GENERALIZED HEXAGONS AS AMALGAMATIONS OF GENERALIZED QUADRANGLES

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Abstract

We define the notion of *regular point* p in a generalized hexagon and show how a *derived geometry* at such a point can be defined. We motivate this by proving that, for finite generalized hexagons of order (s, t), this derivation is a generalized quadrangle if and only if $s = t$. If moreover the generalized hexagon has also a regular line incident with p, then one can amalgamate the two corresponding generalized quadrangles and reconstruct in this way the generalized hexagon. The small Moufang hexagons of order 3^h, for small h, are characterized in this manner.

1 Introduction.

A **generalized** n**-gon** of order (s, t), $n \geq 2, s, t \geq 1$ is a point-line incidence geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, I)$ with the following properties

(GP1) Every line is incident with $s + 1$ points;

(GP2) Every point is incident with $t + 1$ lines;

(GP3) The diameter of the *incidence graph* (i.e. the graph whose vertices are the points and lines of $\mathcal{S}$ and edges the incidence relation) is equal to n;

(GP4) The *girth* (i.e. the length of the smallest non-trivial circuit) of the incidence graph is $2n$.

Generalized polygons, which are generalized n-gons for some n, were introduced by Tits [12]. For $n = 2, 3, 4, 6, 8$, we call them respectively *generalized digons, (generalized) projective planes, generalized quadrangles, generalized hexagons, generalized octagons.* A

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generalized digon is a trivial geometry where every point is incident with every line. A generalized n-gon has a finite number of points or lines if and only if both s and t are finite, in which case it has a finite number of points *and* lines. By a theorem of FEIT & HIGMAN [2], finite generalized n-gons of order (s,t) with $s,t \geq 2$ exist only for $n = 2,3,4,6,8$. A number of other parameter restrictions are well known and we refer to an excellent survey of KANTOR [5] for more details. Finite examples are known for each $n \in \{2,3,4,6,8\}$, e.g. arising from Chevalley groups with a BN-pair of rank two.

Let us look at one particular example. Consider a symplectic polarity in $PG(3,q)$ and consider the geometry of the points and the totally singular lines. This is a generalized quadrangle $W(q)$ of order (q,q). It is natural to look for conditions in order to be able to reconstruct the projective space from the quadrangle and to do so, one could first try to reconstruct the planes. Every plane Π corresponds to exactly one point p of the quadrangle and the lines of Π consist of points collinear to p in $W(q)$. The existence of the plane Π now depends on a condition on p, which leads us to the definition of a *regular* point. Denote by $x^\perp$ the set of points collinear to x including x. A point p is called **regular** if for all pairs of non-collinear points $x, y \in p^\perp$ the set $p^\perp \cap z^\perp$ does not depend on the choice of $z \in x^\perp \cap y^\perp \setminus \{p\}$. In this case, the set $p^\perp \cap z^\perp$ is called a **hyperbolic line**. From this definition, one can prove that, if a generalized quadrangle $\mathcal{S}$ of order (s,t) contains some regular point p, then $s \geq t$ and if $s = t < +\infty$, then the geometry of the points collinear to p, the lines through p and the hyperbolic lines in $p^\perp$, is a projective plane (this is still true if $s = t = \infty$ and $\mathcal{S}$ is a topological generalized quadrangle in which the line pencils have the same dimension as the point rows, see SCHROTH [11]). We call this plane the **derivation** of $\mathcal{S}$ in p. If every point of $\mathcal{S}$ is regular, then we can reconstruct all lines of $PG(3,q)$ and show that $\mathcal{S}$ is isomorphic to $W(s)$. This is probably the oldest characterization result of any class of generalized quadrangles. For more information related to this, consult the monograph by PAYNE & THAS [9].

Now, for q even, $W(q)$ is isomorphic to its dual, and so, with a dual definition, $W(q)$ also has regular lines in this case. PAYNE [6] showed that, more generally, if a generalized quadrangle $\mathcal{S}$ contains a regular point p incident with a regular line L, then one can reconstruct $\mathcal{S}$ by amalgamating the two corresponding projective planes. This "amalgamation" procedure was explained algebraically and it is one of our goals to give a geometric description below.

The main purpose of this paper however is to generalize this procedure to generalized hexagons. So we consider the generalized hexagon $H(q)$ of order (q,q) arising from the adjoint Chevalley group $G_2(q)$. TITS [12] constructs this hexagon on a parabolic quadric $Q(6,q)$ in six-dimensional projective space. Reconstruction of that quadric would imply the reconstruction of the point-residues, which are generalized quadrangles. We axiomatize the conditions under which it is possible to define a generalized quadrangle "at a point", i.e. we define *regular points* and the *derivation in a regular point*. Afterwards, we reconstruct the known self-dual generalized hexagons by amalgamating two generalized quadrangles, which will turn out to be the derivations in some point, resp. some line, incident with each other. We use the computer to say something more about small cases.

It should be noted that the celebrated paper [10] by Ronan provided a solid basis to build the present paper on.

2 Regular Points and Derivations of generalized hexagons.

Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{L}, I)$ is a generalized hexagon of order (s,t) and assume $s,t \geq 2$. The **distance** $d(x,y)$ between two elements $x, y \in \mathcal{P} \cup \mathcal{L}$ is the length of a chain of minimal length connecting x to y. The set of elements at distance i from some x is denoted by $\mathcal{S}_i(x)$. If $d(x,y) = 4$, there is a unique element z at distance 2 from both x, y and we denote z by $x * y$, see Ronan [10]. We call a point p **half regular** if for all pairs x, y of non-collinear points in $\mathcal{S}_2(p)$, the set $p^z := \mathcal{S}_4(z) \cap \mathcal{S}_2(p)$ is independent of the choice of $z \in \mathcal{S}_4(x) \cap \mathcal{S}_4(y) \cap \mathcal{S}_6(p)$, in which case the set p^z is called an **ideal line**. Note the resemblance with the definition of a regular point and hyperbolic lines in a generalized quadrangle as given above. But for a generalized hexagon, it turns out that we need a little more (basically because the girth is larger). A point p is called **regular** if it is half regular and if for every $u \in \mathcal{S}_4(p)$ the set $(p * u)^z := \mathcal{S}_4(z) \cap \mathcal{S}_2(p * u)$ is independent of the choice of $z \in \mathcal{S}_4(p) \cap \mathcal{S}_4(u) \cap \mathcal{S}_6(p * u)$, in which case the set $(p * u)^z$ is again called an **ideal line**. Note that these ideal lines contain p. An immediate consequence of these definitions is the following result, proved by Ronan [10]:

