

Groups of projectivities and Levi subgroups in spherical buildings of simply laced type

Sira Busch, Jeroen Schillewaert and Hendrik Van Maldeghem*

Communicated by Christopher W. Parker

Abstract. We introduce the special and general projectivity groups attached to a simplex F of a thick, irreducible, spherical building of simply laced type. If the residue of F is irreducible, we determine the permutation group of both projectivity groups of F , acting on the residue of F and show that the special projectivity group determines the precise action of the Levi subgroup of a parabolic subgroup on the corresponding residue. This reveals three special cases for the exceptional types E_6 , E_7 , E_8 . Furthermore, we establish a general diagrammatic rule to decide when exactly the special and general projectivity groups of F coincide.

1 Introduction

The theory of buildings evolved during the search for analogues of exceptional simple Lie groups over arbitrary fields; traditionally, people only worked over the fields $\mathbb{C}$ and $\mathbb{R}$. This was of interest, since working over arbitrary fields would allow the field to be finite, and with that, one could find new families of finite simple groups. In 1955, Chevalley managed to construct these analogues and the groups he found are now known as *Chevalley groups*. After Chevalley published his work, Jacques Tits developed the theory of buildings, attaching geometric structures to these groups (see [1, page 335]).

Chevalley groups defined over arbitrary fields are known to be *groups of Lie type* (as in [7]). Groups of Lie type have BN-pairs and are hence associated to buildings (see [7, page 108, Proposition 8.2.1]). They can be described as groups of automorphisms of spherical buildings (i.e. buildings with finite Weyl groups; see [1, §6.2.6]). Chevalley groups are always simple, except in the cases $A_1(2)$, $A_1(3)$, $B_2(2)$, $G_2(2)$ (see [7, page 172, Theorem 11.1.2]).

The first author is funded by the Claussen-Simon-Stiftung and by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy EXC 2044 – 390685587, Mathematics Münster: Dynamics–Geometry–Structure. All authors were supported by the New Zealand Marsden Fund grant UOA-2122 of the second author. This work is part of the PhD project of the first author.

In this article, we will focus on buildings of simply laced type and rank at least 3. Such buildings automatically admit so-called *root elations* (see [31]). Then we can define the Chevalley group attached to such a building Δ as the group of automorphisms generated by all root elations, which we will denote by $\text{Aut}^\dagger(\Delta)$. This agrees with what is known as the *adjoint Chevalley group* (see [7, page 198]), and is also called the *little projective group of Δ* . It is always simple in our cases, since we assume the rank to be at least 3 (compare with [29, Main Theorem]).

Parabolic subgroups of Chevalley groups have attracted much attention in the literature. They can be written as semi-direct products of a *unipotent subgroup* and a *Levi subgroup* (see [7, page 118]). So far, a lot of research focussed on the unipotent subgroups (see for example [12, 19]). In this article, we aim to shed some light on the Levi subgroups.

Let Σ be an apartment of Δ and C a chamber in Σ that we will consider to be the fundamental chamber. Let F be a face of C . A Levi subgroup of the parabolic subgroup G_F of $\text{Aut}^\dagger(\Delta)$ is a subgroup L_F such that G_F is the semi-direct product of L_F and a unipotent subgroup. This matches with how it has been traditionally defined in the literature (see [2, page 158, Definition 11.22]). The parabolic subgroups opposite G_F correspond bijectively to the Levi subgroups of G_F (see [2, page 199, Proposition 14.21]). Hence a Levi subgroup fixes a simplex and a unique opposite simplex pointwise, and it acts as a group of automorphisms on the residue (also sometimes called the *star*, for instance in [30]) of each of these simplices. In the present paper, we determine the precise action of the Levi subgroup on that residue. To the best of our knowledge, this was not recorded before. However, it is known how it can be obtained by means of characters and co-characters of the torus of the corresponding Chevalley group, using the theory of roots and co-roots in Chevalley groups. We explain and apply this approach in Section 8.1. It provides an algebraic answer to the problem of determining the precise action of the Levi subgroup on the corresponding residue of the building.

However, approaching the problem in a geometric way via the theory of buildings gives rise to a new development of the theory of *special and general projectivity groups*. The connection with the problem of the previous paragraph is given by our Theorem A: it shows that the special projectivity group of F coincides with the faithful permutation group induced by the stabiliser $\text{Aut}^\dagger(\Delta)_F$ of F in $\text{Aut}^\dagger(\Delta)$ on the residue $\text{Res}_\Delta(F)$ of F in Δ . This connects the special projectivity group of a simplex F to the Levi subgroup of F . Since we determine all general and special groups of projectivities, this determines the precise action of the Levi subgroup of a parabolic subgroup on the corresponding residue geometrically.

As mentioned above, we also develop some basic and general theory about the projectivity groups. In projective geometry, the *groups of projectivities*, or *projectivity groups*, play an important role in many proofs. For instance, projectivities

between lines in a projective plane can be used to define non-degenerate conics (Steiner's approach) and prove properties of them. In [20], Knarr defined groups of projectivities and groups of even projectivities for generalised polygons and determined them in the finite case. This was further generalised to large infinite classes in [32], where the group of projectivities was called the *general projectivity group* and the group of even projectivities the *special projectivity group* related to a point or line. A generalisation of the definitions to all spherical buildings is obvious and natural questions are, for instance,

- when does the general projectivity group coincide with the special projectivity group, and
- can one determine the various general and special projectivity groups, particularly in the case where the residues are irreducible?

In the present paper, we answer these questions for irreducible spherical buildings Δ with a simply laced diagram (see Remark 8.34 for the other cases). It will turn out that the special and general projectivity groups of residues of rank 1 are always $\mathrm{PGL}_2(\mathbb{K})$ in its natural permutation group action. This is Theorem D. For (irreducible) residues R of rank at least 2, in most cases, we generically obtain the maximal linear (algebraic or projective) group, including possible dualities, if opposition in the Coxeter diagram of the ambient building is trivial, and the one in the Coxeter diagram of R is not trivial. There are only these four classes of exceptions.

- (i) If Δ has type D_n and the type of R contains the types $n - 1$ and n (hence R is of type D_ℓ for some $\ell < n$), then the projectivity groups are contained in $\mathrm{PGO}_{2\ell}(\mathbb{K})$. Here, $\mathbb{K}$ is the underlying field. (Hence there are no proper similitudes in the projectivity groups.)
- (ii) If Δ has type E_6 and R has type A_5 , then the special and general projectivity group consists of those members of $\mathrm{PGL}_6(\mathbb{K})$ which correspond to matrices, for which the determinant is a third power in the field $\mathbb{K}$ of definition.
- (iii) If Δ has type E_7 and R has type A_5 containing the type 2 (in Bourbaki labelling), then the special projectivity group consists of those members of $\mathrm{PGL}_6(\mathbb{K})$ which correspond to matrices, for which the determinant is a square in the field $\mathbb{K}$ of definition. The general projectivity group extends this group with a duality, for instance a symplectic polarity, with corresponding matrix of square determinant.
- (iv) If Δ has type E_7 and R has type D_6 , then the special and general projectivity group are the simple group $\mathrm{P}\Omega_{12}(\mathbb{K})$, extended with a class of diagonal automorphisms.

This is Theorem E. A complete list in tabular form of all special and general projectivity groups acting on irreducible residues of buildings of type E_6, E_7, E_8 and D_n (for $n \geq 4$) is included in Section 8. In our arguments, the so-called polar vertices of the diagram will play a crucial role, and our results will entail a new combinatorial characterisation of the polar type. Theorems B and C below show that these polar types are basically the only ones responsible for the special and general projectivity groups to coincide.

We essentially provide two proofs for Theorem E. The algebraic one, outlined in Section 8.1, and a geometric one in the remainder of Section 8. However, the algebraic proof will make clear that the case of type E_8 is, in a certain way, trivial – one always gets the full linear groups –, so we skip this case in our geometric approach (however, it could be included and the geometric arguments will be contained in the first author’s PhD thesis). The purpose of still providing the detailed geometric arguments for the other cases is the following. Firstly, we consider it interesting to see where the exceptions mentioned above come from in a geometric way and how they emerge from the geometry of buildings. The geometry sometimes provides a different “reason” for a certain group to be the special projectivity group. Secondly, we also want to determine the general projectivity groups, which we cannot obtain from the algebraic considerations. Thirdly, some geometric lemmas that we need, such as Lemma 8.22, can be useful in other contexts, and lastly, it prepares us for handling the cases of non-simply laced types, where the question for non-split buildings is not so easily solved using the algebraic approach (think of buildings of so-called *mixed type* that do not really admit a well-defined root system).

Exceptions (i) to (iv) show that the questions stated above are not trivial and that the answer is rather peculiar, with exactly three special cases for the exceptional groups.

We now get down to definitions and statements of our main results.

2 Preliminaries and statement of the main results

We will need some notions and notation related to spherical buildings, and of point-line geometries related to those. Excellent references for buildings are the books [1, 30], since it will be convenient to consider buildings as simplicial complexes. Standard references for the point-line approach to (spherical) buildings are [4, 24].

2.1 Spherical buildings

Let Δ be a spherical building. We will assume, as in [30], that Δ is a thick numbered simplicial chamber complex, and we will usually denote the type set by

$I = \{1, 2, \dots, r\}$, where r is the rank of Δ , and the set of chambers by $\mathcal{C}(\Delta)$. The type $\text{typ}(F)$ of a simplex F is the set of types of its vertices. A *panel* is a simplex of size $r - 1$. *Adjacent* chambers are chambers intersecting in a panel. This defines in a natural way the *chamber graph*. The (*gallery*) *distance* $\delta(C, C')$ between two chambers C and C' is the distance in the *chamber graph* of the vertices corresponding to C and C' .

One of the defining axioms of a spherical building is that every pair of simplices is contained in an *apartment*, which is a thin simplicial chamber subcomplex isomorphic to a finite Coxeter complex $\Sigma(W, S)$ with associated Coxeter system (W, S) , where W is a Coxeter group with respect to the generating set S of involutions. If $S = \{s_1, \dots, s_r\}$, then let $P_i = \langle s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_r \rangle$, $i \in I$, be the maximal parabolic subgroups. The vertices of $\Sigma(W, S)$ of type $i \in I$ are the right cosets of P_i . The chambers are the sets of cosets of maximal parabolic subgroups containing a given member w of W . For each pair (C, C') of adjacent chambers, there exists exactly one *folding*, that is, a type-preserving idempotent morphism of $\Sigma(W, S)$ mapping C' to C , and such that each chamber in the image has two chambers in its pre-image. The image α of a folding is called a *root*. The root associated to the *opposite folding*, namely, the folding mapping C to C' is called the *opposite root*, and is denoted by $-\alpha$. The intersection $\alpha \cap (-\alpha)$, called a *wall*, is denoted by $\partial\alpha$ (and hence also by $\partial(-\alpha)$), and is also referred to as the *boundary of α* . Every root contains a unique simplex that is fixed under each automorphism of $\Sigma(W, S)$ preserving α (and not necessarily type-preserving). This simplex is called the *centre* of the root. If $\Sigma(W, S)$, or equivalently, Δ , is irreducible (see below), the type of this simplex is called a *polar type* of Δ . In the reducible case, the polar types of the connected components will be called *polar types* of the building.

For each vertex v of $\Sigma(W, S)$, there exists a unique other vertex v' of $\Sigma(W, S)$ with the property that every wall containing v also contains v' (and then automatically, every wall containing v' contains v); then v and v' are called *opposite vertices*. *Opposite* simplices of $\Sigma(W, S)$ are two simplices A, B with the property that the vertex opposite to any vertex in A is contained in B , and vice versa. We denote $A \equiv B$. Opposition defines a permutation, also denoted by $\equiv$, of order at most 2 on the type set I . A subset $J \subseteq I$ is called *self-opposite*, if $J^\equiv = J$. The permutation $\equiv$, acting on I , induces an automorphism of the corresponding Coxeter diagram. Recall that the vertices of the Coxeter diagram correspond to the types, that is, the elements of I , and two vertices i and j are connected by an edge of weight $m_{ij} - 2$, where m_{ij} is the order of $s_i s_j$ in W . Throughout, we use the Bourbaki labelling of connected spherical Coxeter diagrams [3]. The Coxeter diagram, and by extension the chamber complex $\Sigma(W, S)$ and the building Δ , are called *simply laced* if $m_{ij} \in \{2, 3\}$ for all $i, j \in \{1, 2, \dots, r\}$, $i \neq j$. The building Δ is irreducible if the Coxeter diagram is connected. The polar type in the simply

laced and irreducible case is unique. It is the set of nodes to which the additional generator is joined, when constructing the affine diagram. Hence it is $\{1, r\}$ in case A_r , it is 2 in case of D_r , and 2, 1, 8 for E_6, E_7, E_8 , respectively.

Opposite simplices in Δ are simplices that are opposite in some apartment, and then the building axioms guarantee that they are opposite in every apartment in which they are both contained.

We say that a vertex v and a simplex F are *joinable* if $v \notin F$ and $F \cup \{v\}$ is a simplex, notation $v \sim F$. (Note that we denote simplices with capital letters, such as F , since the letter S already has a meaning. The letter F stands for “flag”, which is a synonym of simplex in the language of geometries.) The simplicial complex induced on the vertices joinable to a given simplex F of a building Δ forms a building called the *residue of F in Δ* and is denoted by $\text{Res}_\Delta(F)$. It is well known that the Coxeter diagram of that residue is obtained from the Coxeter diagram of the building by deleting the vertices with type in $\text{typ}(F)$. The opposition relation in $\text{Res}_\Delta(F)$ will be denoted by $\equiv_F$ (also on the types), and two simplices of $\text{Res}_\Delta(F)$, opposite in $\text{Res}_\Delta(F)$, will occasionally be called *locally opposite at F* . The cotype $\text{cotyp}(F)$ of a simplex F is $I \setminus \text{typ}(F)$, and the *type* of the residue $\text{Res}_\Delta(F)$ is the cotype of F .

Now let F and F' be two opposite simplices. Let $C \in \mathcal{C}(\Delta)$ be such that $F \subseteq C$. Then there exists a unique chamber $C' \supseteq F'$ at minimal gallery distance from C . The chamber C' is called the *projection of C from F onto F'* and denoted $\text{proj}_{F'}^F(C)$. That mapping is a bijection from the set of chambers through F to the set of chambers through F' and preserves adjacency in both directions. It follows that it defines a unique isomorphism from $\text{Res}(F)$ to $\text{Res}(F')$, which we denote by $\text{proj}_{F'}^F$ (as it is a special case of the projection operator; see [30, Paragraph 3.19]); see also [30, Theorem 3.28]. When the context makes F clear, we sometimes remove the F from the notation for clarity and simply write $\text{proj}_{F'}$. This projection has the following property.

Proposition 2.1 ([30, Proposition 3.29]). *Let F and F' be opposite simplices of a spherical building Δ . Let v be a vertex of Δ such that $v \sim F$, and set*

$$i := \text{typ}(v) \in I.$$

Then the type i' of the vertex $\text{proj}_{F'}^F(v)$ is the opposite in $\text{Res}(F')$ of the opposite type of i in Δ , that is, $i' = (i^{\equiv})^{\equiv_{F'}}$. Also, vertices $v \sim F$ and $v' \sim F'$ are opposite in Δ if and only if $v' \equiv_{F'} \text{proj}_{F'}^F(v)$.

Now let Δ be irreducible and of rank at least 2. Let α be a root of Δ . Let U_α be the group of automorphisms of Δ pointwise fixing every chamber that has a panel in $\alpha \setminus \partial\alpha$. The elements of U_α are called (*root*) *elations* and U_α itself is called a *root group*. An element of U_α is called a *central elation* if it belongs to U_β

for each root β having the same centre as α . If U_α acts transitively on the set of apartments containing α , then we say that α is *Moufang*. If every root is Moufang, then we say that Δ is Moufang. The automorphism group of Δ is denoted by $\text{Aut } \Delta$ and, if Δ is Moufang, then the subgroup generated by the root elations is denoted by $\text{Aut}^\dagger \Delta$ and called the *little projective group of Δ* . It is also generated by all central elations (in the simply laced case, all elations are central). Also, we denote by $\text{Aut}^\circ(\Delta)$ the subgroup of type-preserving automorphisms of Δ . In the literature, this is also sometimes denoted as $\text{Spe}(\Delta)$. Finally, each irreducible spherical building Δ of rank $r \geq 3$ is Moufang.

2.2 Groups of projectivity

Let Δ be a spherical building and F, F' two simplices which are opposite, and which are not chambers. Then we call the isomorphism $\text{proj}_F^{F'}$, a *perspectivity (between residues)* and denote $F \bar{\cap} F'$. If $F_0, F_1, \dots, F_\ell$ is a sequence of consecutively opposite simplices, then the isomorphism $\text{Res}(F_0) \rightarrow \text{Res}(F_\ell)$ given by

$$\text{proj}_{F_\ell}^{F_{\ell-1}} \circ \dots \circ \text{proj}_{F_2}^{F_1} \circ \text{proj}_{F_1}^{F_0}$$

is called a *projectivity (of length ℓ)*. If ℓ is even, it is called an *even projectivity*, and if $F_0 = F_\ell$, it is called a *self-projectivity*. The set of all self-projectivities of a simplex F is a group called the *general projectivity group of F* and denoted $\Pi(F)$. Likewise, the set of all even self-projectivities of a simplex F is a group called the *special projectivity group of F* and denoted $\Pi^+(F)$. Note that $\Pi(F) = \Pi^+(F)$ whenever $(\text{typ}(F))^\equiv \neq \text{typ}(F)$.

Let $\Pi(F)$ be the general projectivity group of the simplex F of a spherical building Δ , with F not a chamber. Then, as an abstract permutation group, $\Pi(F)$ only depends on the type of F . Likewise, the special projectivity group $\Pi^+(F)$ only depends on the type of F . We have the natural inclusion $\Pi^+(F) \trianglelefteq \Pi(F)$ and $[\Pi(F) : \Pi^+(F)] \leq 2$. We denote the number $[\Pi(F) : \Pi^+(F)]$ by $n(J)$, where the type of F is J . We trivially have $n(J) = n(J^\equiv)$, because it is 1 if $J^\equiv \neq J$.

In the case that Δ has rank 2, that is, Δ is the building of a generalised polygon, F is necessarily a single vertex and can be thought of as either a point (type 1) or a line (type 2) of the generalised polygon. Knarr [20] shows that if Δ is Moufang, then for every point or line x of Δ , the group $\Pi^+(x)$ coincides with the stabiliser of x in the little projective group of Δ , that is, the group generated by the root groups. We generalise this to arbitrary simplices in arbitrary Moufang spherical buildings of simply laced type. This is our first main result, Theorem A. The strategy of the proof is the same as for the rank 2 case. However, the proof requires that the unipotent radical of a parabolic subgroup in a Moufang spherical building pointwise stabilises the corresponding residue, and acts transitively on the sim-

plices opposite the given residue. This follows from the Levi decomposition of parabolic subgroups in Chevalley groups. We provide a brief introduction.

2.3 The Levi decomposition in Chevalley groups

Let Δ be a building and F a simplex of type J . Suppose Δ is of irreducible simply laced type and has rank at least 3. Then, by the classification in [30], Δ is Moufang. Its little projective group $\text{Aut}^\dagger(\Delta)$ is either a Chevalley group, or, in case Δ corresponds to a projective space of dimension d defined over a non-commutative skew field $\mathbb{L}$, it is $\text{PSL}_{d+1}(\mathbb{L})$ (in its natural action). The stabiliser P_F of F is called a *parabolic subgroup* and, if $\text{Aut}^\dagger(\Delta)$ is a Chevalley group, admits a so-called *Levi decomposition* $P_F = U_F L_F$ (see [7, Section 8.5]), where U_F is the so-called *unipotent radical* of P_F and L_F is called a *Levi subgroup*.

