Half Regular and regular points in compact polygons

A. E. Schroth
and
H. Van Maldeghem

Institut für Analysis; Abt. f. Topologie u. Grundlagen d. Analysis
preliminary version 14. Dec. 2010

An N&C production

1 Introduction

The present paper was inspired by [12], where compact symlectic quadrangles were characterized by their derivations, and by [11], where the notion of regular points in generalized hexagons was introduced. We discuss the notion of half regular and regular points in compact generalized polygons. Regularity and half regularity enables one to define derived structures. The point is that, even though every regular point is by definition also a regular point, the derived structure in a half regular point is defined differently as in a regular point and in general yields also different geometries. However, in generalized quadrangles half regular points are also regular and the derived structures are isomorphic. Section 3 is devoted to the investigation of half regular points and the properties of the derived structure in a half regular point. In section 4 we treat regular points and study the derived structure in a regular point. Besides an extension of the main theorem of [12] we proof the following

(1.1) THEOREM: *For a compact connected generalized hexagon $\mathcal{S}$ where pointrows and linepencils are manifolds the following properties are equivalent:*

1) $\mathcal{S}$ is the split Cayley hexagon over $\mathbb{R}$ or $\mathbb{C}$ considered as topological fields.

2) $\mathcal{S}$ is the split Cayley hexagon over a topological commutative field.

3) $\mathcal{S}$ is the split Cayley hexagon over a commutative field.

4) $\mathcal{S}$ is regular.

5) $\mathcal{S}$ is half regular.

6) For every point p the derivation $\mathcal{A}_p$ is a topological projective plane.

7) For every point p the derivation $\mathcal{A}_p$ is a projective plane.

8) For every point p the derivation $\mathcal{S}_p$ is a topological quadrangle.

9) For every point p the derivation $\mathcal{S}_p$ is a quadrangle.

2 Preliminaries

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ be an incidence structure with a set $\mathcal{P}$ of **points**, a set $\mathcal{L}$ of **lines** disjoint from $\mathcal{P}$ and a set $\mathcal{F} \subset \mathcal{P} \times \mathcal{L}$ of **flags**. If $(p, l) \in \mathcal{F}$ we say that p and l are incident. The elements of $\mathcal{V} = \mathcal{P} \cup \mathcal{L}$ are called **vertices**. Two vertices x, y are said to be **collinear** if there is a vertex incident with both, x and y. An **s-path**, $s \in \mathbb{N}$, is a sequence $(x_0, x_1, \ldots, x_s)$ of $s+1$ vertices such that x_i is incident with x_{i+1} for $0 \leq i \leq s-1$ and $x_i \neq x_{i+2}$ for $0 \leq i \leq s-2$. We say that the s-path joins x_0 to x_s.

For $x, y \in \mathcal{V}$ let $d(x, y)$ denote the smallest number $s \in \mathbb{N}$ such that there exists an s-path joining x to y. This defines a distance function $d : \mathcal{V}^2 \to \mathbb{N}$ on $\mathcal{V}$. We define $\mathcal{S}_i(x) := \{y \in \mathcal{V} \mid d(x, y) = i\}$ for $x \in \mathcal{V}$ and $i \in \mathbb{N}$. The **pointrows** and **linepencils**

$\mathcal{S}_1(x)$, $x \in \mathcal{L}$ resp. $x \in \mathcal{P}$, are also denoted $\mathcal{S}(x)$. The set of vertices collinear with a given vertex $v \in \mathcal{V}$ is denoted $v^\perp$. Provided that $\mathcal{S}(v)$ is nonvoid, the relation $v^\perp = \{v\} \cup \mathcal{S}_2(x)$ holds.

DEFINITION: An incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ is a **generalized n-gon** with $n \in \mathbb{N}$ if the following holds.

(1) If x, $y \in \mathcal{V}$, then $d(x, y) \leq n$.
(2) If x, $y \in \mathcal{V}$ are two vertices with $d(x, y) = s < n$, then the s-path joining x to y is unique.
(3) For every $v \in \mathcal{V}$ the set $\mathcal{S}(v)$ contains at least th ree vertices.

Property (3) is sometimes referred to as thickness. Generalized polygons, which are generalized n-gons for some n, were introduced by Tits [15]. For $n = 2, 3, 4, 6, 8$, we call them respectively **generalized digons, (generalized) projective planes, generalized quadrangles, generalized hexagons, generalized octagons**. A generalized digon is a trivial geometry where every point is incident with every line. Throughout this paper we will assume $n \geq 3$.

For every $1 < s < n$ Axiom (2) yields mappings

$$F_s : \{(x, y) \in \mathcal{V}^2 \mid d(x, y) = s\} \to \mathcal{V}^{s-1} : (x, y) \mapsto (x_1, x_2, \ldots, x_{s-1})$$

where $(x, x_1, \ldots, x_{s-1}, y)$ is the unique s-path joining x to y. Furthermore, for $1 \leq k \leq s-1$ let $f_s^k(x, y)$ denote the k^{th} vertex of $F_s(x, y)$. Some of these maps have also other, more traditional notations. Let x, y be two points or two lines at distance two, then $f_2^1(x, y)$ is usually denoted $x \vee y$ or $x \wedge y$ respectively. If n is at least 5 and x, y are two vertices at distance 4, then $f_4^2(x, y)$ is also denoted $x * y$.

The points of a three dimensional projective space over a commutative field F together with the totally isotropic lines with respect to a symplectic polarity form a generalized quadrangle. It is called the *symplectic quadrangle* over F and denoted $W(F)$. A description in terms of coordinates may be given as follows (see [6]): the points are all elements of the form (∞), (a), (k, b), (a, l, a') where ∞ is some symbol not contained in F and $a, k, b, l, a' \in F$; the lines are all elements of the form $[\infty]$, $[k]$, $[a, l]$, $[k, b, k']$ where $k, a, l, b, k' \in F$; incidence is given by the following rules, where I denotes the incidence relation:

(1) $[k, b, k']\ \mathrm{I}\ (k, b)\ \mathrm{I}\ [k]\ \mathrm{I}\ (\infty)\ \mathrm{I}\ [\infty]\ \mathrm{I}\ (a)\ \mathrm{I}\ [a, l]\ \mathrm{I}\ (a, l, a')$ for all $a, a', b, k, k', l \in F$
(2) $[k, b, k']\ \mathrm{I}\ (a, b, a') \iff$

$$b = ak + a' \qquad (*)$$

$$k' = k'a^2 + l + 2aa' \qquad (**)$$

(3) no other incidences occur.

The points of a (parabolic) quadric $Q(6,F)$ in the six dimensional projective space over a commutative field F together with all the lines of $Q(6,F)$ whose Grassmann coordinates satisfy $p_{12}=p_{34}$, $p_{20}=p_{35}$, $p_{01}=p_{36}$, $p_{03}=p_{56}$, $p_{13}=p_{64}$ and $p_{23}=p_{45}$ (where $Q(6,f)$ has the equation $X_0X_4+X_1X_5+X_2X_6={X_3}^2$) form a generalized hexagon. We will call this the *split Cayley hexagon* (because it arises from a split Cayley algebra over F) and denote it $H(F)$. There is also a description in terms of coordinates (see [1]): The points are all elements of the form (∞), (a), (k,b), (a,l,a'), (k,b,k',b'), (a,l,a',l',a'') where ∞ is some symbol not contained in F and all other parameters are from F; the lines are all elements of the form $[\infty]$, $[k]$, $[a,l]$, $[k,b,k']$, $[a,l,a',l']$ $[k,b,k',b',k'']$ where again all parameters exept ∞ are from F; incidence is given by

(1) $[k,b,k',b',k'']$ I (k,b,k',b') I $[k,b,k']$ I (k,b) I $[k]$ I (∞) I $[\infty]$ I (a) I $[a,l]$ I (a,l,a') I $[a,l,a',l']$ I (a,l,a',l',a'') for all $a,a',a'',b,b',k,k',k'',l,l'\in F$
(2) $[k,b,k',b',k'']$ I (a,b,a',l',a'') $\Longleftrightarrow$

$$\begin{aligned} a'' &= ak+b && (***)\\ a' &= a^2k+b'+2ab && (****)\\ k'' &= ka^3+l-3a''a^2+3aa' \\ k' &= k^2a^3+l'-kl-3a^2a''k-3a'a''+3aa''^2 \end{aligned}$$

(3) no other incidences occur.

Definition: A generalized n-gon is called a **compact n-gon**, if $\mathcal{P}$ and $\mathcal{L}$ carry compact topologies such that $\mathcal{F}$ is closed in $\mathcal{P}\times\mathcal{L}$.

