}$ similarly defined as above is collinear to L_π in Γ_z. Consider the affine 3-space $\langle\langle x, z\rangle, L_\pi, L_{\pi'}\rangle\rangle$. It is clear that M_π and $M_{\pi'}$ generate a plane in this affine space, none of whose affine singular lines through x intersect Y. This shows that Y satisfies the condition of Lemma 6.2, hence completing the proof. □

Proposition 6.4 *Let $x, y \in V(\Gamma)$ be mutually at distance 2 in Γ. Then there exists a geometric hyperplane H_y of a symp ξ_y of Γ_x such that $x^\perp \cap y^\perp$ is the union of singular affine lines disjoint from H_y containing exactly one point of each singular affine line through x contained in $\xi_y \setminus H_y$.*

Proof The previous lemma implies that the set of singular affine lines through x containing points of $y^\perp$ is contained in a symp ξ_y in which it is the complement of a geometric hyperplane H_y of ξ_y in Γ_x. In view of Lemma 6.2, it remains to prove that, if a singular affine plane π through x contains two points of $y^\perp$, then $\pi \cap y^\perp$ is a singular affine line. This follows from Corollary 4.10. □

Notation 6.5 We will adopt the notation ξ_y and H_y for any point $y \in \Gamma_2(x)$ and call it the symplecton and geometric hyperplane, respectively, of Γ_x *associated to* y. Strictly speaking the notation ξ_y and H_y should involve x, too, but there will be no confusion.

Now note that, with this notation, it follows from Proposition 6.4 that

$$|x^\perp \cap y^\perp| = \begin{cases} \mu = q^2+q, & \Gamma_x \cong \mathsf{A}_{1,1}(q) \times \mathsf{A}_{d-1,1}(q), d \geq 2, H_y \text{ non-degenerate}, \\ \mu' = q^2, & \Gamma_x \cong \mathsf{A}_{1,1}(q) \times \mathsf{A}_{d-1,1}(q), d \geq 2, H_y \text{ degenerate}, \\ \mu = q^4+q^2, & \Gamma_x \cong \mathsf{A}_{4,2}(q), H_y \text{ non-degenerate}, \\ \mu' = q^4, & \Gamma_x \cong \mathsf{A}_{4,2}(q), H_y \text{ degenerate}, \\ \mu = q^6+q^3, & \Gamma_x \cong \mathsf{D}_{5,5}(q), H_y \text{ non-degenerate}, \\ \mu' = q^6, & \Gamma_x \cong \mathsf{D}_{5,5}(q), H_y \text{ degenerate}. \end{cases} \tag{6.6}$$

Lemma 6.7 *Let* $x, y, z \in V(\Gamma)$ *with* $y, z \in \Gamma_2(x)$. *Suppose* $\xi_y = \xi_z$ *and* $H_y = H_z$, *using Notation 6.5. Then* $x^\perp \cap y^\perp \cap z^\perp = \emptyset$ *as soon as* $y \neq z$. *Moreover, there are at most* $(q-1)$ *vertices* u *in* $\Gamma_2(x)$ *such that* $H_y = H_u$.

Proof Suppose for a contradiction that $w \in x^\perp \cap y^\perp \cap z^\perp$. Let $v \in y^\perp$ such that $\langle x, v, w\rangle = \pi$ is an affine singular plane. Let L_π be the singular affine line of π contained in H_y. Then by Proposition 6.4 the singular affine line $\langle w, v\rangle$ is parallel to L_π and hence, also belongs to $z^\perp$.

Since the complement of geometric hyperplane of a polar space is known to be connected, this shows that $x^\perp \cap y^\perp = x^\perp \cap z^\perp$. By Lemma 4.16, we conclude that $y = z$.

