A CHARACTERIZATION OF SOME RANK 2 INCIDENCE GEOMETRIES BY THEIR AUTOMORPHISM GROUP

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Abstract

Using the classification of the finite simple groups, we classify all finite point-line geometries with a diameter exceeding the gonality by at most 1 and having an automorphism group acting transitively on the set of maximal geodesics of each given type.

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1 INTRODUCTION.

1.1 History and Motivation.

Efforts in order to classify rank 2 geometries (bipartite graphs) with large automorphism groups have been made in various contents such as the Moufang polygons (see TITS [85, 86, 87, 88, 89], WEISS [94], THAS, PAYNE & VAN MALDEGHEM [82]), distance transitive graphs (BROUWER, COHEN & NEUMAIER [10]), distance transitive generalized polygons (BUEKENHOUT & VAN MALDEGHEM [15, 16]) flag-transitive designs (KANTOR [58], BUEKENHOUT, DE LANDTSHEER, DOYEN, KLEIDMAN, LIEBECK & SAXL [14]). A synthesis and further deepening of these efforts seems suitable and possible. The present paper takes this direction. In particular, we want to generalized the results of [15, 16] on generalized polygons to a larger class of geometries. We consider a rank 2 geometry Γ which is a (g, d_p, d_l)-gon with $2 \leq g \leq d_p \leq d_l \leq g+1$ (see 1.2.1). That situation includes the generalized polygons, the linear spaces, the partial geometries, the Moore geometries and the symmetric 2-designs. Some very interesting geometries escape to it, such as partial geometries. We assume that Γ is finite and that it is equipped with an automorphism group acting transitively on the ordered maximal geodesics of each possible type. This is a weakening of the Moufang condition and so we get somewhat better results in that direction (our proof however uses the classification of the finite simple groups). On the other hand, our condition is much stronger than the flag-transitivity condition used successfully in some of the earlier work. We get a complete classification. This provides an objective basis for classes of geometries enlarging the class of classical generalized polygons. It may be useful for extensions to geometries of rank greater than or equal to three.

1.2 Definitions and Notation.

1.2.1 Regular (g, d_p, d_l)-gons.

A rank 2 point-line incidence geometry Γ consists of a triple $(\mathcal{P}, \mathcal{L}, I)$, where $\mathcal{P}$ is the set of **points**, $\mathcal{L}$ is the set of **lines** and $I \subseteq (\mathcal{P} \times \mathcal{L}) \cup (\mathcal{L} \times \mathcal{P})$ is a symmetric incidence relation. The elements of $\mathcal{P}$ and $\mathcal{L}$ are also called **varieties** and the **type** $\mathrm{typ}(x)$ of a variety is its name (the appropriate "point" or "line"). The elements of I are usually called **flags**. Two points incident with some line are called **collinear** and two lines incident with some point are **concurrent**. A **path** γ of length n **based at** a variety x is a sequence $(x = x_0, x_1, x_2, \ldots, x_n)$ of $n+1$ varieties with $x_{i-1} I x_i$ for $1 \leq i \leq n$. If $x_0 = x_n$ and if $x_i \neq x_{i+2}$ (for all i to be taken modulo n), then n is even and γ is called a **circuit of diameter** $n/2$. We call Γ **connected** if every two varieties can be joined by a path. The **distance** $d(x, y)$ between two varieties x and y is the length of the shortest path joining x to y (well defined by connectedness). A **geodesic (based at)** x is a path γ based at x such that the length of γ is equal to the distance between the extremeties of γ. A **maximal geodesic** is a geodesic that is not properly contained in another one. The **gonality** g of Γ is the diameter of the smallest circuit (i.e. a circuit of minimal diameter) in Γ. The **local diameter** $d(x)$ of some variety x is the length of the longest geodesic based at x. The maximal value of $d(x)$ for x a point (resp. a line) is the **point-** (resp. **line-**) **diameter** and it is denoted by d_p (resp. d_l). With this notation, Γ is called a (g, d_p, d_l)**-gon**. The dual $\Gamma^D = (\mathcal{L}, \mathcal{P}, I)$ of Γ is obviously a (g, d_l, d_p)-gon (see BUEKENHOUT [12], where this notion is introduced).

Let x be any variety of Γ. Then we denote by $\Gamma_i(x)$ the set of varieties at distance i from x. We call the geometry Γ **regular** if $|\Gamma_i(x) \cap \Gamma_j(y)| = |\Gamma_i(z) \cap \Gamma_j(u)|$ for all positive integers i, j and all varieties x, y, z, u whenever $d(x, y) = d(z, u)$ and $\mathrm{typ}(x) = \mathrm{typ}(z)$ (and hence $\mathrm{typ}(y) = \mathrm{typ}(u)$). In particular putting $i = 1, j = 2$ and $d(x, y) = 1$ (i.e. x and y are incident), we see that in a regular geometry Γ the number of points (resp. lines) incident with a given line (resp. point) is a constant, say $s+1$ (resp. $t+1$). In that case, we call (s, t) the **order** of Γ. If $s > 1$ and $t > 1$, then we say that Γ is **thick** (terminology of *buildings*, see e.g. TITS [84]). Also, it is straightforward to see that in a regular point-line geometry Γ the length of a maximal geodesic only depends on the types of its extremeties, in other words, the local diameter in every point (resp. line) x is equal to the point-diameter (resp. line-diameter). Note that if $d_p = d_l$ and this is even, then there are two types of maximal geodesics: one kind has points as extremeties and the other kind has lines. In any case we trivially have the inequality $|d_p - d_l| \leq 1$ and by point-line duality, we may assume $g \leq d_p \leq d_l$, in other words we assume that the diameter (which is in general the larger value among d_p, d_l) is equal to d_l. Note that if d_l is odd, then $d_p = d_l$ (this is obvious, see also BUEKENHOUT [12]).

Finally, a **graph** is a geometry in which every line is incident with exactly 2 points. In this case, the points are called **vertices** (**adjacent** if they are collinear) and the lines **edges**. The **incidence graph** of an arbitrary geometry $\Gamma = (\mathcal{P}, \mathcal{L}, I)$ is the graph whose set of vertices is $\mathcal{P} \cup \mathcal{L}$ and in which two vertices form an edge if they form a flag in Γ. We denote the incidence graph of Γ by Γ^I.

In this paper, we will always assume that Γ is finite, connected and regular.

1.2.2 Some classical examples.

The following particular cases provide the main motivation for (g, d_p, d_l)-gons. In most cases that we mention $g \leq d_p \leq d_l \leq g + 1$.

A **generalized** n**-gon** is a regular (n, n, n)-gon and conversely. These were introduced by TITS [83]. If $n = 3$, they are **projective planes**. For $n = 4, 6, 8$, they are called **generalized quadrangles**, respectively **generalized hexagons**,**generalized octagons**. A generalized 2-gon (or **generalized digon**) is a trivial incidence geometry (every point is incident with every line). By a theorem of FEIT & HIGMAN [33], a generalized n-gon of order (s, t) with $s, t \geq 2$ can only exist if $n \in \{2, 3, 4, 6, 8\}$.

A **linear space** is any geometry with gonality 3 and point-diameter 3, hence in a linear space, two points determine uniquely a line. So a linear space is either a *generalized* projective plane (i.e. a $(3, 3, 3)$-gon) or a $(3, 3, 4)$-gon.

A **partial geometry** with parameters (s, t, a), as introduced by BOSE [6], is a point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L}, I)$ of order (s, t) (defined as above) having the properties that (1) every point x is collinear to exactly $a+1$ points incident with any given line not incident with x and (2) two points determine at most one line. Hence the diameter is at most 4 and we have the following possibilities:

1. Γ is a regular $(3,3,3)$-gon, i.e. a projective plane, so Γ has parameters (s,s,s) for some positive integer s.

2. Γ is a regular $(3,3,4)$-gon or a regular $(3,4,3)$-gon, i.e. Γ is a regular *proper* linear space or a regular proper dual linear space.

3. Γ is a regular $(3,4,4)$-gon. Among these, we have the nets and the dual nets (see later). The other members in this class are called the **proper partial geometries**.

4. Γ is a generalized quadrangle and has parameters $(s,t,0)$ for some positive integers s,t.

A **net** of order a and degree b is a partial geometry with parameters $(a-1,b-1,b-2)$. If $a=b$, then it has been called a **helicopter plane** in Van Maldeghem [93]; it is an affine plane with one parallel class of lines removed.

A **partial quadrangle**, as introduced by Cameron [17], with parameters (s,t,a) is a point-line geometry $\Gamma=(\mathcal{P},\mathcal{L},I)$ of gonality $g\geq 4$ such that every two non-collinear points are collinear with exactly $a+1$ points. In general, this is a $(4,5,6)$-gon, but if $a=t$, then we have a generalized quadrangle; if $a=0$, then we have a $(5,5,6)$ or $(5,5,5)$-gon; if Γ is also a dual partial quadrangle, then it is a regular $(4,5,5)$-gon. Partial quadrangles which are not generalized quadrangles are also known as **near pentagons**, see e.g. Brouwer, Cohen & Neumaier [10].

A **Moore geometry** is a $(g,g,g+1)$-gon for $g\geq 3$ and g odd (see Buekenhout [12]). As for generalized polygons, there is here a restriction on g, see subsection 4.3.3. A Moore geometry was originally defined as a point-line geometry of order (s,t) such that every two points are joined by a unique geodesic, see Bose & Dowling [7].

A **symmetric design** or **square design** with parameters (v,k,λ) is an incidence structure $\Gamma=(\mathcal{P},\mathcal{L},I)$ of order $(k-1,k-1)$ such that 2 blocks intersect in λ points and 2 distinct points determine exactly λ blocks. The positive integer v is the total number of points. Γ is also called a $2-(v,k,\lambda)$-design. If $1<\lambda<k$, then Γ is a regular $(2,3,3)$-gon. If $\lambda=k$, it is a generalized digon and if $\lambda=1$, then Γ is a projective plane, hence a regular $(3,3,3)$-gon. The **complementary** design of a symmetric design Γ is the symmetric design Γ^C obtained from Γ by replacing each block by its complement.

1.2.3 Automorphisms, collineations and correlations.

A **collineation** of the geometry $\Gamma=(\mathcal{P},\mathcal{L},I)$ is a permutation on $\mathcal{P}\cup\mathcal{L}$ preserving $\mathcal{P}$, $\mathcal{L}$ and I. A **correlation** is a permutation on $\mathcal{P}\cup\mathcal{L}$ preserving I and interchanging $\mathcal{P}$ and $\mathcal{L}$. An **automorphism** is either a collineation or a correlation.

In the same way one defines **isomorphisms** and **anti-isomorphisms** between geometries and of course we are only interested in the isomorphism class of a geometry. If Γ is anti-isomorphic to itself, i.e. there is a correlation of Γ, then we call Γ **self-dual**. The group of all collineations, resp. automorphisms of Γ will be denoted by $\mathrm{Col}(\Gamma)$, resp. $\mathrm{Aut}(\Gamma)$.

Now let G be an automorphism group of the geometry Γ and suppose Γ is a (g, d_p, d_l)-gon. We shall use the following terminology.

1. Suppose G acts transitively on the set of pairs (x, y) of points at distance i from each other, for all even positive integers i, then we call (Γ, G) a **point distance transitive** (g, d_p, d_l)-pair, dually a **line distance transitive** (g, d_p, d_l)-pair. If (Γ, G) is both point distance transitive and line distance transitive, then we call (Γ, G) a **weakly distance transitive** (g, d_p, d_l)-pair. If G acts transitively on each set of pairs of *varieties* at distance j from each other and having fixed type, for all positive integers j, then (Γ, G) is called a **distance transitive** (g, d_p, d_l)-pair. Moreover if G contains moreover a correlation, then (Γ, G) is a **full distance transitive** (g, d_p, d_l)-pair.

2. Suppose G acts transitively on each set of geodesics based at some point x of Γ and ending in a point y at maximal distance, for all points $x \in \mathcal{P}$, then we call (Γ, G) a **point geodesic transitive** (g, d_p, d_l)-pair. Similarly as for *distance* transitive (g, d_p, d_l)-pairs, we can define **line geodesic transitive** (g, d_p, d_l)-pairs, respectively **weakly geodesic transitive**, **geodesic transitive** and **full geodesic transitive** (g, d_p, d_l)-pairs.

3. If G acts transitively on each set of geodesics of length i based at some fixed variety x, for all varieties x, then (Γ, G) is called a **locally i-arc transitive** (g, d_p, d_l)-pair. If G acts transitively on the full set of geodesics of length i, then (Γ, G) is called **i-arc transitive**. This generalizes a similar notion for graphs, see e.g. Weiss [95], as it was first introduced by Tutte [91].

It is easy to see that, if Γ is a (g, d_p, d_l)-gon and if $2 \leq g \leq d_p \leq d_l \leq g + 1$, then each of the above assumptions on G implies that Γ is regular. Hence from now on we assume that all geometries are finite and regular unless the contrary is explicitly mentioned.

1.3 Main Results.

Now we are ready to formulate our main results and some immediate corollaries. For a description of the geometries and groups under consideration, we refer to section 2, in particular, we write *appropriate automorphism group* in order not to overload the formulation, but all these groups are described in section 2. The proof of theorem 1 follows from propositions 1 up to 8 of section 4.

The symbol q will always denote a prime power. For the notation of groups, we follow the Atlas [22].

THEOREM 1. *Let (Γ, G) be a finite geodesic transitive (g, d_p, d_l)-pair, $2 \leq g \leq d_p \leq d_l \leq g + 1$, then one of the following holds:*

1. Γ *is a thick generalized polygon related to an irreducible finite adjoint or twisted adjoint Chevalley group* $X_n(q)$ *of relative rank 2 and* $X_n(q) \leq G \leq \mathrm{Aut}(X_n(q))$, *or* Γ *is the flag complex of the self-dual thick generalized polygon related to* $L_3(q)$, $S_4(2^e)$ *or* $G_2(3^e)$ *and* G *is as above with the additional condition that it contains a graph automorphism, or* $G \cong A_6$ *and* Γ *is the unique generalized quadrangle of order* $(2,2)$, *or* Γ *is an ordinary polygon and* G *is the corresponding dihedral group;*

2. Γ *can be identified with the Petersen graph on 10 points, resp. Hoffman-Singleton graph on 50 points; the lines are the edges of the graph and* $G \cong S_5$, *resp.* $U_3(5) \trianglelefteq G \leq U_3(5):2$. *Here,* Γ *can be considered as a Moore geometry, in particular as a* $(5,5,6)$*-gon of order* $(1,2)$, *resp.* $(1,6)$*;*

3. Γ *is a* $(3,4,4)$*-gon. The following cases occur:*

   3.1. Γ *is the helicopter plane* $HG(2,q)$ *obtained from the Desarguesian projective plane* $PG(2,q)$ *by deleting a flag* (x,l) *and all varieties incident with one of* x,l *and* G *contains the stabilizer in* $PGL_3(q)$ *of the flag* (x,l) *in* $PG(2,q)$*;*

   3.2. Γ *is the net* $(\mathcal{H}_q^{n+1})^D$ *of order* q^n *and degree* $q+1$ *and* G *contains a group isomorphic to the semi-direct product of an elementary abelian group* q^{2n} *with a group isomorphic to*

   (a) $(SL_2(q) \times SL_n(q))/Z(SL_2(q) \times SL_n(q))$ *if* $n > 2$, *or*
   (b) $(SL_2(q) \times GL_2(q))/Z(SL_2(q) \times GL_2(q))$ *if* $n = 2$, *or*
   (c) $SL_2(2) \times A_7$ *if* $(n,q) = (4,2)$.

   3.3. Γ *is the dual of 3.2.;*

   3.4. Γ *is the net* $Ne(2^8)$*: its points can be identified with the points of an affine space* $AG(8,2)$ *and its lines are the affine 4-subspaces whose 3-spaces at infinity constitute a 2-transitive spread of a hyperbolic quadric in* $PG(7,2)$*;* G *contains the full translation group of* $AG(8,2)$ *and its kernel "at infinity" is* A_9*;*

   3.5. Γ *is the dual of 3.4.*

4. Γ *is a* $(3,3,4)$*-gon. Three cases occur:*

   4.1. Γ *is the linear space consisting of the points and lines of* $PG(d,q), q \geq 3$ *and* $L_{d+1}(q) \trianglelefteq G \leq P\Gamma L_{d+1}(q)$*;*

   4.2. Γ *is the affine Desarguesian plane* $AG(2,q)$, G *contains all translations and induces at the line at infinity a group containing* $L_2(q)$.

   4.3. G *is a group acting 4-transitively on the set of points of* Γ *and the lines of* Γ *can be identified with the pairs of points;*

5. Γ *is a* $(2,3,3)$*-gon. Here,* Γ *is a symmetric 2-design with* $\lambda > 1$ *and four cases occur:*

5.1. Γ *can be identified with* $PG(d,q), d \geq 3$, *the blocks are either the hyperplanes or their complements and* $L_{d+1}(q) \trianglelefteq G \leq P\Gamma L_{d+1}(q) : 2$ *or* $G \cong A_7, S_7$ *(if* $(d,q) = (3,2)$ *and blocks are the hyperplanes);*

5.2. Γ *is the Paley (or Hadamard) design on 11 points and* $L_2(11) \trianglelefteq G \leq L_2(11) : 2$;

5.3. Γ *is isomorphic to one of Kantor's designs* $\mathcal{S}^{\pm}(n)$ *and* $G \cong 2^{2n} : Sp_{2n}(2)$;

5.4. G *acts 3-transitively on the set of points of* Γ *and the blocks are the complements of the points;*

6. Γ *is a generalized quadrangle of order* $(1,s)$ *or* $(s,1)$ *and* G *is appropriate;*

7. Γ *is a generalized digon.*

An immediate corollary is the following:

COROLLARY 1. *Let* (Γ, G) *be a full geodesic transitive* (g, d_p, d_l)*-pair,* $2 \leq g \leq d_p \leq d_l \leq g+1$, *then one of the following holds:*

1.1. Γ *is the generalized quadrangle of order* (q,q) *appearing in conclusion 1 of theorem 1 for every even* q *and* G *is the appropriate group containing a correlation;*

1.2. Γ *is the generalized hexagon of order* (q,q) *appearing in conclusion 1 of theorem 1 for every* q *divisible by 3 and* G *is the appropriate group containing a correlation;*

2 Γ *is the helicopter plane* $HG(2,q)$ *as in conclusion 3 of theorem 1 and* G *is appropriate but containing a correlation;*

3 (Γ, G) *is as in conclusion 5 of theorem 1 (case of symmetric 2-designs) with the only restriction that* G *contains a correlation;*

4 Γ *is generalized digon.*

For various sub-classes of geometries, we obtain more general results by weakening the hypothesis on G. We refer to section 4 for the precise statements.

As a byproduct of our proof, we obtain a result on partial quadrangles (and they do not necessarily satisfy $d_l \leq g+1$, so they are not covered by theorem 1), see also section 4, proposition 4 for a more detailed statement and the proof.

THEOREM 2. *Let* (Γ, G) *be a point geodesic transitive* (g, d_p, d_l)*-pair with* Γ *a partial quadrangle of order* (s,t). *Then one of the following possibilities occur:*

1. Γ *is a generalized quadrangle (and* (Γ, G) *is one of the examples in the conclusion of theorem 2);*

2. $s = 1$ *and* Γ *is one of the following graphs: the pentagon, Petersen, Clebsch, Hoffman-Singleton, Higman-Sims on 100 (resp. 77) vertices, Gewirtz. The group* G *is an appropriate automorphism group containing respectively* D_{10}, A_5, $2^4 : (5 : 4)$, $U_3(5)$, HS, M_{22}, $L_3(4)$;

3. Γ *is a partial quadrangle with parameters* $(2, 10, 1)$ *constructed in* $AG(5, 3)$ *and* G *is appropriate;*

4. Γ *is isomorphic to the partial quadrangle* $T_3^*(\mathcal{O})$ *with* $\mathcal{O}$ *an elliptic quadric or the Suzuki-Tits ovoid in* $PG(3, q)$ *and* G *is appropriate.*

In cases 3 and 4, G *acts on an affine space, contains all translations and the stabilizer of a point contains a normal subgroup isomorphic to one of* $M_{11}, L_2(q^2), Sz(q)$.

2 THE EXAMPLES.

In this section, we give all the examples mentioned in theorems 1 and 2 and proposition 1 below and we briefly comment the properties of their automorphism group.

2.1 Generalized Polygons.

2.1.1 The Moufang generalized polygons.

The *classical* examples (i.e. those which have the Moufang property, see above and section 5) of generalized polygons, due to TITS [83], arise from Chevalley groups. We give a brief description. Let $G = X_n(q)$ be a Chevalley group, $q = p^h$ where p is a prime and let B be the normalizer of a Sylow p-group in G (B is called a *Borel-subgroup*). If G is one of the groups of table 1, there are exactly two maximal subgroups containing B, denote them by P_1 and P_2; they are called the *maximal parabolic subgroups* of the pair (G, B). We define a geometry $\Gamma = (\mathcal{P}, \mathcal{B}, I)$ as follows: the points are the left cosets of P_1 in G and the lines are the left cosets of P_2 in G, a point and a line being incident exactly when the corresponding cosets are not disjoint. The geometry Γ thus obtained is a thick generalized n-gon of order (s, t) as listed in table 1 (where the set of points is chosen in the usual "classical" way).

In every case the pair (Γ, G) has both the Moufang and the Tits property and every group $G^* \leq \mathrm{Aut}(\Gamma)$ acting point distance transitively on Γ contains G (for the "right" choice of the points in the above construction), except if G is not simple, i.e. if (1) $G \cong S_4(2) \cong O_5(2)$, (2) $G \cong G_2(2)$ and (3) $G \cong {}^2F_4(2) \cong R(2)$; in these cases also the derived group G' acts point distance transitively on Γ. In case (1), G' acts distance transitively on Γ, but (Γ, G') does not have the Tits property nor the Moufang property, it is not even half Moufang. In cases (2) and (3), G' does not act line distance transitively and consequently it does not induce the Tits nor the Moufang property; but it does induce the half Moufang property.

Note that for a generalized n-gon Γ, n even, the pair (Γ, G) is distance transitive if and only if it is special distance transitive. This is an immediate consequence of the definition.