In a compact n-gon the maps F_s and f_s^k are continuous [4], thus compact n-gons are topological n-gons in the sence of [4]. If x, y are any two points or any two lines, then the spaces $\mathcal{S}(x)$ and $\mathcal{S}(y)$ are homeomorphic (cf. [15]: p.56). We will frequently use the following fact shown in [4]: (2.1.b):

(2.1) Lemma: *In a topological n-gon the distance function $d:\mathcal{V}^2\to\{0,1,\cdots,n\}$ is lower semi-continuous. In particular if $(x_i,y_i), i\in\mathbb{N}$, is a sequence converging to (x,y) in $\mathcal{V}^2$ such that $d(x_i,y_i)=k$ for every $i\in\mathbb{N}$, then $d(x,y)\le k$.* □

(2.2) Lemma: *Suppose p,q are two vertices of a topological n-gon at maximal distance. Then for every $1\le k\le n-1$ the set $\mathcal{S}_k(p)\cap\mathcal{S}_{n-k}(q)$ is homeomorphic to $\mathcal{S}_1(p)$.*

Proof: The map $\alpha := f_{n-1}^{n-k}(q, \cdot) : \mathcal{S}_1(p) \to \mathcal{S}_k(p) \cap \mathcal{S}_{n-k}(q) : v \mapsto f_{n-1}^{n-k}(q, v)$ is continuous and the inverse map is $\alpha^{-1} := f_k^1(p, \cdot) : \mathcal{S}_k(p) \cap \mathcal{S}_{n-k}(q) \to \mathcal{S}_1(p) : w \mapsto f_k^1(p, w)$. Thus α is a homeomorphism. $\square$

If F is a locally compact topological field, then the topology on F induces topologies on the point- and lineset of the symplectic quadrangle (resp. spit Cayley hexagon) over F such that it becomes a compact quadrangle (resp. hexagon). We will call it the **symplectic quadrangle (resp. spit Cayley hexagon) over the topological field** F. For more details we refere to [12] and [4].

In this note we will mainly consider compact n-gons whose pointrows and linepencils are topological manifolds. For then there are parameters $s, t \in \mathbb{N}$ such that all pointrows are homeomorphic to the s-sphere and all linepencils to the t-sphere ([7]). We say the polygon is of order (s, t). By a theorem of Knarr [7], compact topological n-gons exist only for $n = 2, 3, 4, 6$.

The pointspace $\mathcal{P}$ of a compact n-gon, where pointrows and linepencils are manifolds is a compact manifold ([7]: 2.7) thus also metrizable and we therefore can endow the set of all nonempty closed subsets of $\mathcal{P}$ with the topology induced by the Hausdorff metric ([9]: Section 0). In this topology a sequence of closed sets $A_i \subset \mathcal{P}$ converges to a closed set $A \subset \mathcal{P}$ if A consists precisely of those points $a \in \mathcal{P}$ for which there is a sequence of points $a_i \in A_i$ converging to a ([9]: (0.6)). To avoid cumbersome notation and since we have only defined a topology on closed sets the term 'closed' is often omitted.

With regard to this topology we can show that the sets introduced in Lemma 2.2 depend continuously on the pair (p, q):

(2.3) Lemma: *Suppose p and q are two vertices of maximal distance in a metrizable compact n-gon and that $(p_i, q_i) \in \mathcal{V}^2, i \in \mathbb{N}$, is a sequence converging to (p, q), then for every $0 \leq k \leq n$ the sets $\mathcal{S}_k(p_i) \cap \mathcal{S}_{n-k}(q_i), i \in \mathbb{N}$, converge to $\mathcal{S}_k(p) \cap \mathcal{S}_{n-k}(q)$.*

Proof: Let $W_i = \mathcal{S}_k(p_i) \cap \mathcal{S}_{n-k}(q_i), i \in \mathbb{N}$, and $W = \mathcal{S}_k(p) \cap \mathcal{S}_{n-k}(q)$. By Lemma 2.1 we may assume $d(p_i, q_i) = n$ for every $i \in \mathbb{N}$.

Let $w_i \in W_i$, $i \in \mathbb{N}$. Then $d(p_i, w_i) = k$ and $d(q_i, w_i) = n - k$ for every $i \in \mathbb{N}$, thus $d(p, w) \leq k$ and $d(q, w) \leq n - k$ by Lemma 2.1. On the other hand $n = d(p, y) \leq d(p, w) + d(q, w) \leq k + n - k = n$. This is only possible if $d(p, w) = k$ and $d(q, w) = n - k$, i.e. if $w \in W$. Thus $\lim_{i \to \infty} W_i \subset W$.

To see the converse inclusion let $w \in W$ and $g := f_k^1(p, w)$. Choose $v \in \mathcal{V}$ such that $d(p, v) = d(p_i, v) = n$ for every $i \in \mathbb{N}$ and let $h := f_{n-1}^1(v, g)$. Then the vertices $g_i := f_{n-1}^1(p_i, h)$, $i \in \mathbb{N}$, converge to $g = f_{n-1}^1(p, h)$. Therefore the vertices $w_i := f_{n-1}^{k-1}(g_i, q_i) \in$

W_i, $i \in \mathbb{N}$, converge to $w = f_{n-1}^{k-1}(g, q) \in W_i$. Thus with the above we get $\lim_{i\to\infty} W_i = W$. $\square$

If we identify every line $l \in \mathcal{L}$ with the closed set $\mathcal{S}(l) \in \mathcal{P}$ of points incident with l, then the Hausdorff metric induces a new topology on $\mathcal{L}$. However, using Lemma 2.3 and continuity of joining one readily obtains

(4.3) COROLLARY: *The topology on $\mathcal{L}$ induced by the Hausdorff metric is equivalent to the original topology.* $\square$

If p, q are two vertices of a generalized n-gon at maximal distance, let p^q denote the set $\mathcal{S}_2(p) \cap \mathcal{S}_{n-2}(q)$. So in particular Lemma 2.2 and Lemma 2.3 imply

(2.4) COROLLARY: *In a compact n-gon the map that sends every pair of vertices (p, q) of maximal distance to the set p^q is continuous. Furthermore the set p^q is homeomorphic to $\mathcal{S}_1(p)$.* $\square$

3 Half regular points and derivations of generalized polygons

We call a point p of a generalized n-gon **half regular** if for all pairs x, y in $\mathcal{S}_2(p)$ with $x \vee p \neq y \vee p$, the set p^z is independent of the choice of $z \in \mathcal{S}_{n-2}(p) \cap \mathcal{S}_{n-2}(q) \cap \mathcal{S}_n(p)$, in which case the set p^z is called an **ideal line**. A half regular generalized polygon is a generalized polygon where every point is half regular. In a generalized 3-gon, i.e. a projective plane, every point is half regular.

If p is a half regular point of a generalized n-gon $\mathcal{S}$, then every pair x, y of points collinear to p is contained in either a line or an ideal line of $\mathcal{S}$. We denote that line, considered as a subset of $\mathcal{P}$ by $<x, y>$. We now define the following derived incidence geometry $\mathcal{A}_p$: The points are the points collinear to p, the lines are the sets p^q with $q \in \mathcal{S}_n(p)$ together with all customary lines through p considered as subsets of $\mathcal{P}$ and incidence is the natural one.

As an immediate consequence of the definition of half regular points we get the following characterization:

(3.1) Lemma: *A point p of a generalized n-gon $\mathcal{S}$ is half regular if and only if the derived structure $\mathcal{A}_p$ is a linear space, i.e. an incidence structure where any two points can be joined by a unique line.* □

If $\mathcal{S}$ is a projective plane, then $\mathcal{A}_p$ is isomorphic to $\mathcal{S}$. Together with some additional condition, half-regularity is a rather strong condition as will reveal the following

(3.2) Theorem: *Let p be a half regular point of a compact n-gon $\mathcal{S}$ of order (s, t). Then $s \geq t$ and $\mathcal{A}_p$ is a projective plane if and only if $t = s$.*

Proof: Since in a topological projective plane pointrows and linepencils are homeomorphic, as can be deduced from Lemma 2.3, the result holds rather trivially for $n = 3$. Therefore we will assume $n \geq 4$ for the rest of the proof.

Let $q \in \mathcal{S}_n(p)$ and $g \in \mathcal{S}_{n-1}(p)$ such that $f^2_{n-1}(p, g) \cap p^q = \emptyset$. Then $d(x, g) = n - 1$ for every $x \in p^q$. Since p is half regular, the continuos map $\mu : p^q \to \mathcal{S}_1(g) : x \mapsto f^1_{n-1}(g, x)$ is injective. For otherwise let x_1 and x_2 be two points such that $\mu(x_1) = \mu(x_2) =: z$, then $f^2_{n-1}(p, g) \in p^z \setminus p^q$ therefore p^z and p^q are different lines of $\mathcal{A}_p$ joining x_1 to x_2, a contradiction to Lemma 3.1. Since p^q is compact, μ is an immmersion and the dimension of $\mathcal{S}_1(g)$ has to be greater or equal to the dimension of p^q. However $\mathcal{S}_1(g) \cong \mathbb{S}_s$ and $p^q \cong \mathcal{S}_1(p) \cong \mathbb{S}_t$.

If the dimension of pointrows is greater than the dimension of linepencils the map μ is not surjective. Thus for any point $z \in \mathcal{S}_1(g) \setminus \mu(p^q)$ the sets p^q and p^z do not intersect. Hence $\mathcal{A}_p$ is not a projective plane.

