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An Essay on the Ree Octagons

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Abstract. We coordinatize the Moufang generalized octagons arising from the Ree groups of type ${}^2F_4$. In this way, we obtain a very concrete and explicit description of these octagons. We use this to prove some results on suboctagons, generalized homologies, Suzuki-Tits ovoids and groups of projectivities of the Ree octagons. All our results hold for arbitrary Ree octagons, finite or not.

Keywords: Tits building, generalized octagon, group of projectivities, coordinatization

1 Introduction

Let $\Delta = (\mathcal{P}, \mathcal{L}, I)$ be a rank 2 incidence geometry with point set $\mathcal{P}$, line set $\mathcal{L}$ and incidence relation I. A *path* of *length d* is a sequence $v_0, \ldots, v_d \in \mathcal{P} \cup \mathcal{L}$ with $v_i I v_{i+1}$, $0 \leq i < d$. Define a function $\delta: (\mathcal{P} \cup \mathcal{L}) \times (\mathcal{P} \cup \mathcal{L}) \to \mathbb{N} \cup \{\omega\}$ by $\delta(v, v') = d$ if and only if d is the minimum of all $d' \in N$ such that there exists a path of length d' joining v and v', and $\delta(v, v') = \omega$ if there is no such path.

Then Δ is a *generalized n-gon*, $n \in \mathbb{N}\backslash\{0, 1, 2\}$, or a *generalized polygon* if it satisfies the following conditions:

(GP1) There is a bijection between the sets of points incident with two arbitrary lines. There is also a bijection between the sets of lines incident with two arbitrary points.
(GP2) The image of $(\mathcal{P} \cup \mathcal{L}) \times (\mathcal{P} \cup \mathcal{L})$ under δ equals $\{0, \ldots, n\}$. For $v, v' \in \mathcal{P} \cup \mathcal{L}$ with $\delta(v, v') = d < n$ the path of length d joining v and v' is unique.
(GP3) Each $v \in \mathcal{P} \cup \mathcal{L}$ is incident with at least 2 elements.

Generalized polygons were introduced by Tits [12]. Note that we have excluded the trivial case of generalized digons here.

As an immediate consequence of the definition a generalized polygon is a partial linear space and a dual partial linear space.

We will mainly be concerned with the case $n = 8$, the generalized *octagons*, and also with the case $n = 4$, the generalized *quadrangles*. Generalized polygons are in fact the (weak) buildings of rank 2. We will use some of the building terminology below. For instance, we

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will call a generalized octagon (or any generalized polygon) *thick* if every line contains at least three points and every point is incident with at least three lines. Note also that the dual $\Delta^D = (\mathcal{L}, \mathcal{P}, I)$ of a generalized n-gon $\Delta = (\mathcal{P}, \mathcal{L}, I)$ is again a generalized n-gon.

If Δ is finite, i.e. if it has finitely many points and lines, then there are constants $s, t \in \mathbb{N}$ such that every line is incident with $1+s$ points and every point is incident with $1+t$ lines. In this case, we say that Δ has order (s, t). If $s = t = 1$, then we have an ordinary n-gon. For all finite generalized polygons restrictions on the parameters (s, t) are known. If $n = 8$ and $s, t \geq 2$, a result of Feit and Higman [2] states that $2st$ is a square and that $s \leq t^2 \leq s^4$. If Δ is an octagon with $t = 1$ and $s \geq 2$, then Δ can be obtained by "doubling" (taking the flag complex of) a generalized quadrangle. We will meet this situation later.

The number $n = 8$ plays a very special role in the theory of generalized n-gons. Sometimes it is included in the "nice" cases, sometimes it is not. This can best be seen in the following short list of known results (some of the notions below will be defined later on):

(FH) *A thick finite generalized n-gon must satisfy* $n \in \{3, 4, 6, 8\}$ (Feit and Higman [2]).