	G	n	(s,t)	Remarks
(GP1)	$L_3(q)$	3	(q,q)	Self-dual
(GP2)	$S_4(q)$	4	(q,q)	Self-dual iff q is even
(GP3)	$O_5(q)$	4	(q,q)	Dual to (GP2)
(GP4)	$O_6^-(q)$	4	(q,q^2)	
(GP5)	$U_4(q)$	4	(q^2,q)	Dual to (GP4)
(GP6)	$U_5(q)$	4	(q^2,q^3)	
(GP7)	$G_2(q)$	6	(q,q)	Self-dual iff q is a power of 3
(GP8)	${}^3D_4(q)$	6	(q,q^3)	
(GP9)	${}^2F_4(q)$	8	(q,q^2)	q is an odd power of 2

Table 1: Finite Thick Moufang Generalized n-gons.

The incidence graph Γ^I of a generalized n-gon Γ of order (q, q) is a generalized $2n$-gon of order $(1, q)$. If Γ is Moufang, then also Γ^I is Moufang, but Γ^I has the Tits property, respectively is distance transitive, geodesic transitive only if Γ has the Tits property and is self-dual, respectively is full distance transitive, full geodesic transitive.

2.1.2 The unique generalized quadrangle of order $(3, 5)$.

This example is one out of a class of generalized quadrangles of order $(s, s+2)$ due to Ahrens & Szekeres [1].

Consider the projective plane $PG(2,4)$ and a *complete oval* O in it, i.e. a conic together with its kernel. Embed $PG(2,4)$ as a hyperplane in $PG(3,4)$ and define the following geometry $\Gamma = (\mathcal{P}, \mathcal{L}, I)$: the elements of $\mathcal{P}$ are the points of $PG(3,4)$ not in $PG(2,4)$; the elements of $\mathcal{L}$ are the lines in $PG(3,4)$ meeting O in exactly 1 point; incidence is the natural one. Then Γ is a generalized quadrangle of order $(3,5)$ and it is usually denoted by $T_2^*(O)$. For more information on this interesting quadrangle we refer to a recent paper of Payne [73]. We just mention that the full collineation group of $T_2^*(O)$ contains all translations and homologies of $PG(3,4)$ leaving $PG(2,4)$ pointwise invariant and its kernel on $PG(2,4)$ is the full automorphism group of O which is S_6, the full symmetric group acting in its standard representation on the six points of O. So $\mathrm{Col}(T_2^*(O))$ acts point distance transitively on $T_2^*(O)$, but obviously not line distance transitively.

2.1.3 Non-Thick Generalized Quadrangles.

Consider a generalized quadrangle $\Gamma = (\mathcal{P}, \mathcal{L}, I)$ of order $(s, 1)$. This is actually just an $(s+1) \times (s+1)$-*grid*. Consider a group G acting transitively on that grid and such that the stabilizer G_x of any point x acts transitively on both $\Gamma_2(x)$ and $\Gamma_4(x)$. Then G acts geodesic transitively on Γ. So point distance transitivity implies geodesic transitivity. This case, and the dual one, corresponds to case 6 of theorem 1. The corresponding groups are described in subsection 3.2, class III of the permutation rank 3 groups.

2.2 Partial Quadrangles.

2.2.1 Thick partial quadrangles.

Consider the projective 3-space $PG(3,q)$ and an *ovoid* O in it (an **ovoid** in $PG(3,q)$ is a set of q^2+1 points no three of which are collinear). Embed $PG(3,q)$ as a hyperplane in $PG(4,q)$ and define the following geometry $T_3^*(O)$: the points are the points of $PG(4,q)$ not in $PG(3,q)$ and the lines are the lines of $PG(4,q)$ meeting $PG(3,q)$ in a single point x of O. Then $T_3^*(O)$ is a partial quadrangle with parameters $(q-1, q^2, q^2-q-1)$. There are two cases for which the collineation group of $T_3^*(O)$ acts point distance transitively on $T_3^*(O)$:

1. The first case is when O is a non-ruled quadric (an elliptic quadric), i.e. a set of points satisfying the equation $X_0X_1 = f(X_2, X_3)$, where $f(x,y)$ is an irreducible quadratic form over the field $GF(q)$.

2. The second case is when O is the Suzuki-Tits ovoid for spaces over $GF(q)$ with $q = 2^{2e+1}$. The points of O can be described in coordinates as $\{(x, y, xy + x^{s+2} + y^s, 1)\} \cup \{(0, 0, 1, 0)\}$, $s = 2^{e+1}$.

The case where O is a quadric can also be constructed from the generalized quadrangle (GP4) by deleting all points collinear with one fixed point and deleting all lines through that point (see Cameron [17]). In fact, every partial quadrangle with parameters $(q-1, q^2, q^2-q-1)$ can be constructed that way as proved by Ivanov & Shpectorov [53]. In subsection 4.2, we show that the two examples above are in fact point geodesic transitive.

Consider now $PG(4,3)$, there is a *cap* C in $PG(4,3)$ consisting of 11 points and the Mathieu group M_{11} acts on C in its standard 4-transitive action (a **cap** is a set of points no three of which are collinear). Repeating the construction from the previous paragraph (substituting C for O, $PG(4,3)$ for $PG(3,q)$, etc . . .), one obtains a partial quadrangle with parameters $(2, 10, 1)$ which we shall denote by $Hi(243)$, see Hill [47] and Berlekamp, van Lint & Seidel [4].

The above examples are constructed as *linear representations*, i.e. the points of the geometry Γ are all points of a certain affine space $\mathcal{A}$ and the set of lines of Γ is a union of parallel classes of lines of $\mathcal{A}$.

2.2.2 Strongly regular graphs with $\lambda = 0$.

A graph is called **regular** if every vertex has a constant number k of *adjacent* vertices (**adjacent vertices** are vertices on one edge). A regular graph is called **strongly regular** if every two adjacent vertices are both adjacent to a constant number λ of vertices and if every two non-adjacent vertices are both adjacent to a constant number μ of vertices. In this case the strongly regular graph is said to have parameters (v, k, λ, μ), where v is the total number of vertices.

Note that, if Γ is a strongly regular graph with parameters (v, k, λ, μ), then the graph obtained by interchanging edges with "non-edges" is again a strongly regular graph, called the **complementary** strongly regular graph. If we denote its parameters by $(v, l, \overline{\lambda}, \overline{\mu})$, then we have the relations:

$$k + l = v; \overline{\lambda} = l - k + \mu - 1; \overline{\mu} = l - k + \lambda + 1;$$

and

$$\mu l = k(k - \lambda - 1).$$

For any strongly regular graph, we will use this standard notation without further comments (see Hubaut [48]).

If Γ is a strongly regular graph with $\lambda = 0$, i.e. Γ does not contain triangles, then it is a partial quadrangle with parameters $(1, k-1, \mu-1)$.

Now suppose a group G has a rank 3 permutation representation on a set Γ (for the definition see section 3 below) and let G_x be the stabilizer of an element x of Γ. Let x_i,

$i = 1, 2$, be an element in the orbit i of G_x (where we assume that the three orbits are numbered 0,1,2 and orbit 0 is the trivial one). Then we define edges in Γ by letting G act on $\{x, x_i\}$. If $|G|$ is even, then this defines two mutually complementary strongly regular graphs (see HIGMAN [45]).

Table 2 contains a list, taken from HUBAUT [48], of some strongly regular graphs without triangles constructed in this way, where G is a simple group. The table contains the label for future reference, the group G, the subgroup G_x, the notation for γ (and we make the convention that we choose x_i in the smallest suborbit, with the above notation), the parameters of Γ as a strongly regular graph and the name in the literature.

	G	G_x	Γ	(v, k, λ, μ)	Name
(PQ0)	D_{10}	2	$Pn(5)$	(5,2,0,1)	Pentagon
(PQ1)	A_5	S_3	$Pe(10)$	(10,3,0,1)	Petersen
(PQ2)	$U_3(5)$	A_7	$HS(50)$	(50,7,0,1)	Hoffman-Singleton
(PQ3)	$L_3(4)$	A_6	$Ge(56)$	(56,10,0,2)	Gewirtz
(PQ4)	M_{22}	$2^4 : A_6$	$HS(77)$	(77,16,0,4)	Higman-Sims
(PQ5)	HS	M_{22}	$HS(100)$	(100,22,0,6)	Higman-Sims

Table 2: Rank 3 Graphs with $\lambda = 0$ related to Simple Groups.

We need one further non-trivial strongly regular graph without triangles underlying a rank 3 group, namely the *Clebsch graph* $Cl(16)$. There are several descriptions of it and here is a less usual one: the vertices of $Cl(16)$ are the elements of the field $GF(16)$ of 16 elements and two vertices form an edge if their difference in $GF(16)$ is a third power. The collineation group of $Cl(16)$ is isomorphic to $2^4 : S_5$, but we already have a rank 3 representation if we take the subgroup $2^4 : D_{10}$, where $G_x \cong D_{10}$ is generated by the multiplication with third powers in $GF(16)$ and the involutory automorphism of $GF(16)$, taking $x = 0$. The parameters of $Cl(16)$ are $(16, 5, 0, 2)$ and we label this example (PQ6).

In subsection 4.2, we determine the transitivity properties of the above mentioned strongly regular graphs (see table 20).

2.3 Some Nets.

2.3.1 Helicopter planes.

Consider an affine plane of order q (i.e. there are q points on each line; with our convention, it has order $(q-1, q)$) and delete one entire class of parallel lines. The incidence geometry Γ thus obtained is a net of order q and degree q. Suppose the original affine plane was the Desarguesian plane $AG(2, q)$ and consider the subgroup G of $PGL(2, q)$ stabilizing Γ. Then (Γ, G) is a geodesic transitive $(3, 4, 4)$-pair (for $q \geq 3$).

2.3.2 The net $(H_q^n)^D$.

Consider the following geometry H_q^n: the points are the points of the projective space $PG(n,q)$ which are not contained in a fixed subspace $PG(n-2,q)$ of $PG(n,q)$; the lines are the lines of $PG(n,q)$ disjoint from $PG(n-2,q)$; incidence is the natural one. This yields a dual net with parameters $(q, q^{n-1}-1, q-1)$. The corresponding net can be constructed in another way as follows: Consider a vector space V_2 resp. W_{n-1} of dimension 2 resp. $n-1$ over $GF(q)$. The points of $(H_q^n)^D$ are the elements of the tensor product vector space $V_2 \bigotimes W_{n-1}$ and the lines are the sets of the form $v \otimes W_{n-1}$ and its translates, where v is an arbitrary vector in V_2. By De Clerck & Johnson [27], theorem 4, this constitutes indeed the dual of H_q^n. If a group G acts geodesic transitive on $(H_q^n)^D$, then it is clear that the group induced on W_{n-1} by the stabilizer G_o of the zero vector in $V_2 \bigotimes W_{n-1}$ acts 2-transitively on the vector lines of W_{n-1} (for the definition of 2-transitive group see subsection 3.1) and hence is known (see again subsection 3.1, in particular table 4).

2.3.3 The net $Ne(2^8)$.

Consider the hyperbolic quadric in $PG(7,2)$ and a 2-transitive ovoid (see e.g. Kleidman [60]) in it. Apply triality to obtain a 2-transitive spread S. Embed $PG(7,2)$ as a hyperplane in $PG(8,2)$. Define as the point set of $Ne(2^8)$ the set of points in $PG(8,2) \setminus PG(7,2)$. A line is a 4-subspace of $PG(8,2)$ meeting $PG(7,2)$ in a member of S. This constitutes a net $Ne(2^8)$ with parameters $(15, 8, 7)$. The automorphism group of $Ne(2^8)$ is isomorphic to $2^8 : A_9$, where A_9 is the group acting 2-transitively on the elements of the spread.

2.4 Linear Spaces.

Here, we simply list in table 3 all linear spaces which admit a flag-transitive collineation group (and have $s, t \geq 1$). This result is due to Buekenhout, De Landtsheer, Doyen, Kleidman, Liebeck & Saxl [14]. We will give more information in the proof (paragraph 4.3.2). In the table, a c-geometry is a geometry in which the lines are all pairs of points.

	Linear space	s	t
(LS1)	$PG(n,q)$, $n \geq 2$	q	$\frac{(q^n-q)}{(q-1)}$
(LS2)	Hermitian unital in $PG(2,q^2)$	q	q^2-1
(LS3)	Ree unital arising from $R(q)$, $q = 3^h$, h odd	q	q^2-1
(LS4)	Witt-Bose-Shrikhande space (defined for q even)	$\frac{(q-2)}{2}$	q
(LS5)	$AG(n,q)$, $n \geq 2$	$q-1$	$\frac{(q^n-q)}{(q-1)}$
(LS6)	Some non-Desarguesian translation affine planes	$q-1$	q
(LS7)	Hering spaces	8	90
(LS8)	Affine line spaces		
(LS9)	c-geometry on v points	1	$v-2$

Table 3: Linear Spaces Admitting a Flag-transitive Group.

Note that any set $\mathcal{S}$ defines a unique c-geometry in the obvious way. We denote that linear space by $\Gamma(\mathcal{S})$.

2.5 Symmetric 2-designs.

We mention some symmetric 2-designs for which the collineation group acts 2-transitively on the points; the classification of all such designs is due to Kantor [58].

(SD1) Desarguesian projective spaces. The blocks are the hyperplanes or the complement of the hyperplanes. The collineation groups are the linear or semi-linear groups or A_7 for $PG(3,2)$;

(SD2) The Paley (or Hadamard) design on 11 points, denoted by $Ha(11)$, is a 2-(11,6,2) design. The points are the elements of $\mathbf{Z}$ (mod 11) and the lines are the translates of $\{1,3,4,5,9\}$. Its collineation group is $L_2(11)$. The complementary design is a 2-(11,6,3) design;

(SD3) The "geometry" of Higman on 176 points. This is a 2-(176,50,14) design with collineation group HS, the sporadic Higman-Sims group. We denote this design by $Hi(176)$. The complementary design is a 2-(176,126,90) design;

(SD4) A 2-$(2^{2n}, 2^{n-1}(2^n-1), 2^{n-1}(2^{n-1}-1))$-design $\mathcal{S}^+(n)$ of which there is exactly one for each $n \geq 2$, see Kantor [55]. The complementary design is denoted by $\mathcal{S}^-(n)$. The collineation group of both these designs is a group isomorphic to $2^{2n} : S_{2n}(2)$;

(SD5) Any set Ω with a 2-transitive group acting on it can be turned into a 2-transitive symmetric 2-design by declaring the blocks to be the complements of the points. This can be defined for any set Ω without a 2-transitive group and we denote the corresponding design by $\Gamma(.\Omega)$.

3 PRELIMINARIES.

The proofs of our main results are basically geometric in nature. But we make use of some major group-theoretical results such as the classification of all doubly transitive finite groups (Cameron [18] and Hering [44], the determination of all primitive rank 3 representations of finite groups (Kantor & Liebler [59], Bannai [3], Liebeck & Saxl [67] and Liebeck [65]), the classification of various classes of distance transitive graphs (Ivanov [52], Liebeck, Praeger & Saxl [66], Praeger, Saxl & van Bon [75] and Praeger & Soicher [76]), the determination of all 'large' maximal subgroups of the exceptional groups (Liebeck [68]) and the enumeration of all primitive rank 4 and 5 representations of the Chevalley groups (Cuypers [23]). We now list these results (and some other) for future reference.

We use the classification of the finite simple groups. They fall into five distinct (though non-disjoint) classes:

1. The cyclic groups of prime order;
2. The alternating groups A_n for $n \geq 5$;
3. The classical Chevalley groups, i.e. the linear, symplectic, orthogonal and unitary (simple) groups;
4. The exceptional Chevalley groups (including the Tits group here);
5. The 26 sporadic groups.

The first chapters of the Atlas [22] contain an introduction to this subject.

3.1 Permutation Groups.

Let Ω be a set and G a group acting faithfully on Ω. Then the pair (Ω, G) is said to have **permutation rank** n, $n > 1$, if G is transitive on Ω and if the stabilizer G_x of some element x of Ω has exactly n orbits in Ω. A rank 2 group is also called a **2-transitive** group. The group G acts n**-transitively**, $n > 2$ on Ω if G acts transitively on Ω and G_x acts $(n-1)$-transitively on $\Omega \setminus \{x\}$, for some $x \in \Omega$.

If Ω is an affine space and G contains the full translation group of Ω and is itself contained in the full automorphism group of Ω as an affine space, then we say that (Ω, G) is of **affine type**. If there is a non-abelian simple group S such that $S \trianglelefteq G \leq \mathrm{Aut}(G)$, then we say that G is **almost simple** and (Ω, G) is of **almost simple type**. In this case the group S is the **socle** of G, denoted by $\mathrm{Soc}(G)$ (special case of a more general definition: $\mathrm{Soc}(G)$ is the subgroup of G generated by all minimal normal subgroups of G).

A subset $A \subseteq \Omega$ is a **set of imprimitivity** for (Ω, G) if $A^\theta \cap A$ is either empty or A itself, for all $\theta \in G$. Note that we use exponential notation for the action of G on Ω. The action of G on Ω is called **primitive** if the only sets of imprimitivity are the trivial ones, i.e. the singletons and Ω itself. If (Ω, G) is 2-transitive, then it is automatically primitive (this is easy to see). As a result of the classification of the finite simple groups, all finite 2-transitive groups are known. They are divided into two classes: the almost simple and the affine type. In table 4, we list all 2-transitive groups of almost simple type acting on a set Ω (see e.g. CAMERON [18]) and label them for future reference. We list the minimal group; all other groups are obtained by adjoining group automorphisms.

	G	Ω	Restrictions and Remarks
(TS1)	A_n	n symbols	$n \geq 5$
(TS2)	$L_n(q)$	$\frac{(q^n-1)}{(q-1)}$ points of $PG(n-1,q)$	$(d,q) \neq (2,2),(2,3)$,
			3-transitive if $PGL_2(q) \leq G$
(TS3)	$U_3(q)$	q^3+1 points of a Hermitian unital	$q \geq 3$
(TS4)	$R(q)$	q^3+1 points of the Ree unital	$q = 3^h$, $h \geq 1$ odd
(TS5)	$Sz(q)$	q^2+1 points of The Suzuki Ovoid	$q = 2^h$, $h \geq 3$ odd
(TS6)	$S_{2d}(2)$	$2^{2d-1} \pm 2^{d-1}$ non-degenerate quadrics	$d \geq 3$
(TS7)	$L_2(11)$	11 points of $Ha(11)$	
(TS8)	A_7	15 points of $PG(3,2)$	
(TS9)	M_{11}	11 points of a Steiner system	4-transitive
(TS10)	M_{11}	12 points of a 3-design	
(TS11)	M_{12}	12 points of a Steiner system	5-transitive
(TS12)	M_{22}	22 points of a Steiner system	3-transitive
(TS13)	M_{23}	23 points of a Steiner system	4-transitive
(TS14)	M_{24}	24 points of a Steiner system	5-transitive
(TS15)	HS	176 points of $Hi(176)$	
(TS16)	Co_3	276 points in the Leech lattice	

Table 4: 2-Transitive Representations of Almost Simple Groups.

We will not need an explicit list of the affine 2-transitive groups.

Also, all primitive rank 3 groups are classified (again using the classification of the finite simple groups). They fall into three classes: the almost simple case, the affine case and the "grid case". We briefly describe the results in each of the three cases.

CLASS I. The Almost Simple Case.

First, in order not to mention the same permutation representation twice, we make the four classes of finite simple groups two by two disjoint by deleting those groups in a class that already appeared in a previous class, e.g. we remove $L_2(4) \cong L_2(5)$ from class 2 because this is isomorphic to A_5 in class 1.

The classification has been achieved by various people for the respective classes of simple groups: BANNAI [3] for the alternating groups (table 5), KANTOR & LIEBLER [59] for the classical Chevalley groups (tables 6 and 7), LIEBECK & SAXL [67] for the exceptional Chevalley groups and the sporadic groups (tables 6,8 and 9). So tables 5 to 9 contain all the rank 3 representations of almost simple type. As a general rule, we always list the smallest possible group G; other groups are obtained by adjoining automorphisms of G (provided this larger group still acts on Ω). For some classes, we also list the point-stabilizer (denoted by G_x). If not, we list the set Ω by writing a typical element, the full set is obtained by taking the orbit of the typical element under G.

	G	Ω	v	k;l	λ;μ
(AL1)	A_n	pair in n-set	$\frac{n(n-1)}{2}$	$2(n-2)$;$\frac{(n-2)(n-3)}{2}$	$n-2$;4
(AL2)	A_8	line in $PG(3,2)$	35	16;18	6;8
(AL3)	A_9	$PG(1,8)$ in 9-set	120	56;63	28;24
(AL4)	A_{10}	5|5 splitting of 10 points	126	25;100	8;4

Table 5: Rank 3 Representations of Alternating Groups.

In order to decide whether an element σ in $\mathrm{Aut}(G)$ extends the rank 3 representation, simply let it act on a typical element; if this is possible and the result is inside Ω, then $\{\sigma, G\}$ generates a larger rank 3 group on the same set Ω. We illustrate this with an example that we will need anyway later on: consider example (AL3) (of table 5). There are in total 240 $PG(1,8)$'s in a set of 9 elements and A_9 acts in two orbits twice on 120 of them. So an element of $S_9 \setminus A_9$ interchanges these two orbits and hence S_9 does not act as a rank 3 group on 120 points. But in example (AL1), S_n does act on the set of duads and hence this gives us a rank 3 representation.

In some of the tables, we also list the parameters (v, k, l, λ, μ) of the corresponding strongly regular graphs (see subsection 2.2 for the definitions). These will play a crucial role in our proof. The parameters of the complementary strongly regular graph can be computed easily (see also subsection 2.2) and are not always included in the tables. In the proof of the main result, we will however use these values without further reference. However, the parameters of the strongly regular graphs related to the groups of table 6 can be found in subsection 4.2.

CLASS II. The Affine Case.

The complete classification of this class is due to Liebeck [65]. He subdivides this class into three subclasses. We give a very brief description in all of these cases.

Let us fix our notation: here Ω is an affine space $V_n(q)$ of dimension n over $GF(q)$. We denote by G_o the stabilizer in G of the zero-vector and we choose n minimal with respect to the property $G_o \leq \Gamma L_n(q)$ (as in Liebeck [65].

(A) INFINITE CLASSES.