We provide an explicit description of P_F, U_F and L_F for $\text{PSL}_{d+1}(\mathbb{L})$ in the case that we will need most in the present paper, namely when $\text{Res}_\Delta(F)$ is irreducible. In that case, one chooses the basis in such a way that each subspace of F of dimension i is generated by the first $i + 1$ base points. Also, F consists of i -dimensional subspaces with $0 \leq i \leq d_1 - 1$ and $d - d_3 \leq i \leq d - 1$, where $|F| = d_1 + d_3$. Set $d_2 := d + 1 - d_1 - d_3$. Note that

$$J = \{1, \dots, d_1, d - d_3 + 1, \dots, d\}.$$

Then a generic element of P_F looks like

$$\begin{pmatrix} T_{d_1} & M_{d_1 \times d_2} & M_{d_1 \times d_3} \\ O_{d_2 \times d_1} & M_{d_2 \times d_2} & M_{d_2 \times d_3} \\ O_{d_3 \times d_1} & O_{d_3 \times d_2} & T_{d_3} \end{pmatrix},$$

where $T_{d_i}, i = 1, 3$, is an arbitrary invertible upper triangular matrix over F (needless to say that T_{d_1} and T_{d_3} are independent of each other; even if $d_1 = d_3$, they are considered different), $M_{d_i \times d_j}$ is an arbitrary $d_i \times d_j$ matrix, $i \in \{1, 2\}$ and $j \in \{2, 3\}$ (with a remark similar to the one for the T_{d_i}), and $O_{d_i \times d_j}$ is the $d_i \times d_j$ zero matrix, $i \in \{2, 3\}$, $j \in \{1, 2\}$. Also, the Dieudonné determinant of the whole matrix must be 1. With similar notation, and on top with $U_{d_i}, i \in \{1, 3\}$, an arbitrary unipotent upper triangular $d_i \times d_i$ matrix, $D_{d_i}, i \in \{1, 3\}$ an arbitrary invertible diagonal $d_i \times d_i$ matrix and I_{d_2} the $d_2 \times d_2$ identity matrix, generic elements of U_F and L_F look like (blanks replace zero matrices)

$$\begin{pmatrix} U_{d_1} & M_{d_1 \times d_2} & M_{d_1 \times d_3} \\ & I_{d_2} & M_{d_2 \times d_3} \\ & & U_{d_3} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} D_{d_1} & & \\ & M_{d_2 \times d_2} & \\ & & D_{d_3} \end{pmatrix},$$

respectively (again with the requirement that the Dieudonné determinant of the second matrix is equal to 1). One indeed checks that

$$P_F = U_F L_F \quad \text{and} \quad U_F \cap L_F = \{I_{d+1}\}.$$

Also, the following lemma is easily checked in this case. For Chevalley groups (the case of relevance for the current paper), the lemma follows from the Levi decomposition of parabolic subgroups (see [7, Section 8.5]). We state it in its most general form, as given and proved in [21, Proposition 24.21]

Lemma 2.2. *Let Δ be a spherical Moufang building and let F be a simplex of Δ of type J . Let P_F be the stabiliser of F in $\text{Aut}^\dagger(\Delta)$. Then the unipotent radical $U_F \leq P_F$ acts sharply transitively on the set $F^\equiv$ of simplices opposite F , and pointwise fixes $\text{Res}_\Delta(F)$.*

We will be interested in the faithful permutation group induced by L_F on $\text{Res}_\Delta(F)$.

2.4 Main results

Theorem A. *Let F be a simplex of a Moufang spherical building Δ . Let $\text{Aut}^\dagger(\Delta)$ be the automorphism group of Δ generated by the root groups. Then $\Pi^+(F)$ is permutation equivalent to the faithful permutation group induced by the stabiliser $\text{Aut}^\dagger(\Delta)_F$ of F in $\text{Aut}^\dagger(\Delta)$ on the residue $\text{Res}_\Delta(F)$ of F in Δ .*

Going back to the case where Δ is a Moufang building of rank 2, the results in [32, Chapter 8] show that $n(\{1\}) = n(\{2\}) = 1$, whenever Δ is a so-called ‘‘Pappian polygon’’ (for a definition of the latter, see [32, Section 3.5]). In any case, we always have $1 \in \{n(\{1\}), n(\{2\})\}$ due to [32, Lemma 8.4.6]. One of the goals of the present paper is to generalise this to all spherical buildings. This will be achieved by proving a general sufficient condition in J for $n(J)$ being equal to 1. To state this, we say that the type J of a simplex is *polar closed* if we can order the elements of a partition of J into singletons and pairs, say $J_1, \dots, J_k$, such that, for each $\ell \in \{1, \dots, k\}$, the type J_ℓ is a polar type in the residue of $J_1 \cup \dots \cup J_{\ell-1}$. We then have the following theorem.

Theorem B. *Let Δ be a spherical building with type set I . If either $J \neq J^\equiv$ or $J \subseteq I$ is polar closed, then $n(J) = 1$.*

To see a partial converse of this statement, we restrict to the simply laced case (see also Remark 8.34).

Theorem C. *Let Δ be an irreducible spherical building of simply laced type with type set I . If $J \subseteq I$, $J^\equiv = J$ and $I \setminus J$ has at least one connected component K of size at least 2 such that $I \setminus K$ is not polar closed, then $n(J) = 2$.*

Note that if J is polar closed, then for each connected component K of $I \setminus J$, the type set $I \setminus K$ is polar closed.

This implies the following combinatorial characterisation of the polar type in connected simply laced spherical diagrams. For $K \subseteq I$, we denote by $\overline{K}$ the union of all connected components of K of size at least 2.

Corollary 1. *The polar type of a connected simply laced spherical diagram D_I over the type set I is the unique smallest subset $J \subseteq I$ with the property that opposition in $D_{I \setminus \overline{J}}$ coincides with opposition in D_I .*

Corollary 1 does not hold in the non-simply laced case (since opposition does not determine the direction of the arrow in the Dynkin diagram). Indeed, for types B_n , C_n and F_4 , there are each time two single types satisfying the given condition, reflecting the fact that, in characteristic 2, there are really two choices.

Finally, we consider the case left out in Theorem C above, where $I \setminus J$ has only connected components of rank 1. We reduce the action of $\Pi^+(F)$ on each panel to a case where $|I \setminus J| = 1$ and show the following theorem.

Theorem D. *Let Δ be an irreducible spherical building of simply laced type with type set I . Let $J \subseteq I$ with $|I \setminus J| = 1$, and let P be a panel of type J . Then $\Pi^+(P)$ is permutation equivalent to the natural action of $\mathrm{PGL}_2(\mathbb{K})$ on the projective line $\mathrm{PG}(1, \mathbb{K})$, and equals $\Pi(P)$.*

In view of Theorem D, one could expect that the general and special projectivity groups of simplices, whose residue is isomorphic to $A_r(\mathbb{K})$, are isomorphic to $\mathrm{PGL}_{r+1}(\mathbb{K})$. This is indeed in most cases true, but not always. If it is not true, then necessarily the residue in question is not contained in a larger residue of type A_{r+1} . Our last main result determines the exact permutation representations of the special and general projectivity groups on the corresponding residues of the building.

Theorem E. *Let Δ be an irreducible spherical building of simply laced type with type set I . Let $I \neq J \subseteq I$ with $I \neq I \setminus J$ connected and let F be a simplex of type J . Then $\Pi^+(F)$ and $\Pi(F)$ are*

- (i) *isomorphic to $\mathrm{PGL}_n(\mathbb{L})$ in its natural action; if Δ has type A_r , $r \geq 2$, it is defined over the skew field $\mathbb{L}$, and $|I \setminus J| = n - 1$;*
- (ii) *as in Table 1 and Table 2 for $\mathrm{typ}(\Delta) \in \{D_r, E_m \mid r \geq 4, m = 6, 7, 8\}$.*

The notation used in Tables 1 and 2 is explained in Section 8, where Theorem E is proved.

2.5 Lie incidence geometries

Some arguments – in particular those in Section 8 – will be more efficiently carried out in a specific point-line geometry related to the spherical building in question. We provide a brief introduction here. More details can be found in textbooks like [4, 24].

Point-line geometries, projective spaces, polar spaces and parapolar spaces

Recall that a *point-line* geometry $\Gamma = (X, \mathcal{L})$ consists of a set X , whose elements are called *points*, and a subset $\mathcal{L}$ of the full set of subsets of X , whose members are called *lines* (hence we disregard geometries with so-called repeating lines). The notion of *collinear points* will be used frequently. We denote collinearity of two points x and y with $x \perp y$, and $x^\perp$ has the usual meaning of the set of points collinear to x (including x if there exists a line containing x). A (*proper*) *subspace* is a (*proper*) subset of the point set intersecting each line in either 0, 1 or all of the points of the line. A (*proper*) *hyperplane* is a (*proper*) subspace intersecting each line non-trivially. The *point graph* of Γ is the graph with vertices the points, adjacent when collinear. A subspace is *convex* if its induced subgraph in the point graph is convex (all vertices on paths of minimal length between two vertices of the subspace are contained in the subspace). We will frequently regard a subspace as a subgeometry in the obvious way. A subspace is called *singular* if every pair of points in it is collinear. In our cases, singular subspaces will always be projective spaces. Lines and planes are short for 1- and 2-dimensional projective (sub)spaces, respectively.

The *distance* between points is the distance in the point graph and the *diameter* of the geometry is the diameter of the point graph.

We usually require that Γ is *thick*, that is, each line contains at least three points.

For example, the 1-spaces of any vector space V of dimension at least 3 over some skew field $\mathbb{L}$ form the point set of a thick geometry $\text{PG}(V)$, a generic line of which consists of all the 1-spaces contained in a given 2-space. This geometry is a projective space. The hyperplanes correspond to the codimension 1 subspaces of V .

A *polar space* is a thick point-line geometry such that, for each point x , the set $x^\perp$ is a hyperplane (which we require to be distinct from the whole point set).

A pair of points of a point-line geometry Γ is called *special* if they are not collinear and there is a unique point of Γ collinear to both. Then Γ is called a *para-*

polar space if every non-special pair of points at distance at most 2 is contained in a convex subspace isomorphic to a polar space. Such convex subspaces are called *symplecta*, or *symps* for short. A pair p, q of non-collinear points of a symp is called *symplectic*, in symbols $p \perp\!\!\!\perp q$.

Given an irreducible spherical building Δ of rank r at least 2 of type X_r over the type set I , let $J \subseteq I$ and define X as the set of all simplices of Δ of type J . The set $\mathcal{L}$ of lines consists of the sets of simplices of type J completing a given panel, whose type does not contain J , to a chamber. The geometry $(X, \mathcal{L})$ is usually referred to as the *Lie incidence geometry of type $X_{r,J}$* (where we replace J by its unique element if $|J| = 1$). The main observation here (see the above references), usually referred to as *Cooperstein's theory of symplecta* [10, 11], is that $(X, \mathcal{L})$ is either a projective space, a polar space or a parapolar space.

In the present paper, we will only use projective spaces over arbitrary skew fields (they are related to buildings of type A_r), polar spaces (that are related to buildings of type D_r), some specific parapolar spaces that are related to buildings of types E_6 and E_7 over a field $\mathbb{K}$ and, at the end of this section, the thin exceptional long root subgroup geometries, which can be seen as parapolar spaces with special pairs of points and diameter 3. Polar spaces related to buildings of type D_r will usually be called *polar spaces of type D_r* , or *hyperbolic* polar spaces, since in rank $r \geq 4$, they are in one-to-one correspondence to hyperbolic quadrics in projective spaces. Recall that a *hyperbolic quadric* is the projective null set of a quadratic form of maximal Witt index in a vector space V of even dimension. The standard form (using coordinates $x_{-r}, \dots, x_{-1}, x_1, \dots, x_r$) is given by

$$x_{-r}x_r + x_{-r+1}x_{r-1} + \cdots x_{-2}x_2 + x_{-1}x_1.$$

The automorphisms of Δ induced by elements of $\mathrm{PGL}(V)$ will be called *linear*. They conform to the elements of the corresponding (maximal) linear algebraic group. Note that hyperbolic quadrics contain two natural classes of maximal singular subspaces, characterised by the fact that members of distinct classes intersect in subspaces of odd codimension (the *codimension* is the vector dimension of a complementary subspace).

Concerning types E_r , $r = 6, 7, 8$, we list some basic properties of the Lie incidence geometries of types $E_{6,1}$ and $E_{7,7}$, that we will make use of, in Section 8.5.

We end this section with the following lemma, whose proof makes use of thin exceptional long root subgroup geometries.

Lemma 2.3. *Let α be a root of an irreducible spherical Coxeter complex $\Sigma(W, S)$. Let F be the centre of α . Let v be a vertex joinable to a vertex u that is joinable to F . Then v lies in α .*

Proof. For the classical types A_n , B_n and D_n , this is easily verified. Indeed, for type A_n , viewing $\Sigma(W, S)$ as the point-line geometry of a projective space with two points per line, F is an incident point-hyperplane pair. Then v is either incident with the point or with the hyperplane, and the result follows. For types B_n and D_n , we view $\Sigma(W, S)$ as a polar space with two points per line. There are two possibilities: either F is a point, and then v is a subspace collinear to F and the assertion again follows, or F is a line, and then v is either a subspace collinear to F , or a subspace containing a point of F . In both cases, the assertion follows.

Now let $\Sigma(W, S)$ have exceptional type. The rank 2 case is easy to check, so we may assume the type is E_i , $i = 6, 7, 8$, F_4 or H_i , $i = 3, 4$. The latter cases are not essential for us, as they do not correspond to thick buildings and so we leave this to the reader. In the other cases, we use the representation of $\Sigma(W, S)$ as *thin long root subgroup geometry* Γ , that is, the geometry where points are the long roots of the corresponding root system, and lines (edges if one considers this geometry as a graph) given by pairs of roots making an angle of sixty degrees; see also [4]. Such geometries are depicted for all exceptional types in [34]. The advantage of this description is that F is a point of this geometry (for type F_4 , one has also to consider the same construction with short roots, which gives an isomorphic geometry). Also, α is induced by all points corresponding to roots making an angle of at most 90 degrees with the root corresponding to F , or in other words, points collinear or symplectic to F . Now, every vertex of $\Sigma(W, S)$ corresponds to either a singular subspace of Γ , to a symplecton of Γ or a convex subspace isomorphic to a Lie incidence geometry containing no special pairs and having diameter 2. It is now clear that if u and v are not both convex subspaces distinct from symplecta, then the assertion follows (as v is only incident with points collinear or symplectic to F). The only case where both u and v are convex subspaces occurs for type E_6 , where, up to duality, u is a vertex of type 1 and v of type 6 (the corresponding convex subspaces are geometries of type $D_{5,5}$). Since u and v are incident, they share a symp, and we may assume that symp is opposite F in the convex subspace u . Then one verifies that v contains four points collinear to F , eight points symplectic to F and four points special to F . This is a symmetric configuration with respect to F and its opposite point; hence u lies in $\partial\alpha$. The lemma is proved. $\square$

2.6 A connectivity theorem

We will also need the connectivity of the subgeometry of a Lie incidence geometry of type $E_{6,2}$, $E_{7,1}$ or $E_{7,3}$ induced by the points opposite two given points of the geometry. In an earlier version of the current paper, we proved this inside certain relevant parapolar spaces. The referee made us aware of a more general approach valid for all Lie incidence geometries defined in spherical buildings of simply laced

type. We present this approach here. Hence the aim of this subsection is to prove the following proposition, which more generally also holds for twin buildings of simply laced type, but since we did not define these, we do not insist.

Proposition 2.4. *Let Δ be an irreducible spherical building of simply laced type such that each panel is contained in at least four chambers. Let C, C' be two arbitrary chambers of Δ . Then the subgraph $\Gamma^{C,C'}$ of the chamber graph Γ induced on the set of chambers opposite both C and C' is connected.*

The proof we present is rather similar to the proof of [22, Theorem 5.1], which is essentially the case $C = C'$ of Proposition 2.4. So we first verify the latter for rank 2 residues.

Lemma 2.5. *Let Δ be a spherical building of type $A_1 \times A_1$ or A_2 such that each panel is contained in at least four chambers. Let C, C' be two arbitrary chambers of Δ . Then the subgraph $\Gamma^{C,C'}$ of the chamber graph Γ induced on the set of chambers opposite both C and C' is connected.*

Proof. For $C = C'$, this is straightforward; hence assume that $C \neq C'$.

For type $A_1 \times A_1$, the chamber graph is a grid, and hence the said subgraph is a subgrid, which is always connected.

Suppose now Δ stems from a projective plane with at least 4 points per line. First suppose each line has exactly 4 points. Then one verifies easily that if C and C' are adjacent, then $\Gamma^{C,C'}$ is the chamber graph of a 3×3 grid. If C and C' have distance 2 in the chamber graph, then one verifies that $\Gamma^{C,C'}$ is a cycle of length 18, hence connected. Finally, if C and C' are opposite, then $\Gamma^{C,C'}$ consists of four 3-cliques $\{a_{-2}, a_{-1}, a_0\}$, $\{a_0, a_1, a_2\}$, $\{b_{-2}, b_{-1}, b_0\}$ and $\{b_0, b_1, b_2\}$ and edges $\{c_i, b_i\}$ and $\{c_i, a_i\}$, $i \in \{-2, -1, 1, 2\}$.

So we may suppose that each line has at least five points. Let D, D' be two chambers both opposite to both C and C' . We observe that if D and D' have distance 2, then the unique chamber E adjacent to both D and D' is also opposite both C and C' . (This is most easily seen considering chambers as flags of the corresponding projective plane. Indeed, then E consists of a point of either D and D' , and a line of either D' or D , respectively. Hence, since C is opposite both D and D' , its elements are not incident and do not coincide with any of the elements of D and D' , and hence neither with the elements of E .) Hence we may assume that D and D' are opposite. Let S be the set of chambers sharing their point with D and let S' be the set of chambers sharing their line with D' ; note that S and S' are opposite panels. For each chamber $B \in S$, there exists a unique chamber $B' \in S'$ at distance 2 in the chamber graph. Let C_B be the unique chamber adjacent to both B and B' . Then we observe (with a proof similar to our first observation above)

that any chamber opposite both D and D' is not opposite at most two members of $U = \{C_B \mid B \in S\}$. In particular, C is not opposite at most two members of U , and C' is not opposite at most two members of U . Consequently, there is at least one member of U opposite both C and C' and our first observation now implies that there is a path from D to D' inside $\Gamma^{C,C'}$. $\square$

We can now prove Proposition 2.4.

Proof of Proposition 2.4. Let D and D' be two chambers belonging to $\Gamma^{C,C'}$. Let γ be a path in $\Gamma^{C,C'}$ connecting D with D' such that, among all such paths, the minimal distance d from C' to any member of Γ is the highest, and the number n of chambers attaining that minimal distance is smallest. We claim that d is the diameter of Γ , which shows that γ is inside $\Gamma^{C,C'}$. Indeed, suppose there are elements of γ not opposite C' and let D_1 be the first chamber of γ having distance d to C' . Let D_0 be the element of γ preceding D_1 and D_2 the one following D_1 . Then our assumptions imply that $F = D_0 \cap D_1 \cap D_2$ has corank 2, and so $R := \text{Res}_\Delta(F)$ is a rank 2 building corresponding to either a generalised digon or a projective plane. Since all elements of γ are opposite C , the projection C_R of C onto R is opposite all of D_0, D_1, D_2 . Let C'_R be the projection of C' onto R . Let $n_i, i = 0, 1, 2$, be the distance in the chamber graph of R from C'_R to D_i . Since projections of chambers onto panels are unique, it is straightforward to find paths $\gamma_i, i = 1, 2$, of length $2 - n_i$ (if R corresponds to a generalised digon) or $3 - n_i$ (if R corresponds to a projective plane) connecting D_i with a chamber of R opposite C'_R . Then, using Lemma 2.5, we obtain a path in R connecting D_0 with D_2 having one chamber less at distance d_1 from C'_R , and all other chambers have distance at least $d_1 + 1$. Replacing D_0, D_1, D_2 by this path, we obtain a path in $\Gamma^{C,C'}$ with either higher minimal distance n , or fewer chambers at that minimal distance, a contradiction. The proposition is proved. $\square$

3 General observations and proof of Theorem A

We start this section with a simple, though important observation, used in both [20] and [32, Chapter 8], but not explicitly stated in either. We provide a proof for completeness.

Observation 3.1. *Let Δ be a spherical building over the type set I and let $J \subseteq I$ be self-opposite. Let F be a simplex of type J . Then $n(J) = 1$ if and only if the identity in $\Pi(F)$ can be written as the product of an odd number of perspectivities.*

Proof. If the identity in $\Pi(F)$ can be written as the product of an odd number of perspectivities, then by composing this product with any even projectivity, we

see that we can write any putative member of $\Pi(F) \setminus \Pi^+(F)$ as a product of an even number of perspectivities, that is, as a member of $\Pi^+(F)$, a contradiction. We conclude $\Pi^+(F) = \Pi(F)$ in this case.

