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# ON LOCALLY FINITE ALTERNATIVE DIVISION RINGS WITH VALUATION.

Hendrik Van Maldeghem *

We prove that a locally finite complete alternative division ring with valuation is a field, without using the celebrated theorem of Bruck and Kleinfeld, which classifies all division rings with IP.

## INTRODUCTION

By Bruck and Kleinfeld [ 1 ], a locally compact, totally disconnected non-discrete alternative division ring *D* is an octonion algebra, and hence, since anisotropic quadratic forms over locally compact local fields have at most four variables, *D* is associative. This fact uses the heavy classification theorem of alternative division rings. Our aim is to prove it in a very elementary way without using Bruck and Kleinfeld. As an application, we prove that no proper locally finite Moufang triangle building exists.

I'd like to thank J. Tits for drawning my attention to the above proof.

## 1. DIVISION RINGS WITH VALUATION

DEFINITIONS. A *division ring D* is a set, provided with two binary operations : addition (+) and multiplication (.) such that *D*,+ is

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an abelian group, $D$,. is a loop and both distributive laws hold. In other words : $D$ ,+,. has all defining properties of a skewfield except that the multiplication need not be associative. In what follows, associativity always refers to the multiplication. Also, the identity of the addition (resp. multiplication) is denoted by $0$(resp. 1).

A *discrete valuation* $v$ on $D$ is a surjective map from $D$ to the set of integers $Z$ union the singleton $\{\infty\}$ satisfying (d1), (d2) and (d3) stated below :

(d1) $v(a) = \infty$ iff $a = 0$

(d2) $v(a.b) = v(a)+v(b)$ for all $a,b \in D$

(d3) $v(a-b) = \inf\{v(a),v(b)\}$ if $v(a) \neq v(b)$

In that case $(D,v)$ is a *division ring with valuation* in the sense of [ 8 ]. We abbreviate that to : $(D,v)$ is a *V-DR*. If $D$ is associative, then $v$ is a usual discrete valuation on a skewfield. Now define for n=$0$,1 :

$$D_n^+ = \{r \in D \mid v(r) \geq n\}$$

and let $D^- = D - D_0^+$ and $D_0 = D_0^+ - D_1^+$ . Then the residue structure $K = D_0^+/D_1^+$ is well defined (see [ 8,§2.7.2] ) and will be a division ring. Now $D$ is called *locally finite* when $K$ is finite.

One can check (see [ 8,§2.1] ) the following immediate consequences of the previous definitions :

Let $(D,v)$ be a *V-DR*, then we have :

CONSEQUENCE 1. $v(1) = 0$; $v(0) = \infty$

CONSEQUENCE 2. *(i) If* $x.y = 1$*, then* $v(x) = -v(y)$
*(ii)* $v(x) = v(-x)$

CONSEQUENCE 3. $v(a-b) \geq \inf\{v(a-c),v(c-b)\}$ *for all* $a,b,c \in D$ *and equality holds if* $v(a-c) \neq v(c-b)$.

DEFINITION. A division ring D is said to *have the inverse property* (or briefly (IP)) if it satisfies (RIP) and (LIP) (resp. right (IP) and left (IP)) :

(RIP) For each $y\varepsilon D$, there exists $y^{-1}\varepsilon D$ such that $(xy)y^{-1} = x$ for all $x\varepsilon D$

(LIP) For each $y\varepsilon D$, there exists $y^{-1}\varepsilon D$ such that $y^{-1}(yx) = x$ for all $x\varepsilon D$

A division ring $D$ with (IP) has the following properties :

PROPERTY 1. *Every element $y\varepsilon D$ has a unique two sided inverse $y^{-1}\varepsilon D$.* (see [3, lemma 6.10])

PROPERTY 2. *$D$ is alternative, i.e. $D$ satisfies both alternative laws* (RAL) *and* (LAL), *stated below. Conversely, any alternative division ring has* (IP).

(RAL) $(xy)y = xy^2$ *for all* $x,y\varepsilon D$

(LAL) $y(yx) = y^2x$ *for all* $x,y\varepsilon D$ (see [3,§6])

PROPERTY 3. *Any two elements of $D$ are contained in a subskewfield of $D$.* (see [3,lemma 6.19])

PROPERTY 4. *Any finite alternative division ring $D$ is a field and hence the multiplicative group $D^*$,. of $D$ is cyclic.*

## 2. THE MAIN RESULT

In this section, we denote by $(D,v)$ a locally finite alternative *V-DR*. Note that the residue structure $K$ is also alternative, and hence by the property 4, $K$ is a field. Our aim is to proof the

THEOREM. *$D$ is a skewfield .*

Therefore, we first need the notion of a complete *V-DR*.

DEFINITION. $(D,v)$ is called *complete* if $D$ is complete w.r.t. the metric

$$\delta : (x,y) \to e^{-v(x,y)}$$

Any division ring with valuation can be completed (see [9]). $D$ is a subdivision ring of its completion $D^c$ and clearly $D$ is associative iff $D^c$ is. Hence, to prove the theorem, we can assume that $D$ is complete. Then the result follows from lemma 2 below.

LEMMA 1. *Let $D'$ be a subdivision ring of $D$. Suppose $D'$ contains an $r \varepsilon D$ with valuation 1. then $(D',v')$, where $v'$ is the restriction of $v$ to $D'$, is a V-DR.*

PROOF. (d1), (d2) and (d3) are trivially satisfied. The fact that $v'$ is surjective follows from the observation that $0,1 \varepsilon D'$ together with consequence 1, and $v(r^n) = n$ for all $n \varepsilon Z$ by (d2) and consequence 2.