There are 11 infinite classes and we list them in table 10, where we emphasize the geometric properties of G_o; the exact shape of G_o (as a group) will be given later if necessary. In most of the cases though, the geometric description suffices (since our proof is using rather geometric arguments).

(B) EXTRASPECIAL CLASS.

	G	Ω	v	remark
(CH1)	$L_n(q)$	lines in $PG(n-1,q)$	$\frac{(q^{n+1}-1)(q^n-1)}{(q+1)(q-1)^2}$	$n \geq 4$
(CH2)	$S_{2n}(q)$	(isotropic) points of $PG(2n-1,q)$	$\frac{q^{2n}-1}{q-1}$	$n \geq 2$
(CH3)	$O_{2n+1}(q)$	singular points in $PG(2n,q)$	$\frac{q^{2n}-1}{q-1}$	$n \geq 2$
(CH4)	$O^+_{2n}(q)$	singular points in $PG(2n-1,q)$	$\frac{(q^n-1)(q^{n-1}+1)}{q-1}$	$n \geq 3$
(CH5)	$O^-_{2n}(q)$	singular points in $PG(2n-1,q)$	$\frac{(q^n+1)(q^{n-1}-1)}{q-1}$	$n \geq 3$
(CH6)	$O^+_{10}(q)$	singular 4-spaces in $PG(9,q)$	$\frac{(q^8-1)(q^3+1)}{q-1}$	
(CH7)	$O^+_{2n}(2)$	non-singular points in $PG(2n-1,2)$	$2^{2n-1}+2^{n-1}$	$n \geq 3$
(CH8)	$O^-_{2n}(2)$	non-singular points in $PG(2n-1,2)$	$2^{2n-1}-2^{n-1}$	$n \geq 3$
(CH9)	$O_{2n+1}(3)$	points inside a quadric in $PG(2n,3)$	$\frac{3^n(3^n+1)}{2}$	$n \geq 2$
(CH10)	$O_{2n+1}(3)$	points outside a quadric in $PG(2n,3)$	$\frac{3^n(3^n-1)}{2}$	$n \geq 2$
(CH11)	$O^+_{2n}(3)$	non-singular points $PG(2n-1,3)$	$\frac{3^{n-1}(3^n-1)}{2}$	$n \geq 3$
(CH12)	$O^-_{2n}(3)$	non-singular points $PG(2n-1,3)$	$\frac{3^{n-1}(3^n+1)}{2}$	$n \geq 3$
(CH13)	$O_{2n+1}(4)$	non-singular hyperplanes on $PG(2n,4)$ (one orbit)	$2^{2n-1}(2^{2n}-1)$	$n \geq 2$
(CH14)	$O_{2n+1}(4)$	non-singular hyperplanes on $PG(2n,4)$ (other orbit)	$2^{2n-1}(2^{2n}+1)$	$n \geq 2$
(CH15)	$O_{2n+1}(8):3$	non-singular hyperplanes on $PG(2n,8)$ (one orbit)	$2^{3n-1}(2^{3n}-1)$	$n \geq 2$
(CH16)	$O_{2n+1}(8):3$	non-singular hyperplanes on $PG(2n,8)$ (other orbit)	$2^{3n-1}(2^{3n}+1)$	$n \geq 2$
(CH17)	$U_{2n+1}(q)$	singular points in $PG(2n,q^2)$	$\frac{(q^{2n}-1)(q^{2n+1}+1)}{q^2-1}$	$n \geq 1$
(CH18)	$U_{2n}(q)$	singular points in $PG(2n-1,q^2)$	$\frac{(q^{2n}-1)(q^{2n-1}+1)}{q^2-1}$	$n \geq 2$
(CH19)	$U_5(q)$	singular lines in $PG(4,q^2)$	$(q^5+1)(q^3+1)$	
(CH20)	$U_{2n+1}(2)$	non-singular points in $PG(2n,4)$	$\frac{2^{2n}(2^{2n+1}+1)}{3}$	$n \geq 2$
(CH21)	$U_{2n}(2)$	non-singular points in $PG(2n-1,4)$	$\frac{2^{2n-1}(2^{2n}-1)}{3}$	$n \geq 2$
(CH22)	$E_6(q)$	points of a building	$\frac{(q^{12}-1)(q^9-1)}{q^4-1)(q-1)}$	

Table 6: Rank 3 Representations of Chevalley Groups: Infinite Classes.

	G	G_x	v	$k;l$	$\lambda;\mu$	$\overline{\lambda};\overline{\mu}$
(CG1)	$L_2(8):3$	$7:6$	36	14;21	7;4	10;15
(CG2)	$L_3(4)$	A_6	56	10;45	0;2	36;36
(CG3)	$S_6(2)$	$G_2(2)$	120	56;63	28;24	30;36
(CG4)	$O_7(3)$	$G_2(3)$	1080	351;728	126;108	484;504
(CG5)	$U_3(3):2$	$L_3(2):2$	36	14;21	4;6	12;12
(CG6)	$U_3(5)$	A_7	50	7;42	0;1	35;36
(CG7)	$U_4(3)$	$L_3(4)$	162	56;105	10;24	72;60
(CG8)	$U_6(2)$	$U_4(3):2$	1408	567;840	246;216	488;520

Table 7: Rank 3 Representations of Classical Groups: Exceptional Classes.

	G	G_x	v	$k;l$	$\lambda;\mu$	$\overline{\lambda};\overline{\mu}$
(EG1)	$G_2(3)$	$G_2(2)$	351	126;224	45;45	142;144
(EG2)	$G_2(4)$	J_2	416	100;315	36;20	234;252
(EG3)	$G_2(4)$	$U_3(4):2$	2016	975;1040	462;480	544;528
(EG4)	$G_2(8):3$	$\Gamma U_3(8):2$	130816	32319;98496	7742;8064	74240;73920

Table 8: Rank 3 Representations of Exceptional Groups: Exceptional Classes.

	G	G_x	v	$k;l$	$\lambda;\mu$	$\overline{\lambda};\overline{\mu}$
(SP1)	M_{11}	$M_9.2$	55	18;36	9;4	21;28
(SP2)	M_{12}	$M_{10}.2$	66	20;45	10;4	28;36
(SP3)	M_{22}	$2^4.A_6$	77	16;60	0;4	47;45
(SP4)	M_{22}	A_7	176	70;105	18;34	68;54
(SP5)	M_{23}	$M_{21}.2$	253	42;210	21;4	171;190
(SP6)	M_{23}	$2^4.A_7$	253	112;140	36;60	87;65
(SP7)	M_{24}	$M_{22}.2$	276	44;231	22;4	190;210
(SP8)	M_{24}	$M_{12}.2$	1288	495;792	206;180	476;504
(SP9)	J_2	$U_3(3)$	100	36;63	14;12	38;42
(SP10)	HS	M_{22}	100	22;77	0;6	60;56
(SP11)	McL	$U_4(3)$	275	112;162	30;56	105;81
(SP12)	Suz	$G_2(4)$	1782	416;1365	100;96	1044;1050
(SP13)	Co_2	$U_6(2).2$	2300	891;1408	378;324	840;896
(SP14)	Ru	${}^2F_4(2)$	4060	1755;2304	730;780	1328;1280
(SP15)	Fi_{22}	$2.U_6(2)$	3510	693;2816	180;126	2248;2304
(SP16)	Fi_{22}	$\Omega_7(3)$	14080	3159;10920	918;648	8408;8680
(SP17)	Fi_{23}	$2.Fi_{22}$	31671	3510;28160	693;351	25000;25344
(SP18)	Fi_{23}	$P\Omega_8^+(3).S_3$	137632	28431;109200	6030;5832	86600;86800
(SP19)	Fi'_{24}	Fi_{23}	306936	31671;275264	3510;3240	246832;247104

Table 9: Rank 3 Representations of Sporadic Groups.

	n	G_o
(AI1)	1	
(AI2)	$2m$	stabilizes direct sum $V_{2m}(q) = V_m(q) \bigoplus V'_m(q)$.
(AI3)	$2m$	stabilizes tensor product $V_{2m}(q) = V_m(q) \bigotimes V_2(q)$.
(AI4)	n	stabilizes a subspace over $GF(\sqrt{q})$.
(AI5)	2	stabilizes a subspace over $GF(\sqrt[3]{q})$.
(AI6)	n	stabilizes a non-degenerate Hermitian form in $AG(n, q^2)$.
(AI7)	$2m$	stabilizes a non-degenerate quadratic form of type O^{ϵ}_{2m}, $\epsilon = +1$ or -1.
(AI8)	10	stabilizes a wedge-product $\bigwedge^2(V_5(q))$.
(AI9)	8	$\Omega_7(q).(2, q-1) \trianglelefteq G_o/Z(G_o)$ (spin representation).
(AI10)	16	$P\Omega^+_{10}(q) \trianglelefteq G_o/Z(G_o)$ (spin representation).
(AI11)	4	stabilizes the Suzuki-Tits ovoid, $q = 2^{2h+1}$.

Table 10: Affine Rank 3 Groups: Infinite Classes.

	n	q	k;l	λ;μ	comments
(AE1)	2	p			$p = 7, 13, 17, 19, 23, 29, 31, 47$.
(AE2)	3	2^2	27;36	10;12	$\lvert G_o\rvert = 2^4.3^4$.
(AE3)	2	3^2	32;48	13;12	$\lvert G_o\rvert = 2^4.3^3$.
(AE4)	2	3^3	104;624	31;12	$\lvert G_o\rvert = 2^4.3^2.13$.
(AE5)	4	3	16;64	7;2	$G_o \leq Sp_4(3)$; $l = 4 \times 16$.
(AE6)	4	3	32;48	13;12	$G_o \leq Sp_4(3)$; $k = 2 \times 16$, $l = 3 \times 16$.
(AE7)	4	5	240;384	95;90	$k = 15 \times 16$, $l = 6 \times 64$.
(AE8)	4	7	480;1920	119;90	
(AE9)	8	3	1440;5120	351;306	$\lvert G_o\rvert = 2^{13}.3^4.5$; $k = 45 \times 32$, $l = 40 \times 128$.
(AE10)	4	3	32;48	13;12	either $\frac{\lvert G_o\rvert}{\lvert Z(G_o)\rvert} = 2^7.3^2$ or $\lvert G_o\rvert < 2^8.3^2$ and $G_o \leq GL_2(3) \bigotimes GL_2(3)$.

Table 11: Affine Rank 3 Groups: Extraspecial Class.

Here, G_o is the normalizer of an extraspecial group. We deduce the possibilities in table 11 from Liebeck [65] and Foulser [36]. The comment "$k = a \times b$" means that the suborbit of size k consists of a blocks of imprimitivity of size b.

(C) EXCEPTIONAL CASES.

Here, the group $G_o/Z(G_o)$ is always an almost simple group and again, a list is available. We deduce the information we need from Liebeck [65] and Foulser & Kallaher [37] and list it in table 12. Some parameters are also obtained from Brouwer [9].

CLASS III. The Grid Case.

Here, we have a simple group S with $S \times S \trianglelefteq G \leq S_o \mathrm{wr} 2$, where $S \trianglelefteq S_o \leq \mathrm{Aut}(S)$ and S_o acts 2-transitively on a set of n points. So S_o is one of the groups in table 4. Here, $|\Omega| = n^2$. The notation "wr" means *Wreath product*. In our case, H wr 2 is isomorphic

	$G_o/Z(G_o)$	n	q	k;l	λ;μ
(AF1)	A_5	2	$q = 31, 41, 71, 79$ or 89		
(AF2)	A_5	2	3^2	40;40	19;20
(AF3)	A_5	2	7^2	960;1440	389;380
(AF4)	A_6	3	2^2	18;45	2;6
(AF5)	S_6	4	5	144;480	43;30
(AF6)	A_7	4	2^2	45;210	16;6
(AF7)	A_7	4	7	720;1680	229;220
(AF8)	A_9	8	2	120;135	56;56
(AF9)	A_{10}	8	2	45;210	16;6
(AF10)	$L_2(17)$	8	2	102;153	38;42
(AF11)	$L_3(4).2^2$	6	3	224;504	61;72
(AF12)	$U_4(2)$	4	7	240;2160	59;20
(AF13)	$G_2(4)$	12	3	65520;465920	8559;8010
(AF14)	M_{11}	5	3	22;220	1;2
(AF15)	M_{11}	5	3	110;132	37;60
(AF16)	M_{24}	11	2	276;1771	44;36
(AF17)	M_{24}	11	2	759;1288	310;264
(AF18)	J_2	6	4	1575;2520	614;600
(AF19)	J_2	6	5	7560;8064	3655;3660
(AF20)	Suz	12	3	65520;465920	8559;8010

Table 12: The Affine Rank 3 Groups: Exceptional cases.

to $(H \times H) : 2$; if H acts on a set V, then H wr 2 acts on $V \times V$ as follows: $H \times H$ in the natural way and the outer 2 by switching the two V's. We refer to such a group G as "GRID".

3.2 Distance Transitive Graphs.

Let Γ be a graph and suppose Γ is regular as a (g, d_p, d_l)-gon. Then Γ is called a **distance regular** graph. If (Γ, G) is moreover a point distance transitive (g, d_p, d_l)-pair, then (Γ, G), or briefly Γ is called a **distance transitive graph**.

The **point graph** or **collinearity graph** of a geometry Γ is the graph obtained from Γ by taking as vertices the points of Γ and as edges the pairs of adjacent points. Similarly for the **line graph**. There are obvious connections between the point (resp. line) transitivity properties of a geometry and the transitivity properties of its point (resp. line) graph and there are equally obvious connections between the transitivity properties of a geometry and its incidence graph.

Now note that the point graph, the line graph and the incidence graph of a regular (g, d_p, d_l)-gon is a distance regular graph (almost by definition). So, in view of the assumptions on the geometries we consider, distance transitive graphs will play an important role in this paper. Actually, a complete classification (which seems within reach, see Brouwer, Cohen & Neumaier [10]) would make our proof much easier. But the classification is not yet complete and so we must handle some cases by methods depending on the properties of the "underlying" geometry. But several classes of distance transitive graphs are classified and we will take advantage of such results, except at some places where we can prove a stronger result using geometric properties.

We now summarize the results on distance transitive graphs. So let Γ be a distance transitive graph with corresponding group G. Some standard parameters are defined: fix vertices x and y at distance i from each other, then

1. $a_i = |\Gamma_i(x) \bigcap \Gamma_1(y)|$;
2. $b_i = |\Gamma_{i+1}(x) \bigcap \Gamma_1(y)|$;
3. $c_i = |\Gamma_{i-1}(x) \bigcap \Gamma_1(y)|$;
4. $k_i = |\Gamma_i(x)|$.

We have the obvious relations:

$$a_i + b_i + c_i = k_1; c_{i+1}k_{i+1} = k_i b_i.$$

If we put $d_p = d$, then an **intersection array** of Γ is defined as

$$(b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d).$$

	G	d	v	intersection array
(DC1)	$U_3(3)$	3	63	(6,4,4;1,1,3)
(DC2)	$U_3(3):2$	3	63	(6,4,4;1,1,3)
(DC3)	$U_3(5)$	3	175	(12,6,5;1,1,4)
(DC4)	$U_3(4):2$	3	208	(12,10,5;1,1,8)

Table 13: The Admissible Distance Transitive Graphs related to Classical Groups.

For reasons that will become clear later, we are not interested in distance transitive graphs of diameter $d \geq 3$ with $c_2 \geq 2$ or $a_1 = 0$. Neither are we interested in distance transitive graphs for which the group G acts imprimitively on the set of vertices of Γ. Finally, we assume $d \geq 3$, since $d = 2$ corresponds to strongly regular graphs and rank 3 groups. We will call an element of the class of remaining distance transitive graphs an **admissible distance transitive graph**.

Suppose G is a group acting distance transitively and primitively on a distance regular graph Γ. By a theorem of Praeger, Saxl & Yokayama [74] (see also van Bon [92]), there are three possibilities:

1. Γ is a Hamming graph, but then $c_2 \geq 2$, see e.g. Brouwer, Cohen & Neumaier [10].
2. G is of affine type. Again $c_2 \geq 2$.
3. G is almost simple.

So if we restrict to admissible distance transitive graphs, then the only groups that can occur are almost simple groups.

Suppose now G is almost simple and G acts primitively and distance transitively on a distance regular graph Γ. Then the following cases occur:

CASE I. The Alternating Case.

Here, G is of alternating type. A complete classification has been achieved by Ivanov [52] and Liebeck, Praeger & Saxl [66]. If we restrict to admissible distance transitive graphs, then no examples survive.

CASE II. The Classical Case.

Here, G is a classical Chevalley group. Again, a complete classification has been achieved recently by Praeger, Saxl & van Bon [75], see also Inglis, Liebeck & Saxl [51]. Their result is basically that there are no surprises compared with [10], considering some additional low-dimensional examples listed in Cohen & van Bon [21]. If we restrict to admissible distance transitive graphs then we find the four examples of table 13.

	G	G_x	Restrictions.
(CU1)	$G_2(q)$	$SU_3(q):2$	$q=2,3,4,5,7,8,9,16,32$
(CU2)	$G_2(q)$	$SL_3(q):2$	$q=2,3,4,5,8$
(CU3)	$G_2(4)$	J_2	
(CU4)	${}^2F_4(2)$	$L_3(3):2$	
(CU5)	$F_4(2)$	$O_9(2)$	
(CU6)	${}^2E_6(2)$	$F_4(2)$	
(CU7)	$E_6(2):2$	$F_4(2):2$	G contains a graph automorphism

Table 14: Rank ≤ 5 representations of exceptional groups.

CASE III. The Exceptional Case.

No complete classification has been achieved yet. In order to deal with this situation we use a result of Liebeck [68]. Note that Cuypers [23] has determined all representations of the Chevalley groups of rank ≤ 5. His result, restricted to the exceptional Chevalley groups is summarized in table 14 omitting the case where the corresponding maximal subgroup is a maximal parabolic.

Liebeck [68] has classified all "large" subgroups of the exceptional groups of Lie type and to have an idea of what "large" means here, it follows that all non-parabolic maximal subgroups H of any almost simple group G with exceptional socle such that $|H|^2 \geq |G|$ are known. We present that part of the list in table 15. We only mention the simple exceptional group (G) and the corresponding subgroup (H) inside this simple group.

It is also worth noting that inside $\mathrm{Aut}(G)$ each maximal subgroup H in table 15 gives rise to only one conjugacy class, this follows from a personal communication of Cohen to Cuypers [23] for (ME15) and by Liebeck [68] for the other cases.

CASE IV. The Sporadic Groups.

Here, the situation is completely known for diameter ≤ 4 by work of Praeger & Soicher [76]. The result is, apart from the rank 3 examples given earlier, that there are precisely 6 distance transitive graphs. All of them appear already in Brouwer, Cohen & Neumaier [10]. When restricting to the admissible ones, only one example survives and it is listed in table 16.

In order to handle the sporadic groups in our proof, we will make use of the classification of their maximal subgroups (we will not list them all, but we will give the appropriate references later on) except for M and B, for which this classification is not yet completed.

An important tool will also be the minimum number $P(G)$ of objects on which a given sporadic group G can act non-trivially. We tabulate these values in table 17. The corresponding maximal subgroup is denoted by H. The value for $P(G)$ follows in each case from the classification of the maximal subgroups except if $G \cong M$ or $G \cong B$. In all cases the number $P(G)$ was computed independent from the knowledge of all maximal subgroups by Mazurov [71]. As far as we know, it is not yet proved that $2.B$ is, up to congucacy,

	G	H	restrictions
(ME1)	$G_2(q)$	$SL_3(q):2$	
(ME2)		$SU_3(q):2$	
(ME3)		${}^2G_2(q)$	$q=3^{2h+1}$
(ME4)		$G_2(\sqrt{q})$	q square
(ME5)		$G_2(2)$	$q=3$
(ME6)		J_2	$q=4$
(ME7)	${}^3D_4(q)$	$G_2(q)$	
(ME8)		${}^3D_4(\sqrt{q})$	q square
(ME9)	${}^2F_4(2)'$	$L_3(3):2$	
(ME10)		$L_2(25)$	
(ME11)	$F_4(q)$	$(2,q-1).O_9(q)$	
(ME12)		$((2,q-1)^2.O_8^+(q)).S_3$	
(ME13)		${}^3D_4(q).3$	
(ME14)		$F_4(\sqrt{q})$	q square
(ME15)		${}^2F_4(q)$	$q=2^{2h+1}$
(ME16)	${}^2E_6(q)$	$F_4(q)$	
(ME17)		$(((4,q^5+1).O_{10}^-(q)).(q+1)/(3,q+1)).(4,q+1)$	
(ME18)		$(SL_2(q).U_6(q)).(2,q-1)$	$\lvert Z(H)\rvert=(2,q-1)$
(ME19)		Fi_{22}	$q=2$
(ME20)	$E_6(q)$	$F_4(q)$	
(ME21)		$(((4,q^5-1).O_{10}^+(q)).(q-1)/(3,q-1)).(4,q-1)$	G contains a graph-automorphism
(ME22)		$(SL_2(q).L_6(q)).(2,q-1)$	$\lvert Z(H)\rvert=(2,q-1)$
(ME23)		$E_6(\sqrt{q})$	q square
(ME24)		${}^2E_6(\sqrt{q})$	q square
(ME25)	$E_7(q)$	$((3,q-1).E_6(q)).(q-1)/(2,q-1)).(3,q-1).2$	
(ME26)		$((3,q+1).\,{}^2E_6(q)).(q+1)/(2,q-1)).(3,q+1).2$	
(ME27)		$(SL_2(q).O_{12}^+(q)).2$	$\lvert Z(H)\rvert=(2,q-1)$
(ME28)		$E_7(\sqrt{q})$	q square
(ME29)	$E_8(q)$	$(SL_2(q).E_7(q)).(2,q-1)$	$\lvert Z(H)\rvert=(2,q-1)$
(ME30)		$O_{16}^+(q).2$	$\lvert Z(H)\rvert=(2,q-1)$
(ME31)		$E_8(\sqrt{q})$	q square

Table 15: Large Non-Parabolic Maximal Subgroups of Exceptional Groups.

	G	d	v	intersection array
(DS1)	J_2	4	315	(10,8,8,2;1,1,4,5)

Table 16: Admissible Distance Transitive Graphs related to Sporadic Groups.