THEOREM 1. *If every point of a generalized hexagon $\mathcal{S}$ is half regular, then every point is regular and $\mathcal{S}$ is known (it is one of the Moufang hexagons arising from algebraic groups).*

Our next goal is to define the notion of a *derivation* in a regular point of a generalized hexagon. Therefore, we need some preparation. So let $\mathcal{S}$ be a generalized hexagon with a regular point p. Every pair x, y of points collinear to p is contained in either a line or an ideal line of $\mathcal{S}$. We denote that line by $< x, y >$. Similarly, we denote an ideal line through p containing some other point u at distance 4 from p by $< p, u >$ or $< u, p >$. The **focus** of an ideal line $< x, y >$ is the point $x * y$.

Now let x, y be points collinear to p determining an ideal line $< x, y >$ and let z be a point collinear to x but not on $< x, p >$. Then z determines a unique ideal line t^z for all $t \in < x, y > \setminus \{x\}$ and we put $\Pi_z^{<x,y>}$ equal to the union of $< z, p >$ and all these ideal lines t^z as t ranges over $< x, y > \setminus \{x\}$.

LEMMA 1. *If $< x, y >$ is an ideal line with focus p, and z is a point collinear to x and at distance 4 from p, then $\Pi_z^{<x,y>} = \Pi_u^{<x,y>}$ for all $u \in \Pi_z^{<x,y>}$, $u \neq p$.*

PROOF. First suppose $u \in < z, p >$, $u \neq z, p$. So $< z, p > = < u, p >$. Let $w \in t^z$ for some $t \in < x, y > \setminus \{x\}$. Since x^w contains p and z, it is by definition of regular point equal to $< p, z >$. So $d(w, u) = 4$ and $w \in t^u$, hence $t^u = t^z$ implying $\Pi_u^{<x,y>} = \Pi_z^{<x,y>}$.

Next, suppose $u \in t^z$ for some $t \in < x, y > \setminus \{x\}$. As in the first part, $< z, p > = x^u$ and $< u, p > = t^z$. Without loss of generality, we can assume $y \neq t$. Now $p^{(z*u)}$ contains x and t and hence it equals $< x, t > = < x, y >$. So $d(y, z * u) = 4$. But this implies that both y^z

and y^u contain p and $y * (z * u)$. So $y^z = y^u$. Since y was arbitrary on $< x, t >$, this shows the result.$\square$

So the set $\Pi_z^{<x,y>}$ is determined by $< x, y >$ and any of its points distinct from p. But if we take a point u of $\Pi_z^{<x,y>}$ such that $u * p \neq z * p$, then $< x, y > = < u * p, z * p >$ and hence $\Pi_z^{<x,y>}$ is determined by u and z. The set $\{p, u, z\}$ forms a triad of points at distance 4 with no common point at distance 2 and we call such a triad a **triangle**. Also, we denote the set $\Pi_z^{<x,y>}$ by $\ll p, u, z \gg$ and call it an **ideal plane** through p with **focusline** $< u * p, z * p >$.

We define the following incidence geometry $\mathcal{S}_p$: the points are the ideal planes through p together with the sets $\mathcal{S}_2(x) \cup \{x\}$ for x collinear to p (including p). The lines are all ideal lines containing p together with all customary lines through p. The incidence relation is the natural one (symmetrized inclusion). We call $\mathcal{S}_p$ the **derivation** of $\mathcal{S}$ in p. To avoid confusion, we will denote points of $\mathcal{S}_p$ by capital letters or we write $x^\perp$ instead of $\mathcal{S}_2(x) \cup \{x\}$ when we view this as a point of $\mathcal{S}_p$. We now have the following theorem:

THEOREM 2. *Suppose the generalized hexagon $\mathcal{S}$ of order (s, t) contains a regular point p. Then $s \geq t$. If moreover $s = t < \infty$, then $\mathcal{S}_p$ is a generalized quadrangle of order (s, s) in which the point $p^\perp$ is regular.*

PROOF. Suppose $s = t < \infty$. It is easy to see that there are exactly $1 + s + s^2 + s^3$ points (resp. lines) in $\mathcal{S}_p$ and every line (resp. point) is incident with $s + 1$ points (resp. lines). Also, clearly, $\mathcal{S}_p$ does not contain digons. Suppose it contains three points P_1, P_2, P_3 not on a line but two by two collinear. Then there are four possibilities:

1. $P_1 = p^\perp$. The lines P_1P_2 and P_1P_3 of $\mathcal{S}_p$ are two lines l_1, l_2 of $\mathcal{S}$ through p. From the incidence relation in $\mathcal{S}_p$ now follows that the points P_2 and P_3 are of the form $x_1^\perp$, resp. $x_2^\perp$. The corresponding sets in $\mathcal{S}$ meet in p and hence they can not share a line or ideal line.

2. $p^\perp \neq P_1, P_2, P_3$ but $p^\perp$ lies on the line P_1P_2. Then P_iP_3 must be an ideal line $< x_i, p >$ with $(x_i * p)^\perp = P_i$, $i = 1, 2$. However, these lines can never be contained in an ideal plane (since the focusline would not be well-defined), nor can they be contained in a set of points collinear to a certain point x collinear to p.

3. P_1 is of the form $x^\perp$, x collinear to p, and P_2 and P_3 are ideal planes. So the line P_1P_i in $\mathcal{S}_p$ is an ideal line $< x_i, p >$ contained in $\mathcal{S}_2(x)$, $i = 2, 3$. But P_2P_3 must be an ideal line $< p, y >$ contained in both ideal planes P_2, P_3. Clearly $p * y \neq x$, so both ideal planes P_2, P_3 contain y and hence they have focusline $< p * y, x >$, but this implies $P_2 = P_3$.

