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On Linear Representations of (α, β)-Geometries

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In [11] P. J. Cameron introduced partial quadrangles and raised the question of finding a characterization of partial quadrangles which have linear representations. An almost complete answer was given in [9]: the proof was a number-theoretic one. In this paper we discuss the question for a more general class of geometries, namely the (α, β)-geometries. We shall specialize to the case of (0, 1)-geometries, and we shall give a geometric characterization of the partial quadrangle $T_2^*(\mathcal{O})$.

1. Introduction

1.1. (α, β)-*Geometries*

An (α, β)-geometry $S = (P, B, I)$ is a connected partial linear space of order (s, t) (i.e. 2 points are incident with at most 1 line, each point is incident with $t + 1$ $(t \geqslant 1)$ lines, and each line is incident with $s + 1$ $(s \geqslant 1)$ points), with the property that for every anti-flag (x, L) of S there are either α or β lines through x intersecting L.

The point graph $\Gamma(S)$ of an (α, β)-geometry is the graph with vertex set the set of points of S: two vertices are adjacent iff they are different and collinear in S.

If $\alpha = \beta$, S is called a partial geometry with parameters (s, t, α) [3] and in this case the graph $\Gamma(S)$ is a strongly regular graph with parameters $v = (s + 1)(st + \alpha)/\alpha$, $k = (t + 1)s$, $\lambda = s - 1 + t(\alpha - 1)$, $\mu = (t + 1)\alpha$. The most important partial geometries are those with parameters $(s, t, 1)$, the generalized quadrangles introduced by J. Tits in [42]. For more information and examples on generalized quadrangles we refer to [28]. For an overview of models of partial geometries we refer to [14].

Another important family of (α, β)-geometries are the so-called $(0, \alpha)$-geometries (i.e. $\beta = 0$) [17, 41]. Here the point graph is not necessarily a strongly regular graph, but the $(0, \alpha)$-geometries which have a strongly regular point graph are called semipartial geometries with parameters (s, t, α, μ) and were introduced in [19]. Note that the parameter μ is the parameter of the strongly regular point graph, which counts the number of vertices adjacent to two non-adjacent vertices. Moreover, if $\alpha = 1$ then these semipartial geometries are nothing other than the partial quadrangles with parameters (s, t, μ), as they were introduced by P. J. Cameron [11]. For an overview of models of semipartial geometries and partial quadrangles we refer again to [14].

1.2. *Affine Embeddings and Linear Representations of Geometries*

An (α, β)-geometry $S = (P, B, I)$ is said to be (fully) embedded in a projective or an affine space if B is a subset of the set of lines of the space and if P is the set of all points of the space on these lines. We will always assume in what follows that the dimension of the space is the smallest possible dimension for an embedding. There exists a complete classification of partial geometries embedded in a projective space: see [8] for

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the generalized quadrangles and [16] for $\alpha > 1$. The classification of partial geometries embeddable in an affine space is also known [39]. In the case of generalized quadrangles some sporadic embeddings even occur. The complete classification of semipartial geometries embeddable in a projective space is known for $\alpha > 1$ and for $s > 2$. If S is a semipartial geometry with $\alpha = s = 2$, then S is a cotriangle space, and those are clasisfied [32, 33]. However, as explained in [21], it is impossible to classify the projective embeddings of all the cotriangle spaces, it is only known for small dimensions. The embedding of a semipartial geometry in an affine space is unsolved. The classification is only known for the dimensions 2 and 3 [18].

An easier question to handle is the complete classification of geometries which have a linear representation in an affine space.

A linear representation of an (α, β)-geometry of order (s, t) in $AG(n+1, s+1)$ is an embedding of $S = (P, B, I)$ in $AG(n+1, s+1)$ such that the line set B of S is a union of parallel classes of lines of $AG(n+1, s+1)$ and hence the point set P of S is the point set of $AG(n+1, s+1)$. These lines of S define in the hyperplane at infinity Π_∞ a set of points $\mathcal{K}$ of size $t+1$ such that every line of Π_∞ intersects $\mathcal{K}$ in either 0, $\alpha+1$ or $\beta+1$ points. A line intersecting $\mathcal{K}$ in m points will be called an m-secant. A 1-secant will also be called a tangent line, while a line not intersecting $\mathcal{K}$ will be called a passant.