G	$\lvert\mathrm{Out}(G)\rvert$	$\lvert G\rvert$	H	$P(G)$
M	1	$2^{46}.3^{20}.5^{9}.7^{6}.11^{2}.13^{3}.17.19.23.29.$ 31.41.47.59.71	$[2.B]$	927.10^{17}
B	1	$2^{41}.3^{13}.5^{6}.7^{2}.11.13.17.19.23.31.47$	$(2.\,{}^2E_6(2)):2$	135.10^{8}
Fi_{24}'	2	$2^{21}.3^{16}.5^{2}.7^{3}.11.13.17.23.29$	Fi_{23}	306,936
J_4	1	$2^{21}3^{3}5.7.11^{3}.23.29.31.37.43$	$2^{11}.M_{24}$	173,067,389
CO_1	1	$2^{21}.3^{9}.5^{4}.7^{2}.11.13.23$	CO_2	98,280
Fi_{23}	1	$2^{18}.3^{13}.5^{2}.7.11.13.17.23$	$2.Fi_{22}$	31,671
Th	1	$2^{15}.3^{10}.5^{3}.7^{2}.13.19.31$	${}^3D_4(2):3$	143,127,000
Ly	1	$2^{8}.3^{7}.5^{6}.7.11.31.37.63$	$G_2(5)$	8,835,156
HN	2	$2^{14}.3^{6}.5^{6}.7.11.19$	A_{12}	1,140,000
Fi_{22}	2	$2^{17}.3^{9}.5^{2}.7.11.13$	$2.U_6(2)$	3,510
Co_2	1	$2^{18}.3^{6}.5^{3}.7.11.23$	$U_6(2):2$	2,300
Co_3	1	$2^{10}.3^{7}.5^{3}.7.11.23$	$McL:2$	276
$O'N$	2	$2^{9}.3^{4}.5.7^{3}.11.19.31$	$L_3(7):2$	122,760
Suz	2	$2^{13}.3^{7}.5^{2}.7.11.13$	$G_2(4)$	1,782
Ru	1	$2^{14}.3^{3}.5^{3}.7.13.29$	${}^2F_4(2)$	4,060
He	2	$2^{10}.3^{3}.5^{2}.7^{3}.17$	$S_4(4):2$	2,058
McL	2	$2^{7}.3^{6}.5^{3}.7.11$	$U_4(3)$	275
M_{24}	1	$2^{10}.3^{3}.5.7.11.23$	M_{23}	24
J_3	2	$2^{7}.3^{5}.5.17.19$	$L_2(16):2$	6,156
HS	2	$2^{9}.3^{2}.5^{3}.7.11$	M_{22}	100
M_{23}	1	$2^{7}.3^{2}.5.7.11.23$	M_{22}	23
J_2	2	$2^{7}.3^{3}.5^{2}.7$	$U_3(3)$	100
M_{22}	2	$2^{7}.3^{2}.5.7.11$	$L_3(4)$	22
J_1	1	$2^{3}.3.5.7.11.19$	$L_2(11)$	266
M_{12}	2	$2^{6}.3^{3}.5.11$	M_{11}	12
M_{11}	1	$2^{4}.3^{2}.5.11$	$A_6\cdot 2$	11

Table 17: Sporadic Groups: Orders and Largest Maximal Subgroups.

the unique subgroup of M with index $P(M)$. This is made clear by the brackets in table 17. The fact that $(2.{}^2E_6(2)):2$ is the unique maximal subgroup of B with minimal index follows from Wilson [97].

In the tables, we also list the order of the simple group and the index $\lvert\mathrm{Out}(G)\rvert$ of it in its full automorphism group (see Atlas [22]).

This completes the list of known results that we will use in our proofs.

4 PROOF OF THE MAIN RESULTS.

4.1 Generalized Polygons.

For generalized polygons, our main result follows from BUEKENHOUT & VAN MALDEGHEM [16]. For completeness' sake we state it here as a proposition:

PROPOSITION 1. *Suppose* (Γ, G) *is a point distance transitive* (g, g, g)*-pair, where* Γ *is a finite generalized* g*-gon,* $g \geq 3$*, and* G *is type-preserving. Then* (Γ, G) *is one of the examples of table 18 below (where* q *denotes an arbitrary prime power). If* G *is not necessarily type-preserving, then in the cases where* Γ *is self-dual (see table 1), one can adjoin a graph automorphism.*

4.2 Rank 3 groups.

In this section, we deal with all geometries of theorems 1 and 3 satisfying $4 \leq d_p \leq 5$. This means that, under the hypothesis of the existence of a point distance transitive automorphism group G, G has rank 3 on the points and the point set is a strongly regular graph. More exactly, we will establish the following results.

PROPOSITION 2. *Let* Γ *be a proper partial geometry and suppose* $G \leq \mathrm{Col}(\Gamma)$ *acts weakly distance transitively on* Γ*. Then the points of* Γ *can be identified with the points of an affine line* $AG(1, q)$ *and* $G \leq A\Gamma L_1(q)$*.*

PROPOSITION 3. *Let* Γ *be a proper partial geometry or a net and suppose* $G \leq \mathrm{Col}(\Gamma)$ *acts weakly geodesic transitively. Then* Γ *is a net and one of the following holds:*

(NE1) Γ *is the helicopter plane* $HG(2, q)$ *obtained from the Desarguesian projective plane* $PG(2, q)$ *by deleting a flag* (x, l) *and all varieties incident with one of* x, l *and* G *contains the stabilizer in* $PGL_3(q)$ *of the flag* (x, l) *in* $PG(2, q)$*;*

(NE2) Γ *is the net* $(\mathcal{H}_q^{n+1})^D$ *of order* q^n *and degree* $q+1$ *and* G *contains a group isomorphic to the semi-direct product of an elementary abelian group* q^{2n} *with a group isomorphic to*

- *(a)* $(SL_2(q) \times SL_n(q))/Z(SL_2(q) \times SL_n(q))$ *if* $n > 2$*, or*
- *(b)* $(SL_2(q) \times GL_2(q))/Z(SL_2(q) \times GL_2(q))$ *if* $n = 2$*, or*
- *(c)* $SL_2(2) \times A_7$ *if* $(n, q) = (4, 2)$*.*

(NE3) Γ *is the net* $Ne(2^8)$ *and* $G \cong 2^8 : A_9$*.*

	g	Γ	G	Restrictions
(GP1)	3	$PG(2,q)$	$L_3(q) \trianglelefteq G \leq P\Gamma L_3(q)$	
(GP2)	4	$W(q)$	$S_4(q) \trianglelefteq G \leq P\Gamma Sp_4(q)$	
(GP3)	4	$Q(4,q)$	$O_5(q) \trianglelefteq G \leq P\Gamma O_5(q)$	
(GP4)	4	$Q(5,q)$	$O_6^-(q) \trianglelefteq G \leq P\Gamma O_6^-(q)$	
(GP5)	4	$H(3,q^2)$	$U_4(q) \trianglelefteq G \leq P\Gamma U_4(q)$	
(GP6)	4	$H(4,q^2)$	$U_5(q) \trianglelefteq G \leq P\Gamma U_5(q)$	
(GP7)	6	$H(q)$	$G_2(q) \trianglelefteq G \leq \mathrm{Aut}(G_2(q))$	G contains no graph automorphism
(GP8)	6	$H(q,q^3)$	${}^3D_4(q) \trianglelefteq G \leq \mathrm{Aut}({}^3D_4(q))$	
(GP9)	8	${}^2F_4(q)$	${}^2F_4(q) \trianglelefteq G \leq \mathrm{Aut}({}^2F_4(q)$	q odd power of 2
(GP10)	4	$H(4,q^2)^D$	$U_5(q) \trianglelefteq G \leq P\Gamma U_5(q)$	
(GP11)	4	$W(2)$	A_6	
(GP12)	4	$T_2^*(O)$	$2^6 : 3 : A_6 \leq G \leq 2^6 : 3 : S_6$	O a complete oval in $PG(2,4)$
(GP13)	4	$(s+1) \times (s+1)$-grid	GRID	
(GP14)	4	dual grid		
(GP15)	6	$H(q)^D$	$G_2(q) \trianglelefteq G \leq \mathrm{Aut}(G_2(q))$	G contains no graph automorphism
(GP16)	6	$H(q,q^3)^D$	${}^3D_4(q) \trianglelefteq G \leq \mathrm{Aut}({}^3D_4(q))$	
(GP17)	6	$H(2)$	$U_3(3) \cong G_2(2)'$	
(GP18)	6	$PG(2,q)^I$	$L_3(q) : 2 \leq G \leq P\Gamma L_3(q) : 2$	G contains a graph automorphism
(GP19)	6	$(PG(2,q)^I)^D$	$L_3(q) : 2 \leq G \leq P\Gamma L_3(q) : 2$	G contains a graph automorphism
(GP20)	8	${}^2F_4(q)^D$	${}^2F_4(q) \trianglelefteq G \leq \mathrm{Aut}({}^2F_4(q)$	q odd power of 2
(GP21)	8	${}^2F_4(2)$	$T \cong R(2) \cong^2 F_4(2)'$	
(GP22)	8	$W(q)^I$	$S_4(q).2 \leq G \leq P\Gamma Sp_4(q).2$	q even G contains a graph automorphism
(GP23)	8	$(W(q)^I)^D$	$S_4(q).2 \leq G \leq P\Gamma Sp_4(q).2$	q even G contains a graph automorphism
(GP24)	8	$W(2)^I$	$A_6 : 2$	G contains a graph automorphism
(GP25)	12	$H(q)^I$	$G_2(q).2 \leq G \leq \mathrm{Aut}(G_2(q))$	q is a power of 3 G contains a graph automorphism
(GP26)	12	$(H(q)^I)^D$	$G_2(q).2 \leq G \leq \mathrm{Aut}(G_2(q))$	q is a power of 3 G contains a graph automorphism

Table 18: Point Distance Transitive Generalized Polygons.

	Γ	G	(s,t,α)	Remarks
(PQ0)	$Pn(5)$	D_{10}	$(1,1,0)$	G is geodesic transitive; $G:2$ is full geodesic transitive
(PQ1)	$Pe(10)$	$A_5 \trianglelefteq G \leq S_5$	$(1,2,0)$	G is point geodesic transitive; S_5 is geodesic transitive
(PQ2)	$HS(50)$	$U_3(5) \trianglelefteq G \leq U_3(5):2$	$(1,6,0)$	G is point geodesic transitive and geodesic transitive
(PQ3)	$Ge(56)$	$L_3(4) \trianglelefteq G \leq L_3(4):2_2$	$(1,9,1)$	G is point geodesic transitive but *not* geodesic transitive
(PQ4)	$HS(77)$	$M_{22} \trianglelefteq G \leq M_{22}:2$	$(1,15,3)$	G is point geodisic transitive but *not* geodesic transitive
(PQ5)	$HS(100)$	$HS \trianglelefteq G \leq HS:2$	$(1,21,5)$	G is point geodesic transitive and geodesic transitive
(PQ6)	$Cl(16)$	$2^4:D_{10} \leq G \leq 2^4:S_5$	$(1,4,1)$	$2^4:(5:4)$ is point geodesic transitive; $2^4:A_5$ is geodesic transitive

Table 19: Point Distance Transitive Partial Quadrangles with $s=1$.

	Γ	$AG(n,q)$	S	Restrictions.
(PQ7)	$T_3^*(\mathcal{Q})$	$AG(4,q)$	$L_2(q^2)$	$\mathcal{Q}$ an elliptic quadric in $PG(3,q)$
(PQ8)	$T_3^*(\mathcal{O})$	$AG(4,q)$	$Sz(q)$	$\mathcal{O}$ the Suzuki-Tits ovoid in $PG(3,q)$, $q=2^{2e+1}$
(PQ9)	$Hi(243)$	$AG(5,3)$	M_{11}	

Table 20: Point Geodesic Transitive Partial Quadrangles with $s>1$.

PROPOSITION 4. *Let Γ be a partial quadrangle which is not a generalized quadrangle and let $G \leq \mathrm{Col}(\Gamma)$ act point distance transitively on Γ. Then either the pointset of Γ can be identified with the affine line $AG(1,q)$ and $G \leq A\Gamma L_1(q)$, or Col(Γ) acts point geodesic transitively and one of the following possibilities occurs:*

1. *$s=1$ and Γ is a rank 3 strongly regular graph. The possibilities for Γ, G are listed in table 19;*

2. *Γ has a linear representation in the affine space $AG(n,q)$, G acts point geodesic transitively, it contains the full translation group of $AG(n,q)$ and modulo its center, the stabilizer of a point is an almost simple group S, where the possibilities for Γ, n, q, S are given in table 20.*

PROOF OF PROPOSITIONS 2,3 AND 4 STARTED.

So suppose $\Gamma = (\mathcal{P}, \mathcal{L}, I)$ is a proper partial geometry, a net or a dual net (PG) or a partial (PQ) quadrangle with a collineation group G satisfying the respective conditions of propositions 1, 2 and 3.

First suppose that G acts imprimitively on $\mathcal{P}$. Suppose Γ is a PQ and let A be a non-trivial set of imprimitivity. If A contains two collinear points, then by connectedness, we obtain a contradiction. So let $x, y \in A$ with x and y non-collinear. Let Γ have parameters (s, t, α). Then there are $s(t-\alpha)$ points z collinear to y and not collinear to x. But $t > \alpha$, otherwise Γ is a GQ and A must be trivial. By the transitivity of G, we can fix x and map y to such a point z. But now A contains two collinear points.

Suppose now Γ is a PG with parameters (s, t, a). Again, let A be a set of imprimitivity with $x, y \in A$. As earlier, x and y are non-collinear. If we can find an element z collinear to y but not collinear to x, then we obtain a contradiction as above. The number of such points z is $(t+1)(s-a-1)$. So we may assume $s = a+1$. Since the assumptions for PG are self-dual, we can consider the dual Γ^D and if this has a non-trivial set of imprimitivity, then $t = s+1$. Hence, the only case to consider here is the case where Γ is both a net and a dual net, hence a helicopter plane. A similar argument as in lemma 6 below shows that the corresponding projective plane $\mathcal{P}$ is a translation plane and a dual translation plane with respect to the special line resp. point (for which all elements incident with it do not belong to Γ). From the transitivity follows that the autotopism group has a unique orbit on the set of points off the autotopism triangle (see e.g. Hughes & Piper [50] for the definitions). But in this case, Kallaher [54] shows that $\mathcal{P}$ must be Desarguesian and it is an elementary excercise to verify that the group G is as claimed in proposition 2. This gives us example (NE1).

So from now on, we may assume that G acts primitively on $\mathcal{P}$. Throughout, x will denote an arbitrary point of Γ. The parameters of Γ will be denoted by (s, t, α), resp. (s, t, a) for a PQ, resp. a PG. The parameters of the strongly regular point-graph are (v, k, λ, μ). Note that $\Gamma_2(x)$ is exactly the set of vertices of the point-graph adjacent to x. Proving the propositions amounts to check whether the rank 3 graphs mentioned in section 3.1 are the point graph of a PG or a PQ. Usually a number-theoretical argument or an easy geometric one suffices to kill a given case. Let us summarize the most common arguments:

(PQ) In the case of PQ, we have $s \leq t$ and $|\Gamma_2(x)| < |\Gamma_4(x)|$, i.e. $k \leq l$. Here, $\lambda + 1 = s$, $\mu = \alpha + 1$, $k = s(t+1)$ and $l = \frac{s^2 t(t+1)}{\alpha+1}$. So (s, t, α) is determined:

$$(s, t, \alpha) = (\lambda + 1, \frac{k - \lambda - 1}{\lambda + 1}, \mu - 1).$$

The restrictions here are (1) $\lambda + 1$ divides l, (2) $\mu - 1 \leq \frac{k-\lambda-1}{\lambda+1}$ and in case Γ is not a GQ, (3) $\lambda + 1 \leq \frac{k-\lambda-1}{\lambda+1}$ (the last two inequalities because $\alpha \leq t$ and $s \leq t$ respectively). When one of these conditions is not satisfied for a certain strongly regular graph, we say that *its parameters do not fit a PQ*. A geometric argument which will often be used is the fact that two points y and z which are collinear determine a unique line: indeed, the (shadow of the) line yz is the set $\Gamma_2(y) \cap \Gamma_2(z)$. This can be used to verify that Γ does not contain triangles. When it does, we say that the strongly regular graph *induces triangles*. Or in some cases, $\Gamma_2(y) \cap \Gamma_2(z)$ contains non-collinear points.

In this case, we say that *lines cannot be well-defined*. Other arguments will make use of the group G.

(PG) In this case, one can also calculate the parameters of Γ as a function of the parameters of its point-graph. If no contradiction arises, then the graph is called **pseudo-geometric**. But for a given graph, we have also to consider its complementary graph here, since $|\Gamma_2(x)|$ can be larger or smaller than $|\Gamma_4(x)|$. The calculations first show that $D = (\mu - \lambda)^2 + 4(k - \mu)$ must be a perfect square. Furthermore:

$$t = \frac{1}{2}(\mu - \lambda - 2 + \sqrt{D});$$

$$s = \frac{k}{t+1};$$

$$a = s - \frac{1}{2}(\lambda - \mu + 2 + \sqrt{D}).$$

From now on we adopt D as a standard notation.

Note that lines are maximal cliques of the point-graph, but it is not the case that every maximal clique is a line. Note also that $s, t > 1$, otherwise we have either a linear or dual linear space, or a non-thick GQ.

Another useful argument will be the fact that G_x acts 2-transitively on the set of lines through x and dually, the stabilizer G_L of a line acts 2-transitively on the set of points on L (follows easily from our assumptions).

Since a generalized quadrangle is a partial quadrangle, we will find the examples of table 18 for $g = 4$ back along our way when dealing with partial quadrangles. This provides a more detailed version of the proof in Buekenhout & Van Maldeghem [16].

We are now ready to go through the list of all primitive rank three representations of finite groups (see 3.1).

CLASS I. The Almost Simple Case.

CASE Ia. The Alternating Groups.
We refer to table 5 and treat the different cases (PG, PQ) separately.

(AL1).

(PQ). For $n \geq 7$, $|\Gamma_2(x)| = 2(n-2)$ and the graph induces triangles. For $n = 6$, we obtain example (GP11) of table 18 and for $n = 5$, we get example (PQ1).

(PG). If adjacent vertices are intersecting pairs, then the parameters imply $t = 1$, a contradiction. In the other case, $s + 1 = n/2$ and the points on a line are certain $n/2$-sets of disjoint pairs. We may suppose that $n \geq 8$ (otherwise we have a GQ). The stabilizer of two collinear points (so two disjoint pairs) obviously acts transitively on the set of points collinear to both, never preserving the line through the points.

(AL2).
By the isomorphism $A_8 \cong L_4(2)$, this is a special case of (CH1).

(AL3).
$\boxed{\text{(PQ).}}$ Here, $\lambda + 1$ does not divide l.

$\boxed{\text{(PG).}}$ If $|\Gamma_2(x)| = 63$, then A_9 acts on 120 $PG(1,8)$'s. These form indeed a partial geometry with parameters $(7,8,3)$ denoted by $PQ^+(7,2)$ in De Clerck & Van Maldeghem [28] and first discovered by Cohen [19], and independently by Haemers & van Lint [43] and De Clerck, Dye & Thas [26]. It was shown by Kantor [57] that A_9 is indeed the full automorphism group of this PG. But there are 135 lines and A_9 does not act as a rank 3 group on any set of that size.
If $|\Gamma_2(x)| = 56$, then the parameters imply $t = 3$ and $a = 5$, a contradiction.

(AL4).
$\boxed{\text{(PQ).}}$ Again, $\lambda + 1$ does not divide l.
$\boxed{\text{(PG).}}$ The parameters do not fit any PG here.

CASE Ib. The Chevalley Groups: Infinite Classes.
We refer to table 6 for this case.

(CH1).

$\boxed{\text{(PQ).}}$ Collinearity of points is always "intersecting" for the corresponding lines in $PG(n-1,q)$, except for $n = 4$ and $q = 2$. But in both cases, lines of Γ cannot be well-defined.

$\boxed{\text{(PG).}}$ If collinearity of points is "intersecting" for the corresponding lines in $PG(n-1,q)$, then the parameters imply $t = q = a$, hence we have a dual linear space. So collinearity is being "skew". Here the parameters imply $s - a = q + 1$, so the lines of Γ are sets of lines of $PG(n-1,q)$ forming a spread of $PG(n-1,q)$. But the stabilizer of two skew lines in $PG(n-1,q)$ never fixes a full spread.

(CH2)-(CH3)-(CH4)-(CH5)-(CH17)-(CH18).

$\boxed{\text{(PQ).}}$ If the rank of the corresponding *building* is larger than 2, then lines cannot be well-defined (since $\Gamma_2(y) \cap \Gamma_2(z)$ is a building of one rank less, for y and z non-collinear points). Hence the rank is 2 and we obtain the examples (GP2) up to (GP6).

$\boxed{\text{(PG).}}$ If collinear points in Γ are collinear points in the graph, then obviously we obtain lines meeting in more then just one point (since the lines of Γ must be the maximal isotropic or maximal singular subspaces), except if the rank of the building is 2, in which case we have a GQ. Suppose now that Γ arises from the complementary graph. We can calculate the parameters of Γ and obtain:

	s	t	a
$S_{2n}(q)$	q^n	$q^{n-1}-1$	$q^n-q^{n-1}-1$
$O_{2n+1}(q)$	q^n	$q^{n-1}-1$	$q^n-q^{n-1}-1$
$O^-_{2n}(q)$	q^n	$q^{n-2}-1$	$q^n-q^{n-1}-1$
$O^+_{2n}(q)$	q^{n-1}	$q^{n-1}-1$	$q^{n-1}-q^{n-2}-1$
$U_{2n+1}(q)$	q^{2n+1}	$q^{2n-2}-1$	$q^{2n+1}-q^{2n-1}-1$
$U_{2n}(q)$	q^{2n+1}	$q^{2n-1}-1$	$q^{2n+1}-q^{2n-1}-1$

In all but one case, $a > t$, a contradiction. The exceptional case is $O^+_{2n}(q)$. Here, $s = q^{n-1}$ and so the lines of Γ are ovoids on the hyperbolic quadric. Since G must induce a doubly transitive permutation group on the set of points on a line, we must only consider 2-transitive ovoids. These were classified by Kleidman [60]. But comparing the order of the group G_L stabilizing a line in Γ computed by $\frac{|G|}{|\mathcal{L}|}$ with the order of the corresponding group in Kleidman's list, no example survives. Alternatively, one can argue as in (CH1), namely, the group fixing two non-collinear points never fixes an ovoid.