4. P_1, P_2 and P_3 all are ideal planes. The line P_iP_j, $i \neq j$, $i, j \in \{1, 2, 3\}$, is an ideal line with some focus x_{3-i-j}. Of course x_i is not collinear with x_j for $i \neq j$ (otherwise they cannot be inside a common ideal plane). It follows that the focusline of P_i is exactly $< x_j, x_k >$ for $\{i, j, k\} = \{1, 2, 3\}$. If $< x_1, x_2 > = < x_2, x_3 >$, then P_1 and P_3 have the same focusline and they have an ideal line in common, so they are equal. Hence we

may assume $x_3 \notin < x_1, x_2 >$. Now choose a point $t_i \neq p$ collinear with x_i, $i = 1, 2$, in the ideal plane P_3. So $P_3 = \ll p, x_1, x_2 \gg$ and $d(t_1, t_2) = 4$. Put $l = \mathcal{S}_1(t_1) \cap \mathcal{S}_3(t_2)$. Clearly, $d(l, x_3) = 5$ (otherwise pentagons in $\mathcal{S}$). Now let $t_3 = \mathcal{S}_2(x_3) \cap \mathcal{S}_3(l)$. So $t_3 \in x_3^{t_1}$, so $P_2 = \ll p, t_1, t_3 \gg$, implying that $P_1 = \ll p, t_2, t_3 \gg$. Hence $d(t_2, t_3) = 4$ which is only possible if $t_1 * t_2 = t_2 * t_3 = t_1 * t_3$, implying $x_1, x_2, x_3 \in p^{t_1 * t_2}$, so $x_3 \in < x_1, x_2 >$, a contradiction. Note that this argument also shows that on each line (here l) through any point (here t_1) of an ideal plane, there is a unique point (here $t_1 * t_2$) at distance 2 from exactly $s + 1$ points of the ideal plane. We will need this in the proof of theorem 4 (see later).

This shows that the girth of the geometry is at least eight. But a counting argument now shows that $\mathcal{S}_p$ is a generalized quadrangle of order (s, s).

We now show that in $\mathcal{S}_p$, the point $p^\perp$ is a regular point. Therefore, let $x^\perp, y^\perp$ be two non-collinear points of $\mathcal{S}_p$ both collinear with $p^\perp$. Any point in $\mathcal{S}_p$ collinear to $x^\perp$ and distinct from $p^\perp$ must be an ideal plane with baseline through x, so all points in $\mathcal{S}_p$ collinear with both $x^\perp$ and $y^\perp$ (and distinct from $p^\perp$), are ideal planes P with baseline $< x, y >$. The set of points collinear to both P and $p^\perp$ consists of points of the form $z^\perp$ with $z \in < x, y >$. Hence the result.

The derivation of $\mathcal{S}_p$ in $p^\perp$ can be constructed directly as follows: the points are the points collinear to p, including p in $\mathcal{S}$, the lines are the lines through p together with the ideal lines with focus p, and incidence is the natural one. For $s = t$, this is indeed a projective plane. Suppose now $s \neq t$. Choose a point $x \neq p$ collinear with p and an ideal line l with focus p not through x. Every point u of l not collinear with x determines an ideal line $< x, u >$ and different choices for u correspond to different ideal lines. Hence, there are at least t distinct ideal lines through x with focus p. But of course there are exactly s such ideal lines, considering instead of l a line l' through p. So $s \geq t$.

This completes the proof of the theorem.$\square$

REMARK 1. The above proof of the fact that the girth is at least eight does not depend on the equality $s = t$. So in general, the derivation is a geometry with girth 8 and with $s + 1$ points on every line, and $s + 1$ or $t + 1$ lines through a point and both values occur.

REMARK 2. We could also define more homogeneously the following geometry $\mathcal{S}_p^*$ in a regular point p: the points are the ideal planes through p, the lines are the ideal lines through p together with the ideal lines with focus p, and incidence is the natural one except that an ideal line with focus p is incident with an ideal plane if it is the focusline of the latter. This way, one obtains a geometry of order $(s - 1, t + 1)$ which has, in all known examples, an automorphism group transitive on the points, and which is in case $s = t$ again a generalized quadrangle. In general, we obtain a $(4, 6, 6)$-gon (with the terminology of Buekenhout [1]).

3 Geometric Amalgamations of Projective Planes.

In this section, we describe in a geometric fashion the amalgamation of two projective planes. The purpose is to provide additional background information in order to deal with the case of amalgamations of generalized quadrangles. It will turn out that algebraic conditions will follow more easily from geometric description rather than from an entirely algebraic description. It provides anyway some more insight in the amalgamation procedure.

Let $\mathcal{S}$ be a generalized quadrangle and suppose p is a regular point of $\mathcal{S}$ and l is a regular line in $\mathcal{S}$ incident with p. So there are projective planes $\mathcal{S}_p$ and $\mathcal{S}_l$, the respective derivations of $\mathcal{S}$ in p and l. Now, the lines through p and the points on l can be viewed as elements of both $\mathcal{S}_p$ and $\mathcal{S}_l$. Abstractly, we can view $\mathcal{S}_p$ and $\mathcal{S}_l$ as disjoint planes; in each of these planes there is an incident point-line pair (p, l^*), resp. (p^*, l) and bijections θ (resp. σ) from the set of lines through p (resp. the set of points on l) to the set of lines through p^* (resp. the set of points on l^*) mapping l^* to l (resp. p^* to p). Given such a system, we can define the following geometry:

The points are of two types: first all points of $\mathcal{S}_p$ and secondly, all pairs (m, m'), where m is a line in $\mathcal{S}_l$ not through p^*, m' is a line in $\mathcal{S}_p$ not through p and $(m \cap l)^\sigma = m' \cap l^*$.

The lines are similarly of two types: first all lines of $\mathcal{S}_l$ and secondly, all pairs (x, x'), where x is a point in $\mathcal{S}_p$ not on l, x' is a point in $\mathcal{S}_l$ not on l^* and $(xp)^\theta = x'p^*$.