Using standard notations, the linear representation of an (α, β)-geometry S in $AG(n+1, s+1)$ will be denoted by $T_n^*(\mathcal{K})$.

Many examples of linear representations of (α, β)-geometries are known. We will pay special attention to the cases $\alpha = \beta$ and $\beta = 0$.

2. Linear Representations of Partial Geometries and Semipartial Geometries

2.1. *Linear Representations of Partial Geometries*

As we already have mentioned, all partial geometries embeddable in an affine space $AG(n+1, s+1)$ are classified in [39]. If we restrict our attention to the linear representations, the result can be summarized as follows:

Theorem 1 ([39]). *If $T_n^*(\mathcal{K})$ $(n > 1)$ is a linear representation of a partial geometry of order (s, t), then either $\mathcal{K}$ is the complement of a hyperplane (hence $\alpha = s$) or $n = 2$.*

Remarks. 1. The partial geometries $T_1^*(\mathcal{K})$ are of course the nets of order $s+1$ and degree $t+1$ constructed by removing $s+1-t$ parallel classes of an affine plane of order $s+1$.

2. If $T_2^*(\mathcal{K})$ is a partial geometry then $\mathcal{K}$ is a set of $(s+2)\alpha+1$ points in the plane Π_∞ at infinity, such that each line of Π_∞ either intersects $\mathcal{K}$ in $\alpha+1$ points or in no point at all. Such a set is called a maximal arc of degree $\alpha+1$. If $\alpha = 1$ then $s+1 = 2^h$ and $\mathcal{K}$ is a complete oval (or hyperoval) $\mathcal{O}$ in the projective plane Π_∞, and this generalized quadrangle $T_2^*(\mathcal{O})$ has order $(2^h-1, 2^h+1)$: see [28] for more information. If $q > \alpha > 1$ then $s+1 = q$ is a multiple of $\alpha+1$ and the partial geometry $T_2^*(\mathcal{K})$ has parameters $(q-1, (q+1)\alpha, \alpha)$: it was first constructed by J. A. Thas [36] and independently by W. D. Wallis [44]. It is known that a necessary condition for the existence of maximal arcs of degree $d \leqslant q$ in a projective plane of order q is that d should divide q. It is known that this condition is also a sufficient condition if $q = 2^h$ [20] and [37]; however, it is not if q is odd, as there cannot exist maximal arcs of degree

3 in a desarguesian projective plane $PG(2, 3^h)$ $(h > 1)$ [12, 38]. The existence of maximal arcs, with $1 < d < q$, in projective planes of odd order $q > 3$ is open.

2.2. *Linear Representations of Semipartial Geometries*

We first consider the semipartial geometries $T_2^*(\mathcal{K})$:

THEOREM 2 ([18]). *Let S be a proper semipartial geometry, i.e. it is not a partial geometry; then S is not embeddable in an affine plane. If it is embedded in an* $AG(3, s+1)$, *then* $s+1$ *is a square* $(s+1 = q^2)$ *and it is necessarily a linear representation* $T_2^*(\mathcal{K})$. *Moreover,* $\mathcal{K}$ *is either a unital or a Baer subplane.*

REMARKS. 1. If $\mathcal{K}$ is a unital in the projective plane $\Pi_\infty = PG(2, q^2)$ at infinity, then $T_2^*(\mathcal{K})$ has parameters $(s = q^2 - 1, t = q^3, \alpha = q, \mu = q^2(q^2-1))$.

2. If $\mathcal{K}$ is a Baer subplane of the projective plane $\Pi_\infty = PG(2, q^2)$ at infinity, then $T_2^*(\mathcal{K})$ has parameters $(s = q^2 - 1, t = q(q+1), \alpha = q, \mu = q(q+1))$.