(CH19).

(PQ). Here example (GP10) arises.

(PG). Collinear points of Γ must be non-intersecting lines of the generalized quadrangle $H(4,q)$, otherwise we obtain the generalized quadrangle itself. The parameters imply $a = q^5 - q^2 - 1 > q^3 - 1 = t$.

(CH6).

The parameters of the graph are (taken from Hubaut [48]):

$$k = \frac{q(q^5-1)(q^2+1)}{q-1};$$

$$l = \frac{q^6(q^5-1)}{q-1};$$

$$\lambda = \frac{q^2(q^3-1)(q+1)}{q-1} + q - 1;$$

$$\mu = \frac{(q^3-1)(q^2+1)}{q-1}.$$

(PQ). Here, $\lambda + 1$ never divides k. A geometric argument goes as follows. Consider four points p_1, p_2, p_3, p_4 on the quadric $O^+_8(q)$ such that p_1 resp. p_2 is collinear to the other three and p_3 is not collinear to p_4. Apply triality and embed this in $O^+_{10}(q)$ to obtain a subgraph on four vertices with five edges, clearly impossible for the point-graph of a PQ (it would induce triangles).

	k	l	λ	μ
(CH7)	$2^{2n-2}-1$	$2^{2n-2}+2^{n-1}$	$2^{2n-3}-2$	$2^{2n-3}-2^{n-2}$
(CH8)	$2^{2n-2}-1$	$2^{2n-2}-2^{n-1}$	$2^{2n-3}-2$	$2^{2n-3}+2^{n-2}$
(CH9)	$3^{n-1}.\frac{3^n-1}{2}$	$3^{2n-1}+2.3^{n-1}-1$	$3^{n-1}.\frac{3^{n-1}-1}{2}$	$3^{n-1}.\frac{3^{n-1}-1}{2}$
(CH10)	$3^{n-1}.\frac{3^n+1}{2}$	$3^{2n-1}-2.3^{n-1}-1$	$3^{n-1}.\frac{3^{n-1}+1}{2}$	$3^{n-1}.\frac{3^{n-1}+1}{2}$
(CH11)	$3^{n-1}.\frac{3^{n-1}-1}{2}$	$3^{2n-2}-1$	$3^{n-2}.\frac{3^{n-1}+1}{2}$	3^{n-1}
(CH12)	$3^{n-1}.\frac{3^{n-1}+1}{2}$	$3^{2n-2}-1$	$3^{n-2}.\frac{3^{n-1}-1}{2}$	3^{n-1}
(CH20)	$(2^{2n}-1)(2^{2n-1}+1)$	$2^{2n-1}.\frac{2^{2n}-1}{3}$	$3.2^{4n-3}+2^{2n-1}-2$	$3.2^{2n-2}(2^{2n-1}+1)$
(CH21)	$(2^{2n-1}+1)(2^{2n-2}-1)$	$2^{2n-2}.\frac{2^{2n-1}+1}{3}$	$3.2^{4n-5}-2^{2n-2}-2$	$3.2^{2n-3}(2^{2n-2}-1)$

Table 21: Parameters of Some Rank 3 Graphs Related to Classical Groups.

$\boxed{\text{(PG).}}$ The parameters of the graph imply $a = q^2+q > q^2 = t$; the parameters of the complementary graph imply $s = \frac{q^3(q^4+q^3+q^2+q+1)}{q^2+q+1}$ which is never an integer.

(CH7)-(CH8)-(CH9)-(CH10)-(CH11)-(CH12)-(CH20)-(CH21).

In table 21, we list the parameters k, l, λ, μ of the strongly regular graphs corresponding to these cases, see Hubaut [48].

$\boxed{\text{(PQ).}}$ We check the condition $\lambda + 1 | k$ (observe that $l \leq k$ occurs in table 22; so sometimes one must consider the complementary graph!). As an example, consider (CH21). Here, $k > l$ and so we have to use the parameters of the complementary graph, i.e. we must check $l - k + \mu | l$. This gives us the condition

$$2^{4n-5} - 2^{2n-3} + 3 | 2^{2n-2}(2^{2n-1}+1),$$

and since the left hand side is always odd, this implies

$$2^{4n-5} - 2^{2n-3} + 3 | (2^{2n-1}+1),$$

which is clearly only possible when $n = 2$. But in this case we indeed obtain the generalized quadrangle $W(3)$ in view of the isomorphism $U_4(2) \cong S_4(3)$ and in the Atlas [22], one can see that the action of $U_4(2)$ on non-singular points is the same as $S_4(3)$ on (isotropic) points.

There is only one further case in which the condition $\lambda + 1 | k$ is satisfied and that is in case (CH9), $n = 2$. Here too, we obtain an example, namely the generalized quadrangle $H(3, 4)$, in view of the isomorphism $O_5(3) \cong U_4(2)$ (atlas [22]).

So we have found special cases of (GP2) and (GP5). Since these examples were already found, we could have assumed $n \geq 3$ in table 6 for (CH9) and (CH21).

$\boxed{\text{(PG).}}$ Consider the stabilizer G_x of a point of Γ. This is (a "small" cover of) an almost simple group with socle $X_m(q)$ if $G \cong X_{m+1}(q)$. But G_x should have a 2-transitive representation on $t+1$ elements. Comparing this with table 4, we obtain the following possibilities.

	k	l	λ	μ
(CH13)	$(2^{2n}+1)(2^{2n-2}-1)$	$2^{2n-2}(2^{2n}+1)$	$2^{4n-3}-3.2^{2n-2}-2$	$2^{2n-1}(2^{2n-2}-1)$
(CH14)	$(2^{2n}-1)(2^{2n-2}+1)$	$2^{2n-2}(2^{2n}-1)$	$2^{4n-3}+3.2^{2n-2}-2$	$2^{2n-1}(2^{2n-2}+1)$
(CH15)	$(2^{3n-3}+1)(2^{3n}-1)$	$3.2^{3n-3}(2^{3n}-1)$	$2^{6n-5}-2^{3n-3}+2^{3n}-2$	$2^{3n-2}(2^{3n-3}+1)$
(CH16)	$(2^{3n-3}-1)(2^{3n}+1)$	$3.2^{3n-3}(2^{3n}+1)$	$2^{6n-5}+2^{3n-3}-2^{3n}-2$	$2^{3n-2}(2^{3n-3}-1)$

Table 22: Parameters of the Graphs (CH13) up to (CH16).

1. $G \cong O^+_{2n}(2)$ and $G_x \cong O_{2n-1}(2) \cong S_{2n-2}(2)$. By the information on table 4, $t+1 = 2^{n-2}(2^{n-1}+1)$ or $t+1 = 2^{n-2}(2^{n-1}-1)$. But the parameters of the graph imply either $t = 2^{n-1}-2$ or $t = 2^{n-2}+1$. This is, for $n \geq 3$, never compatible.
2. $G \cong O^-_{2n}(2)$ and $G_x \cong O_{2n-1}(2) \cong S_{2n-2}(2)$. This is ruled out as the preceding case.
3. $G \cong O_5(3)$. In the case (CH9), we either obtain a generalized quadrangle (see above) or the parameters imply $a = 5 > 3 = t$. In the case (CH10), $G_x \cong S_6$ and this has a 2-transitive action on 6 or 10 points. But the parameters here imply $t = 2$ or $t = 3$.
4. $G \cong 0_7(3)$ and $G_x \cong L_4(3) \cong O^+_6(3)$. So G_x has a 2-transitive action on 40 points (hence $t = 39$), but the parameters imply $t = 8$ or $t = 9$.
5. $G \cong U_4(2)$. This produces $W(3)$ for the complementary graph (see above) and the parameters of the graph itself imply $a = 5 > 2 = t$, a contradiction.

(CH13)-(CH14)-(CH15)-(CH16).

In table 22 we list the parameters of the strongly regular graphs in these classes. We obtain this information from HUBAUT & METZ [49].

$\boxed{\text{(PQ).}}$ One can see immediately that $\lambda + 1$ never divides k (considering the complementary graph for (CH14)).

$\boxed{\text{(PG).}}$ As in the previous case, there are only a few possibilities where G_x has indeed a 2-transitive action ($n = 2$ in all cases, $n = 3$ in cases (CH14) and (CH16)). But again, this is never compatible with the parameter t obtained from the parameters of the graph.

(CH22).

The parameters of the graph are:

$$k = \frac{q(q^8-1)(q^3+1)}{q-1};$$

$$l = \frac{q^8(q^5-1)(q^4+1)}{q-1};$$

$$\lambda = \frac{q^9+q^7-q^4-2q+1}{q-1};$$

$$\mu = \frac{(q^3+1)(q^4-1)}{q-1}.$$

$\boxed{\text{(PQ).}}$ Collinearity in Γ is the same as collinearity in the building, hence lines cannot be well-defined.

$\boxed{\text{(PG).}}$ Suppose first that collinearity in Γ is collinearity in the building. If a point y is collinear to all points of a certain set A contained in a maximal subspace of the building, then y is collinear to all points of the subspace generated by A. So an induction argument shows that the lines of Γ are the maximal subspaces of the building. But they sometimes meet in more than one single point.

Hence collinearity in Γ arises from the complementary graph. The parameters here imply $s = q^8 + q^4$, $t = q^8 + q^7 + q^6 + q^5 + q^4 - 1$ and $a = q^8 + q^4 - q^3 - 1$. Hence lines of Γ are "ovoids" of the building (see the remark below). The number of lines of Γ is

$$\frac{q^4(q^9-1)(q^5-1)}{(q-1)^2}.$$

So the stabilizer of a line in Γ would certainly have order greater than q^{58}, but from table 15 follows that only the maximal parabolics qualify for this. Now only the D_5-parabolic has a rank 3 representation, implying $s = t$, a contradiction.

REMARK. Define an **ovoid** in a (finite) building of type E_6 over $GF(q)$ as a set of q^8+q^4+1 mutually non-collinear points. An interesting problem would be the investigation of the existence and construction of such ovoids.

This completes the proof for case Ib.

CASE Ic. The Chevalley Groups: Exceptional Classes.

Here we refer back to tables 7 and 8.

$\boxed{\text{(PQ).}}$ It is readily verified that $\lambda + 1 | k$ only in the cases (CG2) and (CG6), implying the examples (PQ3) resp. (PQ2).

$\boxed{\text{(PG).}}$ If we calculate the possible parameters for Γ, then only a few cases can occur (the other ones giving either non-integer values, or non-proper partial geometries which are neither nets nor dual nets):

(CG3) The complement of the graph implies the parameters $(s, t, a) = (7, 8, 3)$. Actually, the strongly regular graph is isomorphic to the one in case (AL4) (but with a larger group). Anyway, $G_x \cong G_2(2)$ cannot act transitively on 9 objects.

(CG4) Here, the complement of the graph implies $(s, t, a) = (26, 27, 17)$, but $G_x \cong G_2(3)$ cannot act transitively on 28 objects.

(CG8) The complement of the graph implies $(s, t, a) = (21, 39, 12)$, but $G_x \cong U_4(3) : 2$ cannot act transitively on 40 objects.

	G	G_x	(s,t,a)	$\lvert\mathcal{L}\rvert$
(SP2)^C	M_{12}	$M_{10}:2$	$(5,8,3)$	88
(SP7)^C	M_{24}	$M_{22}:2$	$(11,20,9)$	483
(SP8)^C	M_{24}	$M_{12}:2$	$(22,35,13)$	2016
(SP9)	J_2	$U_3(3)$	$(9,3,2)$	30
(SP9)^C	J_2	$U_3(3)$	$(9,6,5)$	70
(SP11)	McL	$U_4(3)$	$(4,27,1)$	1540
(SP12)	Suz	$G_2(4)$	$(26,15,5)$	1056
(SP13)^C	Co_2	$U_6(2):2$	$(22,63,13)$	6400
(SP14)	Ru	${}^2F_4(2)$	$(27,64,11)$	9425
(SP15)^C	Fi_{22}	$2.U_6(2)$	$(44,63,35)$	4992
(SP16)^C	Fi_{22}	$\Omega_7(3)$	$(39,279,30)$	98560
(SP17)^C	Fi_{23}	$2\cdot Fi_{22}$	$(80,351,71)$	137632
(SP18)	Fi_{23}	$P\Omega_8^+(3).S_3$	$(351,80,71)$	31671
(SP19)	Fi_{24}'	Fi_{23}	$(391,80,39)$	63423

Table 23: Pseudo-geometric Graphs Related to Rank 3 Sporadic Groups

(EG1) Here the graph itself implies $(s,t,a)=(14,8,4)$, but the number of lines would be $\frac{351\times 9}{15}$, which is not an integer.

(EG2) The complement of the graph implies $(s,t,a)=(15,20,11)$, but there is no transitive action of J_2 on 21 objects.

The claims above on the transitive actions of the group G_x follow from the orders of the maximal subgroups given in the Atlas [22]. This completes the case of Chevalley groups.

CASE Id. The Sporadic Groups.
We refer to table 9 here.

$\boxed{\text{(PQ).}}$ Only in cases (SP3) and (SP10), $\lambda+1|k$ and this gives rise to examples (PQ4) and (PQ5).

$\boxed{\text{(PG).}}$ Here 14 different parameter sets could arise from the graphs. We list them in table 23. A superscript "C" in the label means that the parameters are obtained from the complement of the graph given in table 9.

From the information on maximal subgroups in the Atlas [22], we readily see that in the cases (SP2)^C, (SP7)^C, (SP9), (SP9)^C, (SP11), (SP12), (SP13)^C, (SP14), (SP15)^C and (SP19) the group G cannot act transitively on a set of size $|\mathcal{L}|$. In case (SP8)^C, M_{24} acts on 2016 lines in 24 sets of imprimitivity of size 84. The stabilizer of such a set is M_{23}, but this has no transitive action on 84 objects, a contradiction. In case (SP16)^C the group $G_x \cong \omega_7(3) \cong O_7(3)$ must act transitively on 280 lines through x, contradicting the information in the Atlas [22]. Consider now the case (SP17)^C. The group $G_x \cong 2\cdot Fi_{22}$ can act transitively in two ways on a set of n elements: either the involution σ of the normal subgroup fixes everything and the kernel of the action is Fi_{22}, or σ has orbits of length 2 and Fi_{22} acts on the $n/2$ corresponding pairs. Since $t = 351$ we get $n = 352$. But Fi_{22} cannot act transitively on a set of 352 or 176 elements. Finally, consider case (SP18). The group Fi_{23} acts on 31671 lines and by the ATLAS [22], the stabilizer of such a line is $2 \cdot Fi_{22}$, which must act transitively on the 352 points on that line, a contradiction as above. We remark here that SOICHER [80] has shown on a computer that the graph (SP18) does not have cliques of size 352, so no group action argument is in fact needed to kill this case. At the other hand, the graph $(SP17)^C$ does have maximal cliques of size 81. Further investigation is needed here to determine if this graph (without the group action) gives rise to a PG.

This completes the proof in the case of an almost simple group. Before turning to the cases in class II (the affine representations), we make the following remark: the proof above can be adapted for point distance transitive groups (using the minimal number $P(G)$ of objects a simple group can act on, see e.g. KLEIDMAN & LIEBECK [61]) and the only example coming out is the A_9-partial geometry of COHEN [19]. We have not taken this more general point of view because it makes the matter more difficult (though feasible) in the affine case. We conjecture that there are no further examples in this situation.

CLASS II. The Affine Case.

First we prove some lemma's and make some observations. The assumptions are the same as in the statements of propositions 2, 3 and 4 (the weakest assumption for PG, i.e. the one of proposition 2). We assume also that the set of points of Γ is identified with the set of points of an n-dimensional affine space $AG(n, q)$. The projective space $PG(n-1)$ (and its elements) completing the affine space $AG(n, q)$ to a projective space $PG(n, q)$ will briefly be referred to as "*at infinity*".

LEMMA 1. *If Γ is a PQ, then the lines of Γ are affine subspaces of $AG(n, q)$, except possibly in case (AI1).*

PROOF. If G_x has only one orbit at infinity, then by inspection of the list (only one such case appears: case (AF2)) $k = l$, a contradiction. So suppose G_x has two orbits at infinity. Let x and y be collinear in Γ and let L be the set of points of $AG(n, q)$ on the line through x and y. Denote by l the line in Γ incident with x and y. Clearly all points of L are collinear to both x and y, so all of them are incident with l in Γ. Hence l must be an affine subspace.$\square$

LEMMA 2. *If Γ is a proper PG, then case (AI1) arises.*

PROOF. Denote by T the translation group of $AG(n, q)$ and suppose $n > 1$. Since $T \trianglelefteq G$, the orbits in $\mathcal{L}$ under the action of T are sets of imprimitivity, hence since Γ is not trivial, T must act transitive on $\mathcal{L}$. Clearly, no non-trivial element of T can fix every line, so T acts faithfully and hence regular on $\mathcal{L}$. So $s = t$. But now $s(s + 1) = k$ and this gives us a value for s. Also, the parameters of the strongly regular graph determine s and t. By inspection of the list, this is never compatible except possibly in case (AI1) (this computation is trivial for the cases of tables 11 and 12; for the cases of table 10, we will provide the parameters below). Note that in almost all cases, the parameters imply that Γ is a net, see below for each single case.$\square$

In fact, this shows proposition 2. So from now on, we may assume that in case of (PG), we have a net. But then clearly $|\mathcal{P}| = \sqrt{q^n}$. Moreover, we can prove the following lemma.

LEMMA 3. If Γ is a net, then either cases (AI1) or (AF2) arise or the lines of Γ are affine subspaces of dimension $n/2$.

PROOF. Let x ,y, L and l be as in the proof of lemma 1. We again have to show that all elements of L are incident with l. Suppose the contrary and let z be any element of L not incident with l. Choose x as the origin on L and y as the point with coordinate 1. Fixing x, we can map y to any other point of L (by the rank 3 property of the group G). But no group in tables 10, 11 or 12 must contain field automorphisms in order to act with rank 3, hence restricting to L, we have here a homology. So the stabilizer in G of L with the action restricted to L contains all homologies. Another consequence of that argument is that every line l' of Γ incident with x either "meets" L in one (x) point or in a constant number of points, say $c > 1$. But by the transitivity on geodesics, one can map every line l' through x to any other line l'' through x, (provided $l', l'' \neq l$) fixing x, y and l. So every line l' through x meets L in c points. Now identify every element of L with its coordinate in $GF(q)$. If we fix 0 and 1, then we can only use field automorphisms. These fix all elements of the prime field $GF(p)$ $q = p^\alpha$ for some prime p and integer α, hence $GF(p) \subseteq l$, l viewed as the set of points incident with l. Suppose $a, b \in L \cap l$. Consider the homology with center 0 and factor 2. It maps 0 to 0 and 1 to 2, so it preserves $l \cap L$, hence $2a \in l$. Consider now the translation $x \longrightarrow x + a$. It maps 0 to a and a to $2a$, hence also preserves $l \cap L$. So $a + b \in l$. Similarly $a - b \in l$, $a.b \in l$ and $a.b^{-1} \in l$, hence $l \cap L$ is a subfield $GF(q')$ of $GF(q)$. Suppose $q = q'^h$. By the above mentioned argument, there are still $\frac{q-1}{q'-1}$ other lines through x and they all meet L in $c = q'$ points. Fixing 0 and a primitive element r of $GF(q')$, there must be a transitive group acting on the set of lines through x distinct from l. But clearly such a collineation fixes every element of $GF(q')$, hence there is at most a cyclic group of order h available. This implies

$$h \geq \frac{q'^h - 1}{q' - 1} - 1,$$

and this is only possible for $q' = h = 2$ (note $h > 1$ otherwise $t = 0$). But this means $t = 1$, a contradiction. Hence the assertion.$\square$

PROOF OF PROPOSITIONS 3 AND 4 CONTINUED.

We start the inspection of the distinct cases.

CASE IIa. The Infinite Classes.

Here we refer to table 10.

(AI1).

$\boxed{\text{(PQ).}}$ Let $q = p^d$, p prime. Here we may suppose that G acts point geodesic transitive. That means that the stabilizer of two collinear points acts transitively on st points, so $st|d$. Hence

$$\begin{array}{rcl} p^d = |\mathcal{P}| & = & 1 + st + s + \frac{s^2t^2+s^2t}{\alpha+1} \\ & \geq & 1 + \sqrt{d} + d + d\sqrt{d} + d^2, \end{array}$$

only giving us solutions when $p \leq 5$ and then also $d \leq 5$. Suppose first $s = t = 1$, then $|\mathcal{P}| = 5$, so $p = 5$ and we obtain example (PQ0). Next, suppose $s = 1 < t$. Note that $t \leq 5$, but in view of the fact that $(s-\alpha)^2 + 4st$ is a perfect square (because $(\lambda-\mu)^2 + 4(k-\mu)$ is a perfect square for strongly regular graphs which are no *conference graphs*, but the latter one have $k = l$, impossible for proper non-trivial partial quadrangles, see e.g. Brouwer, Cohen & Neumaier [10] or Hubaut [48]), the only possible sets of parameters for a PQ are $(1,2,0)$ and $(1,4,1)$. The first one gives us the Petersen graph, but here $v = 10 \neq p^d$ for any prime p and integer d; the second one gives us the unique strongly regular graph with parameters $(16,5,0,2)$, the Clebsch graph $Cl(16)$. This can indeed be realized in $AG(1,16)$, giving us the construction as explained in subsection 2.2.2 and we obtain example (PQ6).

If $s \geq 2$, then $s = t = 2$, but no $\alpha < 2$ gives us possible parameters for a PQ.

$\boxed{\text{(PG).}}$ As above, we have $t(s-a)|d$. Stabilizing two non-collinear points, we obtain in the same way $(t+1)(a+1)|d$, so since $t > a \geq 1$, we have $d \geq 6$. Using these conditions, we obtain

$$\begin{array}{rcl} p^d = |\mathcal{P}| & = & 1 + s(t+1) + \frac{st(s-a)}{a+1} \\ & \leq & 1 + \frac{sd}{a+1} + \frac{sd}{a+1} \\ & \leq & 1 + 2\frac{sd}{a+1} \\ & < & 1 + 2\frac{d(d+a)}{a+1}, \end{array}$$

hence, since $d \geq 6$ and consequently $p^d - 1 - 2d > 0$,

$$a \leq \frac{2d^2 + 1 - p^d}{p^d - 1 - 2d}.$$

Now since $a \geq 1$, this implies finally

$$p^d \leq 1 + d + d^2$$

and this can never happen for $d \geq 6$.