Incidence is defined as follows (with the above notation): a point x of $\mathcal{S}_p$ is incident with a line m of $\mathcal{S}_l$ if and only if either x is on l^* and $x^{\sigma^{-1}}$ is on m, or m contains p^* and $m^{\theta^{-1}}$ contains x; every point x is incident with the pair (x, x'); every line m is incident with the pair (m, m'); finally, the pair of points (x, x') is incident with the pair of lines (m, m') if and only if x is on m' in $\mathcal{S}_p$ and x' is on m in $\mathcal{S}_l$. This amalgamated geometry $\mathcal{S}_p \uplus \mathcal{S}_l$ (along σ, θ) is isomorphic to $\mathcal{S}$ and we give the isomorphism on the set of points: a point of the form x in $\mathcal{S}_p \uplus \mathcal{S}_l$ is mapped in the natural way onto itself; a point (m, m') defines a line m of $\mathcal{S}$ meeting l and a hyperbolic line m' through $m \cap l$. We map (m, m') to the unique point u of m collinear to all points of m' (well defined by regularity of p). Dually, we define a bijection from the set of lines of $\mathcal{S}_p \uplus \mathcal{S}_l$ to the set of lines of $\mathcal{S}$. It is easy to check that these bijections induce an isomorphism.

We can now forget about the generalized quadrangle $\mathcal{S}$ and consider two projective planes $\mathcal{S}_1$ and $\mathcal{S}_2$ with distiguished flags (p, l^*), resp. (p^*, l) and bijection as above. We carry out the amalgamation procedure just described and if this yields a generalized quadrangle, then we can call $(\mathcal{S}_1, \mathcal{S}_2, \sigma, \theta)$ **admissible** and we call the generalized quadrangle the **amalgon** (this is a contamination of words *amalgam* and *polygon*). If both projective planes are Desarguesian, one can give an algebraic condition using the underlying planar ternary ring structure of the coordinatizing fields and Payne [8] shows that every known generalized quadrangle with an incident regular point-line pair arises in this way. Also, still under the condition that the planes are Desarguesian, the field must have characteristic 2. Let q be the order of the field, then for $q = 2$ and $q = 4$, only one generalized quadrangle can be constructed (by uniqueness of the quadrangle in question, see Payne & Thas [9].

If $q = 8$, only Desarguesian planes can be used and we have the following result:

THEOREM 3. *There are, up to isomorphism, exactly two generalized quadrangles of order (8,8) with an incident point-line pair each of which is regular.*

PROOF. This has been checked by computer and the two examples can be described as follows: amalgamate the two projective planes in such a way that, after an identification of both planes (identifying p with p^* and l with l^* using the notation of above) the map σ is the identity and the map θ is either the square or the fourth power (Frobenius automorphism). No other possibilities occur. In the first case, we obtain the classical symplectic quadrangle $W(8)$ described in the introduction, and in the second case we have the generalized quadrangle of order $(8, 8)$ related to a non-classical oval in $PG(2, 8)$, see PAYNE [7]. Both examples are self-dual (even self-polar). □

4 Amalgamations of Generalized Quadrangles.

In this section, we give a geometric description of how to amalgamate two generalized quadrangles to obtain a generalized hexagon. In the next section, we derive the algebraic conditions under which this operation is succesful.

So let $\mathcal{S}_p = (\mathcal{P}_p, \mathcal{L}_p, I_p)$ and $\mathcal{S}_l = (\mathcal{P}_l, \mathcal{L}_l, I_l)$ be two finite generalized quadrangles of order (s, s). Suppose $\mathcal{S}_p$ has a regular point, say P, and $\mathcal{S}_l$ has a regular line L. Let L^* (resp. P^*) be an arbitrary but fixed line (resp. point) incident with P (resp. L). Furthermore, let θ (resp. σ) be a bijection from the set of lines (resp. points) on P (resp. L) to the set of lines (resp. points) on P^* (resp. on L^*) mapping L^* to L (resp. P^* to P). We use the notation "$\perp$" to denote collinear points and concurrent lines; the intersection of two concurrent lines x, y will be denoted by $x \cap y$ and the line joining two collinear points u, v by uv, as usual. We define the following geometry $\mathcal{H} = (\mathcal{P}, \mathcal{L}, I)$

The elements of $\mathcal{P}$ are of four distinct types:

(P1) the points of $P^\perp$ in $\mathcal{S}_p$;

(P2) the pairs $(M, M^*) \in \mathcal{L}_l \times \mathcal{L}_p$ with $M \perp L$, $M^* \perp L^*$ and $(M \cap L)^\sigma = M^* \cap L^* \neq P$;

(P3) the pairs $(Q^*, M^*) \in \mathcal{P}_l \times \mathcal{L}_p$ with M^* meeting a line L_0^* through P distinct from L^* and $Q^* \neq P^*$ incident with $(L_0^*)^\theta$;

(P4) the pairs $(Q, Q^*) \in \mathcal{P}_p \times \mathcal{P}_l$ with Q (resp. Q^*) not collinear with P (resp. P^*) and $(P_0^*)^\sigma = P_0$, where P_0 (resp. P_0^*) is the unique point on L^* (resp. L) collinear with Q resp. Q^*.

The lines are dually of four types as well:

(L1) the lines of $L^\perp$ in $\mathcal{S}_l$;

(L2) the pairs $(Q, Q^*) \in \mathcal{P}_p \times \mathcal{P}_l$ with $Q \perp P$, $Q^* \perp P^*$ and $(QP)^\theta = Q^*P^* \neq L$;

(L3) the pairs $(M^*, Q^*) \in \mathcal{L}_p \times \mathcal{P}_l$ with Q^* collinear to a point P_0^* on L distinct from P^* and $M^* \neq L^*$ incident with $(P_0^*)^\sigma$;

(L4) the pairs $(M, M^*) \in \mathcal{L}_l \times \mathcal{L}_p$ with M (resp. M^*) not meeting L (resp. L^*) and $(L_0^*)^\theta = L_0$, where L_0 (resp. L_0^*) is the unique line through P^* (resp. P) meeting M (resp. M^*).