If pointrows and linepencils have the same dimension, then the theorem on the invariance of domain ([8]) ensures that $\mu(p^q)$ is not only a closed, but also an open subset of the connected space $\mathcal{S}_1(g)$, hence the whole. So for every $z \in \mathcal{S}_1(g) \setminus \{f^1_{n-1}(g,p)\}$ the preimage $\mu^{-1}(z)$ lies in $p^z \cap p^q$. If $y \in \mathcal{S}_n(p)$, then either $f^2_{n-1}(p,l) \in p^q$ for every $l \in \mathcal{S}_1(y)$ and thus $p^y = p^q$ or there is a vertex $g \in \mathcal{S}_1(y)$ such that $f^2_{n-1}(p,g) \notin p^q$ and $\mu^{-1}(y) \in p^y \cap p^q$ by the above. This shows that any two lines of the derivation of type p^q with $q \in \mathcal{S}_n(p)$ intersect. That any other pair of lines of $\mathcal{A}_p$ intersect is a trivial consequence of the definition of a generalized n-gon and needs no topological argument. $\square$

REMARK: An analogous result holds for finite generalized n-gons, namely, if p is a half regular point then the number of lines incident with a point is less or equal to the number of points incident with a line and $\mathcal{A}_p$ is a projective plane if and only if the number of lines incident with a point is the same as the number of points incident with a line. This can be proved much along the same lines replacing topological arguments by counting arguments.

Next we want to show that for a half regular point p of a compact n-gon of order (s,t) the derivation $\mathcal{A}_p$ can be made a topological linear space, i.e. a linear space were joining and intersecting are continuous operations. We define the topologies on $\mathcal{A}_p$ as follows: The pointset of $\mathcal{A}_p$ inherits its topology from the pointset of $\mathcal{S}$ and a sequence L_i, $i \in \mathbb{N}$, of lines of $\mathcal{A}_p$ converges to a line L of $\mathcal{A}_p$ if and only if, with respect to the topology induced by the Haussdorf metric, every convergent subsequence of the given sequence converges to a subset of L. If the polygon is of order (s,s), we endow the lineset of $\mathcal{A}_p$ with the topology induced by the Haussdorf metric. This makes sense since the ordinary lines considered as subsets of $\mathcal{P}$ as well as the ideal lines are closed in $\mathcal{P}$. The following theorem is an extension of [12]: Corollary 2 and the proof runs much along the same lines.

(3.3) THEOREM: *If p is a half regular point of a compact n-gon $\mathcal{S}$ of order (s,t), then $\mathcal{A}_p$ is a topological linear space.*

PROOF: We use sequences to prove our result. First we show that joining is continuous. So we have to prove that the map that sends every pair (a,b) of different points to $<a,b>$ is continuous. Suppose $(a_i,b_i) \in p^\perp \times p^\perp$, $i \in \mathbb{N}$, is a sequence converging to a pair $(a,b) \in \mathcal{S}_2(p) \times p^\perp$ wit $a \neq b$. We may assume $p \neq a_i \neq b_i$ for every $i \in \mathbb{N}$. Take $g \in \mathcal{S}_{n-1}(p) \cap \mathcal{S}_{n-1}(a) \cap \mathcal{S}_{n-1}(b)$. We my assume that also $d(g,a_i) = d(g,b_i) = n-1$ for every $i \in \mathbb{N}$. The line $f^2_{n-1}(g,a_i)$ is at distance $n-1$ to b_i for every $i \in \mathbb{N}$, so the point $x_i := f^1_{n-1}(f^2_{n-1}(g,a_i),b_i)$ is welldefined and $\lim_{i\to\infty} x_i = x := f^1_{n-1}(f^2_{n-1}(g,a),b)$.

If $a \vee p \neq b \vee p$, then, by continuity of joining in $\mathcal{S}$, we may assume $a_i \vee p \neq b_i \vee p$ for every $i \in \mathbb{N}$. But then $<a_i,b_i> = p^{x_i}$, hence $\lim_{i\to\infty}<a_i,b_i> = \lim_{i\to\infty} p^{x_i} = p^x = <a,b>$ by Corollary 2.4.

If $a \vee p = b \vee p$ and if moreover $a_i \vee p = b_i \vee p$ for every $i \in \mathbb{N}$, then $<a_i, b_i> = \mathcal{S}(p \vee a_i)$. Thus $\lim_{i\to\infty} <a_i, b_i> = \lim_{i\to\infty} \mathcal{S}(p \vee a_i) = \mathcal{S}(p \vee a) = <a, b>$ by continuity of joining in $\mathcal{S}$ and by Corollary 4.3.

In the remaining case we may assume $a \vee p = b \vee p$ but $a_i \vee p \neq b_i \vee p$ for every $i \in \mathbb{N}$. To see that $\lim_{i\to\infty} <a_i, b_i>$ is a subset of $<a, b> = \mathcal{S}(a \vee p)$ suppose that there is some $y \in \lim_{i\to\infty} <a_i, b_i> \setminus <a, b>$. Then a is not collinear to y and there is a sequence $y_i \in <a_i, b_i>$, $i \in \mathbb{N}$, converging to y. However, then $<a_i, b_i> = <a_i, y_i>$ and $\lim_{i\to\infty} <a_i, b_i> = \lim_{i\to\infty} <a_i, y_i> = <a, y>$ by the above. This is impossible since on the one hand $a \vee p = b \vee p$ and on the other hand $b = \lim_{i\to\infty} b_i \in <a, y>$ and $<a, y> \cap \mathcal{S}(a \vee p) = \{a\} \neq \{b\}$. Thus $\lim_{i\to\infty} <a_i, b_i> \subset <a, b>$.

If $\mathcal{S}$ is a polygon of order (s, s) we also have to prove the converse implication. So let $z \in <a, b>$ and choose $w \in \mathcal{S}_2(p) \setminus <a, b>$. We may assume $w \notin <a_i, b_i>$ for every $i \in \mathbb{N}$, thus $<z, w>$ meets $<a_i, b_i>$ in exactly one point, denoted y_i. Since $<z, w>$ is compact we may assume that the points y_i converge to some point $y \in <z, w>$. But then $y \in <z, w> \cap <a, b> = \{z\}$. Therefore $\lim_{i\to\infty} <a_i, b_i> = <a, b>$ as claimed.

To see that intersecting is continuous as well suppose that (g_i, h_i), $i \in \mathbb{N}$, is a sequence of pairs of lines of $\mathcal{A}_p$ such that for every iN the lines g_i and h_i intersect in a point x_i converging to a pair (g, h) of different lines. We may assume that there is some point x with $\lim_{i\to\infty} x_i = x$. However, by the definition of the topology on the lineset, $x \in g \cap h$ hence $g \cap h = \{x\}$. $\square$

Remark: The above proof also shows that if $s \neq t$ the space of pairs of lines which intersect is closed. Also the topology on the lineset of $\mathcal{A}_p$ is almost the one induced by the Haussdorf metric. Only if lines not incident with p converge to a line through p we have to use the above definition of the topology.

4 Regular points and derivations of generalized polygons

DEFINITION: A point p is called **regular** if it is half regular and if for every $u \in \mathcal{S}_4(p)$ the set y^z is independent of the choice of $y \in \mathcal{S}_2(p) \cap \mathcal{S}_2(u)$ and $z \in \mathcal{S}_{n-2}(p) \cap \mathcal{S}_{n-2}(u) \cap \mathcal{S}_n(y)$, in which case the set y^z is again called an **ideal line**. Note that these ideal lines contain p. It is easily checked that for $n = 4$ every half regular point is also regular. If $n > 4$, then the set $\mathcal{S}_2(p) \cap \mathcal{S}_2(u)$ contains only the point $p * u$. A regular generalized polygon is a generalized polygon where every point is regular.

REMARK: One is tempted to define a even stronger regularity for octagons. However, the Moufang octagons do not have half regular points or lines and hence neither regular ones.

Our next goal is to define the notion of a *derivation* in a regular point. Therefore, we need some preparations. So let $\mathcal{S}$ be a generalized polygon with a regular point p. Again every pair x, y of points collinear to p is contained in either a line or an ideal line of $\mathcal{S}$ which we denote by $<x, y>$. Similarly, we denote an ideal line through p containing some other point u at distance 4 from p by $<p, u>$ or $<u, p>$. The **focus** of an ideal line $<x, y>$ is the set $\mathcal{S}_2(x) \cap \mathcal{S}_2(y)$. So for generalized quadrangles the focus of an ideal line is again an ideal line and for generalized n-gons with $n > 4$ the focus of an ideal line $<x, y>$ is identified with the point $x * y$.

For $n = 6$ we can also define ideal planes. So, let x, y be points collinear to p determining an ideal line $<x, y>$ and let z be a point collinear to x but not on $<x, p>$. Then z determines a unique ideal line t^z for all $t \in <x, y> \setminus \{x\}$ and we put $\Pi_z^{<x,y>}$ equal to the union of $<z, p>$ and all these ideal lines t^z as t ranges over $<x, y> \setminus \{x\}$.

(4.1) LEMMA: *If $<x, y>$ is an ideal line with focus p, and z is a point collinear to x and at distance 4 from p, then $\Pi_z^{<x,y>} = \Pi_u^{<x,y>}$ for all $u \in \Pi_z^{<x,y>}$.*

PROOF: First suppose $u \in <z, p>$, $u \neq z, p$. So $<z, p> = <u, p>$. Let $w \in t^z$ for some $t \in <x, y> \setminus \{x\}$. Since x^w contains p and z, it is by definition of regular point equal to $<p, z>$. So $d(w, u) = 4$ and $w \in t^u$, hence $t^u = t^z$ implying $\Pi_u^{<x,y>} = \Pi_z^{<x,y>}$.