Conversely, if $\Pi^+(F) = \Pi(F)$, then consider any odd projectivity θ . Our assumption implies that we can write θ^{-1} as an even projectivity. Composing those two products of perspectivities, we obtain the identity written as the product of an odd number of perspectivities. $\square$

We can now prove Theorem A.

Proof of Theorem A. (I) First we want to show that every even self-projectivity of $\text{Res}(F)$ is induced by a product of elations that stabilises F . In fact, we are going to show that any even projectivity

$$\theta: \text{Res}(F) \rightarrow \text{Res}(T)$$

that maps F to a simplex T is induced by an elation. Since self-projectivities are products of projectivities, it then follows that every even self-projectivity is induced by a product of elations that stabilises F .

So let $\theta: \text{Res}(F) \rightarrow \text{Res}(T)$ be an even projectivity that maps F to a simplex T . It suffices to prove the assertion for the case that θ is a product of two perspectivities. Then there exists a simplex R opposite both F and T such that $\theta = \text{proj}_T^R \circ \text{proj}_R^F$. Since Δ is Moufang, it follows with Lemma 2.2 that there exists an elation g , which maps F to T and fixes R pointwise. For an element f in $\text{Res}(F)$, f^g is exactly the projection of $\text{proj}_R(f)$ onto T , since elations preserve incidence. That means $g|_{\text{Res}(F)} = \text{proj}_T^R \circ \text{proj}_R^F$.

(II) Now let $g: \Delta \rightarrow \Delta$ be a central elation. Let c be the centre of any root corresponding to g . Let T be a simplex either containing c or joinable to it. Then, by Lemma 2.3, every vertex u of Δ joinable to T is contained in a root with centre c (consider an apartment containing $\{c\} \cup T$ and $\{u\} \cup T$). It follows that g fixes $\text{Res}(T)$ pointwise and moves a simplex F of the same type to a simplex F^g . First, we claim that the restriction $g|_{\text{Res}(F)}$ is an even projectivity from $\text{Res}(F)$ to $\text{Res}(F^g)$.

Since Δ is Moufang, Δ is thick and therefore there exists a simplex R in Δ opposite both F and T . Since elations preserve incidence, the image R^g is opposite both F^g and $T^g = T$.

Now, for every $f \in \text{Res}(F)$, we have

$$f^g = \text{proj}_{F^g}^{R^g} \circ \text{proj}_{R^g}^T \circ \text{proj}_T^R \circ \text{proj}_R^F(f),$$

proving the claim. For an element h of the little projective group that stabilises F , h is a product of central elations and every such central elation gives rise to an even projectivity like above. $\square$

4 Projective spaces

In this section, we completely settle the case of type A_r regarding the number $n(J)$. The proof will also contain a warm up for a general statement we will prove later on; see Lemma 5.2. The main reason for treating this case separately is that we can provide an elementary proof only using projective geometry independent from building-theoretic notions (we do refer to Proposition 2.1, but this can easily be verified for projective spaces).

Theorem 4.1. *For buildings of type A_r with type set I and with $J \subseteq I$, we have $n(J) = 1$ if and only if either $J^\equiv \neq J$, or $|J| = 2k$ for some $k \leq \frac{r-1}{2}$ and*

$$J = \{1, 2, \dots, k, r - k + 1, r - k + 2, \dots, r\},$$

that is, J is polar closed.

Proof. First note that, for both the “if” and the “only if” parts, we may assume that J is self-opposite. First suppose $n(J) = 1$. Let F and F' be two opposite simplices of type J . Let $j \in I \setminus J$ be minimal with respect to the Bourbaki labelling of the diagram and let v be a vertex of type j incident to F . Since J is self-opposite, $j \leq \frac{r}{2}$. Then, according to Proposition 2.1, the type j' of $\text{proj}_{F'}^F(v)$ is the opposite type in $\text{Res}_\Delta(F')$ of type $r + 1 - j$ (which belongs to $I \setminus J$ since $I \setminus J$ is self-opposite). If $n(J) = 1$, we should have $j = j'$. This is only possible if the integer interval $[j, r + 1 - j]$ belongs to $I \setminus J$. Putting $k = j - 1$, we obtain the “only if” part of the statement.

Now we show the “if” part. We establish the identity projectivity as a product of three perspectivities. Let F be any simplex of type J . Suppose

$$F = \{U_i \mid i \in J, \dim U_i = i - 1\}.$$

Note that, since F is a simplex, $U_i \leq U_j$ for $i \leq j$, with $i, j \in J$. Select a simplex F' opposite F and set $F' = \{U'_i \mid i \in J, \dim U'_i = i - 1\}$. Choosing a basis $\{p_0, p_1, \dots, p_r\}$ well, we may assume

$$U_i = \langle p_0, \dots, p_{i-1} \rangle \quad \text{and} \quad U'_i = \langle p_r, p_{r-1}, \dots, p_{r-i+1} \rangle.$$

Let, for $0 \leq i \leq k - 2$, the point q_i be an arbitrary point on the line $\langle p_i, p_{r-i} \rangle$ distinct from both p_i and p_{r-i} . Define $U''_i = \langle q_0, \dots, q_{i-1} \rangle$ for $1 \leq i \leq k - 1$, and $U''_i = \langle U_{r-i+1}, p_{r-i+1}, \dots, p_{i-1} \rangle$. Then the simplex $F'' = \{U''_i \mid i \in J\}$ is easily checked to be opposite both F and F' . Let W be an arbitrary subspace of dimension k containing U_k and contained in U_{r-k+1} . Then W is generated by U_k and a point $p \in \langle p_k, \dots, p_{r-k} \rangle$. The point p belongs to $U'_{r-k+1} \cap U''_{r-k+1}$. Consequently, $\text{proj}_{F'}^F(W) = \langle U'_k, p \rangle =: W'$, $\text{proj}_{F''}^{F'}(W') = \langle U''_k, p \rangle =: W''$ and

$\text{proj}_F^{F''}(W'') = W$. This implies that $\text{proj}_F^{F''} \circ \text{proj}_{F''}^{F'} \circ \text{proj}_{F'}^F$ is the identity and, by Observation 3.1, the assertion is proved. $\square$

5 Proof of Theorem B

The following lemma is basically the gate property of buildings.

Lemma 5.1. *Let Δ be a spherical building over the type set I and let F_J be a simplex of type $J \subseteq I$. Let $K \subseteq J$ and let F_K be the face of F_J of type K . Let F_J^* be opposite F_J and let $F_K^* \subseteq F_J^*$ be opposite F_K . Set $F_J' := \text{proj}_{F_K}(F_J^*)$. Let $C \supseteq F_J$ be a chamber. Then*

$$\text{proj}_{F^*}(C) = \text{proj}_{F^*}(\text{proj}_{F_J'}(C)).$$

Proof. This follows from the gate property of residues. Since $F_J' = \text{proj}_{F_K}(F_J^*)$,

$$F_J' \subseteq \text{proj}_{F_K}(\text{proj}_{F_J^*}(C)).$$

The latter is on every minimal gallery joining $\text{proj}_{F_J^*}(C)$ with C and hence equals $\text{proj}_{F_J'}(C)$. The assertion follows. $\square$

In the next lemma, we use the following terminology. A triple of pairwise opposite simplices S_1, S_2, S_3 is called a *projective 3-cycle* if

$$\text{proj}_{S_1}^{S_3} \circ \text{proj}_{S_3}^{S_2} \circ \text{proj}_{S_2}^{S_1} = \text{id}.$$

Note that if the triple S_1, S_2, S_3 is a projective 3-cycle, then so is S_i, S_j, S_k , with (i, j, k) any permutation of $(1, 2, 3)$. Also, if S_1, S_2, S_3 form a projective 3-cycle, then they all have the same self-opposite type, say J , and projections between two opposite simplices of type J are type-preserving.

Lemma 5.2. *Let Δ be a spherical building over the type set I and let S_1, S_2, S_3 be a projective 3-cycle of type $J \subseteq I$. Let $K \subseteq I \setminus J$ be such that, for each pair of S_3 -opposite simplices $T_3, T_3' \in \text{Res}(S_3)$, there exists a simplex T_3'' such that T_3, T_3', T_3'' is a projective 3-cycle in $\text{Res}_\Delta(S_3)$. Then $n(J \cup K) = 1$. More exactly, if T_1 is a simplex of type K adjacent to S_1 , then there exist simplices $T_2' \sim S_2$ and $T_3'' \sim S_3$ of type K such that the triple $S_1 \cup T_1, S_2 \cup T_2', S_3 \cup T_3''$ is a projective 3-cycle.*

Proof. Let T_1 be a simplex of type K adjacent to S_1 . We want to write the identity in $\text{Res}_\Delta(S_1 \cup T_1)$ as the product of three projections.

Since S_1, S_2, S_3 is a projective 3-cycle,

$$\text{proj}_{S_2}^{S_1} T_1 = \text{proj}_{S_2}^{S_3} T_3,$$

where $T_3 = \text{proj}_{S_3}^{S_1} T_1$. Hence we have

$$T_2 = \text{proj}_{S_2}(T_1) = \text{proj}_{S_2}(T_3),$$

$$T_3 = \text{proj}_{S_3}(T_2) = \text{proj}_{S_3}(T_1),$$

$$T_1 = \text{proj}_{S_1}(T_3) = \text{proj}_{S_1}(T_2).$$

Let T'_2 be a simplex locally opposite T_2 at S_2 . Then, by Proposition 2.1, the simplices T_1 and T'_2 are opposite in Δ . Set $T'_3 = \text{proj}_{S_3}(T'_2)$. Then T'_3 is opposite T_2 in Δ (again by Proposition 2.1). Since $T_3 = \text{proj}_{S_3}(T_2)$, this implies, again using Proposition 2.1, that T'_3 is locally opposite T_3 at S_3 . Our assumption allows us to choose a simplex $T''_3 \sim S_3$ of type K such that T_3, T'_3, T''_3 is a projective 3-cycle in $\text{Res}_\Delta(S_3)$. Since, in particular, T''_3 is locally opposite both T_3 and T'_3 at S_3 , we have, like before (using Proposition 2.1), the following opposite relations:

$$T_1 \equiv T'_2 \equiv T''_3 \equiv T_1,$$

$$T_3 \equiv_{S_3} T'_3 \equiv_{S_3} T''_3 \equiv_{S_3} T_3.$$

Let v_1 be an arbitrary vertex adjacent to $S_1 \cup T_1$. We want to see that if we project v_1 first onto $S_2 \cup T'_2$, then onto $S_3 \cup T''_3$ and back to $S_1 \cup T_1$, then we get v_1 again. Define

$$v_2 := \text{proj}_{S_2 \cup T'_2}(v_1),$$

$$v_3 := \text{proj}_{S_3 \cup T''_3}(v_1) = \text{proj}_{S_3 \cup T''_3}(v_2),$$

$$v'_2 := \text{proj}_{S_2 \cup T'_2}(v_2); \text{ then } v'_2 = \text{proj}_{S_2 \cup T'_2}(v_3),$$

$$v'_3 := \text{proj}_{S_3}(v'_2); \text{ then } v'_3 \sim T'_3,$$

$$v''_3 := \text{proj}_{S_3 \cup T''_3}(v'_2).$$

We have that v_i is adjacent to T_i for $i \in \{1, 2, 3\}$, that v'_j is adjacent to T'_j for $j \in \{2, 3\}$ and that v''_3 is adjacent to T''_3 , since incidences are preserved under projection.

By Lemma 5.1, the projection of v'_2 from $S_2 \cup T'_2$ onto $S_3 \cup T''_3$ is the same as the projection onto T''_3 of the projection of v'_2 from S_2 onto S_3 and this is the same as $\text{proj}_{T''_3}(v'_3)$ (namely v''_3).

If we project v'_3 onto T_2 , we get the vertex v'_2 . If we project further onto T_2 , we get the vertex v_2 . The converse shows that v_3 maps to v'_3 under the projection locally at S_3 from T_3 to T'_3 .

Now the projection of v_1 onto $S_3 \cup T''_3$ is obtained by first projecting onto S_3 (and this is v_3), and then projecting v_3 locally at S_3 onto T''_3 . But since the triple T_3, T'_3, T''_3 is a projective 3-cycle, we have, locally at S_3 ,

$$\text{proj}_{T''_3}(v_3) = \text{proj}_{T''_3}(\text{proj}_{T'_3}(v_3)) = \text{proj}_{T''_3} v'_3 = v''_3,$$

which shows that the triple $S_1 \cup T_1, S_2 \cup T'_2, S_3 \cup T''_3$ is a projective 3-cycle. This concludes the proof of the lemma. $\square$

In view of Lemma 5.2, and in order to prove Theorem B, it suffices to show that, for any irreducible building Δ , there exists a triple of simplices of polar type which is a projective 3-cycle.

Proposition 5.3. *Let Δ be a spherical building. Let F and F' be two opposite simplices of polar type. Then F and F' are contained in a projective 3-cycle.*

Proof. Let C be a chamber containing F , let Σ be an apartment containing C and F' , let α be the root in Σ with centre F (and so containing C) and let $C' = \text{proj}_{F'}(C)$. Then F' is the centre of the opposite root $-\alpha$ of α in Σ . Let $\theta \in U_\alpha$ be a non-trivial root elation and set $F'' = F'^\theta$. Let $(C_0, C_1, \dots, C_\ell)$ be a minimal path in the chamber graph of Δ connecting $C = C_0$ with $C' = C_\ell$. By symmetry, $\ell = 2k$ is even and $C_0, \dots, C_k$ all belong to α , whereas $C_{k+1}, \dots, C_{2k}$ belong to $-\alpha$. The root $(-\alpha)^\theta$ has centre F'' and contains $C_{k+1}^\theta, \dots, C_{2k}^\theta =: C''$. Moreover, since θ fixes $\partial\alpha = \partial(-\alpha)$ pointwise, the union $(-\alpha) \cup (-\alpha)^\theta$ is an apartment and the chambers C_{k+1} and C_{k+1}^θ are adjacent. Hence F'' is opposite F' and $\delta(C, C') = \delta(C, C'') = \delta(C', C'')$. All this yields

$$\text{proj}_{F'}^{F''}(C'') = C'.$$

This shows that $\{F, F', F''\}$ is a projective 3-cycle. $\square$

Proof of Theorem B. If $J \neq J^\equiv$, then there are no odd self-projectivities and we have $n(J) = 1$. If J is the polar type, then $n(J) = 1$ by Proposition 5.3, and if J is polar closed, then $n(J) = 1$ by Lemma 5.2. $\square$

6 Proof of Theorem C

The following is a direct consequence of Observation 3.1.

Lemma 6.1. *Let $J \subseteq I$ be such that $n(J) = 1$ and $J = J^\equiv$. Then the opposition relation in $I \setminus J$ coincides with the restriction to $I \setminus J$ of the opposition relation in I .*

We then prove Theorem C by verifying that if J is not polar closed and $I \setminus J$ contains a connected component of rank at least 2, then for some connected component of $I \setminus J$, the opposition relation of that component does not coincide with the global opposition relation. We first treat the exceptional cases and then the infinite class of type D_n . The case A_n follows from Theorem 4.1.

6.1 Type E_6

Out of the $2^6 - 2 = 62$ possible types of a non-empty non-maximal simplex, there are exactly $2^4 - 2 = 14$ self-opposite ones. Out of these 14, there are precisely seven for which $I \setminus J$ has a connected component of rank at least 2. We present the possibilities pictorially, colouring the vertices of types in J black. For the other seven, $I \setminus J$ is the union of isolated vertices.

 and  are polar closed.

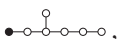
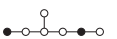
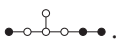
: According to Proposition 2.1, types 3 and 5 are interchanged by a perspectivity.

 and : Opposition in D_4 is trivial, whereas opposition in E_6 interchanges types 3 and 5.

 and : Opposition in E_6 interchanges the two rank 2 residues.

6.2 Type E_7

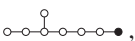
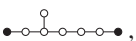
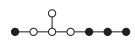
All of the $2^7 - 2 = 126$ possible types of non-empty non-maximal simplices are self-opposite, as opposition is trivial here. There are 18 polar closed types of which only three with a residue containing a connected component of rank at least 2. These components are of types D_4 and D_6 ; the three cases are

,  and .

Now, the only connected subdiagrams of size at least 2 admitting trivial opposition are precisely the ones of types D_4 and D_6 . The above choices for J are the only ones for which $I \setminus J$ has a connected component of size at least 2 admitting trivial opposition. In all other cases, it follows from Lemma 6.1 that $n(J) = 2$.

6.3 Type E_8

Here, opposition is also trivial. There are 19 polar closed types, but for only four of them, the corresponding residue contains a connected component of rank at least 2. These components are of types D_4 , D_6 and E_7 ; the four cases are

, ,  and .

There is actually exactly one more type with a residue of rank 4 admitting trivial opposition:



Here, the unique connected component $K = \{2, 3, 4, 5\}$ of $I \setminus J = \{2, 3, 4, 5, 8\}$ has the property that $I \setminus K = \{1, 6, 7, 8\} = \bullet\text{---}\circ\text{---}\bullet\text{---}\bullet$ is polar closed.

Since all other connected subdiagrams of size at least 2 are either of type $A_2, \dots, A_7, D_5, D_7$ or E_6 , we see that, for all other types J such that $I \setminus J$ has a connected component of size at least 2, we have $n(J) = 2$.

6.4 Type $D_n, n \geq 4$

Obviously, the only connected subdiagrams of size at least 2 of a diagram of type $D_n, n \geq 4$, where opposition agrees with the opposition in D_n are of type D_{n-2k} for $k \in \mathbb{N}$ such that $n - 2k \geq 3$. So a counterexample J to the assertion has

$$\max J = n - (n - 2k) = 2k$$

and the connected component K of size at least 2 of $I \setminus J$ is unique. Clearly, $I \setminus K$, which consists of the vertices of types $1, 2, \dots, 2k$, is polar closed (indeed, consider the ordering $2, 1; 4, 3; \dots; 2k, 2k - 1$).

7 Projectivity groups of panels – Proof of Theorem D

7.1 A basic lemma

The next lemma will enable us to pin down the special and general projectivity groups for residues which have the full linear group as respective projectivity group in a residue.

Lemma 7.1. *Let Δ be a spherical building over the type set I and let F_K be a simplex of type $K \subseteq I$. Let $K \subseteq J \subset I$ and let F_J be a simplex of type J containing F_K . Let $\Pi_K^+(F_J)$ be the special projectivity group of $F_J \setminus F_K$ in $\text{Res}_\Delta(F_K)$. Then $\Pi_K^+(F_J) \leq \Pi^+(F_J)$.*

Proof. Let F'_J and F''_J be two simplices containing F_K such that $F'_J \setminus F_K$ is opposite both $F_J \setminus F_K$ and $F''_J \setminus F_K$ in $\text{Res}_\Delta(F_K)$. We have to show that the product of the two perspectivities in $\text{Res}_\Delta(F_K)$ from $\text{Res}_\Delta(F_J)$ to $\text{Res}_\Delta(F'_J)$, then to $\text{Res}_\Delta(F''_J)$ coincides with the product of two perspectivities in Δ . To that aim, let F_K^* be a simplex in Δ opposite F_K , and let F_J^* be the projection of F'_J onto F_K^* ; hence

$$F_J^* = \text{proj}_{F_K^*}^{F'_J}(F'_J).$$

Let C be any chamber containing F_J . Set

$$C' = \text{proj}_{F'_J}(C),$$

$$C'' = \text{proj}_{F''_J}(C') = \text{proj}_{F''_J}(\text{proj}_{F'_J}(C)),$$

$$C^* = \text{proj}_{F_J^*}(C').$$

Then, according to Lemma 5.1, we have

$$C^* = \text{proj}_{F_J^*}(C) \quad \text{and} \quad C'' = \text{proj}_{F_J''}(C^*),$$

which implies that C'' is indeed equal to the image of C under the product of two perspectivities in Δ . $\square$

Recall that an automorphism of a spherical building Δ of simply laced type is called *linear* if it belongs to $\text{PGL}_{r+1}(\mathbb{L})$ in case Δ corresponds to $\text{PG}_r(\mathbb{L})$, or if it belongs to the linear algebraic group corresponding to the building if Δ has type D_r , $r \geq 4$, or E_6, E_7, E_8 . (For a more precise definition using the corresponding Chevalley group, see Section 8.1.) The next result is an immediate consequence of Lemma 7.1.