Q.E.D.

REMARK. If we do not insist that $v(r) = 1$, but we only demand $v(r) \neq 0$ then $D'$ still admits a valuation $v'$ with

$$v'(x,y) = \frac{v(x,y)}{k} \quad \text{for all } x,y \varepsilon D'$$

where

$$k = \inf\{n>0 \mid v(s)=n \text{ and } s \varepsilon D'\}$$

LEMMA 2. *Let $a \varepsilon D_0^+$ be such that $a+D_1^+$ generates the cyclic multiplicative group of $K$. Let $b \varepsilon D$ have valuation 1. Let us denote by $D'$ the completion of the subskewfield of $D$ (with valuation) generated by $\{a,b\}$. If $D$ is complete, then $D'$ is isomorphic to $D$.*

PROOF. * Clearly $D' \subset D$.

* Suppose $z \varepsilon D$ and let $v(z) = n_1 \neq \infty$ (otherwise $z = 0 \varepsilon D'$). Then $v(z.b^{-n_1}) = 0$ and hence there exists $k_1 \varepsilon Z$ such that

$$v(zb^{-n_1} - a^{k_1}) > 0$$

so

$$v(z - a^{k_1} b^{n_1}) = n_2 > n_1$$

Suppose $n_2 \neq \infty$, otherwise $z = a^{k_1}.b^{n_1} \varepsilon D'$. We do the same with $z - a^{k_1}b^{n_1}$ and get $k_2$ and $n_3$ such that

$$v(z - a^{k_1}b^{n_1} - a^{k_2}b^{n_2}) = n_3 > n_2 > n_1$$

Again we suppose $n_3 \neq \infty$ , and we do the same, etc...

This way, we get a sequence $(u_\ell)_{\ell\in IN}$ with

$$u_\ell = \sum_{i=1}^{\ell} a^{k_i} b^{n_i}$$

and such that $v(z-u_\ell) = n_{\ell+1}$ where $(n_\ell)_{\ell\in IN}$ is a *strictly* increasing sequence of integers. Note that

$$v(u_{\ell+q}-u_\ell) = n_{\ell+1} \quad \textit{for all } \ell, q\in IN,\ q\neq 0$$

Hence $(u_\ell)_{\ell\in IN}$ is a Cauchy sequence in $D'$. By the completeness of $D'$, there exists a limit $c\in D'$. Now by consequence 3,

$$v(u_{\ell+1}-c) \geqslant \inf\{v(u_{\ell+1}-u_\ell),\ v(u_\ell-c)\}$$

So if $v(u_\ell-c) < n_{\ell+1}$, then $v(u_{\ell+1}-c) = v(u_\ell-c) < n_{\ell+2}$ and by an inductive arguement,

$$v(u_{\ell+q}-c) = v(u_\ell-c) \quad \textit{for all } q\in IN$$

But then c cannot be the limit of $(u_\ell)_{\ell\in IN}$ ,a contradiction. Hence we have $v(u_\ell-c) \geqslant n_{\ell+1}$. But then

$$v(z-c) \geqslant \inf\{v(z-u_\ell),\ v(u_\ell-c)\} = n_{\ell+1}$$

Since $(u_\ell)_{\ell\in IN}$ diverges to $+\infty$, $v(z-c) = \infty$, so $z = c\in D'$.

Q.E.D.

APPLICATION TO BUILDINGS .

By Tits' classification theorem of affine buildings of rank [ 4 ] (see [ 7 ]) we know that all these buildings are classical, i.e. they arise from an algebraic group over a local field. One of the three families of affine buildings of rank 3 (on which Tits' theorem is not applicable) is the family of the so called 'triangle buildings' : their Buekenhout diagram is a triangle. By another (*BIG*) theorem of Tits [ 6 ] and a result of Ronan [ 4, corollary (2.3)] , one can define a triangle building as a *simply connected rank 3 geometry which is locally a projective plane* (all residues in the sense of Buekenhout [ 2 ] are projective planes). Tits also defines the building at infinity of an affine building Δ (see [ 7 ]). In the triangle case, this is always the (spherical) building corresponding with a projective plane, which

we denote by *PG*(Δ). By definition the type of Δ is the type of *PG*(Δ), e.g. Δ is called *classical* if *PG*(Δ) is a classical projective plane, i.e. a Desarguesian plane. Similar, we call Moufang if *PG*(Δ) is a Moufang plane, etc.. A building Δ is called *locally finite* if one (and hence all) residue is finite. Now it is known (e.g. see [5], [8], [9]) that a triangle building need not be Desarguesian and there are examples of locally finite proper 'translation' buildings, division ring buildings, nearfield buildings in [9]. The next corollary however shows that, if a locally finite triangle building is Moufang, then it is Desarguesian.
For detailed information and background material about triangle buildings, we refer to [7] and [9].

COROLLARY. *A locally finite Moufang triangle building Δ is classical.*
PROOF. *PG*(Δ) is coordinatized by an alternative division ring (*D*,*v*) with valuation (by [3,9]). The residue ring *K* of *D* coordinatizes a certain residue of in the sense of Buekenhout [2] by [3,9]. Hence *K* is finite and *D* is locally finite. But then, *D* is a skewfield by the theorem and so Δ is Desarguesian.

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State University of Ghent
Seminar of Geometry and Combinatorics
Krijgslaan 281
B-9000 Gent
Belgium

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