Since a singular affine line not contained in H_y, contains $q-1$ points distinct from x, there are at most $q-1$ points z not adjacent to x, such that $H_y = H_z$. □

Lemma 6.8 *Let* $x \in V(\Gamma)$. *Then the number of vertices* y *in* $\Gamma_2(x)$ *with* H_y *degenerate (or, equivalently, with* $|x^\perp \cap y^\perp| = \mu'$*) is at most*

$$\begin{cases} \frac{(q^d-1)(q^{d-1}-1)(q+1)}{q-1} & \text{if } \Gamma_x \cong \mathsf{A}_{1,1}(q) \times \mathsf{A}_{d-1,1}(q), d \geq 2, \\ (q^5-1)(q^2+1)(q^2+q+1) & \text{if } \Gamma_x \cong \mathsf{A}_{4,2}(q), \\ \frac{(q^8-1)(q^5-1)(q^3-1)}{q-1} & \text{if } \Gamma_x \cong \mathsf{D}_{5,5}(q). \end{cases}$$

Proof By Lemma 6.7, the number of vertices y in $\Gamma_2(x)$ with H_y degenerate is at most $q-1$ times the number of pairs in Γ_x consisting of a symp ξ and a degenerate hyperplane H, that is, a point $p \in \xi$ (for which $H = p^\perp \cap \xi$). The results of this easy count are the given numbers. □

Corollary 6.9 *Let $x \in V(\Gamma)$. Then there exists $y \in \Gamma_2(x)$ such that corresponding geometric hyperplane H_y is non-degenerate, or, equivalently, such that $|x^\perp \cap y^\perp| = \mu$. Consequently, H_y is non-degenerate for all $y \in \Gamma_2(x)$.*

Proof Let k be the valency of each vertex of Γ and λ be the number of vertices adjacent to any given pair of adjacent vertices of Γ. Let μ and μ' be as above. Let n be q^{2d}, q^{10} and q^{16} in the cases where the local geometry Γ_x is isomorphic to $\mathsf{A}_{1,1}(q) \times \mathsf{A}_{\mathsf{d}-1,1}(q)$, $\mathsf{A}_{4,2}(q)$ and $\mathsf{D}_{5,5}(q)$, respectively. Then (n, k, λ, μ) are the parameters of the strongly regular graphs $\mathsf{H}_q(2, d)$, $V\mathsf{J}_q(5, 2)$ and $V\mathsf{D}_{5,5}(q)$, respectively.

We bound $|\Gamma_2(x)|$ as follows. Each neighbour y of x has $k - \lambda - 1$ neighbours z at distance 2 from x. Each of those vertices z is counted μ or μ' times. Since $\mu > \mu'$ by Eq. 6.6, there are at least $\frac{k(k-\lambda-1)}{\mu} = n - k - 1$ vertices in $\Gamma_2(x)$ (this is a standard identity in the parameters of a strongly regular graph). Using the values of n above, those of k in Table 3 of Section 2.3, and also using Lemma 6.8, we see that $|\Gamma_2(x)|$ is strictly larger than the number of vertices y in $\Gamma_2(x)$ with H_y degenerate.

Now the assertion follows from the assumption that $|x^\perp \cap z^\perp|$ is constant for $z \in \Gamma_2(x)$. □

We will write $\Gamma_{\leq 2}(x)$ to denote the set of all vertices of Γ at distance at most 2 from a fixed vertex $x \in V(\Gamma)$ (i.e., the ball of radius 2 in Γ centered at x).

Remark 6.10 The counting argument in the proof of Corollary 6.9 in fact shows that $|\Gamma_{\leq 2}(x)| \geq n$ (where n is defined as in that proof), with equality if, and only if, each vertex in $\Gamma_2(x)$ has exactly μ neighbours in $\Gamma_1(x)$. So, under our general assumption that $|x^\perp \cap y^\perp|$ is a constant for all x, y at mutual distance 2 in Γ, we conclude that $|\Gamma_{\leq 2}(x)| = n$.

If for a moment we drop the general additional assumption that $|x^\perp \cap y^\perp|$ is constant for all $x, y \in V(\Gamma)$ at mutual distance 2 and only work with the local assumption, then we can prove the following.