Corollary 7.2. *Let Δ be a spherical building over the type set I and let F_K be a simplex of type $K \subseteq I$. Let $K \subseteq J \subset I$ and let F_J be a simplex of type J containing F_K . Let $\Pi_K^+(F_J)$ be the special projectivity group of $F_J \setminus F_K$ in $\text{Res}_\Delta(F_K)$. Suppose that $\Pi_K^+(F_J)$ is the full linear type-preserving automorphism group of $\text{Res}_\Delta(F_J)$. Then $\Pi^+(F_J)$ also coincides with the full linear type-preserving automorphism group of $\text{Res}_\Delta(F_J)$.*

7.2 End of the proof

Now Theorem D follows from Corollary 7.2 because every vertex of the Coxeter diagram of a simply laced irreducible spherical building of rank at least 3 is contained in a residue isomorphic to the building of a projective plane over some skew field $\mathbb{L}$, and in a projective plane, the special projectivity group of a line is $\text{PGL}_2(\mathbb{L})$ acting naturally on $\text{PG}(1, \mathbb{L})$.

8 General and special projectivity groups of irreducible residues of rank at least 2

In this section, we determine the exact projectivity groups for irreducible residues. We start with the algebraic approach to the special projectivity groups.

8.1 The special projectivity groups

The arguments in this section were generously suggested to us by the referee.

Let Δ be an irreducible spherical Moufang building with associated Coxeter system (W, S) . Let C be a chamber and let Σ be an apartment containing C . For each $J \subset S$, let $\bar{J} := S \setminus J$ and let R_J be the J -residue containing C . Furthermore, let R'_J denote the unique residue of Σ that is opposite R_J .

For each s , let T_s denote the group of all automorphisms of Δ stabilising $\Sigma \cup R_{\bar{s}}$ pointwise. The groups T_s , $s \in S$, obviously normalise each other and we put

$$T := \langle T_s \mid s \in S \rangle \leq \text{Aut}^\circ(\Delta).$$

For a root α in Σ , let U_α be the root group associated with α . As usual, we denote by $-\alpha$ the opposite root.

Furthermore, let α_s be the root of Σ containing C such that R_s is on the boundary of α_s , and let $L_s := \langle U_{\alpha_s}, U_{-\alpha_s} \rangle$.

We abbreviate

$$G^\dagger := \text{Aut}^\dagger(\Delta) = \langle U_\alpha \mid \alpha \text{ is a root of } \Sigma \rangle \leq \text{Aut}^\circ(\Delta).$$

Let $H \leq G^\dagger$ be the pointwise stabiliser of Σ in $G^\dagger$ and let $H_s := H \cap L_s$ for $s \in S$. Again, the H_s , $s \in S$, normalise each other and $H = \langle H_s \mid s \in S \rangle$.

We now restrict to the case of Chevalley groups over fields, that is, $G^\dagger$ is a Chevalley group over a field $\mathbb{K}$, and we assume that the corresponding spherical building Δ is irreducible.

For each $s \in S$, the group T_s acts regularly on $R_s \setminus \Sigma$, which yields an isomorphism $\theta_s: \mathbb{K}^* \rightarrow T_s$ given by $\lambda \mapsto \theta_s(\lambda)$. These isomorphisms provide a canonical isomorphism $T \cong (\mathbb{K}^*)^{|S|}$.

Also, for each $s \in S$, there is a canonical homomorphism $h_s: \mathbb{K}^* \rightarrow T$ given by $\lambda \mapsto h_s(\lambda)$ whose image is H_s .

Note that $H \leq T$. By [25, Lemma 27], the group H corresponds to the root lattice of $G^\dagger$ and, if the root lattice coincides with the weight lattice of $G^\dagger$, then the groups T and H coincide (see for instance [8, §1.11]). In general, the group $\hat{G} = G^\dagger T$ is the group belonging to the type-preserving linear algebraic group corresponding to the building Δ (see previous section).

Let $J \subseteq S$ (and we may think of J as corresponding to a connected subdiagram of the Coxeter diagram of Δ). Let L_J be the corresponding Levi subgroup, that is, L_J is the stabiliser in $G^\dagger$ of $R_J \cup R'_J$. By Theorem A, $\Pi^+(R_J)$ is the image of L_J in $\text{Aut}^\circ(R_J)$. Setting $T_J := L_J \cap T$, we find $L_J = \text{Aut}^\dagger(R_J)T_J$. So, to obtain $\Pi^+(R_J)$, we have to determine T_J . This boils down to understanding the action of $h_s(\mathbb{K}^*)$ on R_J for all $s \in S \setminus J$ (for $s \in J$, this action is already in $\text{Aut}^\dagger(R_J)$). If $s \in S \setminus J$ is not connected to J in the Coxeter diagram, then the action of H_s on R_J is trivial, as both U_{α_s} and $U_{-\alpha_s}$ pointwise fix R_J , and H_s is inside L_s . If $s \in S \setminus J$ is connected to a vertex $j \in J$ (and we write $s \sim j$), then j is uniquely determined. We claim that the action induced by H_s on R_J is the action of T_j . Indeed, since by uniqueness of $j \sim s$, both H_s and T_j pointwise fix $R_{J \setminus \{j\}}$, since T_j acts faithfully on R_j , and $H_s \leq L_s$, it suffices to look in $R_{\{s,j\}}$, which is a projective plane. We may choose coordinates such that R_j corresponds to the line $(*, *, 0)$ and R_s to the line pencil with vertex the point $(1, 0, 0)$ (thus

$C \cap R_{\{s,j\}} = \{(1, 0, 0), (*, *, 0)\}$. Then

$$H_s/R_{\{s,j\}} = \left\{ \begin{pmatrix} 1 & & \\ & k & \\ & & k^{-1} \end{pmatrix} \middle| k \in \mathbb{K}^\times \right\},$$

$$T_j/R_{\{s,j\}} = \left\{ \begin{pmatrix} \ell & & \\ & 1 & \\ & & 1 \end{pmatrix} \middle| \ell \in \mathbb{K}^\times \right\}.$$

Restricting to the first two coordinates and putting $\ell = k^{-1}$, the claim follows. Hence $\Pi^+(R_J)$ is generated by $\text{Aut}^\dagger(R_J)$ and all T_j , $j \in J$, such that there exists $s \in S \setminus J$ with $s \sim j$. It remains to determine $\text{Aut}^\dagger(R_J)T_j$ for these values of $j \in J$ (and the groups they generate for suitable different j). These follow from straightforward computations in the corresponding Chevalley groups. Let us explain the main examples such that the reader can verify the tables.

We first introduce some notation.

Notation 8.1. For each $a \in \mathbb{N}$, let

$$\text{PSL}_n(\mathbb{K}, a) := \{M \in \text{GL}_n(\mathbb{K}) \mid \det M = k^a, k \in \mathbb{K}\} \cdot \text{Sc}_n(\mathbb{K}) / \text{Sc}_n(\mathbb{K}),$$

where $\text{Sc}_n(\mathbb{K})$ is the group of all scalar matrices over $\mathbb{K}$. We get $\text{PGL}_n(\mathbb{K})$ by putting $a = 1$ and $\text{PSL}_n(\mathbb{K})$ by putting $a = n$. Note that we can always choose a as a divisor of n since $\text{PSL}_n(\mathbb{K}, a) = \text{PSL}_n(\mathbb{K}, g)$, with $g = \gcd(a, n)$.

(1) Let R_J be the building of a projective space $\text{PG}(d, \mathbb{K})$. We relabel

$$J = \{1, 2, \dots, d+1\}$$

according to Bourbaki [3]. With respect to the standard chamber in $\text{PG}(d, \mathbb{K})$, we then have

$$T_j = \left\{ \begin{pmatrix} kI_j & & \\ & I_{d-j+1} \end{pmatrix} \middle| k \in \mathbb{K}^\times \right\},$$

where I_ℓ is the $\ell \times \ell$ identity matrix. It follows that

$$\text{PSL}_{d+1}(\mathbb{K})T_j = \text{PSL}_{d+1}(\mathbb{K}, j) = \text{PSL}_{d+1}(\mathbb{K}, g),$$

with $g = \gcd(d+1, j)$. This has the following consequences.

- (i) If some endpoint of the subdiagram of the diagram of Δ corresponding to R_J has a neighbour outside J , then $\Pi^+(R_J)$ is $\text{PGL}_{d+1}(\mathbb{K}) = \hat{G}$.

- (ii) If $j \in J$ corresponds to a Bourbaki label that is relatively prime to $d + 1$, and it is connected to a vertex outside J , then again, we have

$$\Pi^+(R_J) = \mathrm{PGL}_{d+1}(\mathbb{K}) = \hat{G}.$$

(Note (i) is a special case of this.)

- (iii) There are precisely three cases which are not applicable to either (i) or (ii). The first one is $J = \{1, 2, \dots, n-1\}$ in a building of type D_n , with n even. Here, $j = n-2$ and we get $\mathrm{PSL}_n(\mathbb{K}, 2)$. The second one is $J = \{1, 3, 4, 5, 6\}$ in $E_6(\mathbb{K})$. Here, node 4 is joint to a vertex outside J , namely 2. The former node has in Bourbaki labelling for A_5 label 3 and hence

$$\Pi^+(R_J) = \mathrm{PSL}_6(\mathbb{K}, 3).$$

The third situation is when $J = \{2, 4, 5, 6, 7\}$ in $E_7(\mathbb{K})$. Here, the second node is joint to an outside node, leading to $\Pi^+(R_J) = \mathrm{PSL}_6(\mathbb{K}, 2)$.

(2) Let R_J be the building of type D_n over $\mathbb{K}$. Due to the existence of a branching vertex of valency 3 in the Dynkin diagram, only the end vertices can be joint to vertices outside this diagram when viewed as a subdiagram of the Dynkin diagram of Δ . So we are only interested in $G^\dagger T_1$, $G^\dagger T_{n-1}$, $G^\dagger T_n$ and the groups generated by any combination of those. Referring forward to Notations 8.8 and 8.13, it follows from Lemma 8.14 that $G^\dagger T_1 = \mathrm{PGO}_{2n}^\circ(\mathbb{K})$, $G^\dagger T_n T_{n-1} = \overline{\mathrm{PGO}}_{2n}^\circ(\mathbb{K}) = \hat{G}$ (but $T_n = T_{n-1}$ if n is odd), and

$$G^\dagger T_n = G^\dagger T_{n-1} = \overline{\mathrm{P}\Omega}_{2r}(\mathbb{K}) \quad \text{if } n \text{ is even.}$$

Lemma 8.14 (iii) now implies $G^\dagger T_1 T_n = \overline{\mathrm{PGO}}_{2n}^\circ(\mathbb{K}) = \hat{G}$. All this implies that there are exactly two cases where $\Pi^+(R_J)$ is not the full linear group $\hat{G}$.

- (i) Δ is the building of type D_{n+1} over $\mathbb{K}$ and $S \setminus \{J\} = \{1\}$. Then the above implies $\Pi^+(R_J) = \mathrm{PGO}_{2n}^\circ(\mathbb{K})$.
- (ii) Δ is the building of type E_7 over $\mathbb{K}$ and $J = \{2, 3, 4, 5, 6, 7\}$. It follows from the above immediately that $\Pi^+(R_J) = \overline{\mathrm{P}\Omega}_{12}(\mathbb{K})$.

(3) Let R_J be the building of type E_6 over $\mathbb{K}$. We are only interested in $G^\dagger T_1$, which equals $\hat{G}$ by Lemma 8.19. Note that, in this case, the geometric approach obtains $\Pi^+(R_J) = \hat{G}$ in a quite different way, namely using Lemma 8.18 and symplectic polarities.

(4) Let Δ be the building of type E_8 over $\mathbb{K}$. Then $\hat{G} = G^\dagger$, as the weight lattice and root lattice coincide. Hence, for every $J \subseteq S$, we have that $\Pi^+(R_J)$ is the full linear group.

We now proceed with the geometric proof (including the determination of the general projectivity groups) and start with some general results.

8.2 General considerations

The *fix set* of an automorphism ρ of a building Δ is the set of simplices fixed under ρ . Two automorphisms φ_1 and φ_2 are called *congruent* if their fix sets are isomorphic (using a type-preserving automorphism of the building). Conjugate (with respect to a type-preserving automorphism) automorphisms are examples of congruent automorphisms. Clearly, congruence is an equivalence relation and an equivalence class Π is called a *geometric set* of automorphisms. We are going to use this notion only for rather large fix sets. We will mention the examples that we will need in our proofs at appropriate places (see Examples 8.9, 8.15 and 8.17). Here, we provide two examples.

Example 8.2. Let U and U' be two complementary subspaces of the projective space $\text{PG}(r, \mathbb{K})$ (that is, U and U' generate the whole space and are disjoint). Then the set of non-trivial collineations pointwise fixing $U \cup U'$, together with all their conjugates, is a geometric set of automorphisms of $\text{PG}(r, \mathbb{K})$.

Example 8.3. The set of all symplectic polarities of a given projective space is a geometric set of dualities. (This follows from the fact that such polarities are characterised by the property that each point is mapped onto a hyperplane it is contained in.)

Lemma 8.4. *Let Δ be a spherical building over the type set I and let $J \subseteq I$ be a self-opposite type. Suppose that, for each quadruplet of simplices of type J , there exists a simplex of type J opposite all the given simplices. Let F, F', F'' be three pairwise opposite simplices of type J and denote by θ_0 the projectivity $F \bar{\wedge} F' \bar{\wedge} F'' \bar{\wedge} F$. Denote with $\Pi_3(F)$ the set of all self-projectivities of F of length 3 and suppose that $\Pi_3(F)$ is geometric. Then $\Pi(F) = \langle \Pi_3(F) \rangle$ and $\Pi^+(F) = \langle \theta_0^{-1} \theta \mid \theta \in \Pi_3(F) \rangle$.*

Proof. It is clear that the said groups are subgroups of the respective projectivity groups. Now we claim that every self-projectivity of F of length ℓ is the product of $\ell \bmod 2\mathbb{Z}$ members of $\Pi_3(F)$. First note that if F^* is a simplex of type J and $\theta: \text{Res}(F) \rightarrow \text{Res}(F^*)$ is an isomorphism, then $\theta \Pi_3(F^*) \theta^{-1} = \Pi_3(F)$, by the fact that $\Pi_3(F)$ is geometric.

Now let $F = F_0 \bar{\wedge} F_1 \bar{\wedge} \cdots \bar{\wedge} F_\ell = F$ be a self-projectivity of length ℓ . Suppose $\ell \geq 5$. Let G be opposite all of F_0, F_1, F_2 and F_3 . Denote $\theta_i: F_0 \bar{\wedge} G \bar{\wedge} F_i$ and $\rho_i = F_i \bar{\wedge} G \bar{\wedge} F_{i-1} \bar{\wedge} F_i, i = 1, 2, 3$. Then we have

$$F_0 \bar{\wedge} F_1 \bar{\wedge} F_2 \bar{\wedge} F_3 = \theta_1 \rho_1 \theta_1^{-1} \cdot \theta_2 \rho_2 \theta_2^{-1} \cdot \theta_3 \rho_3 \theta_3^{-1} \cdot \theta_3.$$

Hence we can replace $F_0 \bar{\wedge} F_1 \bar{\wedge} F_2 \bar{\wedge} F_3$ by the product of three members of $\Pi_3(F)$ and the projectivity $\theta_3 = F \bar{\wedge} G \bar{\wedge} F_3$ of length 2. So the claim will fol-

low inductively if we show it for $\ell = 4$, that is, in the above, we have the additional perspectivity $F_3 \bar{\wedge} F_0$. Hence we have, with the same notation, and denoting additionally $\rho_4 = F_0 \bar{\wedge} G \bar{\wedge} F_3 \bar{\wedge} F_0$, which belongs to $\Pi_3(F)$,

$$F_0 \bar{\wedge} F_1 \bar{\wedge} F_2 \bar{\wedge} F_3 \bar{\wedge} F_0 = \theta_1 \rho_1 \theta_1^{-1} \cdot \theta_2 \rho_2 \theta_2^{-1} \cdot \theta_3 \rho_3 \theta_3^{-1} \cdot \rho_4,$$

which is a product of four, hence an even number of, elements of $\Pi_3(F)$. Now the assertions are clear, noting that every product $\theta_1 \theta_2$ of members of $\Pi_3(F)$ can be written as the product $(\theta_0 \theta_1^{-1})^{-1} \cdot (\theta_0 \theta_2)$ of two automorphisms of the form $\theta_0 \theta$, where $\theta \in \Pi_3(F)$. $\square$

We will usually apply this lemma to the case where all members of $\Pi_3(F)$ are type-interchanging involutions, and so $\Pi^+(F)$ will also be the intersection of $\Pi(F)$ with the group of type-preserving collineations. It is precisely this method that provides an alternative “reason” for the special projectivity groups to be what they are versus the algebraic approach explained in Section 8.1.

In Lemma 8.4, there is the condition that we find a simplex opposite four given simplices. It is well known that one can find a chamber opposite two given chambers; see [30, Proposition 3.30]. We can generalise this so that the condition in Lemma 8.4 becomes automatic for buildings with thickness at least 5; for the simply laced case, this just means that the building is not defined over the fields $\mathbb{F}_2$ or $\mathbb{F}_3$.

We say that a building *has thickness at least t* if every panel is contained in at least t chambers. The following generalises [30, Proposition 3.30]. The proof is also a rather obvious generalisation.

Proposition 8.5. *If a spherical building has thickness at least $t + 1$, then there exists a chamber opposite t arbitrarily given chambers. In particular, there exists a vertex opposite t arbitrarily given vertices of the same self-opposite type.*

Proof. We will prove the claim by induction. First consider the case that $t = 2$. Then the condition that every panel is contained in at least $t + 1 = 3$ chambers is equivalent to Δ being a thick building and the assertion follows with [30, Proposition 3.30]. Now suppose $t > 2$. Suppose we know we can find a chamber opposite $t - 1$ given chambers. Let $C_1, \dots, C_{t-1}$ be $t - 1$ different chambers in Δ and let C_t be another chamber in Δ . Among all the chambers in Δ opposite to each C_i , $i \in \{1, \dots, t - 1\}$, let E be a chamber with maximal distance to C_t . Assume that C_t and E are not opposite. Then $\text{dist}(C_t, E) \neq \text{diam } \Delta$. Let Σ be an apartment containing both C_t and E . With [30, Proposition 2.41], it follows that there exists a face A of codimension 1 of E such that $E = \text{proj}_A(C_t)$.

As every panel is contained in $t + 1$ chambers, we can find a chamber E' having A as a face that is not equal to E and not equal to $\text{proj}_A(C_i)$ for $i \in \{1, \dots, t - 1\}$.

With [30, Proposition 3.19.7 and Lemma 2.30.7], it follows that

$$\begin{cases} \text{dist}(C_i, E') = \text{dist}(\text{proj}_A(C_i), C_i) + 1 = \text{dist}(C_i, E) \\ \quad = \text{diam}(\Delta) \quad \text{for } i \in \{1, \dots, t - 1\}, \\ \text{dist}(C_t, E') = \text{dist}(C_t, E) + 1. \end{cases}$$

That means E' is opposite to each C_i for $i \in \{1, \dots, t - 1\}$ and has a strictly greater distance to C_t than E . That contradicts the fact that E has maximal distance to C_t among the chambers opposite each C_i for $i \in \{1, \dots, t - 1\}$. It follows that C_t and E are opposite. $\square$

This proposition takes care of all situations where the field has order at least 4. Over $\mathbb{F}_2$, the projectivity groups will always be determined already by Theorem A. So it remains to deal with $\mathbb{F}_3$. In this case, we will prove in the situations we need and more generally that if the simply laced spherical building is defined over the finite field $\mathbb{F}_q$, then we can find a simplex opposite $q + 1$ given simplices of certain given types (see the next paragraphs).

8.3 Projective spaces

Here, Δ is a projective space over a skew field $\mathbb{L}$. We will show that the special projectivity groups of any irreducible residue of rank ℓ is isomorphic to $\text{PGL}_{\ell+1}(\mathbb{L})$. The general group always coincides with the special group, either because the type of the simplex is not self-opposite, or the type is polar closed.