Next, suppose $u \in t^z$ for some $t \in <x, y> \setminus \{x\}$. As in the first part, $<z, p> = x^u$ and $<u, p> = t^z$. Without loss of generality, we can assume $y \neq t$. Now $p^{(z*u)}$ contains x and t and hence it equals $<x, t> = <x, y>$. So $d(y, z * u) = 4$. But this implies that both y^z and y^u contain p and $y * (z * u)$. So $y^z = y^u$. Since y was arbitrary on $<x, t>$, this shows the result. $\square$

So the set $\Pi_z^{<x,y>}$ is determined by $<x, y>$ and any of its points distinct from p. But if we take a point u of $\Pi_z^{<x,y>}$ such that $u * p \neq z * p$, then $<x, y> = <u * p, z * p>$ and hence

$\Pi_z^{<x,y>}$ is determined by u and z. The set $\{p, u, z\}$ forms a triad of points at distance 4 with no common point at distance 2 and we call such a triad a **triangle**. Also, we denote the set $\Pi_z^{<x,y>}$ by $\ll p, u, z \gg$ and call it an **ideal plane** through p with **focusline** $<u * p, z * p>$. Clearly any two different points on an ideal plane are at distance 4.

(4.2) LEMMA: *Any three different points $z_1, z_2, z_3 \in \mathcal{S}_4(p)$ with $d(z_i, z_j) = 4$ and $z_i * z_j \notin \mathcal{S}_2(p)$, $i \neq j$, are contained in a unique ideal plane $\ll z_1, z_2, z_3 \gg$ through p, i.e. $\ll z_1, z_2, z_3 \gg = \ll p, z_i, z_j \gg$, $i \neq j$.*

PROOF: Since any two of the points are contained in precisely one ideal plane, we only have to show that these planes coincide. If any two of the points $z_1 * z_2, z_1 * z_3, z_3 * z_2$ are collinear, then they all coincide, for otherwise there would be a pentagon in $\mathcal{S}$. Thus $p^{z_1 * z_2} = <z_i * p, z_j * p>$ and $\ll p, z_1, z_2 \gg = \Pi_{z_1}^{<z_1 * p, z_2 * p>} = \Pi_{z_1}^{<z_1 * p, z_3 * p>} = \ll p, z_1, z_3 \gg = \Pi_{z_3}^{<z_1 * p, z_3 * p>} = \Pi_{z_3}^{<z_2 * p, z_3 * p>} = \ll p, z_2, z_3 \gg$.

If the three points $z_1 * z_2, z_1 * z_3, z_3 * z_2$ are mutually non-collinear, then the line $h := f_4^1(z_1, z_2)$ is at distance 5 to $z_3 * p$. Let $z_3' := f_5^2(p * z_3, h)$. Then $z_3' * p = z_3 * p$, hence $\ll p, z_i, z_3 \gg = \Pi_{z_i}^{<z_i * p, z_3 * p>} = \Pi_{z_i}^{<z_i * p, z_3' * p>} = \ll p, z_i, z_3' \gg$, $i \in \{1, 2\}$. Thus $d(z_i, z_3') = 4$ for $i = 1, 2$ and $z_1 * z_3'$ is collinear to $z_1 * z_2$, therefore by the above considerations the result holds. $\square$

Now suppose p is a regular point of a generalized n-gon $\mathcal{S}$ where $n \geq 4$. We define the following incidence geometry $\mathcal{S}_p$: the points are the ideal planes through p together with the sets $x^\perp$ for x collinear to p (including p). The lines are all ideal lines containing p together with all customary lines through p. The incidence relation is the natural one (symmetrized inclusion). Let $\mathcal{P}_p$ denote the pointset, $\mathcal{L}_p$ the lineset, and $\mathcal{F}_p \subset \mathcal{P}_p \times \mathcal{L}_p$ the set of incident point-line pairs of $\mathcal{S}_p$. We call $\mathcal{S}_p$ the **derivation** of $\mathcal{S}$ in p.

If p is a regular point of a generalized quadrangle, then $\mathcal{S}_p$ is isomorphic to $\mathcal{A}_p$ via the map that leaves every customary line through p fixed and that sends every ideal line to its focus and every set $x^\perp$ with $x \in p^\perp$ to x. Using Lemma 2.3 and the fact that the sets $x^\perp$ are closed ([2]: 2.2) one can verify readily that this map is also a topological isomorphism. Since locally compact connected quadrangles are compact ([3]: 3.4) Lemma 3.1, Theorem 3.2 and Theorem 3.3 imply the following results.

(4.3) LEMMA: *A point p of a generalized quadrangle $\mathcal{S}$ is regular if and only if the derived structure $\mathcal{S}_p$ is a linear space.* $\square$

(4.4) THEOREM: *Let p be a regular point of a locally compact quadrangle $\mathcal{S}$ of order (s, t) with $0 < s, t$. Then $s \geq t$ and $\mathcal{S}_p$ is a projective plane if and only if $t = s$.* $\square$

The main theorem of [12] extends to the following characterization of compact topological quadrangles.

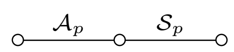
(4.5) THEOREM: *For a locally compact connected generalized quadrangle $\mathcal{S}$ of order (s, t) where $0 < s, t$ the following properties are equivalent:*

1) $\mathcal{S}$ is the symplectic quadrangle over $\mathbb{R}$ or $\mathbb{C}$ considered as topological fields.
2) $\mathcal{S}$ is the symplectic quadrangle over a topological commutative field.
3) $\mathcal{S}$ is the symplectic quadrangle over a commutative field.
4) $\mathcal{S}$ is regular and $t = s$.
5) $\mathcal{S}$ is half regular and $t = s$.
6) For every point p the derivation $\mathcal{A}_p$ is a topological projective plane.
7) For every point p the derivation $\mathcal{A}_p$ is a projective plane.
8) For every point p the derivation $\mathcal{S}_p$ is a topological projective plane.
9) For every point p the derivation $\mathcal{S}_p$ is a projective plane. $\square$

REMARK: In fact we might look at a symplectic quadrangle $\mathcal{S}$ as being embedded in the projective three space $PG(3, F)$, i.e. a geometry with Buekenhout diagramm

points lines planes

If we fix a point p of $PG(3, F)$, then the procedure to obtain $\mathcal{A}_p$ amounts to the reconstruction of the polar space of p with regard to the symplectic polarity. Knowlege of all such planes allowes one to reconstruct the whole space $PG(3, F)$ via all its subplanes. On the other hand, the procedure to obtain $\mathcal{S}_p$ amounts to the reconstruction of the residue in p of $PG(3, F)$. Knowledge of all such residues allows one, dually, to reconstruct the whole space $PG(3, F)$. We may picture this information in the diagram



points lines planes

and the isomorphism between $\mathcal{A}_p$ and $\mathcal{S}_p$ is induced by the symplectic polarity belonging to the underlying projective space.

Algebraically, these projective planes $\mathcal{A}_p$ and $\mathcal{S}_p$ appear in the coordinatization of $W(F)$ in the ternary operation $(*)\, b = ak + a'$ (coordinatization method of Hall [5]).

We now restrict our attention to compact hexagons. Since compact n-gons exist only for $n = 2, 3, 4, 6$ ([7]) this is the only interesting case left. So suppose p is a regular point of a compact hexagon $\mathcal{S}$. As for compact quadrangles we want to define topologies on the point- and lineset of the derivation $\mathcal{S}_p$ using the Hausdorff metric on the set of closed subsets of $\mathcal{P}$. In order to do so, we have to check, that the sets in question are closed in $\mathcal{P}$.

If x is a point or a line, then the set $\mathcal{S}_1(x)$ is closed in $\mathcal{L}$ or $\mathcal{P}$ respectively by the definition of compact n-gons and $x^\perp$ is closed in $\mathcal{P}$ or $\mathcal{L}$ respectively, since the distance function is lower semi-continuous. Also, if p and q are two points at maximal distance, then $p^q = p^\perp \cap \{x \in \mathcal{P} \mid d(x,q) \leq n-2\}$ is closed. So ideal lines are closed subsets of $\mathcal{P}$. Suppose $Q = \ll p, z_1, z_2 \gg$ is an ideal plane. Let V_1 and V_2 be two closed hence compact subsets of $\mathcal{S}_1(p)$ such that $z_i * p \notin V_i$, $i \in \{1,2\}$, and $V_1 \cup V_2 = \mathcal{S}_1(p)$. Then $Q_i := \{f_5^1(f_5^1(f_5^1(l, z_1 * z_2), g), z_i) \mid l \in V_i, g \in \mathcal{S}_1(z_i)\}$, $i \in \{1,2\}$, are compact thus closed subsets of $\mathcal{P}$, hence $Q = Q_1 \cup Q_2$ is compact and closed in $\mathcal{P}$.