Let us settle some notation: if x is a point or line of $\mathcal{H}$, then we e.g. briefly write "$x = (Q, Q^*)$ **is of type** (P4)" to abbreviate "*x is a point of $\mathcal{H}$ and $x = (Q, Q^*) \in \mathcal{P}_p \times \mathcal{P}_l$ with Q (resp. Q^*) not collinear with P resp. P^* and $(P_0^*)^\sigma = P_0$, where P_0 (resp. P_0^*) is the unique point on L^* (resp. L) collinear with Q (resp. Q^*)*". Similarly for the other types and for lines. We now define the incidence relation I in $\mathcal{H}$: let $(x, d) \in \mathcal{P} \times \mathcal{L}$, then xId if and only if one of the following occurs:

(I1) $x = Q$ is of type (P1), $d = M$ is of type (L1) and either QI_pL^* and $MI_lQ^{\sigma^{-1}}$ or MI_lP^* and $QI_pM^{\theta^{-1}}$ (or both if $Q = P$ and $M = L$);

(I2) $d = (Q, Q^*)$ is of type (L2) and $x = Q$ is of type (P1);

(I2)' $x = (M, M^*)$ is of type (P2) and $d = M$ is of type (L1);

(I3) $x = (M, M^*)$ is of type (P2), $d = (M^*, Q^*)$ is of type (L3) and Q^*I_lM;

(I3)' $d = (Q, Q^*)$ is of type (L2), $x = (Q^*, M^*)$ is of type (P3) and M^*I_pQ;

(I4) $x = (Q^*, M^*)$ is of type (P3), $d = (M, M^*)$ is of type (L4) and Q^*I_lM;

(I4)' $d = (M^*, Q^*)$ is of type (L3), $x = (Q, Q^*)$ is of type (P4) and M^*I_pQ;

(I5) $x = (Q, Q^*)$ is of type (P4), $d = (M, M^*)$ is of type (L4), Q^*I_lM and M^*I_pQ.

If this geometry $\mathcal{H}$ is a generalized hexagon (necessarily of order (s, s)), then we call the quadruple $(\mathcal{S}_p, \mathcal{S}_l, \theta, \sigma)$ **admissible**. In this case, we call the generalized hexagon an **amalgon**.

THEOREM 4. *Let, with the above notation, $(\mathcal{S}_p, \mathcal{S}_l, \theta, \sigma)$ be an admissible quadruple. Then we can view the point P of $\mathcal{S}_p$ as a point of $\mathcal{H}$ and this is a regular point of $\mathcal{H}$. Dually, L is a regular line of $\mathcal{H}$. Furthermore, the line L^* (resp. point P^*) of $\mathcal{S}_p$ (resp. $\mathcal{S}_l$) is not regular in the respective generalized quadrangle and s is odd. The derivation of $\mathcal{H}$ in P (resp. L) is in a natural way isomorphic to $\mathcal{S}_p$ (resp. $\mathcal{S}_l$). Moreover, every generalized hexagon with a regular point incident with a regular line can be constructed in this way.*

PROOF. First we show that P is regular in $\mathcal{H}$. Let x and y be two non-collinear points both collinear with P in $\mathcal{H}$. Denote be X and Y the corresponding points in $\mathcal{S}_p$. Suppose X nor Y is incident with L^*. A point in $\mathcal{H}$ at distance 4 from both x and y and at distance 6 from P has necessarily type (P4) and so we can write (Q, Q^*) for it. Distance 4 from both X and y then just means that Q and X resp. Y are collinear in $\mathcal{S}_p$. So now it easily follows that $P^{(Q,Q^*)}$ is exactly the set of points z of $\mathcal{H}$ collinear with P whose corresponding point Z in $\mathcal{S}_p$ is collinear to both P and Q. By regularity of P in $\mathcal{S}_p$, this corresponds to the hyperbolic line through X and Y and hence is independent of the choice of (Q, Q^*). Similar argument if one of X or Y is incident with L^*. So P is half regular in $\mathcal{H}$.

Now let x be a point at distance 4 from P in $\mathcal{H}$. Suppose $x * P$ is not incident with L. Then we have $x = (Q^*, M^*)$ has type (P3) and $x * P$ corresponds to the unique point P_0 on M^* collinear to P (in $\mathcal{S}_p$). Now a point u at distance 4 from both P and x and at distance 6 from $x * P$ could have type (P2) or (P3). We leave (P2) to the reader and assume type (P3), which is the most general case anyway. So let $u = (Q_0^*, M_0^*)$ be of type (P3). Then $d(x, u) = 4$ just means that M^* meets M_0^* in $\mathcal{S}_p$, say in the point P_1 (*not* collinear with P) and that, in $\mathcal{S}_l$, the three points Q^*, Q_0^* and P_2^* (where $P_2^* I L$ and $(P_2^*)^\sigma$ is collinear to P_1) are collinear to a common point different from P^*. Now any point y collinear to $x * P$ and not on L has type (P3), say $y = (Q_1^*, M_1^*)$ with $P^0 I M_1^*$. So if y is at distance 4 from u, then, as above, first of all M_0^* and M_1^* meet, but this implies $M^* = M_1^*$ (otherwise we have a triangle, remember that M^* and M_1^* meet in $P_0 \perp P$); and secondly the three points Q_1^*, Q_2^* and P_2^* are collinear to a common point different from P^* in $\mathcal{S}_l$ (since now M_1^* meets M_0^*). The first condition just says that, given the line $y(x * P)$, the point y is independent from u and the second condition means that, in the notation of Payne & Thas [9], *every triad of points collinear to P^* has exactly* **one** *other center* and this implies in turn that the point P^* cannot be regular in $\mathcal{S}_l$ and s is odd (see [9, propositions 1.3.6.(iii) and 1.7.1.(i)]).

We now show that the respective derivations of $\mathcal{H}$ are isomorphic to $\mathcal{S}_p$ and $\mathcal{S}_l$.

Well, there is a natural bijection between the set of points collinear to P in $\mathcal{H}$ and the set of points collinear to P in $\mathcal{S}_p$, hence there is a natural bijection between the set of points collinear with P in $\mathcal{H}$ and the set of points collinear to $P^\perp$ in $\mathcal{H}_p$ (with the notation of the previous sections). A point X in $\mathcal{H}_p$ not collinear with $P^\perp$ is determined by a point x of $\mathcal{H}$ collinear with P and a point u of $\mathcal{H}$ at distance 4 from P and 6 from x. Hence x corresponds to some point Q in $\mathcal{S}_p$ collinear to P and u corresponds to a line M^* in $\mathcal{S}_p$ (yes, $u = (Q^*, M^*)$ is of type (P3)). Expressing $d(x, u) = 6$ gives us Q is not incident with M^* and we let X correspond to the unique point on M^* collinear with Q. We leave it as a tedious exercise to the reader to verify that this is well-defined (use the fact that an ideal line with focus P in $\mathcal{H}$ corresponds to a hyperbolic line in $(\mathcal{S}_p)_2(P)$). Similarly, we can identify the lines and then it is easy to see that incidence is preserved by this identification.