Theorem 8.6. *Let Δ be a building of type A_r , defined over the skew field $\mathbb{L}$. Let F be any simplex such that $I \setminus \text{typ}(F)$ is connected in the Coxeter diagram (say of type A_ℓ). Then both $\Pi^+(F)$ and $\Pi(F)$ are permutation isomorphic to $\text{PGL}_{\ell+1}(\mathbb{L})$.*

Proof. Applying Theorem A and Corollary 7.2, it suffices to show that the stabiliser G of a hyperplane H of $\text{PG}(r, \mathbb{L})$ in $\text{PSL}_{r+1}(\mathbb{L})$ acts on H as $\text{PGL}_r(\mathbb{L})$. Let g be an arbitrary element of $\text{PGL}_r(\mathbb{L})$ acting on H . Then we can represent g with respect to an arbitrarily chosen basis B in H with an $r \times r$ matrix M . We have to find a member $g^* \in \text{PSL}_{r+1}(\mathbb{L})$ inducing g in H . We can extend B to a basis B^* of $\text{PG}(r, \mathbb{L})$ by adding one point $p_0 \notin H$ and a suitable unit point. Let d belong to the coset of the (multiplicative) commutator subgroup C of $\mathbb{L}^\times$ given by the Dieudonné determinant of M (see [17]). Then the block matrix $M^* := \begin{pmatrix} d^{-1} & 0 \\ 0 & M \end{pmatrix}$ represents a member g^* of $\text{PGL}_{r+1}(\mathbb{L})$ fixing p_0 , stabilising H and inducing g in H . Moreover, by the properties of the Dieudonné determinant, in particular

those established in the proof of [17, Theorem 1], the determinant of M^* is equal to the product of the coset $d^{-1}C$ and the coset $\det M$. By the definition of d , this product is exactly C , and so $g^* \in \mathrm{PSL}_{r+1}(\mathbb{L})$. The proof is complete. $\square$

8.4 Hyperbolic polar spaces

We first prove some lemmas. When we consider residues of vertices of type 1, that is, the points of the corresponding polar space, we will aim to apply Lemma 8.4. Proposition 8.5 already tells us that we can find a point opposite 4 arbitrarily given points if the underlying field has order at least 4. To handle the case with the field $\mathbb{F}_3$, we recall the following slightly more general results for hyperbolic quadrics, proved in [5].

Lemma 8.7. *If every line of a hyperbolic quadric Q of rank at least 3 contains exactly $s + 1$ points, then*

- (i) *there exists a point non-collinear to each point of an arbitrary set T of $s + 1$ (distinct) points, except if these points are contained in a single line, and*
- (ii) *if Q has even Witt index $2d$, then there exists a maximal singular subspace opposite each member of an arbitrary set T of $s + 1$ (distinct) maximal singular subspaces of common type, except if these maximal singular subspaces contain a common singular subspace of codimension 2 in each.*

Notation 8.8. For a hyperbolic quadric Q of Witt index r , associated to the quadratic form $g: V \rightarrow \mathbb{K}$ with associated bilinear form $f: V \times V \rightarrow \mathbb{K}$, we denote by $\mathrm{PGO}_{2r}(\mathbb{K})$ the group of all elements of $\mathrm{PGL}_{2r}(V)$ preserving f and g . The unique subgroup of index 2 preserving each class of maximal singular subspaces will be denoted by $\mathrm{PGO}_{2r}^\circ(\mathbb{K})$. Note that $\mathrm{PGO}_6^\circ(\mathbb{K})$ is isomorphic to $\mathrm{PSL}_4(\mathbb{K}, 2)$.

A *parabolic polarity* of Q is the involution fixing a given parabolic subquadric P of Witt index $r - 1$ and interchanging each two maximal singular subspaces of Q containing a common maximal singular subspace of P . Each parabolic polarity belongs to $\mathrm{PGO}_{2r}(\mathbb{K})$, as in V , it is given by

$$V \rightarrow V: v \mapsto v - \frac{f(v, w)}{g(w)}w$$

for some $w \in V$ with $g(w) \neq 0$.

Since a parabolic subquadric of Witt index $r - 1$ of a given hyperbolic quadric of Witt index r defines a geometric hyperplane, one deduces the following.

Example 8.9. The set of parabolic polarities of a given hyperbolic quadric is geometric.

The following lemma is a reformulation of [9, Theorem 1.5.1].

Lemma 8.10. *The group $\text{PGO}_{2r}(\mathbb{K})$ is generated by the parabolic polarities.*

Theorem 8.11. *Let Δ be the building (of rank $r \geq 4$) associated to a hyperbolic quadric Q of Witt index $r \geq 4$ over the field $\mathbb{K}$. Let F be a simplex of Δ such that $\text{Res}_\Delta(F)$ is irreducible. Then $\Pi(F)$, $\Pi^+(F)$ are given as in Table 1. In case (A*), the permutation group $\text{PGL}_r(\mathbb{K}, 2).2$ denotes the extension of $\text{PGL}_r(\mathbb{K}, 2)$ by a symplectic polarity acting on $\text{PG}(r-1, \mathbb{K})$ (and coincides with the group generated by all symplectic polarities). A long dash in the table in the column of $\Pi(F)$ means that $\text{typ}(F)$ is not self-opposite and so $\Pi(F)$ is trivially isomorphic to $\Pi^+(F)$; it should be read as a “ditto” symbol. Grey rows correspond to projectivity groups that are not necessarily the full linear groups.*

Ref.	$\text{Res}_\Delta(F)$	$\text{cotyp}(F)$	$\Pi^+(F)$	$\Pi(F)$
(A1)	A_1		$\text{PGL}_2(\mathbb{K})$	$\text{PGL}_2(\mathbb{K})$
(A3)	A_3	$\{r-2, r-1, r\}$	$\text{PGO}_6^\circ(\mathbb{K})$	$\text{PGO}_6(\mathbb{K})$
(A)	$A_\ell, 2 \leq \ell \leq r-2$	$\neq \{r-2, r-1, r\}$	$\text{PGL}_{\ell+1}(\mathbb{K})$	$\text{PGL}_{\ell+1}(\mathbb{K}).2$
(A*)	$A_{r-1}, r \in 2\mathbb{Z}$		$\text{PGL}_r(\mathbb{K}, 2)$	$\text{PGL}_r(\mathbb{K}, 2).2$
(A**)	$A_{r-1}, r \in 2\mathbb{Z} + 1$		$\text{PGL}_r(\mathbb{K})$	—
(D)	$D_{r-2\ell}, 4 \leq r-2\ell \leq r-1$		$\text{PGO}_{2r-4\ell}^\circ(\mathbb{K})$	$\text{PGO}_{2r-4\ell}^\circ(\mathbb{K})$
(D')	$D_{r-2\ell+1}, 4 \leq r-2\ell+1 \leq r-1$		$\text{PGO}_{2r-4\ell+2}^\circ(\mathbb{K})$	$\text{PGO}_{2r-4\ell+2}^\circ(\mathbb{K})$

Table 1. Projectivity groups in buildings of type D_r over $\mathbb{K}$

Proof. First we note that if $\mathbb{K} = \mathbb{F}_2$, then all groups are universal and adjoint (simple) at the same time, so the results follow from Theorem A. Hence we may assume $|\mathbb{K}| \geq 3$. For ease of notation and language, we will speak about plus-type and minus-type of the maximal singular subspaces of Q to distinguish the two different types (arbitrarily).

Also, case (A1) follows from Theorem D, whereas case (A) follows from Lemma 7.1 and Theorem 8.6. We now handle the other, less straightforward, cases.

Case (A).* Let M_1, M_2, M_3 be three mutual opposite maximal singular subspaces of plus-type. Let $p_1 \in M_1$ be arbitrary. The maximal singular subspace N through p_1 intersecting M_2 in a submaximal singular subspace (that is, a singular subspace

of dimension $r - 2$) intersects M_3 in a point p_3 , since N is necessarily of minus-type. Hence the maximal singular subspace of minus-type through p_3 intersecting M_1 in a submaximal singular subspace contains p_1 . This shows that the projectivity $M_1 \bar{\cap} M_2 \bar{\cap} M_3 \bar{\cap} M_1$ is a duality each point of which is absolute. Lemma 3.2 of [27] implies that it is a symplectic polarity. By conjugation, we can obtain every symplectic polarity of M_1 in this way. Applying Lemma 8.4 together with Lemma 8.7, case (A*) follows from Example 8.3 and the fact that the matrix corresponding to a symplectic polarity necessarily has square determinant (and every square can occur).

Case (A3). By Theorem A, every self-projectivity preserves the residual form, hence $\Pi(F) \leq \text{PGO}_6(\mathbb{K})$. Case (A*) for $r = 3$, together with Lemma 7.1 and the fact that $\text{PGO}_6^\circ(\mathbb{K})$ is isomorphic to $\text{PSL}_4(\mathbb{K}, 2)$, conclude this case.

*Case (A**).* By Theorem A, $\Pi^+(F)$ contains $\text{PSL}_r(\mathbb{K})$. Hence it suffices to show that $\Pi^+(F)$ contains an element of $\text{PGL}_r(\mathbb{K})$ whose corresponding matrix has arbitrary determinant.

Let M_1 and M_3 be two maximal singular subspaces of plus-type intersecting in a subspace U_{13} of dimension $r - 3$. Let M_2 be a maximal singular subspace opposite both M_1 and M_3 (then M_2 has plus-type). Let U_{24} be a subspace of M_2 of dimension $r - 3$ opposite U_{13} . Let L_1 be the unique line of M_1 collinear to U_{24} . Let L be an arbitrary line in M_1 joining a point $p_{13} \in U_{13}$ with some point $p_1 \in L_1$. Pick $p, p' \in L \setminus \{p_{13}, p_1\}$ and suppose $p \neq p'$ (this is possible as we assume $|\mathbb{K}| \geq 3$).

Let M be the maximal singular subspace of plus-type containing p and intersecting M_2 in a hyperplane. Denote $W = U_{24} \cap M$. Then W has dimension $r - 4$ and is collinear to L . The intersection of M and M_3 is a point q , as both have the same type. As both p and p_{13} are collinear to q , also p' is collinear to q . Hence p' is collinear to $\langle q, W \rangle$, and $\langle p', q, W \rangle$ is a singular subspace of dimension $r - 2$. Hence there is a unique maximal singular subspace M' of plus-type containing p', q and W . It obviously intersects M_1 in p' and M_3 in q . There is a unique maximal singular subspace M_4 containing U_{24} and intersecting M' in a hyperplane (and hence it is of minus-type). Now, with this set-up, one verifies that the projectivity $M_1 \bar{\cap} M_2 \bar{\cap} M_3 \bar{\cap} M_4 \bar{\cap} M_1$ pointwise fixes both U_{13} and L_1 , and maps p to p' . Choosing a basis in $U_{13} \cup L_1$, the matrix of such a homology in M_1 is diagonal of the form $\text{diag}(k, k, \ell, \ell, \dots, \ell)$, and the arbitrariness of p' implies that k and ℓ are also arbitrary. Set $r = 2s + 1$. Putting $k = \ell^{-s+1}$, we obtain the determinant $\ell^{-2s+2+2s-1} = \ell$. Since ℓ is arbitrary, the assertion follows.

Case (D'). First set $\ell = 1$, that is, $r - 2\ell + 1 = r - 1$ and F is just a point of the polar space or hyperbolic quadric Q . Let p_1, p_2, p_3 be three mutual opposite points. Since $p_1^\perp \cap p_2^\perp$ is a hyperbolic quadric of rank $r - 1$, we have that

$p_1^\perp \cap p_2^\perp \cap p_3^\perp$ is either a parabolic subquadric, or a degenerate quadric. In the latter case, $\{p_1, p_2, p_3\}^{\perp\perp}$ is a degenerate plane conic containing p_1, p_2, p_3 , and hence p_3 is collinear to either p_1 or p_2 , a contradiction. Consequently, we have that $p_1^\perp \cap p_2^\perp \cap p_3^\perp$ is a parabolic quadric and the projectivity $p_1 \bar{\wedge} p_2 \bar{\wedge} p_3 \bar{\wedge} p_1$ is a parabolic polarity. Clearly, every parabolic polarity of $\text{Res}_\Delta(p_1)$ can be obtained this way. Then Lemma 8.4, Lemma 8.7, Example 8.9 and Lemma 8.10 yield $\Pi(p_1) = \text{PGO}_{2r-2}(\mathbb{K})$ and $\Pi^+(p_1) = \text{PGO}_{2r-2}^\circ(\mathbb{K})$.

Now let ℓ be arbitrary (but of course $4 \leq r - 2\ell + 1 \leq r - 1$). Since the stabiliser of F in $\text{PGO}_{2r}(\mathbb{K})$ obviously preserves the residual form (in $\text{Res}_\Delta(F)$), we see that $\Pi^+(F)$ is a subgroup of $\text{PGO}_{2r-4\ell+2}^\circ(\mathbb{K})$, and hence coincides with it by Lemma 7.1 and the case $\ell = 1$. In order to show $\Pi(F) = \text{PGO}_{2r-4\ell+2}(\mathbb{K})$, we only need to exhibit a parabolic polarity as a self-projectivity in $\text{Res}_\Delta(F)$. This is done as in the previous paragraph for the case $\ell = 1$: choose three mutual opposite singular subspaces U_1, U_2, U_3 of dimension $2\ell - 1$ contained in a parabolic subquadric obtained from Q by intersecting Q in its ambient projective space with a subspace of dimension 4ℓ . Suppose also $U_1 \in F$. Then, as before, the projectivity $U_1 \bar{\wedge} U_2 \bar{\wedge} U_3 \bar{\wedge} U_1$ is a parabolic polarity of $\text{Res}_\Delta(F)$.

Case (D). This is completely similar to the case $\ell > 1$ of case (D'), noting that $\Pi^+(F)$ coincides with $\Pi(F)$ by Theorem B. $\square$

8.5 Exceptional cases

Also here, we first prove some lemmas. First we recall the following result from [6] in order to deal with the case of a field of order 3 for simplices of type 7 in E_7 .

Lemma 8.12. *If every line of a parapolar space Γ of type $E_{7,7}$ contains exactly $s + 1$ points, then there exists a point at distance 3 from each point of an arbitrary set T of $s + 1$ (distinct) points, except if these points are contained in a single line.*

Notation 8.13 (Similitudes – groups of type D_n). For a hyperbolic quadric Q of Witt index r , associated to the quadratic form $g: V \rightarrow \mathbb{K}$ with associated bilinear form $f: V \times V \rightarrow \mathbb{K}$, we denote by $\overline{\text{PGO}}_{2r}(\mathbb{K})$ the group of all elements of $\text{PGL}_{2r}(V)$ preserving f and g up to a scalar multiple. It is the complete linear (algebraic) group of automorphisms of Q , seen as a building of type D_r . The unique subgroup of $\text{PGO}_{2r}(\mathbb{K})$ of index 2 preserving each class of maximal singular subspaces will be denoted by $\overline{\text{PGO}}_{2r}^\circ(\mathbb{K})$. It is elementary to see that $\overline{\text{PGO}}_{2r}(\mathbb{K})$ is obtained from $\text{PGO}_{2r}(\mathbb{K})$ by adjoining the appropriate *diagonal automorphisms*, that is, if we assume g in standard form (after introducing coordinates)

$$\begin{aligned} g: \mathbb{K}^{2r} \rightarrow \mathbb{K}: (x_{-r}, x_{-r+1}, \dots, x_{-2}, x_{-1}, x_1, x_2, \dots, x_{r-1}, x_r) \\ \mapsto x_{-r}x_r + x_{-r+1}x_{r-1} + \dots + x_{-2}x_2 + x_{-1}x_1, \end{aligned}$$

then we adjoin the linear automorphisms of Q induced by

$$\begin{aligned} \varphi_k: \mathbb{K}^{2r} &\rightarrow \mathbb{K}^{2r}: (x_{-r}, x_{-r+1}, \dots, x_{-2}, x_{-1}, x_1, x_2, \dots, x_{r-1}, x_r) \\ &\mapsto (x_{-r}, x_{-r+1}, \dots, x_{-2}, x_{-1}, \\ &\quad kx_1, kx_2, \dots, kx_{r-1}, kx_r) \end{aligned}$$

for all $k \in \mathbb{K}^\times$ (and we may assume k is not a square as otherwise the given automorphism is already in $\text{PGO}_{2r}(\mathbb{K})$). We denote the commutator subgroup of $\text{PGO}_{2r}^\circ(\mathbb{K})$ by $\text{P}\Omega_{2r}(\mathbb{K})$. The latter is the simple group $\text{D}_r(\mathbb{K})$ of type D_r over the field $\mathbb{K}$ (see [18]). The group obtained from $\text{P}\Omega_{2r}(\mathbb{K})$ by adjoining the diagonal automorphisms as above is denoted by $\overline{\text{P}\Omega}_{2r}(\mathbb{K})$.

If r is even and $\mathbb{K}$ is not quadratically closed, then $\overline{\text{P}\Omega}_{2r}(\mathbb{K})$ does not coincide with $\overline{\text{PGO}}_{2r}^\circ(\mathbb{K})$ as we will demonstrate later (see Remark 8.32).

Let us define a *homology* of a hyperbolic quadric Q as in Notation 8.13 to be any automorphism of Q pointwise fixing two opposite maximal singular subspaces. The automorphisms $\varphi_k, k \in \mathbb{K}^\times$, above are homologies. If r is even, then there are two types of such according to which kind of maximal singular subspace is fixed pointwise (if r is odd, then one always pointwise fixes one maximal singular subspace of each type). We now have the following result, which can be proved using standard arguments like, but simpler than, those of Lemma 8.10 and Lemma 8.18.

Lemma 8.14. *Let Q be a (non-degenerate) hyperbolic quadric of Witt index r corresponding to the building of type D_r over the field $\mathbb{K}$. Then the following hold.*

- (i) *The set of all homologies generates $\overline{\text{PGO}}_{2r}^\circ(\mathbb{K})$.*
- (ii) *If r is even, then the set of homologies pointwise fixing two opposite maximal singular subspaces of only one given type generates $\text{P}\Omega_{2r}(\mathbb{K})$.*
- (iii) *If r is even, then the homologies pointwise fixing two opposite maximal singular subspaces of only one given type, and the elements of $\text{PGO}_{2r}^\circ(\mathbb{K})$ together generate $\overline{\text{PGO}}_{2r}^\circ(\mathbb{K})$.*
- (iv) *The set of all automorphisms fixing two opposite points p, q and pointwise fixing $p^\perp \cap q^\perp$ generates $\text{PGO}_{2r}^\circ(\mathbb{K})$.*

Let U and U' be two opposite maximal singular subspaces of a hyperbolic quadric. Then it is well known that every point not contained in either U or U' is contained in a unique line joining a point of U and a point of U' . One can use this property to deduce that if a non-trivial homology pointwise fixing $U \cap U'$ fixes a subspace S , then either $S \cap U$ and $S \cap U'$ generate S . This, in turn, implies the following.

Example 8.15. The set of all non-trivial homologies of a hyperbolic quadric pointwise fixing two opposite maximal singular subspaces of a given type, is geometric.

We now introduce some notation concerning the exceptional groups of type E_6 . There does not seem to be standard notation (some people use $\hat{E}$ and $\hat{E}$, others $SE_6(\mathbb{K})$ for some of the following groups). The following is partly based on [26].

Notation 8.16 (Groups of type E_6). Let V be a 27-dimensional vector space over the commutative field $\mathbb{K}$, written as the direct sum of three 1-dimensional subspaces and three 8-dimensional subspaces, each of them identified with a split octonion algebra $\mathbb{O}$ over $\mathbb{K}$. We thus write $V = \mathbb{K} \oplus \mathbb{K} \oplus \mathbb{K} \oplus \mathbb{O} \oplus \mathbb{O} \oplus \mathbb{O}$. Let $\mathbb{C}: V \rightarrow \mathbb{K}$ be the cubic form defined as

$$\mathbb{C}(x, y, z; X, Y, Z) = -xyz + xX\bar{X} + yY\bar{Y} + zZ\bar{Z} - (XY)Z - \overline{(XY)Z}.$$

Then we denote by $GE_6(\mathbb{K})$ the similitudes of $\mathbb{C}$, that is, the subgroup of $GL(V)$ preserving $\mathbb{C}$ up to a multiplicative constant. The subgroup of $GE_6(\mathbb{K})$ preserving $\mathbb{C}$ is denoted by $SE_6(\mathbb{K})$ (and is a subgroup of $SL(V)$) and the quotients with the respective centres (consisting of scalar matrices) are $PGE_6(\mathbb{K})$ and $PSE_6(\mathbb{K})$. The latter is also denoted briefly by $E_6(\mathbb{K})$ and is simple. The group $PGE_6(\mathbb{K})$ is the full linear group. The group obtained by adjoining a graph automorphism is denoted by $PGE_6(\mathbb{K}).2$.