For the cases (AI2) up to (AI11), we first list the parameters of the corresponding strongly regular graphs, see table 24. We derive these from the proof of the classification in Liebeck [65].

$\boxed{\text{(PQ).}}$ Since the lines of Γ are subspaces of the affine space $AG(n,q)$, and since $\lambda+2 = s+1$, this number must be a power of q. By inspection of the list, this is only possible for (AI2), (AI7) (case $O_4^-(q)$) and (AI11) (beware of the fact that for some small values of q and n, one has $k > l$ and hence one must consider the complementary graph). In case (AI2), we clearly obtain a grid ($t = 1$), contradicting our assumptions; for cases

	k	l
(AI2)	$2(q^m-1)$	$(q^m-1)^2$
(AI3)	$(q+1)(q^m-1)$	$q(q^m-1)(q^{m-1}-1)$
(AI4)	$(\sqrt{q}+1)(\sqrt{q}^n-1)$	$\sqrt{q}(\sqrt{q}^n-1)(\sqrt{q}^{n-1}-1)$
(AI5)	$(\sqrt[3]{q}+1)(q-1)$	$\sqrt[3]{q}(\sqrt[3]{q}^2-1)(q-1)$
(AI6)	$(q^n-(-1)^n)(q^{n-1}+(-1)^n)$	$q^{n-1}(q-1)(q^n-(-1)^n)$
(AI7)	$(q^m-\epsilon)(q^{m-1}+\epsilon)$	$q^{m-1}(q-1)(q^m-\epsilon)$
(AI8)	$(q^5-1)(q^2+1)$	$q^2(q^3-1)(q^5-1)$
(AI9)	$(q^4-1)(q^3+1)$	$q^3(q-1)(q^4-1)$
(AI10)	$(q^8-1)(q^3+1)$	$q^3(q^5-1)(q^8-1)$
(AI11)	$(q^2+1)(q-1)$	$q(q-1)(q^2+1)$
	λ	μ
(AI2)	q^m-2	2
(AI3)	q^m+q^2-q-2	$q(q+1)$
(AI4)	$\sqrt{q}^n+q-\sqrt{q}-2$	$\sqrt{q}(\sqrt{q}+1)$
(AI5)	$q+\sqrt[3]{q}^2-\sqrt[3]{q}-2$	$\sqrt[3]{q}(\sqrt[3]{q}+1)$
(AI6)	$q^{2n-2}+(-q)^n+(-q)^{n-1}-2$	$q^{n-1}(q^{n-1}+(-1)^n)$
(AI7)	$q^{2m-2}+\epsilon q^m-\epsilon q^{m-1}-2$	$q^{m-1}(q^{m-1}+\epsilon)$
(AI8)	$q^5+q^4-q^2-2$	$q^2(q^2+1)$
(AI9)	$q^6+q^4-q^3-2$	$q^3(q^3+1)$
(AI10)	$q^8+q^6-q^3-2$	$q^3(q^3+1)$
(AI11)	$q-2$	$q(q-1)$

Table 24: Parameters of the Infinite Classes of Affine Rank 3 Graphs.

(AI7) and (AI11), partial quadrangles arise, namely examples (PQ7) and (PQ8). Since the (pointwise) stabilizer in G_x of a pair of points a, b on the corresponding ovoid (elliptic quadric for $G_x \trianglerighteq \Omega_4^-(q).2$; Suzuki-Tits ovoid for $G_x \trianglerighteq Sz(q)$) acts transitively on the other points of the line ab (in $PG(3, q)$), the group G acts point geodesic transitively on Γ.

So we can assume from now on that Γ is a net.

$\boxed{\text{(PG).}}$ The dimension $n = 2m$ is even and $s = q^m - 1$. The parameters can now be easily deduced in each case.

(AI2).

The parameters imply here either $t = 1$ or $s = q^m - 1$, $t = q^m - 2$ and $a = q^m - 3$. So we have here a net obtained from an affine plane by deleting two parallel classes of lines (indeed, the two parallel classes are the translates of $V_m(q)$ and $V'_m(q)$). We refer to lemma 9 where we treat this case in full generality (see below).

(AI3).

If $|\Gamma_2(x)| = k$, then we obtain example (NE2) (see 2.3.2). Consider the model H_q^{n+1}. Fix a line L of H_q^{n+1} and two points P_1 and P_2 on L. A line L' through P_1 determines a unique point in the chosen subspace $PG(n-1, q)$ (see 2.3.2). So point geodesic transitivity implies a 2-transitive group on $PG(n-1, q)$. Fixing L, the group G_L is the direct product of two 2-transitive groups, one acting on the points of L, the other on the points of $PG(n-1, q)$. Using the matrix form for the elements, computations show that for $n > 2$ this is enough to have point geodesic transitivity. If $n = 2$, then one needs to have $GL_2(q)$ acting on $PG(1, q)$. Hence the result.

Now suppose $|\Gamma_2(x)| = l$. If this would constitute a net, then the subspaces of dimension $n-1$ at infinity of a line of Γ together with the ones from the net (NE2) would make up a spread of $PG(2n-1, q)$ and so Γ would be embedded in a translation affine plane $\mathcal{T}$. Coordinatizing this plane by the method of Hughes & Piper [50] such that the parallel classes (point at infinity) labeled $(\infty), (0)$ and (1) do *not* yield lines of Γ, we obtain a quasifield Q and $GF(q)$ is clearly a subfield of the nucleus. Hence we can consider Q as a vectorspace over $GF(q)$. It is also clear from the definition of the lines in $(H_q^{n+1})^D$ that $GF(q)$ itself corresponds to the *non-lines* of Γ. By the transitivity assumption of G, we can now fix (∞), stabilize $GF(q)$ and still act 2-transitively on the other vectors of Q. This is certainly impossible if the dimension of Q over $GF(q)$ is larger than 2. If this dimension m is equal to 2, then if $q > 2$, one cannot map two vectors whose difference is inside $GF(q)$ to two vectors whose difference is outside $GF(q)$. But if $q = 2$ and $m = 2$, then it follows that $a = 0$, a contradiction.

(AI4).

Suppose first that $|\Gamma_2(x)| = k$. The dimension of the maximal subspaces is here 1 (a Baer subspace at infinity does not contain full lines), hence $s = q - 1$ and $n = 2$. This gives us actually a rank 3 net, but if this were a point geodesic transitive net, then we must be able to fix (∞) and (O) (at infinity), stabilize $GF(\sqrt{q})$ and map any element of $GF(q) \setminus GF(\sqrt{q})$ to any other such. The group doing this (multiplication with elements of $GF(\sqrt{q})$ and field automorphisms is clearly to small for such a transitive action except when $q = 4$. But this corresponds to example (NE2) for $q = 2$ and $n = 2$.

Now suppose $|\Gamma_2(x)| = l$. Maximal cliques here have dimension $n/2$ and this is precisely what we want. Now, a point P at infinity not in the Baer subspace lies on exactly 1 Baersubline L. The transitivity on maximal geodesics implies that we can fix P and act transitively on the points of the Baer subspace. But clearly the points of L form an orbit, hence there are no other points, so $n = 2$. But except if we have a grid ($q = 4$), we rule this case out similarly as above (we must be able to fix (∞), stabilize $GF(\sqrt{q})$ and act 2-transitively on the points of $GF(q) \setminus GF(\sqrt{q})$). Note that from a result of BLOKHUIS & METSCH [5] follows that for $n = 4$ (and presumely for every n), these graphs are not geometric.

(AI5).

This is entirely similar to case (AI4) for $n = 2$. No examples survive here, even not $q = 8$.

(AI6).

First suppose that $|\Gamma_2(x)| = k$. Note that n still must be even. But in this case, the group acts transitively on the maximal singular subspaces of the Hermitian variety at infinity and these subspaces can meet in at least one point if $n > 2$. Hence for $n > 2$, lines of Γ would meet in more than just one point (an affine liine of $AG(n, q^2)$), a contradiction. But if $n = 2$, then we have the case of a Baer subline again, see (AI4), $n = 2$.

Now suppose $|\Gamma_2(x)| = l$. At infinity, all lines meet the Hermitian variety, hence no planes forming a clique can be found in $AG(n, q^2)$. Consequently $n = 2$, but this is again case (AI4) for $n = 2$.

(AI7).

Note that here we must have the case of $O^+_{2m}(q)$, otherwise $q^m - 1$ does not divide k nor l. Note also that the case $n = 2$ (or equivalently $m = 1$) corresponds to case (AI2), $n = 2$. But now we have the same arguments as in case (AI6), except that, if $|\Gamma_2(x)| = l$, all *planes* meet the quadric, hence $n = 4$ could still occur. But we should be able to fix at infinity a non-singular point P and act transitively on the points at infinity on a tangent or secant through P (indeed, fix the origin o in $AG(4, q)$ and let a be a point such that the point at infinity of oa is exactly P. Let P' be any point on a secant or tangent through P at infinity. Let a' be the affine point on the line through o with direction P' and such that the point at infinity of aa' is a point of the quadric, then a and a' are not collinear in Γ, but a and o are, as well as a' and o, so one could stabilize o and a and hence act transitively on all such a''s), but this would not preserve tangent lines, nor secant lines.

(AI8).

First suppose $|\Gamma_2(x)| = k$. Maximal cliques here have dimension 3 (a set $\{v \wedge w | v, w \in V_3(q)\}$, where $V_3(q)$ is a fixed 3-dimensional subspace of $V_5(q)$) or 4 (a set $\{v \wedge w | v$ is fixed, $w \in V_5(q)\}$) and they should have dimension 5, a contradiction.

Now suppose $|\Gamma_2(x)| = l$. Let v_1, v_2, v_3, v_4, v_5 span $V_5(q)$ and denote by o the origin. Then in Γ, the following pairs of points are collinear:

$$\begin{array}{rcl} o & \text{and} & v_1 \wedge v_2 + v_3 \wedge v_4, \\ v_1 \wedge v_4 & \text{and} & v_1 \wedge v_2 + v_3 \wedge v_4, \\ v_1 \wedge v_5 & \text{and} & v_1 \wedge v_2 + v_3 \wedge v_4, \end{array}$$

while o is not collinear with $v_1 \wedge v_4$ nor with $v_1 \wedge v_5$. Hence there must exist a collineation of Γ fixing o and $v_1 \wedge v_2 + v_3 \wedge v_4$ and mapping $v_1 \wedge v_4$ to $v_1 \wedge v_5$. This collineation clearly must preserve $v_1 \wedge v_2$ and hence also $v_3 \wedge v_4$ and so it can never map $v_1 \wedge v_4$ to $v_1 \wedge v_5$.

(AI9)-(AI10).

Here G_x has no (suitable) 2-transitive representation.

(AI11).

Here neither k nor l is divisible by $q^2 - 1$.

CASE IIb. The Extraspecial Class.

Here we refer to table 11.

$\boxed{\text{(PQ).}}$ The only case where $n > 2$ (otherwise we have triangles) and $\lambda + 2$ is a power of q is (AE5) and in this case $s = 8$. But $k = 16$, hence $t = 1$, a contradiction.

$\boxed{\text{(PG).}}$ We consider each case in turn.

(AE1).

Here $n = 2$ and q is a prime. We show that this can never happen. Clearly, though, we have rank 3 nets, but let us assume that the group G is geodesic transitive. Denote by K the set of points at infinity of lines in $AG(2, q)$ which are also lines of Γ and let L be the set of points at infinity of lines in $AG(2, q)$ which are not lines in Γ (but they are lines in the net obtained from the complementary graph). Expressing geodesic transitivity (as in (AI7) above for example), we obtain a group of collineation fixing two points of K and acting transitively on L. Since no field automorphisms are involved here, L is an orbit of a subgroup of a homology group (which is isomorphic to the multiplicative group of $GF(q)$), $|L|$ divides $p - 1$ and since we exclude the case $t = 1$ (corresponds to a non-thick GQ), $|L|$ is at most $\frac{p-1}{2}$. On the other hand, there must be a collineation group fixing one point of K and one point of L and acting transitively on the remaining points of K. So as before, $|K| - 1 \leq \frac{p-1}{2}$ (we exclude here the case of a helicopter plane already treated before). Hence $|K| + |L| \leq \frac{p+1}{2} + \frac{p-1}{2} = p < p + 1$, a contradiction. Note that this also rules out case (AF1) as we shall remark later.

(AE2).

Here n is odd, a contradiction.

(AE3). The possible parameters arising here are $(8, 3, 2)$ and $(8, 5, 4)$. The number of maximal geodesics based at the fixed point x is equal to $st(t + 1)(s - a)$. Here this is $2^6.3^2$ resp. $2^6.3.5$. But this never divides $|G_x|$ as given in table 11.

(AE4).

Similarly as above, the parameters are $(26, 3, 2)$ or $(26, 23, 22)$ and the number of maximal geodesics based at x never divides $|G_x|$.

(AE5)-(AE6).

As indicated in table 11, the blocks of imprimitivity of G_x all have size 16. But that means that either $16|s$ or $t+1 = 16/(q-1)$ (indeed, a block must be contained in a line of Γ or meet it in $q-1$ points otherwise we have smaller blocks). But $s = 3^2 - 1 = 8$, hence $t = 7$ and $a = 6$. But in this case t does not divide the order of $G_x \cong Sp_4(3)$.

(AE7)-(AE8)-(AE10).

The argument we present here also covers the sporadic cases (AF7) and (AF12) and partially (AF5). Lines of Γ are planes of $AG(4, q)$. We call the line at infinity of such planes *good* lines and the other lines of $PG(3, q)$ *bad* lines. Points on good, resp. bad lines will be called likewise. We stabilize a good point P and a good line L not through P. By the geodesic transitivity, there is a transitive action on the bad points in the plane $< P, L >$ generated by P and L. Note that there is at least one bad point Q in $< P, L >$ since by transitivity, otherwise there would be no bad point at all. On the line PQ, we can repeat the argument used in (AE1) to obtain 2 possibilities (since q is a prime here): (a) there is exactly one bad point on PQ; (b) there are $q-1$ bad points on PQ. But by the transitivity again, the number of bad points on every line through P is either 0 or a constant. Hence the possibilities are:

1. Every line through P contains either 0 or exactly one bad point. By transitivity, this is through for every bad line containing at least two good points and one bad. Joining two bad points, we see that all points on this line must be bad points except for the intersection with L. So the number of bad points in $< P, L >$ is q. By transitivity (vary the plane through L), $l = q(q+1)(q-1)$ and this is never the case in the examples (do not forget the complementary graph!).

2. Every line through P contains either 0 or $q-1$ bad points. If there were only one line through P in $< P, L >$ containing $q-1$ bad points, then as above, $l = (q-1)(q+1)(q-1)$, never occuring in the examples. So there are at least two such lines and it easy to see that this implies either $l = (q^2 - q)(q+1)(q-1)$ or $l = (q^2-1)(q+1)(q-1)$. Only the first case actually occurs: for $q = 3$, case (AE10) and $q = 5$, case (AF5). The parameters of the first case are (8,3,2). The number of maximal geodesics based at x is $2^6.3^2 = |L_2(3)|^2$, hence, if $|G_x| < 2^8.3^2$, then $G_x = L_2(3) \bigotimes L_2(3)$ and this is case (AI3) for $q = 3$. So suppose $\frac{|G_x|}{|Z(G_x)|} = 2^7.3^2$. Fixing the four lines in Γ through x, there remains a group whose order is divisible by $2^7.3^2/|S_4| = 2^4.3$. Moreover fixing the four blocks of imprimitivity (the lines in $AG(4, 3)$) on one of these lines, we have left a group containing an involution σ. Now look at the action of σ on the projective 3-space at infinity of $AG(4, 3)$. It fixes a line L pointwise and three (mutually skew) other lines L_1, L_2, L_3 not necessarily pointwise. This forces σ to be the identity. Indeed, consider the *residue* in a point

P of L (for this terminology, see BUEKENHOUT [11]). This is a projective plane and σ fixes three non-concurrent lines and one point not on that line, hence it induces the identity. Hence every line through P is fixed, so also all points on L_1, L_2, L_3 are fixed, a contradiction. We handle case (AF5) later (see below). Note that here we only have to consider the graph and not the complement anymore. So the only set of parameters for Γ is $(24, 5, 4)$.

(AE9).

Here the parameters of Γ are either $(80, 17, 16)$ or $(80, 63, 62)$ and in both cases t does not divide $|G_x|$.

CASE IIc. The Exceptional Cases.

Here we refer to table 12.

$\boxed{\text{(PQ).}}$ The only cases where $n > 2$ and $\lambda + 2$ is a power of q are (AF4) and (AF14). The first one provides us example (GP12). The second one gives us the construction of example (PQ9) (see HILL [47]).

$\boxed{\text{(PG).}}$ We first eleminate some immediate cases and consider afterwards each other remaining case in turn.

(AF1)-(AF4)-(AF7)-(AF10)-(AF11)-(AF12)-(AF14)-(AF15)-(AF16)-(AF17)-(AF19).

Case (AF1) is ruled out similarly as (AE1) noting q is a prime here. Cases (AF7) and (AF12) are ruled out above together with (AE7),(AE8) and (AE10). In cases (AF10), (AF11) and (AF19), k nor l is divisible by $q^{n/2} - 1$. In the other cases, n is odd.

(AF2).

This is the only case where G_o has only one orbit at infinity. The parameters of Γ are $(8, 4, 3)$. Consider three points x, y, z on a line in $AG(2, 9)$ such that x and y are collinear in Γ as well as y and z, but x and z are not. We claim that such points. Indeed, every affine line through x contains 4 points collinear to x in Γ and also 4 points not collinear to x in Γ; applying the translation group fixing that affine line and noting that every translation has order 3, this group cannot preserve this $5|4$ splitting, hence the claim. But fixing x and z, there must be a transitive action on the 20 points collinear to both. But the orbit of y has size at most 4 because it must be a subset of the affine line through x and z, a contradiction.

(AF3).

Consider the affine line $AG(1, 7^2)$ and the automorphism H group fixing the zerovector 0. The only subgroup of H which could have an orbit on the non-zero vectors of size $> \frac{q-1}{2} = 24$ is the group generated by a homology h with factor θ^3 (where θ is a primitive element of $GF(7^2)$) and some automorphism involving the field automorphism. But the field automorphism preserves the orbits of h, hence either all orbits collapse or no orbit collapses and so all subgroups of H have orbits of size either $q - 1 = 48$ or $\leq \frac{q-1}{2} = 24$

and hence the argument of (AE1) kills also this case. Alternatively, FOULSER [36] shows that $|G_x| = 2^7.3^2.5$ and the parameters imply $t = 19$ ot $t = 29$; so t cannot divide $|G_x|$ and consequantly G_x cannot act transitively on the maxial geodesics based at x.

(AF5).

By the argument in (AE7)-(AE8)-(AE10), the parameters here are $(24, 5, 4)$. But in this case the number of maximal geodesics based at x is $2^6.3^2.5^2$ and this does not divide $|G_x|$.

(AF6).

The possibilities here are $t = 14$ or $t = 34$. Since A_7 is 2-transitive only on 7 or 15 elements, we must have $t = 14$ and the parameters of Γ are $(48, 14, 13)$. Hence $st(t+1)(s-a)$ contains a factor 5^2 and this does not divide $|G_x|$.

(AF8).

Since A_9 is 2-transitive on 9 objects, the parameters here are $(15, 8, 7)$. Since $A_9 \leq O_8^+(2)$, it fixes a spread of the hyperbolic quadric at infinity of $AG(8, 2)$. Since this is the unique spread it fixes, the lines of Γ are the 4-spaces of $AG(8, 2)$ whose plane at infinity is a member of that spread. In Γ, the stabilizer of an ordered pair of lines (l_1, l_2) through x in Γ is a group isomorphic to A_7 and this should act transitively on the 15 points on both these lines except x. If we stabilize a further point y on l_1, then this group $L_3(2)$ can only act in two different ways on l_2. One possibility is that it also fixes a point u on l_2 and is transitive on the other points of l_2. But this is impossible since the points collinear to y must be a union of orbits. The second possibility is that it stabilizes a "plane", i.e. 7 points and acts transitively on the remaining 8. The latter are the points on l_2 not collinear with y and hence example (NE3) arises.

(AF9).

In this case, we deduce from the information in the ATLAS [22] that $G_x \cong A_{10}$ acts primitively on $\Gamma_2(x)$ (in both cases: the graph and its complement), so since the lines through x always induce blocks of imprimitivity of size s, we have here $s = 1$, a contradiction.

(AF18)-(AF20).

Here G_x has no 2-transitive representation.

This completes the affine case. We now consider the last class of primitive rank 3 groups: the grid case.

CLASS III. The Grid Case.

$\boxed{\text{(PQ).}}$ In this case, the geometry is clearly a grid itself ($t = 1$) and so we obtain example (GP13).

$\boxed{\text{(PG).}}$ If Γ is not a grid, then $|\mathcal{P}| = n^2$, $|\Gamma_2(x)| = (n-1)^2$ and $|\Gamma_4(x)| = 2(n-1)$. This implies $s = n - 1$, $t = n - 2$ and $a = n - 3$. Adding the lines of the grid, we obtain an affine plane which can be further completed to a projective plane. We will call such a net a **cogrid plane** of order n. In lemma 4 we classify all geodesic transitive cogrid planes without the assumptions on the rank 3 group of the Grid Case.