We now consider a semipartial geometry $T_n^*(\mathcal{K})$. The above construction of $T_2^*(\mathcal{K})$ with $\mathcal{K}$ a Baer subplane of the plane Π_∞ at infinity can be generalized for any dimension. Indeed, if $\mathcal{K}$ is a Baer subspace of the hyperplane $\Pi_\infty = PG(n, q^2)$ at infinity, then $T_n^*(\mathcal{K})$ is a semipartial geometry with parameters

$$\left(s = q^2 - 1,\, t = \frac{q^{n+1}-1}{q-1} - 1,\, \alpha = q,\, \mu = q(q+1)\right).$$

This semipartial geometry has the same parameters as the semipartial geometry $H_q^{(n+2)\star}$, constructed in [19] in the following way.

Consider an n-dimensional subspace H of the projective space $PG(n+2, q)$ $(n \geqslant 1)$. The point set of this geometry is the set of lines of $PG(n+2, q)$ which have no point in common with H, while the line set of this geometry is the set of planes of $PG(n+2, q)$ which have exactly 1 point in common with H: the incidence is the natural incidence. One can easily prove that $T_n^*(\mathcal{K})$ with $\mathcal{K}$ a Baer subspace is indeed the linear representation in $AG(n+1, q)$ of $H_q^{(n+2)\star}$. We remark that $H_q^{3\star}$ is a well known net, a so-called regulus net, which is indeed embeddable in an affine plane $AG(2, q^2)$. For more information on (t)-regulus nets we refer to [24] and [15].

To our knowledge, $T_n^*(\mathcal{K})$ $(n > 2)$ with $\mathcal{K}$ a Baer subspace is the only semipartial geometry known which is neither a partial geometry nor a partial quadrangle and has a linear representation in $AG(n+1, s+1)$.

LEMMA 1. $T_n^*(\mathcal{K})$ *is a linear representation of a semipartial geometry with parameters* (s, t, α, μ) *iff* $\mathcal{K}$ *is a set of points in the projective space* $PG(n, s+1)$ *at infinity, with the property that each line of this projective space intersects* $\mathcal{K}$ *in either* 0, 1 *or* $\alpha + 1$ *points, and such that any point of the projective space not on* $\mathcal{K}$ *lies on* $\mu/[\alpha(\alpha+1)]$ $(\alpha+1)$*-secants and on* $t + 1 - (\mu/\alpha)$ *tangent lines.*

PROOF. We only have to prove the assumption on the number of $(\alpha+1)$-secants through a point not on $\mathcal{K}$. This assumption is equivalent to the property that the point graph of the semipartial geometry is a strongly regular graph. Indeed, let x and y be 2 points of the affine space $AG(n+1, s+1)$ which are not collinear; then the line xy of the affine space will intersect the hyperplane Π_∞ at infinity in a point p not on $\mathcal{K}$. Every line through p which intersects $\mathcal{K}$ yields $\alpha(\alpha+1)$ points of $AG(n+1, s+1)$ collinear with both x and y; hence p is incident with $\mu/[\alpha(\alpha+1)]$ intersecting lines. As $|\mathcal{K}| = t+1$, there are $t + 1 - (\mu/\alpha)$ tangent lines. □

REMARK. Because of the above lemma, a complete classification of linear representations is far from being trivial. However, for the case $\alpha = 1$, i.e. for the partial quadrangles, there is an almost complete classification, as we will explain in the next section.

3. LINEAR REPRESENTATIONS OF PARTIAL QUADRANGLES

3.1. *Calderbank's Theorem*

If $T_n^*(\mathcal{K})$ is the linear representation of a partial quadrangle with parameters (s, t, μ) then, by the above lemma, $\mathcal{K}$ is a set of points in the projective space at infinity such that every line is either a passant, a tangent or a 2-secant, i.e. it is a $(t+1)$-cap with the property that each point not in $\mathcal{K}$ is on $t + 1 - \mu$ tangents. R. Calderbank [9] has given an almost complete classification of partial quadrangles with a linear representation. His proof is a number-theoretic proof. He lists the possible parameter values of the associated strongly regular graph.