Suppose p is a regular point of a compact hexagon. Let $\Gamma_1(p) := \{\mathcal{S}_1(g) \mid g \in \mathcal{S}_1(p)\}$, let $\Gamma_2(p) := \{x^\perp \mid x \in \mathcal{S}_2(p)\}$, let $\Gamma_3(p)$ denote the space of ideal lines containing p and let $\Gamma_4(p)$ denote the space of ideal planes through p, always equipped with the topology induced by the Hausdorff metric on the space of closed subsets of $\mathcal{P}$. For $g \in \Gamma_1(p)$ let $\Gamma_1'(g) := \{x^\perp \mid x \in \mathcal{S}_1(g) \setminus \{p\}\} \subset \Gamma_2(p)$ and for $g \in \Gamma_3(p)$ let $\Gamma_1'(g) \subset \Gamma_4(p)$ denote the space of ideal planes which contain g. One checks easily that the maps $\mathcal{S}_1 : \mathcal{S}_1(p) \to \Gamma_1(p) : g \mapsto \mathcal{S}_1(g)$ and ${}^\perp : \mathcal{S}_2(p) \to \Gamma_2(p) : x \mapsto x^\perp$ are homeomorphisms. Thus for $g \in \Gamma_1(p)$ the set $\Gamma_1'(g)$ is homeomorphic to $\mathcal{S}_1(g) \setminus \{p\}$.

We now investigate the topology on $\Gamma_4(p)$:

(4.6) Lemma: *The ideal plane* $\Pi_z^{<z*p,t>}$ *depends continuously on* (z,t).

Proof: Suppose $(z_i, t_i) \in \mathcal{S}_4(p) \times \mathcal{S}_2(p)$ with $f_4^1(p, z_i) \neq t_i \vee p$, $i \in \mathbb{N}$, is a sequence converging to $(z,t) \in \mathcal{S}_4(p) \times \mathcal{S}_2(p)$ with $f_4^1(p,z) \neq t \vee p$. Fix $h \in \mathcal{S}_1(z) \setminus \{f_4^2(z,p)\}$, let $v_i := f_5^2(z_i, f_5^1(t_i, h))$, $i \in \mathbb{N}$, and $v := f_5^2(z, f_5^1(t,h)) = f_5^2(z,h)$. Then $\lim_{i\to\infty} v_i = v$. Moreover, p^{v_i} is the focusline of $Q_i := \Pi_{z_i}^{<z_i*p,t_i>}$ and p^v is the focusline of $Q := \Pi_z^{<z*p,t>}$. Thus the focuslines of Q_i, $i \in \mathbb{N}$, converge to the focusline of Q. Now let $z_i^1 := z_i$ and $z_i^2 := t_i * v_i$. Then $\lim_{i\to\infty} z_i^1 = z^1 := z$ and $\lim_{i\to\infty} z_i^2 = z^2 := v * t$. As in the proof of the compactness of Q, let V_1 and V_2 be two closed hence compact subsets of $\mathcal{S}_1(p)$ such that $z^j \notin V_j$, $j \in \{1,2\}$, and $V_1 \cup V_2 = \mathcal{S}_1(p)$.

We may assume $z_i^j \notin V_j$, for $j \in \{1,2\}$ and every $i \in \mathbb{N}$. Then for $j \in \{1,2\}$ the closed sets $Q_i^j := \{f_5^1(f_5^1(f_5^1(l, v_i), f_5^1(z_i^j, g)), z_i^j) \mid l \in V_j, g \in \mathcal{S}_1(z^{1-j} * p)\}$, $i \in \mathbb{N}$, converge to $Q^j := \{f_5^1(f_5^1(f_5^1(l,v), f_5^1(z^j, g)), z^j) \mid l \in V, g \in \mathcal{S}_1(z^{1-j} * p)\}$, therefore the sequence $Q_i = Q_i^1 \cup Q_i^2$, $i \in \mathbb{N}$, converges to $Q = Q^1 \cup Q^2$. $\square$

(4.7) Corollary: *A sequence* $Q_i \in \Gamma_4(p), i \in \mathbb{N}$, *converges to* $Q \in \Gamma_4(p)$ *if and only if one of the following conditions holds:*

a) *There exists a sequence of triangles* (p, z_i, u_i), $i \in \mathbb{N}$, *with* $\{z_i, u_i\} \in Q_i$ *converging to a triangle* (p, z, u) *with* $\{z, u\} \in Q$.

b) *There exists a sequence of points* $z_i \in Q_i \setminus \{p\}$, $i \in \mathbb{N}$, *converging to a point* $z \in Q \setminus \{p\}$ *and the focuslines of* Q_i *converge to the focusline of* Q.

Proof: The first condition is a trivial consequence of the assumption. Conversely, suppose that (p, z_i, u_i), $i \in \mathbb{N}$, with $\{z_i, u_i\} \in Q_i$ is a sequence of triangles converging to a triangle (p, z, u), with $\{z, u\} \in Q$. Then $Q_i = \ll p, u_i, z_i \gg = \Pi_{z_i}^{<z_i * p, u_i * p>}$, $i \in \mathbb{N}$, converges to $Q = \ll p, u, z \gg = \Pi_z^{<z*p, u*p>}$ by Lemma 4.6.

That the second condition implies the assumption follows directly from Lemma 4.6. Conversely, if Q_i, $i \in \mathbb{N}$, is a sequence of ideal planes converging to an ideal plane Q, then, by the above, there is a sequence of triangles (p, z_i, u_i) with $Q_i = \ll p, z_i, u_i \gg$, $i \in \mathbb{N}$, converging to a triangle (p, z, u) spanning Q. Since the focusline of Q_i is the set $p^{u_i * z_i}$, the result follows from Corollary 2.4. $\square$

As an immediate consequence of Corollary 4.7 we get

(4.8) Corollary: *The map that sends every triangle* (p, z, u) *to the ideal plane* $\ll p, z, u \gg$ *is continuous, i.e. depends continuously on* (z, u). $\square$

Lemma 4.6 also enables us to determine the topological structure of the space $\Gamma_1'(g)$ when g is an ideal line containing p.

(4.9) Corollary: *For* $g \in \Gamma_3(p)$ *the set* $\Gamma_1'(g)$ *is homeomorphic to a pointrow minus a point and closed in* $\Gamma_4(p)$.

Proof: Fix $z \in g \setminus \{p\}$ and $h \in \mathcal{S}_1(p)$ such that $z * p \notin \mathcal{S}_1(h)$. The map $\tau : \mathcal{S}_1(h) \setminus \{p\} \to \Gamma_1'(g) : t \mapsto \Pi_z^{<z*p, t>}$ is bijective by the construction of ideal planes and continuous by Lemma 4.6.

Now suppose that $Q_i \in \Gamma_1'(g)$, $i \in \mathbb{N}$, is a sequence of ideal planes converging to some $Q \in \Gamma_4(g)$. Since $g \subset Q_i$ for every $i \in \mathbb{N}$ we get $g \subset Q$, thus $Q \in \Gamma_1'(g)$ and $\Gamma_1'(g)$ is closed in $\Gamma_4(g)$.

Let $u \in Q \setminus \{p\}$, such that $u * p \neq z * p$. Then, by the definition of the topology on $\Gamma_1'(g)$, there are points $u_i \in Q_i \setminus \{p\}$, $i \in \mathbb{N}$, with $u_i * p \neq z * p$ and $\lim_{i \to \infty} u_i = u$. Thus the sequence $\tau^{-1}(Q_i) = f_5^1(h, u_i * z)$, $i \in \mathbb{N}$, converges to $\tau^{-1}(Q) = f_5^1(h, u * z)$. Therefore τ is a homeomorphism. $\square$

Remark: Since the sets $\mathcal{S}_1(x)$ are doubly homogeneous, the above corollary fully determines the topology of $\Gamma_1'(g)$.

Next we investigate the topology on $\Gamma_3(p)$:

(4.10) LEMMA: *The map that sends $y \in \mathcal{S}_4(p)$ to the ideal line $<y,p> \in \Gamma_3(p)$ and the map that sends every ideal line through p to its focus are continuous.*

PROOF: Suppose $y \in \mathcal{S}_4(p)$. Let $t \in \mathcal{S}_2(p)$ with $d(y,t) = 6$, let $h \in \mathcal{S}_1(t) \setminus \{t \vee p\}$ and let U denote a neighbourhood of y in $\mathcal{S}_4(p)$ such that $x * p$ is not collinear to t for every $x \in U$. Then $<x,p> = (p * x)^{f_5^4(x,h)}$ for $x \in U$, thus $<x,p>$ depends continousely on x by Corollary 2.4.

Suppose l_i, $i \in \mathbb{N}$, is a sequence of ideal lines through p converging to an ideal line l through p then there exist points $y_i \in l_i \setminus \{p\}$ converging to some point $y \in l \setminus \{p\}$. Thus $l_i = <y_i,p>, l = <y,p>$ and the foci $y_i * p$ of l_i converge to the focus $y * p$ of l. □

Corollary 4.9 allows us to define another topology on the set of ideal lines containing p. Namely, let $\Gamma'_3(p)$ denote the set $\{\Gamma'_1(g) \mid g \in \Gamma_3(p)\}$ equipped with the topology induced by the Hausdorff metric on the space of closed subsets of $\Gamma_4(p)$. Fortunately, this construction does not yield anything essentially new.