LEMMA 4. *If Γ is a connected cogrid plane of order $n \geq 3$, and G is a geodesic transitive automorphism group, then either Γ has order 3 and hence is a grid, or Γ has order 4 and (Γ, G) is as in example (NE2), $q = 2$, i.e. Γ is isomorphic to $(H_2^2)^D$.*

PROOF. The cases where Γ has order 3 or 4 are easy verifications. So we assume that $n \geq 5$. Denote by $\mathcal{A}$ the corresponding affine plane. Let P_1 and P_2 denote the directions of the lines in $\mathcal{A}$ (or the points at infinity) which do not belong to Γ. Let $x, y, z \in \mathcal{P}$, $l, m \in \mathcal{L}$ and $xIlIyImIz$ be a maximal geodesic. Suppose $|G_{x,l,y,m,z}| = k$, then we have

$$|G_{x,l,y,m}| = 2k,$$

$$|G_{x,l,y}| = 2(n-2)k,$$
$$|G_{x,l,y,P_1}| = (n-2)k,$$
$$|G_{x,l}| = 2(n-1)(n-2)k,$$
$$|G_{x,l,P_1}| = (n-1)(n-2)k,$$
$$|G_x| = 2(n-1)^2(n-2)k,$$
$$|G_l| = 2n(n-1)(n-2)k,$$
$$|G_{l,y,m}| = 2(n-1)k,$$
$$|G| = 2n^2(n-1)^2(n-2)k,$$
$$|G_{P_1,P_2}| = n^2(n-1)^2(n-2)k.$$

We first show that $\mathcal{A}$ is a translation affine plane. The proof goes along the same lines as the proof of Wagner's theorem in HUGHES & PIPER [50]. We adapt the same notation: $p^v \| s$ if p^v divides s and p^{v+1} does not.

(1). First suppose n is even. Let G_2 be a Sylow 2-subgroup of G, then $|G_2| = 2^{1+2u+u'+v}$, where $2^u \| n$, $2^{u'} \| n-2$ and $2^v \| k$. Let λ be a non-trivial element in the center of G_2. Then either λ is a translation, an elation with affine axis or a Baer involution. In the latter two cases, the number of (affine) fixed points is n. Note that G_2 acts on this set, so let f be the length of any orbit of G_2 on the set of fixed points of λ. There holds $|G_2| = f|(G_2)_a|$, where a is a point in the orbit of length f. Since $(G_2)_a \leq G_a$, $|(G_2)_a|$ divides $2^{1+u'+v}$, so $2^{2u}|f$. Hence $2^{2n}|n$, a contradiction. Hence λ is a translation. Let P be its center (P is a point at infinity). By the transitivity of the group G (which can fix a point x and map any other point y on the line joining x to P to any other such point), P is the center of n translations. But the orbit of P under G contains either $n-1$ elements (if $P_1 \neq P \neq P_2$) or exactly 2 (if $P = P_1$ or $P = P_2$), hence by HUGHES & PIPER [50], $\mathcal{A}$ is a translation plane.

(2). Suppose now n is odd. By DEMBOWSKI [29], it suffices to show that every point of $\mathcal{A}$ is the center of an involutory homology, and by our transitivity of G, it suffices to show this for one point x. Let l be any line through x in Γ. Let G_2 be a Sylow 2-subgroup of G_{x,l,P_1}, then $|G_2| = 2^{u+v}$, where $2^u \| n-1$ and $2^v \| k$. Let again λ be an element of the center of G_2 and suppose λ is a Baer involution. Then it fixes exactly $\sqrt{n}-1$ affine points other than x on l. Let y be one of them and let f again be the length of the orbit of y under G_2. Clearly $(G_2)_y \leq G_{x,l,y,P_1}$ and hence $|(G_2)_y|$ divides 2^v. But $|G_2| = f|(G_2)_y|$, hence $2^u|t$ and so $2^u|\sqrt{n}-1$. But since $n-1 = (\sqrt{n}-1)(\sqrt{n}+1)$, this implies $2^{u+1}|n-1$, a contradiction. So λ is an involutory homology. Since it stabilizes at least three lines through x (namely l and the lines through x with directions P_1 and P_2), x is the center. So $\mathcal{A}$ is a translation plane.

Now we coordinatize $\mathcal{A}$ in such a way that the directions of the Y-axis and X-axis are exactly P_1 and P_2. The transitivity of G on maximal geodesics implies that the group fixing the points at infinity with coordinates $(0), (\infty)$ and $(1,1)$ acts transitively on the other points at infinity. Hence in the corresponding quasifield Q (see Hughes & Piper [50]), the group fixing 0 and 1 acts transitively on the other elements. If the kernel contains more than just 0 and 1, then it contains evereything and we have a field. But no field of order $n \geq 5$ has an automorphism group acting transitively on its non-zero and non-one elements. So the kernel of Q is just $\{O, 1\}$ and n is a power of 2.

Let x, l, y be as in the beginning of this proof, then $|G_{x,l,y,P_1}|$ is even. Let σ be an involution in G_{x,l,y,P_1}. Clearly, σ must be a Baer involution, so it fixes a Baer subplane containing P_1 and P_2. In particular,$n = 2^{2e}$ is an even power of 2. Now we invoke the classification of the 2-transitive finite permutation groups. Let Ω be the set of points at infinity of Γ distinct from P_1 or P_2. Then G_{x,P_1} induces a 2-transitive permutation group on Ω of degree $2^{2e}-1$. Since $2^{2e}-1 = (2^e-1)(2^e+1)$, this cannot be a prime power and so Ω is not an affine space. Hence Ω can be identified with one of the sets of table 4. Clearly, (TS7) and (TS9) up to (TS16) is impossible. The degrees of (TS6) and (TS4) are even, a contradiction. If q^3+1 or q^2+1 is odd, then q is even, but then it cannot be equal to $2^{2e}-1$ unless $q = 2$ and $e = 1$, but then $n = 4$, contradicting our earlier hypothesis. Hence the only possibilities left are (TS1), (TS2) and (TS8). Note that $n \geq 16$.

(TS1) Consider an element in G_x inducing an involution λ on Ω with $n-5$ fixed points. We can assume that the points with coordinate (1) is one of them and so λ is a quasifield automorphism. We let λ act on $\mathcal{A}$ by coordinates and obtain an automorphism fixing a subplane of order $n-4 > 2^e$, a contradiction (the largest subplane of $\mathcal{A}$ has order 2^e, see Hughes & Piper [50] or Dembowski [29]).

(TS2) Here, Ω can be identified with the points of $PG(d, q)$. First let $d \geq 2$. Then we consider an elation in $PG(d, q)$ fixing all points of a certain hyperplane. As above, this can again be viewed as a quasi field automorphism and extended to an automorphism of $\mathcal{A}$ fixing a subplane which order is too large. Now let $d = 1$. Then $q+1 = 2^{2e}-1$, hence q is even and so $q = 2$ is the only possibility, a contradiction.

(TS8) There are several arguments here. We could appeal to the classification of all translation planes of order 16 by Dempwolff & Reifart [30, 31] and observe that none of them has a coordinatzing quasifield with an automorphism group isomorphic to $L_3(2)$ (which is the stabilizer of the point (1) of Ω in A_7). Or we could argue as follows: take any element g of G_x which swappes P_1 and P_2. If its action on Ω is inside A_7, then by composing g with a suitable element of G_{x,P_1}, we obtain an automorphism α of $\mathcal{A}$ fixing 15 points at infinity and at least one affine point. By raising α to the power half of its order, we obtain a Baer involution fixing too many points. So A_7 is a normal subgroup of index 2 of some larger group acting on $PG(3,2)$, and this is a contradiction as S_7 does not act type-preserving on $PG(3,2)$.

This completes the proof of the lemma.□

So we have proved propositions 2, 3 and 4. An immediate corollary is

COROLLARY 1. *No locally* 4*-arc transitive* $(4,5,5)$*-pair exists.*

PROOF. A locally 4-arc transitive $(4,5,5)$-gon is a partial quadrangle and a dual partial quadrangle, hence $s=t$. It has an automorphism group acting point geodesic transitively. But by proposition 4, no such partial quadrangle exists.□

4.3 Symmetric Designs, Linear Spaces and Moore Geometries.

4.3.1 Symmetric 2-designs.

PROPOSITION 5. *If* (Γ, G) *is a geodesic transitive (or equivalently a locally 3-arc transitive)* $(2,3,3)$*-pair with* G *type-preserving, then it is one of the examples (ST1) up to (ST7) of table 25. If* (Γ, G) *is a weakly geodesic transitive (or equivalently a locally 2-arc transitive)* $(2,3,3)$*-pair with* G *type preserving, then it is one of the examples (ST1) up to (ST10) of table 25 (the notation of table 26 is the one of subsection 2.5; a "C" in the exponent means complementary design; for* $PG(n,q)$ *we take the hyperplanes as blocks). Conversely, every example listed in table 25 is a geodesic transitive (resp. weakly geodesic transitive)* $(2,3,3)$*-pair.*

PROOF. Kantor [58] classified all 2-transitive symmetric 2-designs. So it is only a matter of checking which ones have the stronger transitivity properties mentioned in the proposition. Kantor's list is essentially (SD1) up to (SD4) of subsection 2.5, but we have to consider also the complementary designs and the class (SD5).

(SD1).

The result for $L_{d+1}(q) \trianglelefteq G \leq P\Gamma L_{d+1}(q)$ is obvious when $\Gamma = PG(n,q)$. If $\Gamma = PG(n,q)^C$, then it is clear that G is transitive on triples (x,l,y), where x and y are distinct points incident with a line l (so l is the complement of a hyperplane in $PG(n,q)$, in other words, G is transitive on triples (x,H,y), where H is a hyperplane not containing x nor y). Now let H' and H'' be two hyperplanes through x not containing y. Consider the unique hyperplane U spanned by the line xy and the intersection $H' \cap H''$. Then it is easy to see that one can find a point z in U and a translation with center z and axis U

	Γ	G	Restrictions and Remarks.
(ST1)	$PG(d,q)$	$L_{d+1}(q) \trianglelefteq G \leq P\Gamma L_{d+1}(q)$	$d \geq 3$
(ST2)	$PG(d,q)^C$	$L_{d+1}(q) \trianglelefteq G \leq P\Gamma L_{d+1}(q)$	$d \geq 3$
(ST3)	$PG(3,2)$	A_7	G is regular on maximal geodesics
(ST4)	$Ha(11)$	$L_2(11)$	G is regular on maximal geodesics
(ST5)	$\mathcal{S}^+(n)$	$2^{2n} : S_{2n}(2)$	$n \geq 2$
(ST6)	$\mathcal{S}^-(n)$	$2^{2n} : S_{2n}(2)$	$n \geq 2$
(ST7)	$\Gamma(.\Omega)$	G	G acts 3-transitively on Ω
(ST8)	$Ha(11)^C$	$L_2(11)$	
(ST9)	$Hi(176)^C$	HS	
(ST10)	$PG(3,2)^C$	A_7	

Table 25: Geodesic Transitive and Weakly Geodesic Transitive Symmetric 2-designs.

mapping H' to H'' (z is just a point of U not in $H' \cap H''$). Hence the examples (ST1) and (ST2).

Consider now $PG(3,2)$ with $G \cong A_7$. The stabilizer in A_7 of a plane Π in $PG(3,2)$ acts as $L_3(2)$ on Π, so it acts 2-transitively on the points of Π. Now fix two points y, z of Π. Then the group $G_{\Pi,y,z}$ is transitive on the lines in Π through z distinct from yz. Consider such a line L. It is fixed by a unique involution σ in $G_{\Pi,y,z}$. This involution does not fix the residue in z pointwise (because the stabilizer of z is a group isomorphic to $L_3(2)$ and hence this group acts faithfully on the residue), hence it does not fix every plane through z. Likewise, it does not fix every plane through y, but σ induces a translation in the residue of y and it is easily seen that yz is the center, so all planes through yz are fixed by σ. Hence σ acts transitively on the planes through L distinct from Π. This explains example (ST3).

Consider now $\Gamma = PG(3,2)^C$ with $G \cong A_7$. The stabilizer in A_7 of a plane Π in $PG(3,2)$ acts as $L_2(7)$ on the complement of Π, so it acts 2-transitively on the points off Π. So (Γ, G) is point geodesic transitive. Let y be a point off a plane Π, then the stabilizer $G_{y,\Pi}$ is the Frobenius group of order 21 and acts regularly on the flags of Π. But every line of Π determines a unique plane distinct from Π and not containing y, hence a unique line of Γ incident with y and distinct from the complement of Π. So (Γ, G) is line geodesic transitive (this follows also by duality). Hence it is weakly geodesic transitive and we obtain example (ST10). The number of maximal geodesics in Γ is $15.8.7.4 = 7.6.5.4.4$ and so this does not divide $|A_7|$ which implies that (Γ, G) is not geodesic transitive.

(SD2).

Here $\Gamma = Ha(11)$ and $G \cong L_2(11)$. The stabilizer of a line l of Γ is A_5 (see the ATLAS [22]) acting naturally on the five points of l. Let y, z be two distinct points on l and suppose $G_{l,y,z}$ fixes the three lines of Γ through y not containing z. Each of these lines is determined by its intersection with l and so $G_{l,y,z}$ fixes l pointwise, contradicting the action of A_5 on l. Since $G_{l,y,z}$ has order three, example (ST4) arises.

Suppose now $\Gamma = Ha(11)^C$. The stabilizer of a line in Γ acts on the points of l as

$L_2(5)$ in its natural representation. So (Γ, G) is point geodesic transitive. Dually, (Γ, G) is also line geodesic transitive, hence it is weakly geodesic transitive. Since the number of maximal geodesics is 11.6.5.3 and since this does not divide $|L_2(11)|$, (Γ, G) cannot be geodesic transitive. This gives example (ST8).

(SD3).

First let $\Gamma = Hi(176)$ and $G \cong HS$. The stabilizer of a line is isomorphic to $U_3(5):2$, but this does not have a 2-transitive representation on 50 points, so (Γ, G) is not even point geodesic transitive.

Now let $\Gamma = Hi(176)^C$ and $G \cong HS$. The stabilizer of a line is again $U_3(5):2$ and this acts non-trivially on the 126 points of that line, hence we can identify these points with the points of a Hermitian unital. Since $U_3(5):2$ is 2-transitive on such a unital, this implies that (Γ, G) is points geodesic transitive, dually line geodesic transitive and hence weakly geodesic transitive. Since the number of maximal geodesics is 176.126.125.36 and since this is larger then $|HS|$, we do not obtain a geodesic transitive $(2,3,3)$-pair. So we have (ST9).

(SD4).

Suppose $\Gamma = \mathcal{S}^+(n)$ and $G \cong 2^{2n} : S_{2n}(2)$. Kantor [55] shows that the stabilizer $G_{l,y}$ of an incident point-line pair is a group isomorphic to $O^-_{2n}(2)$ acting on the remaining points of l in its natural way (identifying these points with points of an elliptic quadric in $PG(2n-1,2)$). It acts similarly on the lines through y distinct from l. These actions are rank 3 and hence the stabilizer of a point $z \neq y$ on l in $O^-_{2n}(2)$ stabilizes also another line through y and z, acts transitively on the set of other lines through both y and z and acts also transitively on the set of lines through z not containing y. This shows that (Γ, G) is geodesic transitive. Similarly for $\Gamma = \mathcal{S}^-(n)$. This produces examples (ST5) and (ST6).

(SD5).

Let Ω be a set with a 2-transitive group G acting. Transitivity on geodesics of length 2 means that G is 3-transitive. But there is a unique line not containing a given point, hence 2-arc transitivity implies geodesic transitivity here. Hence example (ST7).

This completes the proof of proposition 5.□

4.3.2 Linear spaces.

PROPOSITION 6. *If (Γ, G) is a geodesic transitive (or equivalently a weakly geodesic transitive) $(3,3,4)$-pair with G type-preserving, then it is one of the examples (LT1) up to (LT3) listed in table 26. If (Γ, G) is a locally 3-arc transitive $(3,3,4)$-pair with G type preserving, then it is one of the examples (LT1) up to (LT8) of table 27. Conversely, every example listed in table 26 gives rise to a geodesic transitive (resp. locally 3-arc transitive) $(3,3,4)$-pair.*

PROOF. We have to check which examples of table 3 have the desired transitivity property.

(LS1).

	Γ	G	Restrictions and Remarks
(LT1)	$PG(n,q)$	$L_{n+1}(q) \trianglelefteq G \leq P\Gamma L_{n+1}(q)$	$n \geq 3$
(LT2)	$AG(2,q)$	$L_2(q) \trianglelefteq G_o \leq \Gamma L_2(q)$	G contains all translations
(LT3)	$\Gamma(\mathcal{S})$	G	G is almost simple and acts 4-transitively on $\mathcal{S}$
(LT4)	$PG(3,q)$	A_7	G contains all translations
(LT5)	$U_H(q)$	$PGU_3(q) \trianglelefteq G \leq P\Gamma U_3(q)$	(Hermitian unital in $PG(3,q^2)$)
(LT6)	$AG(n,q)$	$L_n(q) \trianglelefteq G_o \leq \Gamma L_n(q)$	$n \geq 3$ and G contains all translations
(LT7)	$AG(4,2)$	$G \cong 2^4 : A_7$	
(LT8)	$\Gamma(\mathcal{S})$	G	G is almost simple and acts 3-transitively on $\mathcal{S}$

Table 26: Geodesic Transitive and locally 3-arc transitive Linear Spaces.

Here Γ is the projective n-space $PG(n,q)$ over $GF(q)$. The lines of Γ are the lines of $PG(n,q)$. So example (LT1) is obvious. Suppose now $(n,q) = (3,2)$ and $G \cong A_7$. This yields clearly a 3-arc transitive example, but the number of maximal geodesics is 35.3.6.2.4 and this exceeds $|A_7|$. Hence (LT4).

(LS2).

Here Γ is the Hermitian unital $U_H(q)$ in $PG(2,q^2)$. Let $G \cong PGU_3(q)$, then it is clear that G acts 2-transitively on the point set of Γ. The stabilizer of a couple (y,z) of points is a group of order q^2-1. Suppose some element σ of this group fixes a line l of Γ through y distinct from yz. Then σ fixes three lines through y (indeed, σ also fixes the tangent line in y), hence it fixes all lines through y. So σ is a homology with center y and axis zu, where yz is the polar line of u. But then σ acts on q remaining points of l, and thus fixes at least one further point w on l. Since w cannot be incident with zu because zu is a tangent line, we obtain $\sigma = 1$. Hence $G_{y,z}$ is regular on the lines of Γ through y distinct from yz.

So $PGU_3(q)$ acts regularly on the set of geodesics of length 3. This implies that $P\Gamma U_3(q)$ is too small to acts transitively on the set of maximal geodesics (there are q^2-q-1 maximal geodesics through a geodesic of length 3).

This explains example (LT5).

(LS3).

Here $\Gamma = U_R(q)$, the Ree Unital and $G \trianglelefteq R(q)' \cong^2 G_2(q)'$. As above, the group fixing two distinct points has order $q-1$ in $R(q)$. But there are q^2 lines through any point, so even an overgroup cannot be transitive on geodesics of length 3.

(LS4).

Witt-Bose-Shrikhande spaces have no 2-transitive collineation group.

(LS5).

Here $\Gamma = AG(n,q)$ with $n \geq 2$, G contains the full translation group of the affine space and G_x, the stabilizer of a point x, acts transitively on the points "at infinity". If G is locally 3-arc transitive, then clearly G_x is 2-transitive on the set of points at infinity, so G_x contains $SL(n,q)$ or A_7 (the latter for $(n,q) = (4,2)$). If $n \geq 3$, this clearly produces examples (LT6) and (LT7). These cannot be geodesic transitive because one can never map a maximal geodesic lying in a subplane to one not lying in any subplane. On the other hand, if $n = 2$, every geodesic of length 3 is contained in a unique maximal geodesic, so here locally 3-arc transitivity is equivalent with geodesic transitivity. Hence example (LT2).

(LS6).

There are three possibilities according to Buekenhout, De Landtsheer, Doyen, Kleidman, Liebeck & Saxl [14].

1. The Lüneburg-Tits planes do not have a 2-transitive collineation group (on the set of points).
2. The Hering plane of order 27 has a 2-transitive collineation group on the set of points, but the stabilizer of a point x is isomorphic to $SL_2(13)$ and this does not act 2-transitively on the 28 lines through x. So here we have point geodesic transitivity but not line geodesic transitivity.
3. The nearfield plane of order 9: similar to the Hering plane of order 27.

(LS8).

Here, the points of Γ can be identified with the points of an affine line over $(GF(q)$ (which can in turn be identified with the elements of the field $GF(q)$) and the collinetion group G is a subgroup of the 1-dimensional semi-linear affine group. Consider the line l in Γ through the points 0 and 1. Note that, if (s,t) is the order of Γ and $v = q$ is the number of points, then

$$1 + s + s^2 \leq v \leq 1 + t + t^2$$

and equality holds in both cases if and only if Γ is a projective plane (which we do not consider here). This implies $t \geq \sqrt{q}$. So if G acts locally 3-arc transitively, then the stabilizer of 0 and 1 must act transitively on the set of t lines through 0 distinct from l. Hence this group, which is a subgroup of the automorphism group of $GF(q)$, must have order at least $\sqrt{q}$. This is only possible if $q = 4$, but then we obtain example (LT2) for $q = 2$.

(LS9).

Here, every line has 2 points, so locally 3-arc transitive groups correspond to 3-transitive groups. The case where G is of affine type is already included in examples (LT2), (LT6) and (LT7), hence we may assume that G is almost simple. Clearly, geodesic transitive groups correspond to 4-transitive groups.

This completes the proof of proposition 6.□

	Γ	G	Remarks
(PQ1)	$Pe(10)$	$A_5 \trianglelefteq G \leq S_5$	S_5 is geodesic transitive
(PQ2)	$HS(50)$	$U_3(5) \trianglelefteq G \leq U_3(5):2$	G is geodesic transitive

Table 27: Point Distance Transitive Moore Geometries of Diameter $d \geq 2$.

4.3.3 Moore geometries.

PROPOSITION 7. *If (Γ, G) is a point distance transitive $(g, g, g+1)$-pair, $g \geq 5$, with G type-preserving, then it is example (PQ1) or (PQ2) of table 27. Moreover, in example (PQ2), G acts geodesic transitively, and in example (PQ1), G acts geodesic transitively if and only if $G \cong S_6$.*

PROOF. By Buekenhout [12], Γ is a Moore geometry of diameter $d = \frac{g-1}{2} =\geq 2$. By Fuglister [38, 39], Damerell & Georgiacodis [25] and Damerell [24], the diameter is equal to 2, hence $g = 5$. By Kantor [56], $s = 1$ and $t = 2, 6$ or 56. Hence Γ is a partial quadrangle and the result now follows directly from proposition 4. Note that Aschbacher [2] showed that no example with $t = 56$ can be point distance transitive.$\square$

COROLLARY 2. *If we allow diameter 1 for Moore geometries, then a point distance transitive $(g, g, g+1)$-pair is one of the examples in tables 26 and 27 or it is one of the following linear spaces:*

(LS2) *The Hermitian Unital with $G \cong U_3(q)$;*

(LS3) *The Ree unital and G is an automorphism group of the corresponding Ree group;*

(LS5) *The affine space $AF(n, q)$ with $AL_1(q^n) \leq G \leq A\Gamma L_1(q^n)$.*

(LS6) *The Hering plane of order 27 or the nearfield plane of order 9;*

(LS7) *A Hering space;*

(LS9) *A c-geometry with a 2-transitive almost simple group acting.*

PROOF. The additional examples follow from the proof of proposition 6. We only have to show that no examples arise where Γ is a linear space in the class (LS8). But as in the proof of lemma 3, one shows that the lines of Γ in this case are translates and homological images of a subfield, hence this produces the examples under (LS5).$\square$

4.4 Geodesic Transitive $(g, g+1, g+1)$-pairs, $g \geq 5$.

Theorem 1 will be proved if we show that no geodesic transitive $(g, g+1, g+1)$-pairs exist for $g \geq 5$. That is the content of this subsection.