There remains to be shown that, given a generalized hexagon $\mathcal{H}$ containing a regular point p incident with a regular line l, the respective derivations can be amalgamated in such a way that the amalgon is naturally isomorphic to $\mathcal{H}$. Again, we give the bijection on the set of points, the bijection on the set of lines is dual and the proof of the fact that the incidence relation is preserved is left to the reader. The maps σ and θ are the identity.

The image of points of type (P1) is clear. Let, in the amalgon, (M, M^*) be a point of type (P2). Then M defines a unique line m in $\mathcal{H}$ concurrent with l and M^* defines a unique ideal line through p with focus $m \cap l$, so we map (M, M^*) to the unique point of that ideal line on m. Now let (Q^*, M^*) be a point of type (P3) in the amalgon. Then M^* defines a unique ideal line m through p by definition, let us say with focus x. The point Q^* defines in $\mathcal{H}$ the dual of an ideal line, i.e. it defines a set of $s+1$ line concurrent to a fixed line l_1 and containing l. But the condition (P3) on (Q^*, M^*) and the choice of θ imply $l_1 = xp$. Hence there is a unique line through x among the $s+1$ lines that constitute the dual of an ideal point, mentioned above. And on that line, there is a unique point of m. That point is by definition the image of (Q^*, M^*). Finally, suppose (Q, Q^*) is a point of type (P4). Then Q defines a set Γ of s^2+s+1 points in $\mathcal{H}$ consisting of $s+1$ ideal lines through p with some focusline m. Denote by x the point of m on l. The Q^* defines the dual of an ideal line, so it defines a set Ω of $s+1$ lines concurrent with some line l_0 through x (by the condition (P4)). Clearly, there exists a unique line l_1 in Ω incident with some point of Γ. By the fourth part of the proof of Theorem 2, there is a unique point u on l_1 at distance 2 from $s+1$ points of Γ. The point u is by definition the image of (Q, Q^*). This way, we have identified the points of the amalgon with the points of $\mathcal{H}$. Dually, one identifies the lines. It is a tiresome but easy exercise to check that this identification preserves incidence. $\square$.

5 Algebraic Conditions.

In this section we amalgamate two classical generalized quadrangles and write down the algebraic conditions under which this operation produces a generalized hexagon.

By Theorem 4, characteristic 2 is ruled out, so we have to consider the classical generalized quadrangle $W(q)$ over the field $GF(q)$ (which has order (q,q) and which has regular points) and its dual $Q(4,q)$ (which has regular lines). Let us first describe these quadrangles algebraically. This is best done by the coordinatization method of Hanssens & Van Maldeghem [3].

In what follows, a, a', b, k, k', l are elements of $GF(q)$ and ∞ is a fixed different symbol. We define a geometry $\mathcal{S}_p = (\mathcal{P}_p, \mathcal{L}_p, I_p)$ as follows: the points are the elements $(\infty), (a), (k,b)$ and (a,l,a'); the lines are the elements $[\infty], [k], [a,l]$ and $[k,b,k']$; incidence is defined by the chain:

$$(a,l,a')I_p[a,l]I_p(a)I_p[\infty]I_p(\infty)I_p[k]I_p(k,b)I_p[k,b,k']$$

and

$$(a,l,a')I_p[k,b,k'] \Leftrightarrow \begin{cases} a' &= ak+b \\ l &= a^2k+k'+2ab \end{cases}$$

We set $P := (\infty)$ and $L^* = [\infty]$. The geometry thus described is the generalized quadrangle $W(q)$ by Hanssens & Van Maldeghem [4]. Now, let $\mathcal{S}_l$ be the dual of this geometry and we denote its elements with double brackets to make clearly the distinction. So points are $((\infty)), ((a))$, etc..., lines are $[[\infty]], [[k]]$, etc... and incidence is as in the chain above or

$$((a,l,a'))I_l[[k,b,k']] \Leftrightarrow \begin{cases} k' &= ak+l \\ b &= ak^2+a'+2kl \end{cases}$$

Now suppose θ is a permutation of $GF(q)$ mapping a line $[k]$ to the line $[[k^\theta]]$ and σ is likewise mapping a point $((a))$ to (a^σ). By the transitivity properties of $W(q)$ and $Q(4,q)$, we can choose the coordinates in such a way that $1^\theta = 1^\sigma = 1$ and $0^\theta = 0^\sigma = 0$. The amalgamated geometry $\mathcal{H}$ has the following elements (and in the meantime we abbreviate notation):

(P1) the point (∞) which we write as $\langle(\infty)\rangle$;
the points (a) which we write as $\langle(a^{\sigma^{-1}})\rangle$;
the points (k,b) which we write as $\langle(k,b)\rangle$;

(P2) the points $([[a,l']],[a^\sigma,l])$ which we write as $\langle(a,l',l)\rangle$;

(P3) the points $(((k^\theta,b')),[k,b,k'])$ which we write as $\langle(k,b,b',k')\rangle$;

(P4) the points $((a^\sigma,l,a'),((a,l',a'')))$ which we write as $\langle(a,l',l,a\text{"},a')\rangle$;

(L1) the line $[[\infty]]$ which we write as $\langle[\infty]\rangle$;
the lines $[[k]]$ which we write as $\langle[k^{\theta^{-1}}]\rangle$;
the lines $[[a,l]]$ which we write as $\langle[a,l]\rangle$;

(L2) the lines $((k,b'),((k^\theta,b)))$ which we write as $\langle[k,b,b']\rangle$;

(L3) the lines $([a^\sigma,l'],((a,l,a')))$ which we write as $\langle[a,l,l',a']\rangle$;

(L4) the lines $([[k^\theta,b,k']],[k,b',k''])$ which we write as $\langle[k,b',b,k\text{"},k']\rangle$;

With this new notation, the incidence relation in the amalgamated geometry becomes:

$$\langle(a,l,a',l',a'')\rangle I\langle[a,l,a',l']\rangle I\langle(a,l,a')\rangle I\langle[a,l]\rangle I\langle(a)\rangle I\langle[\infty]\rangle I$$
$$\langle(\infty)\rangle I\langle[k]\rangle I\langle(k,b)\rangle I\langle[k,b,k']\rangle I\langle(k,b,k',b')\rangle I\langle[k,b,k',b',k'']\rangle,$$

and

$$\langle(a,l,a',l',a'')\rangle I\langle[k,b,k',b',k'']\rangle \Leftrightarrow (*)\left\{\begin{array}{rcl} a'' &=& a^\sigma k+b \\ a' &=& (a^\sigma)^2k+b'+2a^\sigma b \\ k'' &=& ak^\theta+l \\ k' &=& a(k^\theta)^2+l'+2k^\theta l \end{array}\right.$$

This in fact amounts to the coordinatization of generalized polygons as described in Van Maldeghem [13].