THEOREM 3 ([9]). *Assume that $T_n^*(\mathcal{K})$ is a linear representation in* AG$(n+1, q)$ *of a partial quadrangle which is not a generalized quadrangle. Then there exists an integral solution (y, a) of the diophantine equation*

$$y^2 = 4q^{a/2} + 4q + 1,$$

where the parameter a is a function of the dimension $n + 1$.

REMARKS. 1. The diophantine equation has always as solution $(y, a) = (2q + 1, 4)$, and from the proof of the theorem of Calderbank it follows that the partial quadrangle related to this solution has parameters $(s = q - 1, t = q^2, \mu = q(q-1))$ and the linear representation is in AG$(4, q)$. As we will discuss in the next section, a lot of partial quadrangles are known with these parameters.

2. Suppose that $q = 2$. Then the partial quadrangle coincides with its point graph, and R. Calderbank proves that there is only one solution, the strongly regular graph srg$(v = 16, k = 5, \lambda = 0, \mu = 2)$, which is the Clebsch graph and is uniquely defined by its parameters. This example is, of course, the smallest example from the class discussed in 1.

3. Suppose that $q = 3$. Then it is shown in [5] that the diophantine equation, which is then of the form $y^2 = 4 \cdot 3^b + 13$, has only the solutions $(y, b) = (5, 1)$, $(7, 2)$ and $(11, 3)$. The first solution corresponds to the partial quadrangle $T_4^*(\mathcal{K})$ with parameters $(s = 2, t = 10, \mu = 2)$ where $\mathcal{K}$ is an 11-cap in PG$(4, 3)$; see, for instance, [13] and [29] for descriptions. The second solution fits in the infinite class mentioned in 1. Finally, the solution $(11, 3)$ yields the partial quadrangle $T_5^*(\mathcal{K})$ with parameters $(s = 2, t = 55, \mu = 20)$, where $\mathcal{K}$ is the unique 56-cap in PG$(5, 3)$. This cap was first constructed by B. Segre [31], but was also studied by several other authors: see, for example [1, 7, 22, 27, 40].

4. Suppose that $q = 4$. It is then shown in [2] that the diophantine equation, which is now of the form $y^2 = 2^b + 17$, has as its only integer solutions $(y, b) = (5, 3), (7, 5), (9, 6)$ and $(23, 9)$. The solution $(9, 6)$ again fits in the class discussed in 1, while the solution $(5, 3)$ corresponds with the generalized quadrangle of order $(3, 5)$. The solution $(7, 5)$ yields a partial quadrangle $T_5^*(\mathcal{K})$ with parameters $(s = 3, t = 77, \mu = 14)$, where $\mathcal{K}$ is a 78-cap in PG$(5, 4)$ such that each external point is on 7 secants. At least one example exists and was discovered by Hill [23]. The solution

(23, 9) corresponds with a partial quadrangle $T_6^*(\mathcal{K})$ with parameters ($s = 3$, $t = 429$, $\mu = 110$), where $\mathcal{K}$ is a 430-cap in PG(6, 4) such that each external point is incident with 55 2-secants. Up to now, however, the existence of such a cap is not known.

5. Suppose that $q \geq 5$. Then it is proved in [43] that the diophantine equation $y^2 = 4q^{a/2} + 4q + 1$ has only the integer solution $(y, a) = (2q + 1, 4)$.

6. The existence of a cap $\mathcal{K}$ in PG(n, q) such that every exterior point is on a constant number of tangents implies the existence of a uniformly packed $[|\mathcal{K}|, |\mathcal{K}| - n - 1, 4]$ code C, which means that the dual code $C^\perp$ is a $[|\mathcal{K}|, n + 1]$ code over GF(q) with exactly 2 weights [9, 10].