The cubic form $\mathbb{C}$ above can also be written without the use of octonions, but using the unique generalised quadrangle $GQ(2, 4)$ of order $(2, 4)$, that is, polar space of rank 2 with 3 points on each line and 5 lines through each points. An explicit construction of $GQ(2, 4)$ runs as follows; see [23, Section 6.1]. Let $\mathcal{P}'$ be the set of all 2-subsets of the 6-set $\{1, 2, 3, 4, 5, 6\}$, and define

$$\mathcal{P} = \mathcal{P}' \cup \{1, 2, 3, 4, 5, 6\} \cup \{1', 2', 3', 4', 5', 6'\}.$$

Denote briefly the 2-subset $\{i, j\}$ by ij for all appropriate i, j . Let $\mathcal{L}'$ be the set of partitions of $\{1, 2, 3, 4, 5, 6\}$ into 2-subsets and define

$$\mathcal{L} = \mathcal{L}' \cup \{\{i, j', ij\} \mid i, j \in \{1, 2, 3, 4, 5, 6\}, i \neq j\}.$$

Then $\Gamma = (\mathcal{P}, \mathcal{L})$ is a model of $GQ(2, 4)$.

The sets $\{1, 2, 3, 4, 5, 6\}$ and $\{1', 2', 3', 4', 5', 6'\}$ have the property that they both do not contain any pair of collinear points, and that non-collinearity is a paring between the two sets. Such a pair of 6-sets is usually called a *double six*.

The following set $\mathcal{S}$ of lines of $\text{GQ}(2, 4)$ is a *spread*, that is, a partition of the point set $\mathcal{P}$ into lines:

$$\mathcal{S} = \{\{14, 25, 36\}, \{15, 26, 34\}, \{16, 24, 35\}, \{12, 2, 1'\}, \\ \{23, 3, 2'\}, \{13, 1, 3'\}, \{45, 4, 5'\}, \{56, 5, 6'\}, \{46, 6, 4'\}\}.$$

We now have the following equivalent description of the cubic form $\mathfrak{C}$; see [35, Section 2]. Let V be the vector space of dimension 27 over $\mathbb{K}$ where the standard basis B is labelled using the elements of $\mathcal{P}$, say $B = \{e_p \mid p \in \mathcal{P}\}$. We denote a generic vector $v \in V$ by $\sum_{p \in \mathcal{P}} x_p e_p$, with $x_p \in \mathbb{K}$. Then

$$\mathfrak{C}(v) = \sum_{\{p,q,r\} \in \mathcal{S}} x_p x_q x_r - \sum_{\{p,q,r\} \in \mathcal{L} \setminus \mathcal{S}} x_p x_q x_r.$$

The projective null set of $\nabla \mathfrak{C}$ is a set of points denoted $\mathcal{E}_6(\mathbb{K})$, and endowed with the lines contained in it, it is a point-line geometry isomorphic to the Lie incidence geometry of type $E_{6,1}$ over the field $\mathbb{K}$. There is a special type of graph automorphism, called a *symplectic polarity*, which is an involution centralising a split group of type F_4 over the same field $\mathbb{K}$. All symplectic polarities are conjugate (see [13]), and as a consequence of the main results of [33], every graph automorphism having a fix set isomorphic to the fix set of a symplectic polarity is a symplectic polarity. Hence we have the following example.

Example 8.17. The set of symplectic polarities of a type E_6 building is geometric.

With these constructions and notation at hand, we are able to prove the following generation results.

Lemma 8.18. *The products of an even number of symplectic polarities of the building Δ of type E_6 over the field $\mathbb{K}$ generate the Chevalley group $\text{PGE}_6(\mathbb{K})$. The symplectic polarities themselves generate $\text{PGE}_6(\mathbb{K})$.*

Proof. We first claim that diagonal automorphisms φ necessarily have a ninth power as determinant. Indeed, φ acts as

$$\varphi: V \rightarrow V: \sum_{p \in \mathcal{P}} x_p e_p \mapsto \sum_{p \in \mathcal{P}} \lambda_p x_p e_p,$$

with $\lambda_p \in \mathbb{K}$. Then φ is a similitude of $\mathfrak{C}$ only if the product $\lambda_p \lambda_q \lambda_r =: \lambda$ is a constant across all lines $\{p, q, r\} \in \mathcal{L}$. Then the determinant of φ is obtained by multiplying this constant over the spread $\mathcal{S}$ and hence the determinant equals λ^9 . The claim is proved.

Now a symplectic polarity σ of Δ induces a symplectic polarity in every fixed 5-space of the corresponding Lie incidence geometry Γ of type E_6 . Since all symplectic polarities are conjugate, every symplectic polarity of a given 5-space extends to a symplectic polarity of Δ . By the strong transitivity of $\text{Aut } \Delta$, we may even assume that two given opposite 5-spaces are fixed (and then, since the fix building has type F_4 and its polar type corresponds to the fixed 5-spaces (as is apparent from [13]), all 5-spaces in the span of the two given ones in $\text{PG}(V)$ are fixed). Now, two opposite 5-spaces W and W' are given by the span of the base points corresponding to the two respective 6-sets of a double six. It is easily seen that the product of the symplectic polarities corresponding to the symplectic forms

$$x_{-3}y_3 + x_{-2}y_2 + x_{-1}y_1 - x_1y_{-1}x_2y_{-2} - x_3y_{-3}$$

and

$$\lambda x_{-3}y_3 + x_{-2}y_2 + x_{-1}y_1 - x_1y_{-1}x_2y_{-2} - \lambda x_3y_{-3},$$

$\lambda \in \mathbb{K}$, corresponds to the diagonal collineation of $\text{PG}(5, \mathbb{K})$ with diagonal

$$(\lambda, 1, 1, 1, 1, \lambda).$$

Letting these coordinates correspond naturally to the bases

$$(e_1, \dots, e_6) \quad \text{and} \quad (e_{1'}, \dots, e_{6'})$$

of the subspaces of V corresponding to W and W' , respectively, we first derive that the product θ of the corresponding symplectic polarities of Δ acts on $\langle W, W' \rangle$ as

$$(x_1, x_2, \dots, x_6, x_{1'}, x_{2'}, \dots, x_{6'}) \\ \mapsto (\lambda x_1, x_2, x_3, x_4, x_5, \lambda x_6, \lambda x_{1'}, x_{2'}, x_{3'}, x_{4'}, x_{5'}, \lambda x_{6'}).$$

Secondly, since each point $\langle e_{ij} \rangle$ is the unique point of Γ collinear to all $\langle e_\ell \rangle$, except for $\langle e_i \rangle$ and $\langle e_j \rangle$, and to all $\langle e_{\ell'} \rangle$, except for $\langle e_{i'} \rangle$ and $\langle e_{j'} \rangle$ (as follows from [16, Lemma 3.5]), we see that θ is a diagonal automorphism. Now one easily calculates that θ is uniquely determined by its restriction to $\langle W, W' \rangle$ and maps e_{ij} to e_{ij} if $|\{i, j\} \cap \{1, 6\}| = 1$, to λe_{ij} if $\{i, j\} \cap \{1, 6\} = \emptyset$, and to $\lambda^{-1} e_{ij}$ if $\{i, j\} = \{1, 6\}$. Correspondingly, the determinant of θ is λ^9 . Now, clearly, the diagonal automorphisms generate non-trivial elements of $\text{PSE}_6(\mathbb{K})$. Since the latter is simple, and since the subgroup of $E_6(\mathbb{K})$ generated by all symplectic polarities of Δ is normal, the group generated by arbitrary products of an even number of symplectic polarities contains $\text{PSE}_6(\mathbb{K})$. Since it also contains all diagonal automorphisms by the above, the assertions follow. $\square$

Lemma 8.19. *With the notation of Section 8.1, we have $E_6(\mathbb{K}).T_1 = \text{PGE}_6(\mathbb{K})$.*

Proof. A chamber C of a building of type E_6 over $\mathbb{K}$ is given by a 6-tuple of pairwise incident elements of the Lie incidence geometry of type $E_{6,1}$ over $\mathbb{K}$ described above consisting of a point, a line, a plane, a 5-space, a 4-space intersecting the 5-space in a 3-space, and a hyperplane of the 5-space. In the above description, we can take, with obvious notation,

$$C = (\langle e_1 \rangle, \langle e_1, e_2 \rangle, \langle e_1, e_2, e_3 \rangle, \langle e_1, e_2, \dots, e_6 \rangle, \langle e, 1, \dots, e_4, e_{\{5,6\}} \rangle, \langle e_1, \dots, e_5 \rangle).$$

Denote $W = \langle e_1, \dots, e_6, e_{\{5,6\}} \rangle$. An arbitrary diagonal automorphism of W inducing an element of T_1 , that is, acting trivially on the rank 1 residues defined by C , except for the type 1 rank 1 residue, is given by

$$\begin{cases} e_1 \mapsto \lambda e_1, \\ e_i \mapsto e_i, & i = 2, 3, 4, 5, 6, \\ e_{\{5,6\}} \mapsto e_{\{5,6\}}. \end{cases}$$

It is easy to check that this can be extended to a unique diagonal automorphism preserving the cubic form $\mathbb{C}$ by defining

$$\begin{cases} e_{1'} \mapsto e_{1'}, \\ e_{i'} \mapsto \lambda^{-1} e_{i'}, & i = 2, 3, 4, 5, 6, \\ e_{\{i,j\}} \mapsto e_{\{i,j\}}, & 1 \notin \{i, j\} \subseteq \{1, 2, \dots, 6\}, i \neq j, \\ e_{\{1,j\}} \mapsto \lambda^{-1} e_{\{1,j\}}, & j = 2, 3, 4, 5, 6. \end{cases}$$

Now one calculates that the determinant of the corresponding diagonal matrix is λ^{-9} , and the result follows as in the proof of Lemma 8.18 above. $\square$

Lemma 8.4 and Lemma 8.12 can be used to determine $\Pi^+(F)$ and $\Pi(F)$ for simplices of type 7 in buildings of type E_7 . However, in general, the number of possibilities for triangles of mutually opposite simplices is too large to be practical or useful. The following result provides an alternative to Lemma 8.4. The condition that J is a self-opposite type is not essential, but convenient, and we will only need it in that case.

Notation 8.20. For a spherical building Δ of type X_n over I and a type set $J \subseteq I$, we denote by Γ_J the graph with vertices the simplices of type J , adjacent when contained in adjacent chambers. Adjacent vertices F and F' in Γ_J are denoted by $F \sim F'$. An alternative definition of Γ_J is that it is the point graph of the corresponding Lie incidence geometry of type $X_{n,J}$.

Lemma 8.21. *Let Δ be a spherical building over the type set I and let $J \subseteq I$ be a self-opposite type. Suppose that, for each pair of simplices F, F' of type J ,*

the subgraph $\Gamma_J^{\{F, F'\}}$ of Γ_J induced on the vertices opposite both F and F' is connected. Suppose also that there is a simplex of type J opposite any given set of three simplices of type J . Let F be a given simplex of type J . Denote by $\Pi_4(F)$ the set of all self-projectivities $F \bar{\wedge} F_2 \bar{\wedge} F_3 \bar{\wedge} F_4 \bar{\wedge} F$ of F of length 4 with $F \sim F_3$, $F_2 \sim F_4$. Suppose that $\Pi_4(F)$ is geometric. Then $\Pi^+(F) = \langle \Pi_4(F) \rangle$.

Proof. We first prove the following property for four simplices F_1, F_2, F_3, F_4 , where $\text{typ}(F_1) = \text{typ}(F_3) = J$ and both F_2 and F_4 are opposite both F_1 and F_3 .

Property (*). *The projectivity $\rho: F_1 \bar{\wedge} F_2 \bar{\wedge} F_3 \bar{\wedge} F_4$ can be written as a product of a perspectivity $F_1 \bar{\wedge} F_4$ and conjugates of members of $\Pi_4(F_1)$.*

Indeed, let $F_1 = F'_1 \sim F'_2 \sim \dots \sim F'_n = F_3$ be a path in $\Gamma_J^{\{F_2, F_4\}}$. Define

$$\rho_i: F_4 \bar{\wedge} F'_i \bar{\wedge} F_2 \bar{\wedge} F'_{i+1} \bar{\wedge} F_4, \quad i \in \{1, 2, \dots, n-1\}.$$

Denote by ρ_0 the perspectivity $F_1 \bar{\wedge} F_4$. Then it is elementary to see that

$$\rho = \rho_0 \rho_1 \rho_2 \cdots \rho_{n-1}.$$

So, since $\Pi_4(F)$ is geometric, it suffices to show that each ρ_i can be written as the product of conjugates of members of $\Pi_4(F_1)$. Letting $(F_4, F'_i, F_2, F'_{i+1})$ play the role of (F_1, F_2, F_3, F_4) in the previous argument, it follows that ρ_i is a product of conjugates of members of $\Pi_4(F'_{i+1})$. Hence Property (*) is proved.

Now let $\rho: F \bar{\wedge} F_2 \bar{\wedge} F_3 \bar{\wedge} \dots \bar{\wedge} F_{2\ell-1} \bar{\wedge} F_{2\ell} \bar{\wedge} F$ be an arbitrary even projectivity. We prove by induction on $\ell \in \{1, 2, \dots\}$ that ρ is the product of conjugates of members of $\Pi_4(F)$. This is trivial for $\ell = 1$ and it is equivalent to Property (*) for $\ell = 2$. So let $\ell \geq 3$. Select a simplex F'_2 opposite each of F, F_3 and F_5 (since these all have the same type, this is still possible if J is not self-opposite). Setting

$$\begin{cases} \rho_1^*: F \bar{\wedge} F_2 \bar{\wedge} F_3 \bar{\wedge} F'_2 \bar{\wedge} F, \\ \rho_2^*: F'_2 \bar{\wedge} F_3 \bar{\wedge} F_4 \bar{\wedge} F_5 \bar{\wedge} F'_2, \\ \rho': F \bar{\wedge} F'_2 \bar{\wedge} F_5 \bar{\wedge} F_6 \bar{\wedge} \dots \bar{\wedge} F_{2\ell} \bar{\wedge} F, \end{cases}$$

we see that if $\rho_0: F \bar{\wedge} F'_2$, we have $\rho = \rho_1^* \cdot (\rho_0 \rho_2^* \rho_0^{-1}) \cdot \rho'$, where we know by the induction hypothesis that all factors are products of members of $\Pi_4(F)$, using the fact that $\Pi_4(F)$ is geometric (and hence closed under conjugation). $\square$

This method of determining $\Pi^+(F)$ “explains” these groups in a way equivalent to the algebraic approach of Section 8.1.

In order to be able to apply Lemma 8.21, we have to check the conditions in the various cases. It turns out we will use Lemma 8.21 in exactly three different cases, for which we now note down the condition on the corresponding graph Γ_J .

Lemma 8.22. *Let Δ be the spherical building over the field $\mathbb{K}$, $|\mathbb{K}| > 2$, of type either E_6 or E_7 . Let $J = \{2\}$ if $\text{typ } \Delta = E_6$, and $J \in \{\{1\}, \{3\}\}$ if $\text{typ } \Delta = E_7$. Let v and v' be vertices of type J . Then the subgraph $\Gamma_J^{v,v'}$ of Γ_J induced on the vertices opposite both v and v' is connected.*

Proof. Select chambers C and C' of Δ containing v and v' , respectively. Let u be a vertex of $\Gamma_J^{v,v'}$. In $\text{Res}_\Delta(u)$, we can find a chamber opposite the projections of C and C' onto $\text{Res}_\Delta(u)$ (use [30, Proposition 3.30]). That chamber is, by Proposition 2.1, opposite both C and C' . Hence we have found a chamber C_u containing u opposite both C and C' . If u' is another vertex of $\Gamma_J^{v,v'}$, then we can also find a chamber $C_{u'}$ containing u' and opposite both C and C' . Now Proposition 2.4 implies that we can find a sequence of consecutively adjacent chambers, all opposite both C and C' , connecting C_u with $C_{u'}$. The vertices of type J of two consecutive such chambers are either equal, or adjacent in Γ_J . Moreover, since the chambers are opposite both C and C' , their vertices of type J are opposite both v and v' , as J is a self-opposite type. The lemma is proved. $\square$

Remark 8.23. It is clear that the previous lemma holds for all spherical buildings of simply laced type and self-opposite subset J of the types, with the same proof. If J is not self-opposite, then one has to consider the subgraph induced on the set of simplices of type J opposite two given simplices of opposite type of J .

Before we can determine in a geometric way the various projectivity groups in the exceptional buildings of simply laced type, we need some basic properties of Lie incidence geometries of types $E_{6,1}$ and $E_{7,7}$. Most of them can be read off the diagram, and others follow from considering an apartment of the building. They are called “facts” in papers like [15, 16]. For Lie incidence geometries of type $E_{6,1}$, good references are [13, 28], and for Lie incidence geometries of type $E_{7,7}$, a good reference is [14]. In both papers, the basic facts are explained in some more detail.

Lie incidence geometries of type $E_{6,1}$

These geometries have diameter 2 and contain no special pairs of points. Hence every pair of points is contained in a symp. Symps are polar spaces of type D_5 . The basic properties, which we shall use without notice, are summarised in the following lemma.

Lemma 8.24. *Let Γ be a Lie incidence geometry of type $E_{6,1}$ over a field $\mathbb{K}$. Then the following properties hold.*

- (i) *Two distinct symps either meet in a unique point, or share a maximal singular subspace, referred to as a 4-space.*

- (ii) For a point p and a symp ξ , with $p \notin \xi$, we either have $p^\perp \cap \xi = \emptyset$, or $p^\perp \cap \xi$ is a maximal singular subspace of ξ , referred to as a $4'$ -space.
- (iii) The 4-spaces in a given symp form one natural class of maximal singular subspaces of ξ ; the $4'$ -spaces form the other.

In the building, 4-spaces correspond to vertices of type 5, whereas $4'$ -spaces correspond to simplices of type $\{2, 6\}$.

We now mention some other facts. The first one can be read off the diagram. It is also contained as [15, Fact 4.14].

Lemma 8.25. *Let Γ be a Lie incidence geometry of type $E_{6,1}$ over a field $\mathbb{K}$. Then the following hold.*

- (i) A 4-space and a $4'$ -space that have a plane π in common intersect in a 3-space. Consequently, a 4-space and a 5-space that share a plane share a 3-space.
- (ii) Two distinct non-disjoint 5-spaces intersect in either a point or a plane. Consequently, a $4'$ -space that shares a 3-space with a 5-space is contained in it.
- (iii) Two disjoint 5-spaces that are not opposite contain respective planes contained in a common 5-space. Every point of each of the two 5-spaces is collinear to some point of the plane contained in the other 5-space.

Lemma 8.26. *Let Γ be a Lie incidence geometry of type $E_{6,1}$ over a field $\mathbb{K}$. Let ξ be a symp in Γ and let π be a plane in Γ intersecting ξ in a unique point x . Then there exists a unique plane $\alpha \subseteq \xi$, all points of which are collinear to all points of π .*

Proof. Let L be a line in π not intersecting ξ . The lemma follows from [16, Fact A.9]. □

Lie incidence geometries of type $E_{7,7}$

These geometries have diameter 3 and contain no special pairs of points. Points at distance 3 correspond to opposite vertices of type 7 in the corresponding building. Hence every pair of non-opposite points is contained in a symp. Symps are polar spaces of type D_6 . The basic properties, which we shall use without notice, are summarised in the following lemma.

Lemma 8.27. *Let Γ be a Lie incidence geometry of type $E_{7,7}$ over a field $\mathbb{K}$. Let x be a point and ξ a symp. Then either*

- (i) $x \in \xi$, or
- (ii) $x \notin \xi$, x is collinear to each point of a unique $5'$ -space of ξ and symplectic to all other points of ξ , or
- (iii) $x \notin \xi$, x is collinear to a unique point x' of ξ , symplectic to all points of ξ collinear to x' , and opposite each other point of ξ .

In case (ii) above, the point x is said to be *close* to ξ , whereas in case (iii), it is said to be *far* from ξ .

Two distinct symps sharing at least a plane share a 5-space. Again, the 5-spaces in a given symp form one natural class of maximal singular subspaces, whereas the $5'$ -spaces form the other class.