(4.11) LEMMA: *The map $\Gamma'_1 : \Gamma_3(p) \to \Gamma'_3(p) : g \mapsto \Gamma'_1(g)$ is continuous.*

PROOF: Clearly the map is a bijection. To show that Γ'_1 is continuous suppose that $g_i \in \Gamma_3(p)$, $i \in \mathbb{N}$, is a sequence of ideal lines converging to some ideal line $g \in \Gamma_3(p)$. By the definition of the topology on $\Gamma_3(p)$ there exist a sequence of points $y_i \in g_i \setminus \{p\}$, $i \in \mathbb{N}$, converging to some point $y \in g \setminus \{p\}$. Now suppose $q_i \in \Gamma'_1(g_i)$, $i \in \mathbb{N}$, is a sequence of ideal points converging to some ideal plane $q \in \Gamma_4(p)$. Then there is a sequence $z_i \in q_i \setminus \{p\}$, $i \in \mathbb{N}$, converging to some $z \in q \setminus \{p\}$ such that $z * p \neq y * p$ and $z_i * p \neq y_i * p$ for all $i \in \mathbb{N}$. Since $q_i = \ll p, z_i, u_i \gg$ we get $q = \ll p, z, u \gg \in \Gamma'_1(g)$. Conversely, if $q \in \Gamma'_1(g)$ choose a point $u \in q \setminus \{p\}$ such that $u * p$ is not collinear to $y * p$. Then we may assume that also $y_i * p$ is not collinear to $u * p$ for every $i \in \mathbb{N}$. Then the ideal planes $\ll p, y_i, u_i \gg \in \Gamma'_1(g_i)$, $i \in \mathbb{N}$, converge to $q = \ll p, y, u \gg$.

To show that the inverse map is continuous as well suppose that the sequence $\Gamma'_1(g_i) \in \Gamma'_3(p)$, $i \in \mathbb{N}$, converges to $\Gamma'_1(g) \in \Gamma'_3(p)$. Let $q_i \in \Gamma'_1(g_i)$, $i \in \mathbb{N}$, be a sequence of ideal planes converging to some ideal plane $q \in \Gamma'_1(g_i)$ and let t_i denote the focus of g_i, $i \in \mathbb{N}$. We may assume that the sequence t_i, $i \in \mathbb{N}$ converges to some point t in the focusline of q. Furthermore there exist points $z_i \in q_i \setminus \{p\}$ with $z_i * p \neq t_i$ for every $i \in \mathbb{N}$ converging to some $z \in q \setminus \{p\}$ with $z * p \neq t$. Then $g_i = t_i^{z_i}$ converges to t^z, thus there are points $y_i \in g_i \setminus \{p\}$ converging to some $y \in t^z \setminus \{p\}$. Hence $g_i = <p, y_i>$ converges to $<p, y>$. But then $<p,y> \subset v$ for every $v \in \Gamma'_1(g)$, hence $<p,y> = g$. □

Notation: If q is a point of the derivation $\mathcal{S}_p$, let $\Gamma_1(q)$ denote the set of lines of $\mathcal{S}_p$ incident with q. If $q = p$, then $\Gamma_1(q)$ is already defined as a topological space and $\Gamma_1(p) \cong \mathcal{S}_1(p)$. If $q \in \Gamma_4(p)$, then $\Gamma_1(q)$ inherits a topology as a subset of $\Gamma_3(p)$. If $q \in \Gamma_2(p)$ we define the topology on $\Gamma_1(q)$ as follows. The space $\Gamma_1(q) \setminus \{q \vee p\}$ inherits the topology as a subset of $\Gamma_3(p)$. Let $g \in \mathcal{S}_1(q) \setminus \{q \vee p\}$. Then $\mu : \mathcal{S}_1(g) \setminus \{q\} \to \Gamma_1(q) \setminus \{q \vee p\} : y \mapsto <y, p>$ is a homeomorphism by Lemma 4.10 and the definition of the topology on $\Gamma_3(p)$, thus $\Gamma_1(q) \setminus \{q \vee p\}$ is locally compact. Finally we define the topology on $\Gamma_1(q)$ to be the one-point-compactification of $\Gamma_1(q) \setminus \{q \vee p\}$.

(4.12) Lemma: *If q is an ideal plane, then $\Gamma_1(q)$ is homeomorphic to the focusline of q, hence to $\Gamma_1(p) \cong \mathcal{S}_1(p)$.*

Proof: The map that sends every line of $\Gamma_1(p)$ to its focus is bijective by the construction of ideal planes and continuous by Lemma 4.10, thus a homeomorphism. The focusline of q is homeomorphic to $\mathcal{S}_p$ by Corollary 2.4. □

(4.13) Lemma: *Let $q \in \Gamma_2(p)$. A sequence $L_i \in \Gamma_1(q) \setminus \{q \vee p\}$, $i \in \mathbb{N}$, converges to $q \vee p$ if and only if there is a subset $A \subset \mathcal{S}_1(q \vee p)$ such that with respect to the topology induced by the Hausdorff metric the lines L_i converge to A.*

Proof: Suppose $\lim_{i\to\infty} L_i = q \vee p$. Let $y_i \in L_i \setminus \{p\}$, $i \in \mathbb{N}$. We may assume that there is a point y such that $\lim_{i\to\infty} y_i = y$. Since $d(y_i, p) = 4$ for every $i \in \mathbb{N}$ we know $d(y, p) \leq 4$ by Lemma 2.1. Suppose $d(y, p) = 4$. Then $\lim_{i\to\infty} L_i = \lim_{i\to\infty} <y_i, p> = <\lim_{i\to\infty} y_i, p> = <y, p>$ which contradicts our assumption. Thus $y \in p^\perp$. Also by Lemma 2.1 we get $y \in q^\perp$. Hence $y \in p^\perp \cap q^\perp = \mathcal{S}_1(q \vee p)$. Since the converse implication holds trivially this proves the Lemma. □

Remark: It is not clear whether in the above lemma the set A is the whole of $\mathcal{S}_1(a \vee p)$ or not. Therefore we can not simply define the topology on $\Gamma_1(q)$ to be the topology induced by the Hausdorff metric, since then $\Gamma_1(q)$ need not be compact.

We now summarize the topological results which we obtained so far and which we will use later:

(4.14) Corollary: *Let p be a regular point of a compact hexagon of order (s, t). Then $\Gamma_1(p) \cong \mathbb{S}_t$ and for $g \in \Gamma_1(p)$ or $g \in \Gamma_3(p)$ the space $\Gamma_1'(g)$ is homeomorphic to $\mathbb{R}^s$ and closed in $\Gamma_2(p)$ or $\Gamma_4(p)$ respectively.* □

Now we will determine the geometrical structure of $\mathcal{S}_p$.

Suppose p is a regular point of a compact hexagon $\mathcal{S}$ and $Q = \ll p, u, z \gg$ is an ideal plane. Let $x := p * z$ and let $F := <p * z, p * u>$ be the focusline of Q. Take $h \in \mathcal{S}_1(z) \setminus \{z \vee x\}$, $a \in x \vee p \setminus \{p, x\}$ and $g \in \mathcal{S}_1(a) \setminus \{a \vee p\}$. If $t \in F$ is a point different from x and if r, s are two points of t^z, then, by regularity using arguments as in the proof of Lemma 4.1, $a^r = a^s$, in particular $f_5^1(g, r) = f_5^1(g, s)$.

(4.15) Lemma: *In the above situation the map*

$$\tau : <p * z, p * u> \to \mathcal{S}_1(g) : t \mapsto \begin{cases} f_5^1(g, f_5^2(t, h)) & t \neq x \\ a & t = x \end{cases}$$

is injective and continuous.

Proof: We first prove that τ is injective. Let $t \in F \setminus \{x\}$. Then $d(a, f_5^2(t, h)) = 6$ for otherwise the points $t, p, a, a * f_5^2(t, h), f_5^2(t, h)$ would form a pentagon. Therefore $\tau(t) \neq a = \tau(x)$.

Suppose $t_1, t_2 \in F \setminus \{x\}$ such that $\tau(t_1) = \tau(t_2)$. Then the points $s_1 := f_5^2(t_1, h)$, $s_2 := f_5^2(t_2, h)$ and $s_3 := \tau(t_1) = \tau(t_2)$ satisfy $d(s_j, s_i) = 4$ and $s_j * s_i \notin \mathcal{S}_2(p)$, $i \neq j$, hence by Lemma 4.2 they define an ideal plane $\ll s_1, s_2, s_3 \gg$ which coincides with the ideal plane $\ll p, s_1, s_2 \gg$. However, by construction $s_1, s_2 \in Q$, thus $\ll p, s_1, s_2 \gg = Q$ and therefore $\{a\} = (a \vee p) \cap F = (x \vee p) \cap F = \{x\}$, contradicting our assumptions. Thus τ is injective.