3.2 *Partial Quadrangles Constructed from Generalized Quadrangles*

It is knwon that if $\mathrm{S} = (\mathrm{P}, \mathrm{B}, \mathrm{I})$ is a generalized quadrangle, then one can construct a (0, 1)-geometry in the following way. Let p be any point of S, and denote by $p^\perp$ the set of all points of S collinear with p (the trace of p) and by $\mathrm{B}(p)$ the set of lines of S through p. Then the incidence structure $\mathrm{S}_p = (\mathrm{P}_p, \mathrm{B}_p, \mathrm{I}_p)$ with $\mathrm{P}_p = \mathrm{P} \backslash p^\perp$, $\mathrm{B}_p = \mathrm{B} \backslash \mathrm{B}(p)$, and with $\mathrm{I}_p = \mathrm{I} \cap (\mathrm{P}_p \times \mathrm{B}_p)$ is clearly a (0, 1)-geometry of order $(s - 1, t)$.

Lemma 2. *Let S_p be a (0, 1)-geometry of order $(s - 1, t)$ constructed from a generalized quadrangle* S *of order (s, t) in the above way. Then S_p satisfies the following property*:

(*) *If L and M are 2 disjoint lines of S_p then there are either 0, $s - 1$ or s lines of S_p concurrent to both L and M.*

Proof. Let L and M be 2 disjoint lines of S_p. Then as lines of S the following possibilities occur. Either L and M intersect in a point x of $p^\perp$ or they are disjoint in S and intersect in 2 collinear or 2 non-collinear points of $p^\perp$. In the first case there are no lines of S_p concurrent with L and M, while in the second case there are s, and in the last case there are $s - 1$. □

Note that this property is of course trivial in the case $s = 2$. The point graph $\Gamma(\mathrm{S}_p)$ of S_p will be a strongly regular graph with parameter μ iff for any 2 non-collinear points x and y in P_p, the set $\{p, x, y\}^\perp$ of points in S collinear with p, x and y has cardinality $t + 1 - \mu$. It is known (see [4, 11]) that in a generalized quadrangle S, $|\{x, y, z\}^\perp|$ is a constant for any triad $\{x, y, z\}$ of non-collinear points iff S has order (s, s^2); moreover, in this case $|\{x, y, z\}^\perp| = s + 1$. Hence the only partial quadrangles of type S_p have parameters $(s - 1, s^2, s(s - 1))$.

Remark. Many generalized quadrangles of order (s, s^2) are known. In all of them, s is a prime power q and in what follows we will therefore use q instead of s.

First of all, there is the semi-classical example $T_3(\mathcal{O})$, constructed by J. Tits [42] (see also [28]). If p is the special point ∞ in $T_3(\mathcal{O})$ then the resulting partial quadrangle has a linear representation in AG(4, q): it is the partial quadrangle $T_3^*(\mathcal{O})$ with $\mathcal{O}$ an ovoid in the hyperplane Π_∞. If p is any other point of $T_3(\mathcal{O})$ then the resulting partial quadrangle might be non-isomorphic to $T_3^*(\mathcal{O})$. On the other hand, it is known that any flock of a cone in PG(3, q) implies the existence of a generalized quadrangle of order (q, q^2), and these generalized quadrangles give rise to non-isomorphic partial quadrangles with parameters $(q - 1, q^2, q(q - 1))$.

In [26], Ivanov and Shpectorov prove that every partial quadrangle with parameters $(q - 1, q^2, q(q - 1))$ is of type S_p and is uniquely extendable to a generalized quadrangle S. For this they prove that such a partial quadrangle always satisfies property (*). In fact, they prove an even more general result. Indeed, they prove that every strongly regular graph srg(q^3, $(q^2+1)(q-1)$, $q-2$, $q(q-1)$) such that every 2 adjacent vertices are contained in a clique of order q is the point graph of a partial quadrangle of type S_p, and this partial quadrangle is uniquely extendable to a generalized quadrangle of order (q, q^2). We remark that this implies that the srg(81, 20, 1, 6) is unique; see, for instance, [6] in which another proof of the result of Ivanov and Shpectorov is also given.