Lemma 6.11 *Let Γ be a graph which is locally isomorphic to either $\mathsf{H}_q(2, e)$, $e \geq 3$, $V\mathsf{J}_q(5, 2)$ or $V\mathsf{D}_{5,5}(q)$. Let $x \in V(\Gamma)$ be arbitrary. Suppose $|\Gamma_{\leq 2}(x)| = n$, with $n = q^{2e}, q^{10}, q^{16}$, respectively. Let ξ be a symp in the local geometry at x and let H be a non-singular hyperplane of ξ. Then there are exactly $(q - 1)$ vertices u in $\Gamma_2(x)$ such that $\xi_u = \xi$ and $H_u = H$.*

Proof For a given local geometry let α denote the number of symps it contains and let β denote the number of nondegenerate hyperplanes. Then $(\alpha, \beta) = \left(\frac{(q^e-1)(q^{e-1}-1)}{(q^2-1)(q-1)}, q^3 - q\right)$ for $\mathsf{H}_q(2, e)$, $\left(\frac{q^5-1}{q-1}, q^5 - q^2\right)$ for $\mathsf{A}_{4,2}(q)$ and $\left(\frac{q^5-1}{q-1}(q^4 + 1), q^7 - q^3\right)$ for $\mathsf{D}_{5,5}(q)$.

We obtain at most $(q - 1)\alpha\beta$ distinct mu-graphs, with equality if, and only if, for each symp ξ and each nondegenerate hyperplane H of ξ, there are exactly $q - 1$ vertices $y \in \Gamma_2(x)$ with $\xi = \xi_y$ and $H = H_y$. One calculates that, with the above notation, $(q - 1)\alpha\beta = n - k - 1$. This proves the lemma. □

7 Characterising adjacency in $\Gamma_2(x)$ using local geometry

In the sequel, let d be equal to 0 for the case $\mathsf{A}_{1,1}(q) \times \mathsf{A}_{\mathsf{e}-1,1}(q)$ with $e \geq 3$, $d = 1$ for the case $\mathsf{A}_{4,2}(q)$ and $d = 2$ in the case $\mathsf{D}_{5,5}(q)$. Let $e = 2$ for the cases $\mathsf{A}_{4,2}(q)$ and $\mathsf{D}_{5,5}(q)$.

Proposition 7.1 *Let x, z be vertices of Γ and $y \in \Gamma_2(x) \cap \Gamma_2(z)$. Then the vertices x and z are adjacent if, and only if, $x^\perp \cap y^\perp \cap z^\perp$ contains a spanning set of points of a singular affine subspace of dimension $d+1$.*

Proof First suppose that z and x are adjacent. Then the perp $\langle x, z\rangle^\perp$ shares at least one singular subspace U with ξ_y; here U is a singular affine subspace of dimension $d+2$. Since $q \geq 3$, the set $y^\perp \cap U$, which is contained in $x^\perp \cap y^\perp \cap z^\perp$, generates an affine subspace of U of dimension $d+1$ because it coincides with an affine $(d+2)$-space minus a hyperplane of it.

Secondly, suppose that $x^\perp \cap y^\perp \cap z^\perp$ contains a spanning set of points of a singular affine $(d+1)$-space U. Pick $u \in U$. Then $\langle x, U\rangle$, $\langle y, U\rangle$ and $\langle z, U\rangle$ are singular subspaces of dimension $d+1$ in Γ_u containing the singular d-space U. Since x and y are not adjacent in Γ, they are contained in a unique symp $\zeta_{x,y}$ of Γ_u. Then $\langle u, z\rangle$ being collinear to each point of U, is collinear to either $\langle x, u\rangle$ or $\langle y, u\rangle$ as follows from Theorem 15.4.5 of [23]. Our assumption implies $z \perp x$. □

Theorem 7.2 *Let Γ be a graph with local graph isomorphic to the $(q-1)$-clique extension of the point graph of $\mathsf{A}_{1,1}(q) \times \mathsf{A}_{\mathsf{e}-1,1}(q)$ with $e \geq 3$, $\mathsf{A}_{4,2}(q)$ or $\mathsf{D}_{5,5}(q)$. Assume that the number of vertices adjacent to two given non-adjacent vertices at distance 2 is a constant. Then Γ is isomorphic to $\mathsf{H}_q(2, e)$, $V\mathsf{J}_q(5, 2)$ or $V\mathsf{D}_{5,5}(q)$, respectively.*