We now show that τ is continuous. The restriction of τ to $F \setminus \{x\}$ is continuous by construction. To prove continuity at x let $t_i, i \in \mathbb{N}$, be a sequence in $F \setminus \{x\}$ converging to x, let $s \in Q$ such that $s * p \notin \{t_i \mid i \in \mathbb{N}\} \cup \{x\}$ and let $k \in \mathcal{S}_1(s) \setminus \{f_4^1(s, p)\}$. Then $d(x, k) = 5$ and the point $w := f_5^2(x, k)$ is at distance 5 to g and at distance 4 to a. Thus $f_5^1(g, w) = a$. Furthermore ${t_i}^z = {t_i}^s$ implying $\tau(t_i) = f_5^1(g, f_5^2(t_i, h)) = f_5^1(g, f_5^2(t_i, k))$ for every $i \in \mathbb{N}$. Hence $\lim_{i \to \infty} \tau(t_i) = \lim_{i \to \infty} f_5^1(g, f_5^2(t_i, k)) = f_5^1(g, f_5^2(\lim_{i \to \infty} t_i, k)) = f_5^1(g, f_5^2(x, k)) = f_5^1(g, w) = a = \tau(x)$. □

(4.16) Corollary: *Suppose that in the above situation linepencils are homeomorphic to $\mathbb{S}_t$ and pointrows are homeomorphic to $\mathbb{S}_s$. Then $t \leq s$ and τ is bijective and hence a homeomorphism if and only if $t = s$.*

Proof: The focusline F is homeomorphic to $\mathcal{S}_1(p)$ (Corollary 2.4), hence to $\mathbb{S}_t$. Thus $\tau : F \to \mathcal{S}_1(g)$ is equivalent to a continuous and injective map $\tau' : \mathbb{S}_t \to \mathbb{S}_s$. But then $t \leq s$ and τ is surjective if and only if $t = s$. However, a continuous bijection between two compact Hausdorff spaces is a homeomorphism. □

¿From Lemma 4.10 and Corollary 4.16 we deduce readily

(4.17) Corollary: *Suppose $q \in \Gamma_4(p)$ is an ideal plane and $a \in \mathcal{S}_2(p)$ is a point not incident with the focusline F of q. Let x denote the point of the intersection $a \vee p \cap F$ and $l \in \Gamma_1(q)$ the ideal line whose focus is x. Then the map*

$$\sigma : \Gamma_1(q) \to \Gamma_1(a) : g \mapsto \begin{cases} a^m \text{ where } m \in g \setminus \{p\} & g \neq l \\ a \vee p & g = l \end{cases}$$

is injective and continuous. It is a homeomorphism if and only if pointrows and linepencils are of the same dimension. □

(4.18) Theorem: *Suppose p is a regular point of a compact hexagon $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ of order (s,t). Then $t \leq s$ and the derivation $\mathcal{S}_p$ of $\mathcal{S}$ at p is a quadrangle if and only if $t = s$. Moreover if $t = s$, then p is a (half-) regular point of the derived quadrangle and there are topologies on $\mathcal{P}_p$ and $\mathcal{L}_p$ extending the topology on the spaces $\Gamma_i(p)$, $i \in \{1,2,3,4\}$, such that $\mathcal{S}_p$ becomes a compact quadrangle of order (s,s).*

Proof: By Corollary 4.16 we have $t \leq s$. Let $q \in \Gamma_i(p)$, $i \in \{0,2,4\}$, be a point and $l \in \Gamma_j(p)$, $i \in \{1,3\}$, a line of $\mathcal{S}_p$ not incident with q. We will show that there is at most one point $\pi_{i,j}(q,l)$ on l collinear to q and one line $\lambda_{i,j}(q,l)$ through q intersecting l. Several cases have to be considered:

$q \in \Gamma_0(p)$, i.e. $q = p^\perp$: Then $l \in \Gamma_3(p)$, i.e. l is some ideal line $<y,p>$, $y \in \mathcal{S}_4(p)$. Thus $\pi_{0,3}(q,l) = (p * y)^\perp$ and $\lambda_{0,3}(q,l) = p \vee (p * y) = f_4^1(p,y)$ always exist.

$q \in \Gamma_2(p), l \in \Gamma_1(p)$: In this case $q = x^\perp$, $x \in \mathcal{S}_2(p)$ and $l = p \vee y$, $y \in \mathcal{S}_2(p)$, where x is not collinear to y. Thus $\pi_{2,1}(q,l) = p^\perp$ and $\lambda_{2,1}(q,l) = p \vee x$ always exist.

$q \in \Gamma_2(p), l \in \Gamma_3(p)$: Then $q = x^\perp$, $x \in \mathcal{S}_2(p)$ and $l = <y,p>$, $y \in \mathcal{S}_4(p)$, with $x \neq p * y$. If $p*y$ is collinear to x, then $\pi_{2,3}(q,l) = (p*y)^\perp = (x*y)^\perp$ and $\lambda_{2,3}(q,l) = x \vee (x*y) = x \vee p$ always exist. If $p*y$ is not collinear to x, then $\pi_{2,3}(q,l) = \Pi_y^{<y*p,x>}$ and $\lambda_{2,3}(q,l) = x^y$ always exist.

$q \in \Gamma_4(p), l \in \Gamma_1(p)$: Thus $q = \ll p,u,z \gg$, (p,u,z) a triangle and $l = p \vee y$, $y \in \mathcal{S}_2(p)$. The focusline of q intersects the line l in precisely one point s. We may assume $s \neq u * p$. Then $\pi_{4,1}(q,l) = s^\perp$ and $\lambda_{4,1}(q,l) = s^u$ always exist.

$q \in \Gamma_4(p), l \in \Gamma_3(p)$: Then $q = \ll p,u,z \gg$, (p,u,z) a triangle and $l = <y,p>$, $y \in \mathcal{S}_4(p)$. If the focus $p * y$ of l is contained in the focusline of q, then $\pi_{4,3}(q,l) = (p * y)^\perp$ and $\lambda_{4,3}(q,l) = s^u, u \in q \cap \mathcal{S}_6(y * p)$, always exist.

If $p*y$ is not an element of the focusline of q, then consider the map $\sigma : \Gamma_1(q) \to \Gamma_1(p*y)$ as defined in Corollary 4.17. A line $g \in \Gamma_1(q)$ meets l if and only if $\sigma(g) = l$. Thus by injectivity there is at most one line through q meeting l, hence at most one point on l collinear to q. If $t < s$, then σ is not surjective and we can find a line $l' \in \Gamma_1(p*y)$ not in the image of σ. But then there is no line through q meeting l', hence no point on l'

is collinear to q, implying that $\mathcal{S}_p$ is not a quadrangle. However, if $t = s$, then σ is a homeomorphism, thus $\lambda_{4,3}(q,l) = \sigma^{-1}(l)$ and $\pi_{4,3}(q,l) = \ll p, y, v \gg, v \in \sigma^{-1}(l) \setminus \{p\}$, always exist.

This proves that $\mathcal{S}_p$ is a quadrangle if and only if $t = s$. It should be noted that only in the very last case the topological requirements were needed.

We will assume $t = s$ for the rest of the proof. That p is a regular point of $\mathcal{S}_p$ follows immediately from the fact that p is a halfregular point of $\mathcal{S}$.

We now will show that the maps $\pi_{i,j}$ and $\lambda_{i,j}$ are continuous.

Continuity of the maps $\pi_{0,3} : \Gamma_3(p) \to \Gamma_2(p)$, $\lambda_{0,3} : \Gamma_3(p) \to \Gamma_1(p)$ and $\lambda_{2,1} : \Gamma_2(p) \to \Gamma_1(p) : x^\perp \mapsto x \vee p$ follows from Lemma 4.6 and Lemma 4.2.

Lemma 4.6, Lemma 4.2 and Corollary 2.4 imply that the restrictions $\pi_{2,3} : \Gamma_2(p) \times \Gamma_3(p) \setminus X \to \Gamma_4(p)$ and $\lambda_{2,3} : \Gamma_2(p) \times \Gamma_3(p) \setminus X \to \Gamma_3(p)$, where $X := \mathcal{F}_p \cup {\pi_{2,3}}^{-1}(\Gamma_2(p))$ consisits of those $(x^\perp, l) \in \Gamma_2(p) \times \Gamma_3(p)$ where the focus of l is collinear to x, are continuous.

That the maps $\pi_{4,1} : \Gamma_4(p) \times \Gamma_1(p) \to \Gamma_2(p)$ and $\lambda_{4,1} : \Gamma_4(p) \times \Gamma_1(p) \to \Gamma_3(p)$ are continuous is essentialy a consequence of Corollary 4.7.b.