It is now convenient to recoordinatize as follows: substitute $k^{\theta^{-1}}$ for k and set $\theta^{-1}=\tau$. Everything remains the same, except the condition $(*)$ becomes

$$(**)\left\{\begin{array}{rcl} a'' &=& a^\sigma k^\tau+b \\ a' &=& (a^\sigma)^2k^\tau+b'+2a^\sigma b \\ k'' &=& ak+l \\ k' &=& ak^2+l+2kl \end{array}\right.$$

We denote the corresponding amalgamated geometry by $W(q)\,{}_\sigma\!\uplus_\tau Q(4,q)$. For the rest of the paper, we omit the "$\langle$" and "$\rangle$" around the coordinates, since no confusion will be possible (we do not consider other coordinates anymore).

THEOREM 5. *The almagamated geometry $\mathcal{H} := \mathcal{W}(\text{II})\,{}_\sigma\!\uplus_\tau\,\mathcal{Q}(\triangle, \text{II})$ as defined above is a generalized hexagon if and only if, under the given conditions, the following equalities never occur in $GF(q)$ simultanously (where 2 and 5 must hold for $(\phi, \theta) = (\tau, \sigma)$ and the others for $(\phi, \theta) = (\tau, \sigma)$ and $(\phi, \theta) = (\sigma^{-1}, \tau^{-1})$).*

1. *CONDITION: $a \neq A \neq b \neq a$ and $K \neq k \neq L \neq K$.*

$$\begin{cases} (K-k)^2(A-a) = (L-k)^2(A-b) \\ (K^\phi - k^\phi)(A^\theta - a^\theta) = (L^\phi - k^\phi)(A^\theta - b^\theta) \end{cases}$$

2. *CONDITION: $a \neq A \neq b \neq a$ and $K \neq k \neq L \neq K$.*

$$\begin{cases} (K-k)^2(A-a) = (L-k)^2(A-b) \\ (K^\phi - k^\phi)(A^\theta - a^\theta)^2 = (L^\phi - k^\phi)(A^\theta - b^\theta)^2 \end{cases}$$

3. *CONDITION: $a \neq A \neq b \neq a$ and $K \neq k \neq L \neq K$.*

$$\begin{cases} (K-k)^2(A-a) = (L-k)^2(A-b) \\ (L^\phi - K^\phi)(A^\theta - a^\theta)^2 = (L^\phi - k^\phi)(b^\theta - a^\theta)^2 \end{cases}$$

4. *CONDITION: $a \neq A \neq B \neq b \neq a \neq B \neq A \neq b$ and $k \neq K \neq P \neq L \neq k$.*

$$\begin{cases} (K-k)^2(A-a) + (P-k)^2(B-A) + (L-k)^2(b-B) = 0 \\ (K^\phi - k^\phi)(A^\theta - a^\theta) + (P^\phi - k^\phi)(B^\theta - A^\theta) + (L^\phi - k^\phi)(b^\theta - B^\theta) = 0 \\ (K^\phi - k^\phi)((A^\theta)^2 - (a^\theta)^2) + (P^\phi - k^\phi)((B^\theta)^2 - (A^\theta)^2) + (L^\phi - k^\phi)((b^\theta)^2 - (B^\theta)^2) = 0 \end{cases}$$

5. *CONDITION: $B \neq X \neq A \neq a \neq B \neq Y \neq X \neq A \neq Y$ and $k \neq l \neq P \neq k \neq K \neq P \neq L \neq l$.*

$$\begin{cases} (P-K)^2(X-A) + (P-k)^2(A-a) = \\ \quad = (P-L)^2(Y-B) + (P-l)^2(B-a) \\ (P^\phi - K^\phi)(X^\theta - A^\theta) + (P^\phi - k^\phi)(A^\theta - a^\theta) = \\ \quad = (P^\phi - L^\phi)(Y^\theta - B^\theta) + (P^\phi - k^\phi)(B^\theta - a^\theta) \\ (P-K)(X-A) + (P-k)(A-a) = \\ \quad = (P-L)(Y-B) + (P-l)(B-a) \\ (P^\phi - K^\phi)((X^\theta)^2 - (A^\theta)^2) + (P^\phi - k^\phi)((A^\theta)^2 - (a^\theta)^2) = \\ \quad = (P^\phi - L^\phi)((Y^\theta)^2 - (B^\theta)^2) + (P^\phi - k^\phi)((B^\theta)^2 - (a^\theta)^2) \end{cases}$$

Moreover, the automorphism group of $\mathcal{H}$ is transitive on points (resp. lines) with five coordinates and first coordinate fixed.

PROOF. The conditions all follow from expressing that certain pentagons and quadrangles cannot occur in $\mathcal{H}$, similarly to the case of amalgamating projective planes, see e.g. Payne & Thas [9]. The reader can easily reconstruct the calculations with the following information. All circuits of minimal length in $\mathcal{H}$ containing a point of type (P1) or a line of type (L1) have length six. No circuits of length three can occur. The only circuits of length five that could exist are of the following type (and we write the types of the elements instead of the elements themselves; consecutive elements are incident as well as the first and the last):

(C5.1) (P2),(L3),(P4),(L4),(P4),(L4),(P4),(L4),(P4),(L3) and the dual;

(C5.2) (P3),(L4),(P4),(L4),(P4),(L3),(P4),(L4),(P4),(L4) (self-dual);

(C5.3) (P3),(L4),(P4),(L3),(P4),(L4),(P4),(L4),(P4),(L4) and the dual;

(C5.4) (P3),(L4),(P4),(L4),(P4),(L4),(P4),(L4),(P4),(L4) and the dual;

(C5.5) (P4),(L4),(P4),(L4),(P4),(L4),(P4),(L4),(P4),(L4) (self-dual).