Proof Let Γ^* be one of $\mathsf{H}_q(2, e)$, $V\mathsf{J}_q(5, 2)$ or $V\mathsf{D}_{5,5}(q)$, having the same local graph as Γ. Then there is an isomorphism $\beta : \Gamma_x \to \Gamma^*_{x^*}$ which lifts to a graph isomorphism, which we also denote with β, from the subgraph of Γ induced by $\Gamma_1(x)$ to the one of Γ^* by $\Gamma^*_1(x^*)$. Set $\beta(x) = x^*$. Lemma 6.11 permits to extend β to $\Gamma_2(x)$ by mapping the vertex y of $\Gamma_2(x)$ to the unique vertex $y^* \in \Gamma_2(x^*)$ with $\xi^*_{y^*} = \beta(\xi_y)$, $H^*_{y^*} = \beta(H_y)$ and $\beta(z) \in \xi^*_{y^*} \setminus H^*_{y^*}$, for an arbitrary $z \in \xi_y \setminus H_y$ (using self-explaining notation). Note this is independent from the vertex z as $x^\perp \cap y^\perp$ is determined by H_y and the point z and the same holds in Γ^*. By Proposition 7.1, β is a graph isomorphism from $\Gamma_{\leq 2}(x)$ to Γ^* (remember Γ^* has diameter 2). By regularity of both graphs, we conclude $\Gamma_{\leq 2}(x) = V(\Gamma)$ and the proof is complete. □

Now Theorem A follows from Theorem 5.2 and Theorem 7.2, together with Proposition 4.17.

Theorem 7.3 *Let Γ be a graph with local graph isomorphic to the $(q-1)$-clique extension of the point graph of $\mathsf{A}_{1,1}(q) \times \mathsf{A}_{\mathsf{e}-1,1}(q)$ with $e \geq 3$, $\mathsf{A}_{4,2}(q)$ or $\mathsf{D}_{5,5}(q)$. Assume that $|V(\Gamma)| \leq q^{6d+2e}$. Then Γ is isomorphic to $\mathsf{H}_q(2, e)$, $V\mathsf{J}_q(5, 2)$ or $V\mathsf{D}_{5,5}(q)$, respectively.*

Proof In light of Theorem 7.2, it suffices to prove that the number of vertices adjacent to two given non-adjacent vertices at mutual distance 2 is a constant. The assumed upper bound on the number of vertices of Γ lets us apply Remark 6.10. Hence, the conclusions of Corollary 6.9 apply and we obtain that $|x^\perp \cap y^\perp| = \mu$ for all $x \in V(\Gamma)$, $y \in \Gamma_2(x)$ (with the value of μ given in Table 3 depending on the local graph of Γ). The desired conclusion now follows from Theorem 7.2. □

Now Theorem 5.2 and Theorem 7.3 imply Theorem C.

8 Group theoretic application

In this section, we provide a group theoretic application of our local characterisations given by Theorem A. Let Γ be one of the graphs of Table 1. Then $V(\Gamma)$ can be identified with a vector space V over $\mathbb{F}_q$, and adjacency is given in terms of the structure at infinity in $\mathsf{PG}(V)$. A point at infinity belonging to the corresponding Lie incidence geometry Δ at infinity shall be referred to as a singular point (and likewise for singular subspaces); other points are non-singular. Let S be the adjoint Chevalley group naturally associated with and acting on the structure at infinity. Let K be a group of automorphisms of Γ inducing linear transformations on V, containing all dilations (that is, linear transformations corresponding to scalar matrices) and inducing a group $H \cong K/Z(K)$ at infinity, with $S \leq H \leq \mathsf{Aut}(S)$. With this set-up, we have the following theorem.

Theorem 8.1 *Let $q > 2$ be a prime power. Let G be a group and $x \in G$ a non-central element such that the following hold.*

(i) $C_G(x)$ is isomorphic to K.
(ii) For every $y, z \in x^G$, if we set $U := C_G(y)$ and $W := C_G(z)$, then $U \cap W$ induces at infinity the full stabiliser in H of a point p such every vector corresponding to p is an eigenvector with eigenvalue 1 of each member of $U \cap W$, viewed as subgroup of $K \cong U$.
(iii) For every $y, z \in X^G$, with the notation of (ii), $C_G(U \cap W) = \langle Z(U), Z(W)\rangle$ is the Frobenius group of order $q(q-1)$ isomorphic to $\mathsf{AGL}_1(q)$ (that is, the point stabiliser in $\mathsf{PGL}_2(q)$ in its natural action on $\mathsf{PG}(1, q)$).
(iv) For every $y, z \in x^G$, $G = \langle C_G(y), C_G(z)\rangle$ if the point p of (ii) is singular.