Lemma 8.28. *Let Γ be a Lie incidence geometry of type $E_{7,7}$ over a field $\mathbb{K}$. Let M be a maximal 5-space and let ξ and ξ' be two distinct symps containing M . Let $p \in \xi \setminus M$ and $p' \in p^\perp \cap (\xi' \setminus M)$. Then every point on the line $\langle p, p' \rangle$ is contained in a (unique) symp, which contains M .*

Proof. Since $p' \perp p$, we have $p^\perp \cap M = p'^\perp \cap M =: U$ is a 4-space. Then every point $q \in pp'$ is contained in the $5'$ -space generated by q and U , which is itself contained in a unique symp by definition of $5'$ -space. $\square$

We will also need the following two results, which follow from considering a suitable apartment.

Lemma 8.29. *Let Γ be a Lie incidence geometry of type $E_{7,7}$ over a field $\mathbb{K}$. Let ξ and ξ' be two symps. If $\xi \cap \xi' = L$, with L a line, then a point $x \in \xi$ is opposite some point $x' \in \xi'$ if and only if $x^\perp \cap L \cap x'^\perp = \emptyset$. In particular, $\xi \cup \xi'$ does not contain any pair of opposite singular t -spaces for $t \geq 2$.*

Lemma 8.30. *Let Γ be a Lie incidence geometry of type $E_{7,7}$ over a field $\mathbb{K}$. Let U and U' be two opposite 3-spaces and let $W \supseteq U$ and $W' \supseteq U'$ be two 6-spaces, which are not opposite. Then there exists a plane α in W disjoint from U , no point of which is opposite any point of W' .*

We are now ready to determine the various projectivity groups in the exceptional simply laced cases in a geometric way.

Theorem 8.31. *Let Δ be a building of type E_6 , E_7 or E_8 over the field $\mathbb{K}$. Let F be a simplex of Δ such that $\text{Res}_\Delta(F)$ is irreducible. Then $\Pi(F)$ and $\Pi^+(F)$ are given as in Table 2, where the last column contains a checkmark if $\text{typ}(F)$ is polar closed. Again, a long dash in the table in the column of $\Pi(F)$ means that $\text{typ}(F)$*

Reference	$\text{typ}(\Delta)$	$\text{Res}_\Delta(F)$	$\text{cotyp}(F)$	$\Pi^+(F)$	$\Pi(F)$	
(A1)		A_1		$\text{PGL}_2(\mathbb{K})$	$\text{PGL}_2(\mathbb{K})$	
(A2)	E_6	A_2	$\{2, 4\}$	$\text{PGL}_3(\mathbb{K})$	$\text{PGL}_3(\mathbb{K}).2$	
	E_6	A_2	$\neq \{2, 4\}$	$\text{PGL}_3(\mathbb{K})$	—	
	E_7, E_8	A_2		$\text{PGL}_3(\mathbb{K})$	$\text{PGL}_3(\mathbb{K}).2$	
(A3)	E_6	A_3	$\{3, 4, 5\}$	$\text{PGL}_4(\mathbb{K})$	$\text{PGL}_4(\mathbb{K})$	✓
	E_6	A_3	$\neq \{3, 4, 5\}$	$\text{PGL}_4(\mathbb{K})$	—	
	E_7, E_8	A_3		$\text{PGL}_4(\mathbb{K})$	$\text{PGL}_4(\mathbb{K}).2$	
(A4)	E_6	A_4		$\text{PGL}_5(\mathbb{K})$	—	
	E_7, E_8	A_4		$\text{PGL}_5(\mathbb{K})$	$\text{PGL}_5(\mathbb{K}).2$	
(A5)	E_6	A_5		$\text{PSL}_6(\mathbb{K}, 3)$	$\text{PSL}_6(\mathbb{K}, 3)$	✓
	E_7	A_5	$\{2, 4, 5, 6, 7\}$	$\text{PSL}_6(\mathbb{K}, 2)$	$\text{PSL}_6(\mathbb{K}, 2).2$	
	E_7	A_5	$2 \notin \text{cotyp}(F)$	$\text{PGL}_6(\mathbb{K})$	$\text{PGL}_6(\mathbb{K}).2$	
	E_8	A_5		$\text{PGL}_6(\mathbb{K})$	$\text{PGL}_6(\mathbb{K}).2$	
(A6)	E_7, E_8	A_6		$\text{PGL}_7(\mathbb{K})$	$\text{PGL}_7(\mathbb{K}).2$	
(A7)	E_8	A_7		$\text{PGL}_8(\mathbb{K})$	$\text{PGL}_8(\mathbb{K}).2$	
(D4)	E_6	D_4		$\overline{\text{PGO}}_8^\circ(\mathbb{K})$	$\overline{\text{PGO}}_8(\mathbb{K})$	
	E_7, E_8	D_4		$\text{PGO}_8^\circ(\mathbb{K})$	$\overline{\text{PGO}}_8^\circ(\mathbb{K})$	✓
(D5)	E_6	D_5		$\overline{\text{PGO}}_{10}^\circ(\mathbb{K})$	—	
	E_7, E_8	D_5		$\overline{\text{PGO}}_{10}^\circ(\mathbb{K})$	$\overline{\text{PGO}}_{10}(\mathbb{K})$	
(D6)	E_7	D_6		$\overline{\text{P}\Omega}_{12}(\mathbb{K})$	$\overline{\text{P}\Omega}_{12}(\mathbb{K})$	✓
	E_8	D_6		$\overline{\text{PGO}}_{12}^\circ(\mathbb{K})$	$\overline{\text{PGO}}_{12}^\circ(\mathbb{K})$	✓
(D7)	E_8	D_7		$\overline{\text{PGO}}_{14}^\circ(\mathbb{K})$	$\overline{\text{PGO}}_{14}(\mathbb{K})$	
(E6)	E_7, E_8	E_6		$\text{PGE}_6(\mathbb{K})$	$\text{PGE}_6(\mathbb{K}).2$	
(E7)	E_8	E_7		$\text{PGE}_7(\mathbb{K})$	$\text{PGE}_7(\mathbb{K})$	✓

Table 2. Projectivity groups in the exceptional cases E_6, E_7, E_8

is not self-opposite and so $\Pi(F)$ is trivially isomorphic to $\Pi^+(F)$; it should again be read as a “ditto” symbol. Grey rows correspond to projectivity groups which are not necessarily full linear groups.

Proof. In Section 8.1, we noted that the projectivity groups for $\text{typ}(\Delta) = E_8$ are the full linear groups. We see no point in reproving this geometrically, the more

because the E_8 case is geometrically the most intricate case with the longer arguments; the interested reader can consult the first author's thesis for a detailed version of that. The general projectivity groups in the E_8 case can be deduced from Theorem B and Theorem C, or by analogy with the E_7 cases. The E_6 and E_7 cases do reveal some beautiful geometry and complementary views and we provide the detailed proofs.

Case (A1) was handled in Theorem D. We now handle the other cases. Note that we may again assume that $|\mathbb{K}| \geq 3$ as otherwise the linear groups are unique.

Cases (A2) and (A3). Every subdiagram of type A_2 or A_3 of E_r , $r = 6, 7, 8$, is contained in a subdiagram of type A_3 or A_4 , respectively. Then the assertions all follow from Corollary 7.2 and Theorem 8.6.

Case (A4). If $2 \notin \text{cotyp}(F)$ for E_6 , or if $\text{cotyp}(F) \neq \{1, 2, 3, 4\}$ for E_7, E_8 , then we can again embed the diagram of $\text{Res}_\Delta(F)$ in a diagram of type A_5 and use Corollary 7.2 and Theorem 8.6.

Now suppose $2 \in \text{cotyp}(F)$ for E_6 and $\text{cotyp}(F) = \{1, 2, 3, 4\}$ for E_7 and E_8 . Then the assertion follows from Corollary 7.2 and case (A**) for $r = 5$ of Theorem 8.11.

Case (A5). In the Coxeter diagram of type E_7 , every subdiagram of type A_5 not containing the node of type 2 is contained in one of type A_6 and hence the assertion in this case follows from Corollary 7.2.

Now suppose Δ is the building of type E_6 over the field $\mathbb{K}$, and F is a vertex of type 2. We argue in the corresponding Lie incidence geometry of type $E_{6,1}$. There, F is a 5-space. Let F_1, F_2, F_3 be three 5-spaces, with both F_1 and F_3 opposite both F and F_2 , and with F adjacent to F_2 , and F_1 adjacent to F_3 , that is, $\pi_0 := F \cap F_2$ and $\pi_1 := F_1 \cap F_3$ are planes. We also initially assume that π_0 and π_1 are opposite. Consider the projectivity $\rho: F \bar{\cap} F_1 \bar{\cap} F_2 \bar{\cap} F_3 \bar{\cap} F$. We claim that ρ fixes each point of π_0 . Indeed, let $p_0 \in \pi_0$ be such a point. Then, clearly, since $F \cap F_2$ contains p_0 , the projectivities $F \bar{\cap} F_1 \bar{\cap} F_2$ and $F_2 \bar{\cap} F_3 \bar{\cap} F$ fix p_0 ; hence $p_0^\rho = p_0$ and the claim is proved. Likewise, ρ fixes each point of F collinear with a point of π_1 . The set of such points forms a plane π'_0 of F , disjoint from π_0 . Choosing a basis of F in $\pi_0 \cup \pi'_0$, a matrix of ρ is a diagonal matrix with diagonal elements three times 1 and three times some scalar $k \in \mathbb{K}$. We now show that k can be arbitrary. This is equivalent to showing the following property.

Property (*). *Given F_1, F_2 and F_3 as above, given a line L_0 in F containing points $x_0 \in \pi_0$ and $x'_0 \in \pi'_0$, and given two points $p, q \in L_0 \setminus \{x_0, x'_0\}$, we can re-choose F_3 through π_1 such that ρ maps p to q .*

We now prove Property (*). Let p_1 be the projection of p onto F_1 and let p_2 be the projection of p_1 onto F_2 . If p and p_2 were not collinear, then the symp

$\xi(p, p_2)$ would contain p_1 and π_0 , leading to additional points in π_0 collinear to p_1 inside $\xi(p, p_2)$, contradicting the fact that F and F_1 are opposite and hence p_1 is far from F . Hence there is some singular 4-space U containing π_0 , L and p_2 . (Note that, since U intersects F in a 3-space, Lemma 8.25 (ii) implies that U is really a 4-space and not a 4'-space.) Set $\xi := \xi(x_0, p_1)$. Then ξ contains p, q, p_1, p_2 and the unique point $x_1 \in \pi_1$ collinear to x'_0 . It is clear that π_1 intersects ξ in only x_1 , as otherwise there would be a point of π_1 collinear to x_0 , contradicting the fact that π_0 and π_1 are opposite. So Lemma 8.26 yields a plane $\alpha \subseteq \xi$ collinear to π_1 . Lemma 8.25 implies that α and π_1 are contained in a unique 4'-space U_2 , which is itself contained in a unique 5-space F_3 . Now both q and p_2 are (inside ξ) collinear to all points of respective lines of α , implying that they are collinear to a common point $p_3 \in F_3$. Now Property (*) follows.

It now also follows that the set of such projectivities ρ (as above with π_0 and π_1 opposite) is geometric (they are the homologies with two disjoint planes as centres; see Example 8.2). Now we drop the assumption of π_0 being opposite π_1 . We claim that, in this more general case, the projectivity ρ , as defined above, is the product of homologies with disjoint planes as centres. Indeed, set $\pi'_0 := \text{proj}_F^{F_1}(\pi_1)$ as above. If π'_0 is disjoint from π_0 , then by Proposition 2.1, π_0 and π_1 are opposite. Now we treat the other cases. Set $d = \dim(\pi_0 \cap \pi'_0)$ and note that $d = -1$ is precisely the case we already proved.

- $d = 0$. Let π_2 be a plane in F sharing a line with π_0 but disjoint from π'_0 . Then it is easy to check that the unique 4-space U containing the 3-space generated by π_2 and π_0 is disjoint from $\text{proj}_F^{F_1}(\pi_1)$. Hence there exists a 5-space $F'_2 \neq F$ containing π_2 and opposite both F_1 and F_3 , and we have that π_1 is opposite both $F \cap F'_2$ and $F_2 \cap F'_2$. We can now write ρ as the product of $F \bar{\cap} F_1 \bar{\cap} F'_2 \bar{\cap} F_3 \bar{\cap} F$ and the conjugate of $F_3 \bar{\cap} F'_2 \bar{\cap} F_1 \bar{\cap} F_2 \bar{\cap} F_3$ by $F \bar{\cap} F_3$, reducing this case to the case $d = -1$, which we already proved.
- $d = 1$. Let π_2 be a plane in F sharing a line with π_0 and exactly one point (necessarily in π_0) with π'_0 . Then, like in the case $d = 0$, we can choose a 5-space $F'_2 \neq F$ through π_2 opposite both F_1 and F_3 and such that π_1 has a unique point collinear to some point of $F \cap F'_2$, and that point is also the unique point of π_1 collinear to some point of $F_2 \cap F'_2$. We have hence reduced this case to two times the case $d = 0$, which we proved above.
- $d = 2$. This case is similarly reduced to the case $d = 1$. We leave the (straight-forward) details to the reader.

The claim is proved. Hence, thanks to Lemma 8.22, we can apply Lemma 8.21 and obtain that $\Pi^+(F)$ is generated by all homologies with disjoint planes as centres. This group contains $\text{PSL}_6(\mathbb{K})$ and then clearly corresponds to all 6×6 matrices

with a determinant equal to some non-zero third power. Also, $\Pi(F) = \Pi^+(F)$ by virtue of Theorem B.

Now suppose $\text{cotyp}(F) = \{2, 4, 5, 6, 7\}$ in case of E_7 . Here, we can take for F a pair consisting of a 5-space W and a symp ξ containing W in the Lie incidence geometry of type $E_{7,7}$ over the field $\mathbb{K}$. We employ the same method as in the previous case (Case A_5 in E_6), noting that a projectivity $\{W, \xi\} \bar{\cap} \{W', \xi'\} \bar{\cap} \{W'', \xi''\}$, where the simplices $\{W, \xi\}$ and $\{W'', \xi''\}$ are adjacent, is trivial when $W = W''$, and so we may always assume that, in such a (sub)sequence, $W \neq W''$ and $\xi = \xi''$. However, since the action of the projectivity is apparently independent of the symps ξ and ξ' , we may only consider projections from 5-spaces onto 5-spaces. Hence let W_1, W_2, W_3 be three 5-spaces with both W_1 and W_3 opposite both W and W_2 , and $\Sigma_1 := W_1 \cap W_3$ and $\Sigma_0 := W \cap W_2$ 3-spaces such that the symps ξ_0 and ξ_1 containing W, W_2 , and W_1, W_3 , respectively, are also opposite. Like in the previous case (type A_5 inside E_6), we may assume from the beginning that Σ_0 and Σ_1 are opposite 3-spaces. Set $L_0 := \text{proj}_{W_1}^{W_1}(\Sigma_1)$. Then, by Proposition 2.1, L_0 and Σ_0 are disjoint. Set $L_2 := \text{proj}_{W_2}^{W_1}(\Sigma_1)$; then, likewise, L_2 and Σ_0 are disjoint. Let x_0 be an arbitrary point on L_0 . Then, inside ξ_0 , one sees that there is a unique point x_2 on L_2 collinear to x_0 . We claim that the projectivity $\rho_1: W \bar{\cap} W_1 \bar{\cap} W_2$ maps x_0 to x_2 . Indeed, set $W'_1 := \text{proj}_{\xi_0}^{\xi_1}(W_1)$. Then, again by Proposition 2.1, W'_1 is disjoint from both W and W_2 . Set $U_1 := \text{proj}_{W_1}^W(x_0)$ and $U'_1 := \text{proj}_{\xi_0}^{\xi_1}(U_1)$ and note that $\Sigma_1 \subseteq U_1$. Then $U'_1 \subseteq W'_1$. Since x_0 is at distance 2 from each point of U_1 , it follows by Lemma 8.27 that x_0 is collinear to all points of U'_1 . Hence x_0 is contained in the unique S' -space V_0 of ξ_0 containing U'_1 . Likewise, if $x'_2 = \text{proj}_{W_2}^{W_1}(U_1)$, then $x'_2 \in V_0$. Hence x_0 and x'_2 , which is contained in L_2 as U_1 contains Σ_1 , are collinear. Consequently, $x'_2 = x_2$ and the claim is proved.

It now also follows that $\rho_3: W_2 \bar{\cap} W_3 \bar{\cap} W$ maps x_2 back to x_0 , since x_0 is the unique point on L_0 collinear to x_2 . Consequently, the projectivity

$$\rho: W \bar{\cap} W_1 \bar{\cap} W_2 \bar{\cap} W_3 \bar{\cap} W$$

fixes each point of L_0 . It is easy to see that it also fixes every point of Σ_0 . Hence it is a homology corresponding to a diagonal matrix with the diagonal consisting of four times a 1 and two times a scalar $k \in \mathbb{K}^\times$. If we can now show that every non-zero scalar k can occur, then similarly to the case A_5 in E_6 , using Lemma 8.21 and Lemma 8.22, we are done.

But it follows from the arguments in the previous paragraphs that the projectivity ρ_1 coincides with the projectivity $W \bar{\cap} W'_1 \bar{\cap} W_2$ inside ξ_0 . Likewise, the projectivity ρ_3 coincides with the projectivity $W_2 \bar{\cap} W'_3 \bar{\cap} W$ inside ξ_0 , with $W'_3 := \text{proj}_{\xi_0}^{\xi_1}(W_3)$. Now the assertion follows with exactly the same arguments as case (A^{**}) in the proof of Theorem 8.11. This concludes case $(A5)$.

Case (A6). In a Coxeter diagram of type E_7 , a subdiagram of type A_6 necessarily has type $\{1, 3, 4, 5, 6, 7\}$. We work in the Lie incidence geometry Δ of type $E_{7,7}$, where F is a 6-space.

Let $F = W_0$ and W_2 be two 6-spaces in Δ intersecting in a 3-space that we denote by U . Let W_1 be a 6-space opposite both W_0 and W_2 . Then U projects to a plane α in W_1 . Projection here means that each point of U is symplectic to each point of α . Let U' be a 3-space in W_1 that has no intersection with α , and note that U is opposite U' . Let W_3 be a 6-space that intersects W_1 in U' and is opposite both W_0 and W_3 ; this is possible since we assume $|\mathbb{K}| \geq 3$. Set

$$\rho: W_0 \bar{\wedge} W_1 \bar{\wedge} W_2 \bar{\wedge} W_3 \bar{\wedge} W_0.$$

We claim that all points of U are fixed under ρ . Indeed, a point of $W_0 \cap W_2$ first maps to a hyperplane of W_1 , then back to itself, then to a hyperplane of W_3 and again back to itself. That means U is fixed pointwise under ρ . The claim is proved.

The projection of U' onto W_0 is a plane that we will denote by β . Similarly to the proof of the previous claim, we find that β is stabilised by ρ . We now intend to show that it is pointwise fixed. A point p of β maps to a hyperplane H of W_1 that contains U' and intersects α in a line. The projection of H onto W_2 is a point that we will denote by p' . We claim that the point p is collinear to p' .

Indeed, suppose for a contradiction that p is not collinear to p' . Then, since p and p' are both collinear to each point of U , they are contained in a unique symplectic ξ . Also, there is a unique symplectic ξ_H containing the 5'-space H (by definition of 5'-spaces). Since p is not opposite any point of H , it follows from Lemma 8.27 that p is close to ξ_H and hence collinear to some 5'-space V_p of ξ_H . Similarly, p' is collinear to some 5'-space $V_{p'}$ of ξ_H . If $V_p \cap V_{p'}$ were empty, then p and p' would be opposite (as follows from considering an apartment through V_p and $V_{p'}$). If $V_p \cap V_{p'}$ were 3-dimensional, then ξ and ξ' would intersect in a 5-space, and no point of U could be opposite any point of U' (by Lemma 8.27), a contradiction. Now suppose $V_p \cap V_{p'} = K$ is a line. Then, clearly, also $\xi \cap \xi_H = K$. But this contradicts Lemma 8.29 and the fact that $U \subseteq \xi$ is opposite $U' \subseteq \xi_H$. Hence $V_p = V_{p'}$, implying that p and p' are contained in the same unique 6-space containing V_p , and so p and p' are collinear after all. The claim is proved.

Now we claim that all points of β are fixed under ρ . Indeed, the point p projects to the hyperplane H in W_1 . This hyperplane projects to p' in W_2 . This already implies that p and p' are collinear. Likewise, p' and the projection p'' onto W_0 of its projection onto W_3 are collinear. Since $p'' \in \beta$ and the point p is the only point of β that p' is collinear to (as $p'^{\perp} \cap W_0$ is 4-dimensional), we conclude $p = p''$ and the claim is proved.