The types of the circuits of length four that can occur are to be constructed by shortening the above sequences of types of possible circuits of length five. We obtain:

(C4.1) (P2),(L3),(P4),(L4),(P4),(L4),(P4),(L3) and the dual;

(C4.2) (P3),(L4),(P4),(L4),(P4),(L4),(P4),(L4) and the dual;

(C4.3) (P3),(L4),(P4),(L4),(P4),(L4),(P4),(L4) and the dual;

(C4.4) (P3),(L4),(P4),(L4),(P4),(L4),(P4),(L4) and the dual;

(C4.5) (P4),(L4),(P4),(L4),(P4),(L4),(P4),(L4) (self-dual).

Condition 1 then is obtained by expressing circuits of the given types in (C4.1) and (C5.1) cannot occur, condition 2 with (C4.2) and (C5.2), etc.... In choosing a "general" point, one can make use of the fact that the amalgon $\mathcal{H}$ has certain automorphisms. It is easy using $(**)$ to check that the following maps preserve incidence and hence are isomorphisms of $\mathcal{H}$ (since the action on elements with less than five coordinates is determined by the action on the elements with exactly five coordinates, we only give the latter action):

$$\begin{array}{rccl}
\alpha_1 : & (a,l,a',l',a'') & \longrightarrow & (a,l+K,a',l',a'') \\
& [k,b,k',b',k''] & \longrightarrow & [k,b,k'+2kK,b',k''+K] \\
\\
\alpha_2 : & (a,l,a',l',a'') & \longrightarrow & (a,l,a'+B,l',a'') \\
& [k,b,k',b',k''] & \longrightarrow & [k,b,k',b'+B,k''] \\
\\
\alpha_3 : & (a,l,a',l',a'') & \longrightarrow & (a,l,a',l'+L,a'') \\
& [k,b,k',b',k''] & \longrightarrow & [k,b,k'+L,b',k''] \\
\\
\alpha_4 : & (a,l,a',l',a'') & \longrightarrow & (a,l,a'+2aA,l',a''+A) \\
& [k,b,k',b',k''] & \longrightarrow & [k,b+A,k',b',k'']
\end{array}$$

This shows also the last assertion of the theorem.□

Denote by α_3 the permutation of $GF(q)$ mapping x to x^3.

THEOREM 6. *The amalgamated geometry $W(q)\,{}_\sigma\!\uplus_\tau Q(4,q)$ is isomorphic to the classical generalized hexagon arizing from the adjoint Chevalley group $G_2(q)$ if and only if q is a power of 3, both σ and τ are automorphisms of the field $GF(q)$ and $\tau\sigma^{-1} = \alpha_3$, the generating automorphism in $GF(q)$.*

PROOF. Suppose $W(q)\,{}_\sigma\!\uplus_\tau Q(4,q)$ is a generalized hexagon. Expressing that its automorphism group contains a "root elation" mapping $(0,0,0,0,0)$ to $(A,0,0,0,0)$ for some $A \in GF(q)$, e.g. an automorphism fixing all elements incident with one of (∞), $[0]$, $(0,0)$, $[0,0,0]$, $(0,0,0,0)$, we obtain immediately that σ, and dually τ, has to be an automorphism (and the hexagon arising from $G_2(q)$ admits such automorphism since it is Moufang, see Tits [12]. Furthermore, one can recoordinatize in the following way: For $x = a', a'', b, b'$, substitute x^σ by x in $(**)$. This amounts to substitute the identity for σ and $\tau\sigma^{-1}$ for τ. Now if $W(q)\,{}_\sigma\!\uplus_\tau Q(4,q)$ is isomorphic to $G_2(q)$, then its automorphism group should also contain some gemeralized homologies, see Van Maldeghem [14]. Expressing that it contains a generalized homology fixing every point on the line $[\infty]$, stabilizing the hexagon through (∞), $(0,0,0)$ and $(0,0,0,0)$ and mapping the line $[1]$ to $[K]$, for arbitrary $K \in GF(q)^*$, one sees easily that $\tau\sigma^{-1} = \alpha_3$, hence α_3 is an automorphism of $GF(q)$, so q is a power of 3.

Conversely, suppose $W(q)\,{}_\sigma\!\uplus_\tau Q(4,q)$ has the above mentioned properties. The generalized hexagon $G_2(q)$ with q a power of 3 has regular points and regular lines (since it is self-dual) and hence by theorem 4, it must arize as an amalgon of generalized quadrangles, which are readily seen to be classical (indeed, one of them is the point-residue of a non-degenerate parabolic quadric in $PG(6,q)$; the other one is then the dual). Since the only way that this can happen is as mentioned, by the first part of the proof, $W(q)\,{}_\sigma\!\uplus_\tau Q(4,q)$ must be isomorphic to $G_2(q)$ under the given conditions. □

Theorem 6 provides a construction (or representation) of the Moufang generalized hexagon arising from $G_2(q)$, q a power of 3. This construction is extremely useful when dealing with questions where one particular flag (an incident point-line pair) remains fixed.

THEOREM 7. *The amalgamated geometry $W(q)\,{}_\sigma\!\uplus_\tau Q(4,q)$ for $2 \leq q \leq 9$ is a generalized hexagon if and only if $q = 3$ or $q = 9$ and then it is isomorphic to $G_2(q)$.*

PROOF. This has been checked by computer. The cases $q = 2,3,4,8$ of course are trivial. No attempt has been made to include more values for q since obviously only powers of 3 will be of interest.□

FURTHER RESULTS: With the aid of a computer, we have checked that for $q = 27$, $G_2(27)$ always arises granted one of σ or τ is an automorphism. If both are field automorphisms, then the conditions in theorem 5 simplify to a certain extent and we conjecture that also in this case, no other generalized hexagons arise. We have checked this by computer for all $q = 3^h$, $h \leq 10$.

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