In any case, it follows from the theoreom of Calderbank that the linear representation of a partial quadrangle with parameters $(q-1, q^2, q(q-1))$ should be in AG$(4, q)$, and hence it should be $T_3^*(\mathcal{O})$, with $\mathcal{O}$ an ovoid in Π_∞.

We may ask whether there exist other (0, 1)-geometries of type S_p (i.e. constructed from a generalized quadrangle S) with a linear representation. Another question is whether the sporadic partial quadrangles with a linear representation do satisfy property (*). We will consider both questions in the next section.

4. Linear Representations of (0, 1)-Geometries

It is clear that the (0, 1)-geometries of order $(q-1, t)$ of type $T_1^*(\mathcal{K})$ are necessarily the grids of order $q-1$, i.e. the generalized quadrangles of order $(q-1, 1)$. If S is a (0, 1)-geometry of type $T_2^*(\mathcal{K})$, then $\mathcal{K}$ is a set of points in the plane Π_∞ such that every line intersects in 0, 1 or 2 points, i.e. $\mathcal{K}$ is an arc in Π_∞. $T_2^*(\mathcal{K})$ satisfies (*) iff $|\mathcal{K}|$ is $q+1$ or $q+2$. For higher dimensions we will assume property (*), and the main result is proved in the next theorem.

Theorem 4 *Let $T_n^*(\mathcal{K})$ $(n \geq 3)$ be a linear representation of a (0, 1)-geometry, of order $(q-1, t)$, $q > 2$ that satisfies (*). If $\mathcal{K}$ spans the hyperplane Π_∞, then $T_n^*(\mathcal{K})$ is the partial quadrangle $T_3^*(\mathcal{O})$.*

Proof. 1. Suppose first that $n > 4$. We may assume that there are at least $n+1$ linear independent points in $\mathcal{K}$. We take 5 of them and consider the 4-space β that they generate inside Π_∞. Embedding β as a hyperplane in a 5-space which intersects AG$(n+1, q)$ non-trivially, we see that we obtain a linear representation of a (0, 1)-subgeometry in AG$(4, q)$. Hence we may assume that n is either 3 or 4.

2. Suppose that $n = 3$ and let q be odd. Let L_1 and L_2 be 2 non-intersecting lines of $T_3^*(\mathcal{K})$ that define 2 different points p_1, p_2 in Π_∞. Suppose that there are q lines M_j $(j = 1 \cdots q)$, concurrent with both L_1 and L_2; then the q points $M_j \cap \Pi_\infty$ together with p_1 and p_2 define a set of $q+2$ points in the plane $\Pi_\infty \cap \langle L_1, L_2 \rangle$ with no 3 on a line—a contradiction as q is odd. Hence we may assume that every 2 lines of the (0, 1)-geometry which are disjoint and non-parallel have either 0 or $q-1$ common intersecting lines. Hence $\mathcal{K}$ is a set of $t+1$ points in Π_∞ such that every plane intersects in either 0, 1 or 2 points or in a conic. Let p be any point of $\mathcal{K}$: we may assume that there is a plane β in Π_∞ such that $p \notin \beta$ and such that β intersects $\mathcal{K}$ in a conic. Let us first suppose that there exists a plane β_p through p that intersects $\mathcal{K}$ only in p. Let L be any line through p which is not a line of β_p; then $L \cap \beta = p'$. Through p' there is at least 1 line M that intersects the conic $\beta \cap \mathcal{K}$ in 2 points m_1 and m_2. The plane $\langle p, m_1, m_2 \rangle$ intersects $\mathcal{K}$ in a conic C with tangent line at p the line $\langle p, m_1, m_2 \rangle \cap \beta_p$; hence the line L intersects C in another point of $\mathcal{K}$. Hence we have proved that every line through p which is not in β_p intersects $\mathcal{K}$ in another point; hence $\mathcal{K}$ has q^2+1 points, and is an ovoid in Π_∞. Now suppose that there is no tangent plane at $\mathcal{K}$. If we project the set $\mathcal{K}$ from any point p of $\mathcal{K}$ on a plane β not through p, then

this yields a set $\mathcal{K}'$ in β such that every line of β intersects $\mathcal{K}'$ in either 1 or q points. Hence $\mathcal{K}'$ is an affine plane of order q [35], but then $\mathcal{K}$ has a tangent plane at p—a contradiction.