PROPOSITION 8. *There do not exist geodesic transitive* $(g, g+1, g+1)$*-pairs with* $g \geq 4$.

PROOF. The case $g = 4$ is exactly corollary 1. So suppose throughout that (Γ, G) is a geodesic transitive $(g, g+1, g+1)$-pair of order (s, t) with $g \geq 5$. By the transitivity assumption, the number of geodesics joining two elements at distance g only depends on the types of these elements. If g is odd, there is only one possibility since the types have to be distinct and we denote that number by $\alpha + 1$. If g is even, then we denote by $\alpha + 1$ (resp. $\beta + 1$) the number of geodesics joining two points (resp. lines) at distance g. Note that $1 \leq \alpha < t$ and $1 \leq \beta < s$ for g even, and $1 \leq \alpha < s, t$ for g odd.

We first show some lemma's.

LEMMA 5. *If* g *is even, then* $s = t$ *and* $\alpha = \beta$.

PROOF. We count the number of maximal geodesics joining a point x and a line l at distance $g + 1$ in two ways and obtain

$$(s + 1)(\alpha + 1) = (t + 1)(\beta + 1).$$

Similarly, we count the number of maximal geodesics joining a point x and a line l' at distance $g - 1$ in two different ways and obtain

$$s.\alpha = t.\beta$$

from which the statement readily follows.□

LEMMA 8. *The group* G *acts primitively on both the set of points and the set of lines of* Γ.

PROOF. Without loss of generality, we can by way of contradiction assume that G acts imprimitively on the set $\mathcal{P}$ of points of Γ. Let A be a non-trivial set of imprimitivity. Let y, z be two points in A. Let l be a line through z at distance $d(y, z) - 1$ from y. If $d(y, z) \leq g$ or if $\alpha < s - 1$, there exists a point $u \neq z$ on l with $d(y, u) = d(y, z)$. By the transitivity assumption on G, we easily deduce $u \in A$. So all points collinear with any point of A are again in A. Since Γ is connected, $A = \mathcal{P}$, a contradiction.

So we may assume $d(y, z) = g + 1$ for all points $y, z \in A$ and $\alpha = s - 1$ (note that the first condition implies g odd). By transitivity, A is the full set of points at maximal distance from y (and also containing y). Let y, z, l be as above and choose a point w on l distinct from z. Since $t > 1$, there exists a line l' through w distinct from l and at distance $g - 1$ from y (because there is only one line through w at distance $g - 2$ from y). Now there exists a unique point $u \neq w$ on l' at distance $g + 1$ from y and we have $u \in A$. But $d(z, u) = 4$ and both are in A, consequently $g + 1 = 4$, contradicting our hypothesis.

Note that in the case $g = 3$, sets of imprimitivity do occur; this is namely the case for helicopter planes.□

LEMMA 7. *The point and line graph of* Γ *are admissible distance transitive graphs. Moreover, if* g *is even, then* $g \leq 26$.

PROOF. Straightforward counting of the a_i's, b_i's and c_i's of the graph (see 3.2) shows that these numbers are indeed constants and so the graph is distance regular. The intersection array is

$$(s(s+1), s^2, \ldots, s^2, s^2; 1, 1, \ldots, 1, \alpha+1),$$

if g is even, and

$$(s(t+1), st, \ldots, st, t(s-\alpha); 1, 1, \ldots, 1, (t+1)(\alpha+1)),$$

if g is odd. If g is even, then by definition Γ is a generalized Moore geometry of type $GM_{g/2}(s, s, \alpha+1)$ (as defined by ROOS & VAN ZANTEN [77]). By FUGLISTER [40, 41], $g \leq 26$. Since G acts transitively on maximal geodesics, we obtain an admissible distance transitive graph by the preceding lemma and the fact that $s, t > 1$ (and hence $a_1 > 0$).□

LEMMA 8. *Let $x \in \mathcal{P}$. We have $|G_x| \geq \sqrt{|G|}$.*

PROOF. First suppose g is odd. The number of points at distance $2i \leq g$ from x equals $s^i.t^{i-1}(t+1)$ and the number of points at distance $g+1$ from x is

$$\frac{(st)^{\frac{g-1}{2}}(s-\alpha)}{\alpha+1}.$$

Put $n = \frac{g-1}{2}$, then the total number of points is

$$|\mathcal{P}| = 1 + (\sum_{i=1}^{n} s^i t^{i-1}(t+1)) + \frac{s^n t^n (s-\alpha)}{\alpha+1}.$$

Now, G_x is transitive on the set of maximal geodesics based at x. There are precisely $s^n t^n (t+1)(s-\alpha)$ such geodesics. Hence the latter divides $|G_x|$. In particular,

$$s^i t^{i-1}(t+1) \leq \frac{|G_x|}{s^{n-i}t^{n-i+1}(s-\alpha)} \leq \frac{|G_x|}{s^{n-i}t^{n-i+1}}$$

and

$$1 + \frac{s^n t^n (s-\alpha)}{\alpha+1} \leq 1 + \frac{|G_x|}{(t+1)(\alpha+1)} < \frac{|G_x|}{2t}.$$

Adding up these inequalities, we obtain

$$\begin{array}{rcl} |\mathcal{P}| & \leq & \frac{|G_x|}{t}(1 + \frac{1}{st} + (\frac{1}{st})^2 + \ldots + (\frac{1}{st}^{n-1} + \frac{1}{2}) \\ & < & 2\frac{|G_x|}{t}. \end{array}$$

On the other hand, $|G| = |\mathcal{P}|.|G_x|$, hence $|G_x| > \sqrt{\frac{t}{2}|G|}$ and since $t \geq 2$, the result follows.

If g is even, then one shows completely similarly

$$|G_x| \geq \sqrt{\frac{(s^2-1)(\alpha+1)}{s^2+\alpha}|G|}$$

and this completes the proof of the lemma.$\square$

LEMMA 9. *The group G is almost simple and of exceptional Chevalley or sporadic type.*

PROOF. From subsection 3.2 and lemma 7 it follows that G is almost simple and not of alternating type. Suppose G is classical, then the intersection array of the point graph of Γ is one of (DC3) or (DC4) (since (DC1) and (DC2) correspond to generalized hexagons). No integer parameters match these arrays.$\square$

LEMMA 10. *The geometry Γ is not self dual.*

PROOF. Suppose Γ is self-dual and consider the incidence graph Γ^I, which is a distance regular graph. The group G acts on Γ^I as a distance transitive collineation group. The *girth* of Γ^I is $2g \geq 10$, hence by a theorem of WEISS [96], Γ^I is either the *Foster graph* on 90 vertices with intersection array

$$(3,2,2,2,2,1,1,1;1,1,1,1,2,2,2,3)$$

or the incidence graph of the generalized hexagon $H(q)$ (label (GP7)) with q a power of 3. Clearly the Foster graph is not admissible ($a_1 = 0$) and the incidence graph of the generalized hexagon gives a unique solution for (s,t,α), namely (s,s,s), a contradiction.$\square$

LEMMA 11. *Let $x \in \mathcal{P}$. Then G_x induces a 2-transitive permutation group on the set of lines through x.*

PROOF. Let L_i, L_i' be two lines through x, $i = 1,2$. Choose points x_i, x_i' resp. on L_i, L_i', $i = 1,2$, all distinct from x. All these points are at distance 4 from each other (in the incidence graph), so by transitivity, there is an element of G mapping (x_1, x_1') to (x_2, x_2'). Since x is the unique point at distance 2 from both x_1 and x_1', resp. x_2 and x_2', it must be fixed and the assertion follows.$\square$

LEMMA 12. *The group G is not of exceptional Chevalley type.*

PROOF. Let S be the socle of G. If G_x is a parabolic subgroup, then by BROUWER, COHEN & NEUMAIER [10], we obtain a generalized polygon or a non-admissible distance transitive graph (or no distance transitive graph at all). If G_x is not a parabolic subgroup, then by lemma 8, $G_x \cap S$ is one of the groups H in table 15. Let l be a line of Γ, then also $G_l \cap S$ is one of these groups. Moreover, G_x is not isomorphic to G_l by lemma 10 and the fact that isomorphic subgroups appearing in table 15 are always conjugate in the automorphism group of S. Now the only group S having two non-isomorphic subgroups listed in table 15 having a 2-transitive representation is $G_2(q)$. Actually, $G_2(q)$ has three such subgroups if q is an odd power of 3. We list them together with the value of t or s they induce (by their 2-transitive representation) and the number of points ot lines they define (by the relation $|\mathcal{P}| = |G|/|G_x|$ and dually):

H	tor$s$	$\|\mathcal{P}\|$or$\|\mathcal{L}\|$
$SL_3(q):2$	q^2+q	$\frac{1}{2}q^3(q^3+1)$
$SU_3(q):2$	q^3	$\frac{1}{2}q^3(q^3-1)$
${}^2G_2(q)$	q^3	$q^3(q^3-1)(q+1)$

Counting the number of flags, we obtain

$$(t+1)|\mathcal{P}| = (s+1)|\mathcal{L}|,$$

and this can never be satisfied with any combination of the above values. In fact, alternatively, the number of points compared with the value for t induced by a single subgroup H above forces g to be too small compared with the permutation rank of H in G.

This completes the proof of the lemma.$\square$

So we can from now on assume that G is of sporadic type. First note that the intersection array of the graph (DS1) of table 16 can never match the parameters of Γ. Hence we can assume that $g \geq 9$.

First we suppose that $G \neq M$ and $G \neq B$. We explain our strategy by taking as example $Fi_{24}' \trianglelefteq G \leq Fi_{24}$. Since $g \geq 9$, we have that $(st)^4$ divides $|G|$. Considering the primes whose fourth powers divide $|G|$, we get an upper bound ω for st, namely $st \leq 2^5.3^4 = 2592 =: \omega$ (more specific $st|2^5.3^4$). Next, we consider all maximal subgroups H of G such that $|H| \geq \sqrt{|G|}$ and H has a 2-transitive representation. Checking the latter condition is usually not trivial, but it cannot do any harm to ignore it. For example, using the classification of all maximal subgroups of Fi_{24}' and Fi_{24} by Linton & Wilson [70], we obtain 6 maximal subgroups H satisfying $|H| \geq \sqrt{|G|}$ (excluding $Fi_{24}' \leq Fi_{24}$). Two of them, $O_{10}^-(2)$ (in Fi_{24}', and $O_{10}^-(2):2$ in Fi_{24}) and Fi_{23} (in Fi_{24}', and $Fi_{23}\times 2$ in Fi_{24}), have clearly no 2-transitive permutation representation. Let us consider the others and label them H_1, H_2, H_3 and H_4.

	inFi_{24}'	inFi_{24}	$\|G:H\|$
H_1	$2\cdot Fi_{22}:2$	$(2\times 2\cdot Fi_{22}).2$	$2^2.3^7.7^2.17.23.29$
H_2	$(3\times 0_8^+(3):3):2$	$S_3\times O_8^+(3):S_3$	$2^8.3^2.7^2.11.17.23.29$
H_3	$3^7\cdot 0_7(3)$	$3^7\cdot O_7(3):2$	$2^12.5.7^2.11.17.23.29$
H_4	$3_+^{1+10}:U_5(2):2$	$3_+^{1+10}:(2\times U_5(2):2)$	$2^10.5.7^3.13.17.23.29$

First suppose $s \neq t$. Take two of the maximal subgroups above, say H_i and H_j and suppose $H_i \cong G_x$ and $H_j \cong G_l$ for a point x and a line l in Γ. Now the number of flags in Γ is equal to

$$(t+1)|G:H_i| = (s+1)|G:H_j|,$$

hence the smallest common multiple of $|G : H_i|$ and $|G : H_j|$ divides the number of flags of Γ. Dividing by $|G : H_i|$, we obtain

$$\tau := \frac{|G : H_j|}{(|G : H_i|, |G : H_j|)} | t + 1.$$

Similarly

$$\sigma := \frac{|G : H_i|}{(|G : H_i|, |G : H_j|)} | s + 1.$$

Hence $st \geq (\sigma - 1)(\tau - 1)$ and this gives us a lower bound for st. The idea is to contradict the above upper bound. Note that we can take $H_i \ncong H_j$ otherwise Γ is self-dual. Let us tabulate the values for σ and τ in all cases:

(H_i, H_j)	τ	σ	$(\tau - 1)(\sigma - 1)$
(H_1, H_2)	$2^6.11$	3^5	> 2592
(H_1, H_3)	$2^{10}.5.11$	3^7	> 2592
(H_1, H_4)	$2^8.5.7.13$	3^7	> 2592
(H_2, H_3)	$2^4.5$	3^2	720
(H_2, H_4)	$2^2.5.7.13$	$3^2.11$	> 2592
(H_3, H_4)	7.13	$2^2.11$	> 2592

So the only possibility is $80|t+1$ and $9|s+1$. Since $st \leq 2592$, this implies $(s, t) = (79, 8)$, contradicting $st|2^5.3^4$.

Suppose $s = t$, then $|G_x| = |G_l|$ and for most of the sporadic groups, this implies $G_x \cong G_l$ and G_x conjugate to G_l in $\mathrm{Aut}(G)$, hence Γ self-dual. This contradicts lemma 10. The only case where this argument fails is for McLaughlin's group McL. We deal with it later.

We apply this technique for $s \neq t$ to all the sporadic groups (except to Held's group He and Lyons' group Ly since by Cohen & Cuypers [20] resp. Soicher [80], these groups do not act distance transitively on any graph). In table 27, we list the cases where the upper bound ω and lower bound $(\tau - 1)(\sigma - 1)$ are not in conflict. The classification of all maximal subgroups of the sporadic groups we consider here can be found in Kleidman, Parker & Wilson [62] (for Fi_{23}), Kleidman & Wilson [63, 64] (for Fi_{22} and J_4), Linton [69] (for Th) and the Atlas [22] (for the remaining groups).

Five groups remain. Note that in each case $\frac{t+1}{s+1} = \frac{|H_i|}{|H_j|}$. This gives us:

1. $G \cong J_4$ and $\sigma = 1$. Since $s > 1$, we have $t > 96$, contradicting $\omega = 32$.

2. $G \cong Fi_{22}$ or $G \cong Fi_{22} : 2$. Here $st \leq 144$ and this leaves $(s, t) =$ (3,6), (7,13), (3,20), (2,8), (3,11), (4,14), (5,17), (6,20). In view of $st|144$, there remains $(s, t) = (3, 6)$ or $(2, 8)$. In both cases $(st)^5$ does not divide $|G|$, hence $g = 9$. If $(s, t) = (2, 8)$, then $\alpha = 1$, implying $|\mathcal{P}| = 3(1 + 16 + 16^2 + 16^3 + \frac{16^4}{2})$ and this does not divide $|G|$. Similarly, if $(s, t) = (3, 6)$, then $\alpha = 1$ or 2 and again $|\mathcal{P}|$ does not divide $|G|$.

G	ω	$(H_i;H_j)$	$\tau;\sigma$
J_4	32	$(2^{11}:M_{24};2^{10}:L_5(2))$	23;1
Fi_{22}	144	$(2^6:S_6(2);(2\times 2_+^{1+8}:U_4(2)):2)$	7;4
		$(2^6:S_6(2);2^{5+8}:(S_3\times A_6))$	21;4
		$((2\times 2_+^{1+8}:U_4(2)):2;2^{5+8}:(S_3\times A_6))$	3;1
Co_2	48	$((2_+^{1+6}\times 2^4).A_8;2^{4+10}.(S_5\times S_3))$	7;2
Co_3	12	$(U_4(3):(2^2)_{133};2\cdot S_6(2))$	9;2
Suz	24	$(2_-^{1+6}:U_4(2);2^{4+6}:3A_6)$	3;1

Table 28: Non-conflicting Pairs of Maximal Subgroups of Sporadic Groups.

1048575	1680	360	125	64
15624	1023	342	124	63
6560	960	288	120	59
5040	840	275	119	58
4095	728	242	81	49
3480	624	175	80	48
3124	528	168	71	46
2400	512	135	70	40

Table 29: Possible Values for $s, t \geq 40$ in the Case of the Monster.

3. $G \cong Co_2$. Since $s > 1$, the only possibility is $(s,t) = (3,13)$, but then st does not divide 48.

4. $G \cong Co_3$. No value $s > 1$ remains here.

5. $G \cong Suz$ or $G \cong Suz:2$. Here $(s,t) = (2,8)$, but st does not divide 24.

This completes the case $s \neq t$. As remarked above, the case $s = t$ is only possible for $McL \trianglelefteq G \leq McL:2$. Note that the upper bound ω is still well-defined and here we have $\omega = 12$. So since $s = t$, the only possibility is $(s,t) = (2,2)$ and $\alpha = 1$. Note that in every case, 5 divides $|G_x|$. But an element of order 5 fixing a point must fix everything else.

So we may finally assume $G \cong M$ or B. First suppose $G \cong M$ and g odd. We kill this case using a little computer-programme, based on the following observations.

1. The Atlas [22] provides a list of all simple groups (possibly) involved in M. This yields a list of the degrees of all 2-transitive almost simple groups that can act on the set of points (resp. lines) incident with a line (resp. point). A similar list derived from 2-transitive affine groups can be obtained by considering all primes p and positive integers n such that $p^n(p^n-1)$ divides $|M|$. This gives us all possible values for the parameters s and t of Γ. We list all the values larger than 40 in table 29.

2. The number of geodesics based at a fixed point is $(st)^{\frac{g-1}{2}}(t+1)$, so this must divide $|M|$. Similarly $(st)^{\frac{g-1}{2}}(s+1)$ divides $|M|$. This defines an upper bound for g which is in a lot of cases smaller than 9.

3. The above argumant kills $s,t = 59, 58, 47$ and 46. Also, no elements of order 59 or 47 appear in G_x. So if an element θ of order 59 or 47 fixes a point, then it fixes all lines through that point, all points on these lines, etc. Hence θ fixes everything, a contradiction. This implies that θ acts semi-regularly on $\mathcal{P}$. So $|\mathcal{P}|$ is divisible by 59.47. For a given pair (s,t) and a given gonality g, this determines α up to a multiple of 59.47, because

$$59.47|\frac{|\mathcal{P}|.(\alpha+1)}{s+1} = (\alpha+1)(1+st+(st)^2+\ldots+(st)^{\frac{g-3}{2}})+(st)^{\frac{g-1}{2}}.$$

4. Given (s,t), the gonality g is bounded below by the condition $|\mathcal{P}| \geq P(M)$ (see table 17).

This suggests the following programme: Take any pair (s,t) from table 29. Determine all possible g (using the uppser bound from 2. and the lower bound from 4. above) and consider them in turn. Compute $\alpha \pmod{59.47}$. There are a priori two possibilities: (1) $\alpha \geq s$ or t (and this is always the case as it turns out), a contradiction, so we move on to the next case of (s,t,g); (2) $\alpha \leq s,t$ (this never happens when $G \cong M$, but it does when $G \cong B$, see below). Here a unique value for α arises (since both $s,t < 59.47$) and we compute $|\mathcal{P}|$ and check if this divides $|G|$.

So suppose now g is even. Since here $s=t$, we can slightly change the method above so as to be able to do the calculations by hand.

The number of geodesics of length g is $s^{g-1}(s+1)$ and $10 \leq g \leq 26$ (by lemma 7). We again consider the degrees d of all possible 2-transitive groups involved in M and retain those for which $(d-1)^9.d$ divides $|M|$. This gives s = 120, 80, 48, 40, 30, 24, 20, 18, 16, 15, 12, 10, 9, 8, 6, 5, 4, 3 or 2. Now, from table 17, we read $|\mathcal{P}| \geq 972.10^{17}$. Combining this with

$$|\mathcal{P}| = \frac{s^{g-1}-1}{s-1} + \frac{s^{g-1}(s+1)}{\alpha+1} < \frac{s^{g-1}-1}{s-1} + s^{g-1}(s+1) = \frac{s^{g+1}-1}{s-1},$$

this gives us for each s a lower bound for g. If $s \leq 5$, this lower bound is larger than 26 and if $s \geq 6$, then the only cases where this lower bound does not contradict the upper bound (obtained from $s^{g-1}|\ |M|$) are $(s,g) = (120,10)$, (48,12), (24,16) and (12,20). Using the same method to determine α as in the case g odd (but now calculating with 71.59 instead of 59.47), we obtain respectively α = 3039, 1008, 3531, 317 and these all contradict $\alpha < s$. This rules out the case $G \cong M$ completely.

Suppose now $G \cong B$. We rule this out as above.

For g odd, we can consider the product of the primes 47.23 and we run the same computer-programme as for the Monster above (we can use the same orders because $B \leq$

M). It turns out that α is again always too big except if $g = 9$, $s = 20$ and $t = 18$. In this case $\alpha = 17$. But the number of points of Γ is 20578025181 = 3.7.23.47.9064817 and this does not divide $|B|$.

If g is even, then an entirely similar calculation as for $G \cong M$ gives us the following result. We list the possible s, g and the value for α they imply.

s	g	α
48	8	27682
	10	27682
24	8	19019
	10	27682
	12	6980
	14	22231
16	10	27682
12	10	24065
	12	11660
	14	19849
8	12	498
	14	6426
6	14	2865
4	18	11862
	20	13450

We woud like to thank V. De Smet for writing the above mentioned programmes. She used CAYLEY.

This rules out B and the proof of proposition 8 is complete.□

This completes the proof of our our main result (theorem 1).

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