Let xy be a line in W_0 connecting a point $x \in U$ with a point $y \in \beta$. Let a and b be two distinct points on xy not equal to either x or y . We claim that we can

re-define W_3 such that ρ maps a to b . Let a' and y' be the images of a and y , respectively, under $W_0 \bar{\cap} W_1 \bar{\cap} W_2$. Since $y \perp y'$ by one of our previous claims, $\langle a, b, a' \rangle$ is a singular plane and ba' and yy' intersect in a point s .

Let W' be a 6-space through U and s . The projection of W' onto U' is a 6-space that is not opposite W' and that we will denote by W_3 . Since W_3 and W' are not opposite, there exists, by Lemma 8.30, a plane γ_3 in W_3 such that no point of γ_3 is opposite any point of W' .

Since both y and y' are not opposite any point of H , and hence also not opposite any point of U' , the same thing holds for s . It follows that s is not opposite any point of W_3 , as W' and γ_3 generate W_3 . This, in turn, implies that a' and b are not opposite the same points of W_3 , which means, in other words, that b is the image of a' under $W_2 \bar{\cap} W_3 \bar{\cap} W_0$. The claim is proved.

By Theorem A, $\Pi^+(W_0)$ contains $\text{PSL}_7(\mathbb{K})$. By the above, it also contains all diagonal matrices with diagonal $(1, 1, 1, 1, k, k, k)$, $k \in \mathbb{K}^\times$ arbitrarily, and the entries k can be anywhere. This readily implies that $\Pi^+(W_0)$ contains all matrices with determinant a third power, and since 3 and 7 are relatively prime, we conclude $\Pi^+(W_0) = \text{PGL}_8(\mathbb{K})$ and $\Pi(W_0) = \text{PGL}_8(\mathbb{K}).2$.

Case (D5). We first consider the case of a Coxeter diagram of type E_6 . Without loss of generality, we may assume that F has type 6. Hence we consider F as a symp in a geometry of type $E_{6,1}$ over the field $\mathbb{K}$.

Let p_1 be a point in Δ and ξ_0 a symp opposite p_1 in Δ . Let U be a maximal singular subspace in ξ_0 . Then U is a 4-space. Let ξ_2 be another symp through U opposite p_1 . Opposite a 4-space are lines. Let L be a line through p_1 opposite U and $V := \text{proj}_{\xi_0}(L)$. Let p_3 be any point on L opposite both ξ_0 and ξ_2 , so that we have a projectivity $\rho : \xi_0 \bar{\cap} p_1 \bar{\cap} \xi_2 \bar{\cap} p_3 \bar{\cap} \xi_0$. We will show that ρ fixes U and V pointwise.

First let x be a point in U . Then x projects to a symp $\xi(x, p_1)$, then back to x , since $x \in \xi_0 \cap \xi_2$, then to a symp $\xi(x, p_3)$ and then again back to x .

Now let y be a point in V . The point $y = y_0$ projects to a symp $\xi(y_0, p_1) = \xi_y$ and then to a point $y_2 \in \xi_2$. Suppose y_0 and y_2 were not collinear. The symp ξ_y has to contain the closure of y_0 and y_2 . Both y_0 and y_2 are collinear to a 3-space of U . The intersection of these 3-spaces contains a plane. That means that the closure has to contain a plane of U that then had to be contained in ξ_y . But that contradicts the fact that U and p_1 are opposite, because p_1 would have to be collinear to elements of that plane. It follows that $y_0 \perp y_2$. Now, since $V = \text{proj}_{\xi_0}(L)$, we see that $L \subseteq \xi_y$. So y_2 continues mapping to ξ_y and then back to y_0 . Hence points of V are fixed.

Next we want to show that we can always define p_3 on L in a way such that the projectivity ρ defined above maps an arbitrary point p on a line xy , with $x \in U$

and $y \in V$, to another arbitrary point q on xy for $p \notin U, V$ and $q \notin U, V$. Given U, V, L and p_1 as before and a line xy as described above, let p be an arbitrary point on xy not in U or V . Then projecting p to p_1 yields a symp $\xi(p, p_1)$ that projects to a point p_2 onto ξ_2 . Let $y_2 := \text{proj}_{\xi_2}(\text{proj}_{p_1}(y))$. By the previous paragraph, the points x, y, y_2 generate a singular plane, which contains p, q and p_2 . Let $a := p_2q \cap yy_2$. Suppose a were collinear to p_1 . Then a would be in $\xi(p, p_1)$ and $\xi(p, p_1)$ would contain the plane $\langle x, y, y_2 \rangle$ and in particular the line xy . But that contradicts the fact that $\xi(p, p_1)$ intersects ξ_0 only in p . It follows that a is not collinear to p_1 . That means a is collinear to a different point of L that we will define as p_3 . This point p_3 is not collinear to p_2 as otherwise $\xi(p, p_1)$ would contain L , forcing $p \in V$, a contradiction. Since a and p_2 are in $\xi(p_3, p_2)$, $\xi(p_3, p_2)$ contains the whole line ap_2 and hence the point q . With that, it follows that p maps to $\xi(p, p_1)$ to p_2 to $\xi(p_3, p_2) = \xi(p_3, q)$ and finally to q .

Now Lemma 8.14 (i) proves the assertion.

In a Coxeter diagram of type E_7 (or E_8), a subdiagram of type D_5 is always contained in a subdiagram of type E_6 , and so we can apply Corollary 7.2, the previous paragraphs and Theorem B.

Case (D4). Each subdiagram of type D_4 in a diagram of type E_n , $n = 6, 7, 8$, is contained in a subdiagram of type D_5 . It follows that if F is a simplex of cotype D_4 in a building Δ of type E_n , $n = 6, 7, 8$, then there is a subsimplex $F' \subseteq F$ of cotype D_5 . By the previous case and Theorem A, the stabiliser of F' in the little projective group $\text{Aut}^\dagger(\Delta)$ of Δ acts on $\text{Res}_\Delta(F')$ as the full linear (type-preserving) group of automorphisms. Hence the stabiliser of F in $\text{Aut}^\dagger(\Delta)$, acting on $\text{Res}_\Delta(F)$ contains the stabiliser in the full linear type-preserving group of $\text{Res}_\Delta(F')$ of the vertex $F \setminus F'$. This is clearly also the full linear type-preserving group of $\text{Res}_\Delta(F)$.

Now, in case of $\text{typ}(\Delta) = E_6$, it follows from Theorem B that $\Pi^+(F)$ has index 2 in $\Pi(F)$, and so $\Pi(F)$ is the full linear group of the corresponding polar space of $\text{Res}_\Delta(F)$. In case of E_7 or E_8 , Theorem B implies that $\Pi(F) = \Pi^+(F)$.

Case (D6). We treat the case of type D_6 inside type E_7 . Let ξ be a symp of the geometry of type $E_{7,7}$ over the field $\mathbb{K}$. We first claim that $\Pi(\xi)$, which is equal to $\Pi^+(\xi)$ by Theorem B, contains all homologies pointwise fixing two ξ -opposite maximal singular 5-spaces. Let M_{13} and M be two such subspaces of ξ . Let ξ_3 be an arbitrary symp distinct from ξ and containing M_{13} . Let ξ_2 be a symp opposite both ξ and ξ_3 (and note that this implies that each point of ξ_2 is opposite some point of ξ). There is a unique maximal singular subspace M_{24} contained in ξ_2 each point of which is collinear to some point of M , that is, $M_{24} = \text{proj}_{\xi_2}^\xi(M)$. Let L be any given line in ξ joining a point $p_{13} \in M_{13}$ and $p \in M$. Choose two points $q, q' \in L \setminus \{p_{13}, p\}$. Set $q_2 = \text{proj}_{\xi_2}(q)$ and $q_3 = \text{proj}_{\xi_3}(q_2)$.

If q were not collinear to q_3 , then the symp containing them would contain a 3-dimensional subspace of M_{13} and q_2 ; this would imply that q_2 is close to ξ , contradicting Lemma 8.27 in view of our remark in the previous paragraph that says that q_2 is opposite some point of ξ . Hence $\langle q, q_3, q' \rangle$ is a plane π , contained in the symp ζ containing p_{13} and q_2 . Let ξ' be any symp containing M_{24} , but distinct from ξ_2 . Let p_{24} be the unique point of ξ_2 collinear to p , and note $p_{24} \in M_{24}$, and that p_{24} and q_2 are collinear. Hence $p_{24} \in \zeta$. This implies that $\zeta \cap \xi'$ is either a line or a 5-space through p_{24} . In the latter case, p_{13} , being collinear with more than one point of that intersection, is close to ξ' , contradicting Lemma 8.27 and the fact that M_{24} is opposite M_{13} , and hence p_{13} is opposite points of ξ' . Hence $\zeta \cap \xi'$ is a line $K \ni p_{24}$. If q_2 were not collinear to K , then ζ would contain a 3-space of M_{24} , again a similar contradiction (since ζ contains p_{13}). The planes π and $\langle q_2, K \rangle$ are easily seen to be opposite in ζ ; hence there is a unique point $q_4 \in \langle q_2, K \rangle$ collinear to both q_3 and q' . Now let ξ_4 be the symp containing M_{24} and q_4 , whose existence follows from Lemma 8.28. Then one checks that ξ_4 is opposite both ξ and ξ_3 , and the projectivity $\xi \bar{\wedge} \xi_2 \bar{\wedge} \xi_3 \bar{\wedge} \xi_4 \bar{\wedge} \xi$ pointwise fixed both M_{13} and M , and maps q to q' . This proves the claim.

Now, if we want to apply Lemma 8.21, then we have to show that every projectivity

$$\rho: \xi_0 \bar{\wedge} \xi_1 \bar{\wedge} \xi_2 \bar{\wedge} \xi_3 \bar{\wedge} \xi_0,$$

with $M_0 := \xi_0 \cap \xi_2$ and $M_1 = \xi_1 \cap \xi_3$ singular 5-spaces, is the product of similar projectivities, but with M_0 opposite M_1 . So suppose M_0 and M_1 are not opposite. As for the case of type A_5 in type E_6 , there are 3 cases to consider, and they are again all quite similar to each other, so we consider for example the case where the set of points of M_0 collinear to a point of M_1 is a line L (the other possibilities are a 3-space and the whole space M_1). Then we consider a suitable 5-space M_2 in ξ_0 intersecting M_0 in a 3-space contained in M_0 , and disjoint from L . Then we find a symp ξ'_2 containing M_2 , opposite both ξ_1 and ξ_3 , and intersecting ξ_2 in a 5-space opposite M_1 . As in the case of type A_5 in type E_6 , we can now write ρ as the product of $\xi_0 \bar{\wedge} \xi_1 \bar{\wedge} \xi'_2 \bar{\wedge} \xi_3 \bar{\wedge} \xi_0$ and the conjugate of $\xi_3 \bar{\wedge} \xi'_2 \bar{\wedge} \xi_1 \bar{\wedge} \xi_2 \bar{\wedge} \xi_3$ by $\xi_0 \bar{\wedge} \xi_3$.

Now we can use Lemma 8.14, Example 8.15 and, thanks to Lemma 8.22, also Lemma 8.21 to conclude that $\Pi^+(F) = \overline{\text{P}\Omega}_{12}(\mathbb{K})$.

Case (E6). Let Γ be the parapolar space of type $E_{7,7}$ over the field $\mathbb{K}$. Let p_1, p_2 and p_3 be three mutually opposite points of Γ . If we show that the self-projectivity $\rho: p_1 \bar{\wedge} p_2 \bar{\wedge} p_3 \bar{\wedge} p_1$ is always a symplectic polarity, then Lemma 8.4 and Lemma 8.12 imply that $\Pi(p)$ is generated by all the symplectic polarities. By [14, Proposition 6.8(i)], ρ pointwise fixes a subbuilding of type F_4 . More exactly, if $\text{Res}_\Delta(p_1)$ is viewed as a parapolar space Γ_{p_1} of type $E_{6,1}$ with the lines through

p_1 as points, then ρ pointwise fixes a geometric hyperplane inducing in Γ_{p_1} a geometry of type $F_{4,4}$ over the field $\mathbb{K}$. It follows from [13] that ρ is a symplectic polarity. Now Example 8.17 and Lemma 8.18 show that $\Pi^+(p)$ is $\text{PGE}_6(\mathbb{K})$ and $\Pi(p)$ is $\text{PGE}_6(\mathbb{K}).2$. $\square$

This concludes the proofs of all our main results. We conclude the paper with some remarks.

Remark 8.32. It now follows from Theorem 8.31 that $\overline{\text{P}\Omega}_{12}(\mathbb{K})$ does not always coincide with $\overline{\text{PGO}}_{12}(\mathbb{K})$. Indeed, if it did, then the special projectivity groups in the buildings of type E_7 of all irreducible residues of types contained in D_6 would be the full linear groups. This contradicts the second grey row of Table 2 for fields containing non-square elements.

Remark 8.33. In the course of the proof of Theorem 8.31, we do not really need the full strength of Lemmas 8.14 (i) and 8.18, since we know by Theorem A that the little projective group is already contained in the group we want to generate. This knowledge would simplify the proof, since we would only have to prove that the little projective group together with the said homologies generate the full linear group.

Remark 8.34. One could ask what to expect of the case where the diagram is not simply laced. For starters, the description of all spherical buildings is more complicated. Secondly, Theorem D will not hold anymore in full generality. Indeed, there are polar spaces of rank n where $\Pi^+(F)$ is not permutation equivalent to $\text{PGL}_2(\mathbb{K})$, for F of cotype n , even if the set of maximal singular subspaces through a submaximal singular subspace carries in a natural way the structure of a projective line over $\mathbb{K}$ (like a symplectic polar space). However, analogues, appropriately phrased, of Theorems B and C should still hold. Also, Theorem A remains through across all types. In the split case, the algebraic approach via the Chevalley groups and the weight lattices also still works if one performs a computation in the symplectic generalised quadrangles analogously to the one we did in Section 8.1 in $R_{s,j}$ in case it is a projective plane over a field. For type F_4 , all special projectivity groups are the full linear groups as the weight lattice coincides with the root lattice.

Acknowledgments. The authors are grateful to Gernot Stroth for an illuminating discussion concerning the structure and action of Levi complements in Chevalley groups and to the referee for some very helpful comments and suggesting different approaches at various points, in particular the approach using algebraic groups in Section 8.1.

Bibliography

- [1] P. Abramenko and K. S. Brown, *Buildings. Theory and Applications*, Grad. Texts in Math. 248, Springer, New York, 2008.
- [2] A. Borel, *Linear Algebraic Groups*, Grad. Texts in Math. 126, Springer, New York, 1969.
- [3] N. Bourbaki, *Groupes et Algèbres de Lie, Chapitres 4, 5 et 6*, Act. Sci. Indust. 1337, Hermann, Paris, 1968.
- [4] F. Buekenhout and A. M. Cohen, *Diagram Geometry Related to Classical Groups and Buildings*, *Ergeb. Math. Grenzgeb.* (3) 57, Springer, Heidelberg, 2013.
- [5] S. Busch and H. Van Maldeghem, A characterisation of lines in finite Lie incidence geometries of classical type, *Discrete Math.* **349** (2026), no. 2, Article ID 114711.
- [6] S. Busch and H. Van Maldeghem, Lines and opposition in finite Lie incidence geometries of exceptional type, in progress.
- [7] R. W. Carter, *Simple Groups of Lie Type*, Pure Appl. Math. 28, John Wiley & Sons, London, 1972.
- [8] R. W. Carter, *Finite Groups of Lie Type, Conjugacy Classes and Complex Characters*, Wiley Classics Lib., John Wiley & Sons, London, 1985.
- [9] C. Chevalley, *The Algebraic Theory of Spinors*, Columbia University, New York, 1954.
- [10] B. N. Cooperstein, Some geometries associated with parabolic representations of groups of Lie type, *Canad. J. Math.* **28** (1976), no. 5, 1021–1031.
- [11] B. N. Cooperstein, A characterization of some Lie incidence structures, *Geom. Dedicata* **6** (1977), no. 2, 205–258.
- [12] C. W. Curtis, W. M. Kantor and G. M. Seitz, The 2-transitive permutation representations of the finite Chevalley groups, *Trans. Amer. Math. Soc.* **218** (1976), 1–59.
- [13] A. De Schepper, N. S. N. Sastry and H. Van Maldeghem, Split buildings of type F_4 in buildings of type E_6 , *Abh. Math. Semin. Univ. Hambg.* **88** (2018), no. 1, 97–160.
- [14] A. De Schepper, N. S. N. Sastry and H. Van Maldeghem, Buildings of exceptional type in buildings of type E_7 , *Dissertationes Math.* **573** (2022), 1–80.
- [15] A. De Schepper, J. Schillewaert and H. Van Maldeghem, On the generating rank and embedding rank of the hexagonal Lie incidence geometries, *Combinatorica* **44** (2024), no. 2, 355–392.
- [16] A. De Schepper and H. Van Maldeghem, On inclusions of exceptional long root geometries of type E, *Innov. Incidence Geom.* **20** (2023), no. 2–3, 247–293.
- [17] J. Dieudonné, Les déterminants sur un corps non commutatif, *Bull. Soc. Math. France* **71** (1943), 27–45.

-
- [18] J. Dieudonné, *La géométrie des groupes classiques*, 2nd ed., Springer, Berlin, 1963.
 - [19] D. Gorenstein, R. Lyons and R. Solomon, *The Classification of the Finite Simple Groups*, Math. Surveys Monogr. 40, American Mathematical Society, Providence, 1994.
 - [20] N. Knarr, Projectivities of generalized polygons, *Ars Combin.* **25B** (1988), 265–275.
 - [21] B. Mühlherr, H. P. Petersson and R. M. Weiss, *Descent in Buildings*, Ann. of Math. Stud. 190, Princeton University, Princeton, 2015.
 - [22] B. Mühlherr and M. Ronan, Local to global structure in twin buildings, *Invent. Math.* **122** (1995), no. 1, 71–81.
 - [23] S. E. Payne and J. A. Thas, *Finite Generalized Quadrangles*, Res. Notes Math. 110, Pitman, Boston, 1984.
 - [24] E. E. Shult, *Points and Lines: Characterizing the Classical Geometries*, Universitext, Springer, Heidelberg, 2011.
 - [25] R. Steinberg, *Lectures on Chevalley Groups*, Yale University, New Haven, 1968.
 - [26] Y. Stepanov, *On an octonionic construction of the groups of type E_6 and 2E_6* , PhD thesis, Queen Mary University of London, 2020.
 - [27] B. Temmermans, J. A. Thas and H. Van Maldeghem, Domesticity in projective spaces, *Innov. Incidence Geom.* **12** (2011), 141–149.
 - [28] J. Tits, Sur la géométrie des R -espaces, *J. Math. Pures Appl. (9)* **36** (1957), 17–38.
 - [29] J. Tits, Algebraic and abstract simple groups, *Ann. of Math. (2)* **80** (1964), 313–329.
 - [30] J. Tits, *Buildings of Spherical Type and Finite BN-Pairs*, Lecture Notes in Math. 386, Springer, Berlin, 1974.
 - [31] J. Tits, Endliche Spiegelungsgruppen, die als Weylgruppen auftreten, *Invent. Math.* **43** (1977), no. 3, 283–295.
 - [32] H. Van Maldeghem, *Generalized Polygons*, Monogr. Math. 93, Birkhäuser, Basel, 1998.
 - [33] H. Van Maldeghem, Symplectic polarities of buildings of type E_6 , *Des. Codes Cryptogr.* **65** (2012), no. 1–2, 115–125.
 - [34] H. Van Maldeghem and M. Victoor, Some combinatorial and geometric constructions of spherical buildings, in: *Surveys in Combinatorics 2019*, London Math. Soc. Lecture Note Ser. 456, Cambridge University, Cambridge (2019), 237–265.
 - [35] H. Van Maldeghem and M. Victoor, On Severi varieties as intersections of a minimum number of quadrics, *Cubo* **24** (2022), no. 2, 307–331.

Received September 9, 2024; revised September 23, 2025

Author information

Sira Busch, Department of Mathematics, Münster University, Münster, Germany.

E-mail: s_busc16@uni-muenster.de

Jeroen Schillewaert, Department of Mathematics, University of Auckland, Auckland, New Zealand.

E-mail: j.schillewaert@auckland.ac.nz

Corresponding author:

Hendrik Van Maldeghem, Department of Mathematics, Computer Science and Statistics, Ghent University, Ghent, Belgium.

E-mail: hendrik.vanmaldeghem@UGent.be