To prove that the maps $\pi_{4,3} : \Gamma_4(p) \times \Gamma_3(p) \setminus Y \to \Gamma_4(p)$ and $\lambda_{4,3} : \Gamma_4(p) \times \Gamma_3(p) \setminus Y \to \Gamma_3(p)$, where $Y := \mathcal{F}_p \cup {\pi_{4,3}}^{-1}(\Gamma_2(p))$ consisits of those $(q,l) \in \Gamma_4(p) \times \Gamma_3(p)$ where the focus of l is not contained in the focusline of q, are continuous, suppose that $(q_i, l_i) \in \Gamma_4(p) \times \Gamma_3(p) \setminus Y$, $i \in \mathbb{N}$, is a sequence converging to some $(q,l) \in \Gamma_4(p) \times \Gamma_3(p) \setminus Y$. Thus $g_i := \lambda_{4,3}(q_i, l_i)$ is a welldefined ideal line for every $i \in \mathbb{N}$. Let t_i denote the focus of l_i, s_i the focus of g_i, $i \in \mathbb{N}$, and t the focus of l . Then s_i is not collinear to t_i and we may assume that the points s_i, $i \in \mathbb{N}$, converge to some point s contained in the focusline of q. Let $z_i \in l_i \setminus \{p\}$, $i \in \mathbb{N}$, be a sequence converging to some $z \in l \setminus \{p\}$ and let $y_i \in q_i \setminus \{p\}$, $i \in \mathbb{N}$, be a sequence converging to some $y \in q \setminus \{p\}$ such that $y * p \neq s \neq z * p$. We may assume $y_i * p \neq s_i$, thus $g_i = s_i^{z_i} = s_i^{y_i}$ for every $i \in \mathbb{N}$ and the lines g_i, $i \in \mathbb{N}$, converge to $g := s^y$. We can find a sequence of points $u_i \in g_i \setminus \{p\}$, $i \in \mathbb{N}$, converging to some $u \in g \setminus \{p\}$. Since $d(u_i, z_i) = 4$ for every $i \in \mathbb{N}$, the distance from u to z is at most 4. If $d(u,z) < 4$, then $u * p = z * p$, which contradicts the assumption that $z * p = t$ is not contained in the focusline of q. Thus $d(u,z) = 4$ hence $g = s^z$ and the sequence $\pi_{4,3}(q_i, l_i) = \ll p, z_i, u_i \gg$ converges to $\ll p, z, u \gg$. However, $\ll p, z, u \gg$ contains $<p, z> = l$ and $<p, u> = g \in \Gamma_1(q)$. Thus $\ll p, z, u \gg = \pi_{4,3}(q,l)$ and $g = <p, u> = \lambda_{4,3}(q,l)$.

Since the above maps are continuous and since the properties listed in Corollary 4.14 hold, there are topologies on $\mathcal{P}_p$ and $\mathcal{L}_p$ such that on $\Gamma_2(p)$ and $\Gamma_4(p)$ the given topologies are respected and such that $\mathcal{S}_p$ becomes a compact quadrangle of order (s,s) by [14]:?????. The topologies on $\Gamma_1(p)$ and $\Gamma_3(p)$ are induced then by the Hausdorff metric on the set of closed subsets of $\mathcal{P}_p$. However, because of Lemma 4.11 and since the map ${}^\perp : p^\perp \to \Gamma_2(p) \cup \{p^\perp\} \subset \mathcal{P}_p : x \mapsto x^\perp$ is a homeomorphism these topologies on $\Gamma_1(p)$ and $\Gamma_3(p)$ are equivalent to the topologies induced by the Hausdorff metric on the set of closed subsets of $\mathcal{P}$.

This proves our assertions. $\square$

REMARK: Even though for every $i \in \{1,2,3,4\}$ the topology on $\Gamma_i(p)$ is the topology induced by the Haussdorf metric an the set of closed subsets of $\mathcal{P}$, it is not clear, whether the topologies on $\mathcal{P}_p$ and $\mathcal{L}_p$ are induced by the Hausdorff metric an the set of closed subsets of $\mathcal{P}$. So e.g. if $g_i \in \Gamma_3(p)$ is a sequence converging to some $g \in \Gamma_1(g)$ with respect to the topology on $\mathcal{L}_p$, it is not clear, whether $\lim_{i\to\infty} g_i = g$ with respect to the topology induced by the Hausdorff metric. However, if every point of the hexagon $\mathcal{S}$ is regular, then one can show with very little effort, that these topologies coincide. In the general case we think that using results of [13] and algebraic topology, one can prove that both topologies coincide as well.

The above proof shows also that even if $t < s$ the maps $\pi_{i,j}$ and $\lambda_{i,j}$ are defined unless $(i,j) = (4,3)$. The maps $\pi_{4,3}$ and $\lambda_{4,3}$ are in general only defined on a subset of $\Gamma_4(p) \times \Gamma_3(p) \setminus \mathcal{F}_p$. However, all of these maps are still continuous.

The last step towards our goul is to prove that every regular hexagon of order (s,s) is a split Cayley hexagon. Our proof relies heavily on results of Ronan ([11]).

(4.19) LEMMA: *If $\mathcal{S}$ is a regular compact connected hexagon of order (s,s), then $\mathcal{S}$ is the split Cayley hexagon over $\mathbb{R}$ or $\mathbb{C}$.*

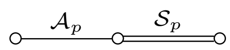
PROOF: Our first aim is to construct the underlying polar space. So consider the following geometry Δ: the points of Δ are the points of $\mathcal{S}$, the lines of Δ are all lines and ideal lines of $\mathcal{S}$ and the planes of Δ are all sets of points of $\mathcal{S}$ collinear to a point of $\mathcal{S}$ and all ideal planes of $\mathcal{S}$. One checks easily that Δ is a polar space of rank 3. To see that Δ is the classical polar space over a commutative field, we define a subgeometry Γ of Δ as follows: Let Q be an ideal plane of $\mathcal{S}$ and Q' its twin, i.e. the ideal plane consisiting of the points $u * v$ with $u, v \in Q$ and$u \neq v$. The points of Γ are all the points on lines of $\mathcal{S}$ having at least (and hence exactly) two points in common with Q and Q', the lines of Γ are all lines of Δ induced and the planes of Γ are all planes of Δ induced. Again it is straight forward to check that this is a D_3 polar space (see also [11]: 8.16). Thus it follows from [11]: 4.9 that Δ is classical. It has diagram C_3, so $\mathcal{S}$ is embedded in $C_3(F)$, where F is some commutative field. By [11]: 8.19, it is the split Cayley hexagon over F. Now the topology on the pointset of $\mathcal{S}$ induces a topology on the field F, such that it becomes a locally connected topological commutative field, thus F is either $\mathbb{R}$ or $\mathbb{C}$ by Pontrjagin's classification ([10]). Since the topology on lines fully determines the topology of $\mathcal{S}$, the topology on $\mathcal{S}$ induced by the topology of F is equivalent to the given one. $\square$

Proof of Theorem 1.1: The implications 1) → 2) → 3) → 4) → 5) are trivial and that every half regular hexagon is regular is almost obvious. By [7] the dimensions of pointrows and linepencils coincide. Thus Theorem 3.2 and Theorem 3.3 imply the equivalence of 5), 6) and 7) and Theorem 4.18 implies the equivalence of 4), 8) and 9). Finally, Lemma 4.19 proves the implication 5) → 1) which closes the last gap. □

Remark: Again one can look at the split Cayley hexagon as being embedded in the quadric $Q(6, F)$, i.e. a geometry with Buekenhout diagramm

points lines planes

If we fix a point p of $Q(6, F)$, then the procedure to obtain $\mathcal{A}_p$ amounts to the reconstruction of the unique projective plane through p all of whose lines containing p belong to the hexagon. This way one constructs all lines of $Q(6, F)$ and this allowes one to reconstruct the whole space $Q(6, F)$. On the other hand, the procedure to obtain $\mathcal{S}_p$ amounts to the reconstruction of the residue in p of $Q(6, F)$. Knowledge of all such residues allows one, dually, to reconstruct $Q(6, F)$ as well (After all, the ideal planes in $H(F)$ are the usual projective planes on $Q(6, F)$). We can picture this information in the diagram



points lines planes

Algebraically, the general quadrangle $\mathcal{S}_p$ appear in the coordinatization of $H(F)$ via the operations $(***)$ and $(****)$.

5 References

[1] De Smet, V. and Van Maldeghem, H.: *The finite Moufang hexagons coordinized*, preprint 1992

[2] Forst, M.: *Topologische 4-Gone*, Mitt. d. Math. Sem. Gießen **147** (1981), 3-129

[3] Grundhöfer, Th. und Knarr, N.: *Topology in generalized quadrangles*, Topology Appl. **34** (1990), 139-152

[4] Grundhöfer, Th. and Van Maldeghem, H.: *Topological Polygons and affine buildings of rank three*, Atti Sem. Mat. Fis. Univ. Modena, XXXVIII, 459-474 (1990)

[5] Hall, M.: *Projective planes*, Trans. Am. Math. Soc. **54** (1943), 229-277

[6] Hanssens, G. und Van Maldeghem, H.: *A New Look at the Classical Generalized Quadrangles*, Ars Combinatoria **24** (1987), 199-210

[7] Knarr, N.: *The nonexistence of certain topological polygons*, Forum Math. **2** (1990), 603-612

[8] Łysko, J.M.: *Some theorems concerning finite dimensional homogeneous ANR-spaces*, Bull. Acad. Polon. Sci. **24** (1976) 491-496

[9] Nadler, S.B.: *Hyperspaces of sets*, Monographs and textbooks in pure and applied mathematics **49**, Marcel Dekker, New York, (1978)

[10] Pontrjagin, L.S.: *Über stetige algebraische Körper*, Ann. Math. **33** (1932) 163-174

[11] Ronan, M.A.: *A geometric characterization of Moufang hexagons*, Invent. Math. **57** (1980), 227-262

[12] Schroth, A.E.: *Characterising symplectic quadrangles by their derivations*, Arch. Math. **58** (1992), 98-104

[13] Schroth, A.E.: *On the structure of topological n-gons*, Preprint

[14] Schroth, A.E.: *Lie geometries of four dimensional Laguerre planes*, Preprint

[15] Tits, J.: *Sur la trialité et certains groupes qui s'en déduisent*, Publ. Math.: I.H.E.S. **2** (1959), 13-60