Then $G \cong V \rtimes K$.

Sketch of the proof. Let Λ be the graph with vertex set $V(\Lambda) = x^G$ and define the vertices $y, z \in V(\Lambda)$ to be adjacent, notation $y \sim z$, if, and only if, the point p in (ii) related to $C_G(y)$ and $C_G(z)$ is singular.

Step 1. It follows from (iii) that, if $x \sim y$, then $\{x\} \cup \{z \in V(\Lambda) \mid C_G(x) \cap C_G(y) = C_G(x) \cap C_G(z)\}$ is a clique of size q, permuted, in a sharply 2-transitive way, by the Frobenius group $\langle Z(C_G(x)), Z(C_G(y))\rangle$. Hence, for given $x \in V(\Lambda)$, each neighbouring vertex y uniquely defines a clique $\mathscr{C}_y$ of size $q-1$ in $\Lambda_1(x)$.

From now on we fix $x \in V(\Lambda)$.

Step 2. We show that two vertices y, z in $\Lambda_1(x)$ are adjacent if the points p_y and p_z related to $C_G(x) \cap C_G(y)$ and $C_G(x) \cap C_G(z)$, respectively, as in (ii), are collinear.

So, suppose that p_y and p_z are collinear singular points (in Δ). Then $C_G(x) \cap C_G(y) \cap C_G(z)$ is contained in the 2-point-stabiliser in Δ corresponding to p_x and p_y. It follows from (ii) that the singular line $p_x p_y$ is fixed pointwise. Since parabolic subgroups of given type form a conjugacy class, the same holds in Δ when $C_G(y)$ is identified with K, if there are no diagram automorphisms related to Δ. Hence, in that case, $C_G(y) \cap C_G(z)$ contains the pointwise stabiliser of a line of Δ and consequently, $C_G(y) \cap C_G(z)$ is a stabiliser of an isotropic point on that line (one can show that the pointwise stabiliser of a line does not fix any other point—singular or not—of $\mathsf{PG}(V)$ off that line).

If there are dualities, $C_G(x) \cap C_G(y) \cap C_G(z)$ could also be the full stabiliser of another singular subspace, say of dimension $d > 1$ in the residual geometry at some point p_x of Δ (here, we still identify K with $C_G(y)$). Hence, it is a stabiliser of a $(d+1)$-space X, acting transitively on all lines of X through p_x. Such a subgroup clearly does not fix any additional isotropic or non-isotropic points.

Step 3. Similarly to Step 2, one shows that two vertices y, z in $\Lambda_1(x)$ are not adjacent if the points p_y and p_z related to $C_G(x) \cap C_G(y)$ and $C_G(x) \cap C_G(z)$, respectively, as in (ii), are not collinear.

Steps 2 and 3 imply that the local graph $\Lambda_1(x)$ is a $(q-1)$-clique extension of the point graph of Δ. Given $y, z \in x^G$, their corresponding point-stabilisers in the action of G on Λ are precisely $C_G(x)$ and $C_G(y)$. If y and z are adjacent, then these point-stabilisers generate the group G by (iv), hence implying that the graph Λ is connected.

Theorem A completes the proof of the assertion if we can show that non-adjacent vertices have a constant number of common neighbours. But this follows from the fact that S acts transitively on the set of non-singular points of $\mathsf{PG}(V)$ together with the observation that, if $C_G(x) \cap C_G(y)$ and $C_G(x) \cap C_G(z)$ fix the same non-singular points, then an argument as in Step 1 shows that some member of the Frobenius group $\langle Z(C_G(y)), Z(C_G(z)) \rangle$ fixes x and conjugates y into z. □

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