3. Suppose that $n = 3$ and q is even. In this case any plane of Π_∞ intersects $\mathcal{K}$ in 0, 1 or 2 points, in an oval or in a complete oval. If there is at least one tangent plane at $\mathcal{K}$ then the same reasoning as for the case q odd yields that $\mathcal{K}$ is an ovoid. Suppose there is no tangent plane at $\mathcal{K}$; then again by projecting $\mathcal{K}$ from any point p of $\mathcal{K}$ on a plane β not through p, one yields a set $\mathcal{K}'$ in β such that every line intersects $\mathcal{K}'$ in either 1, q or $q+1$ points. Hence $\mathcal{K}'$ is either the set of points of the plane β minus 1 point or minus q collinear points [35]. In the first case $\mathcal{K}$ has q^2+q+1 points, whereas in the second case $\mathcal{K}$ has q^2+2 points, i.e. $|\mathcal{K}| > q^2+1$—a contradiction [30].

4. Suppose that $n = 4$. By the same technique as above, the set $\mathcal{K}$ in $\mathrm{PG}(4, q) = \Pi_\infty$ meets every hyperplane of Π_∞ in either the empty set, a single point, 2 points, an oval or an ovoid. Note that we cannot have a complete oval, since we assume that $\mathcal{K}$ spans Π_∞ and no complete oval is contained in an ovoid. Hence the projection $\mathcal{K}'$ of $\mathcal{K}$ from a point p of $\mathcal{K}$ onto some hyperplane β not containing p has the property that the intersection of $\mathcal{K}'$ with any plane in β is either empty, a single point, a set of q points on a line or the complement of a line (call this an affine plane). Since we may assume that $\mathcal{K}$ contains at least 5 linearly independent points, among them p, we see that there are at least 4 linearly independent affine planes α_i $(i = 1, 2, 3, 4)$ in $\mathcal{K}'$. Denote by L_i the line of β completing the plane α_i, $i = 1, 2, 3, 4$ to a projective plane. An arbitrary plane α of β meets the (union of the planes) α_i in at least 2 distinct lines, each of them different from L_i, $i = 1, 2, 3, 4$; hence $\alpha \cap \mathcal{K}'$ contains at least 2 affine lines, and consequently $\alpha \cap \mathcal{K}'$ is an affine plane. Now let L be any line of α: then, clearly, $L \cap (\beta \backslash \mathcal{K}')$ is either a point or L itself. Hence, since L and α are arbitrary, $\beta \backslash \mathcal{K}'$ is a hyperplane, forcing $\mathcal{K}$ to have q^3+1 points, too many for a cap [34] (see also [25]). Alternatively, we can count the number of ovoids contained in $\mathcal{K}$. Since such an ovoid is determined by 4 linearly independent points, this number is equal to

$$\frac{(q^3+1)q^3(q^3-1)(q^3-q)}{(q^2+1)q^2(q^2-1)(q^2-q)} = \frac{q(q+1)(q^2-q+1)(q^2+q+1)}{q^2+1}$$

and this is an integer iff $(q-1)/(q^2+1)$ is—a contradiction. Hence the case $n = 4$ cannot occur. This completes the proof of the theorem. □

COROLLARIES. 1. *None of the sporadic partial quadrangles which have a linear representation satisfy* (∗).

2. *If* S *is a generalized quadrangle and* S_p *has a linear representation, then* $S \simeq T_3(\mathcal{O})$.

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