(TW) *A Moufang generalized n-gon must satisfy* $n \in \{3, 4, 6, 8\}$ (Tits [17] and Weiss [24]).

(K) *A compact connected topological generalized n-gon must satisfy* $n \in \{3, 4, 6\}$ (Knarr [6]).

(VM) *A generalized n-gon with valuation must satisfy* $n \in \{3, 4, 6\}$ (Van Maldeghem [19]).

Also, as far as Moufang generalized n-gons are concerned, there are several classes of examples in each of the cases $n = 3, 4, 6$, but there is only one class of Moufang octagons and this class is related to the Ree groups of type ${}^2F_4$ (Tits [18]). For obvious reasons we will call a member of this class a *Ree octagon*, although these octagons are due to Tits [13].

The purpose of this paper is to give an elementary description of the Ree octagons using coordinates. We will then show that this description can be used to solve some specific problems. It should be noted that Tits' paper [18] is crucial for us; our description depends essentially on the commutation relations given there and so we do not provide a new existence proof of the Ree octagons, though the coordinatization may be used to do so *a posteriori*.

Whereas there is an abundance of literature on generalized quadrangles and hexagons, only a few articles can be found on octagons. So an alternative geometric descripion might be helpful. In fact the octagon coordinates are especially convenient for reasoning with several geometrical objects at a far distance of one particular apartment of reference. Our Theorem D may serve as an example. We would like to point out though that the direct use of the commutation relations sometimes provide a much shorter and elegant proof (cp. Proposition 4.3; its proof was suggested to us by the referee, who we hereby would like to thank). On the other hand, theorems like Theorem D below are much harder to prove without coordinates. Other applications of this coordinatization of the Ree octagons can be found in Van Maldeghem [22].

For the convenience of the reader we will now state our main results (without defining all the notions needed; these can be found below).

Theorem A *Every Ree octagon $O(K, \sigma)$ is (x, y)-transitive for all pairs of opposite points (x, y), and it is (L, M)-quasi-transitive for all pairs of opposite lines (L, M) if and only if K is perfect.*

Theorem B *Every thick suboctagon of any Ree octagon $O(K,\sigma)$ arises in a standard way from a subfield $K' \leq K$ closed under σ.*

Theorem C *Let ST be a set of points of the Ree octagon Δ for which there exists a Suzuki subquadrangle Δ' in which ST can be seen as the set of flags corresponding to a Suzuki-Tits ovoid.*
Then there exists a unique point x_{ST} of Δ such that

(i) *x_{ST} is at distance 4 (within the octagon Δ) from each point of ST,*
(ii) *x_{ST} is fixed by the full group of automomorphisms of Δ stabilizing ST,*
(iii) *x_{ST} is the middle element of the root with respect to which there is a (n involutory) root elation in Δ inducing a polarity in Δ' defining the Suzuki-Tits ovoid corresponding to ST.*

The point x_{ST} is uniquely determined by each of the properties (i), (ii) *or* (iii).

Theorem D *There are no generalized quadrangles arising from the Ree octagons by the method described by Löwe* [7].

Theorem E *The groups of projectivities of the Suzuki quadrangles are given by*

$$\begin{aligned}\Pi(W(K,\sigma)) = \Pi_+(W(K,\sigma)) &\cong \mathrm{PSL}_2K \rtimes (K^\sigma)^\times/(K^2)^\times \\ &\cong \Pi_+^D(W(K,\sigma)) = \Pi^D(W(K,\sigma)).\end{aligned}$$

The action is equivalent to the natural action of $\mathrm{PSL}_2K \rtimes (K^\sigma)^\times/(K^2)^\times$ *on the projective line over K.*

Theorem F *The groups of projectivities of the Ree octagons are given by*

$$\Pi(O(K,\sigma)) = \Pi_+(O(K,\sigma)) \cong \mathrm{PSL}_2K \rtimes K^\dagger/(K^2)^\times$$

and

$$\Pi^D(O(K,\sigma)) = \Pi_+^D(O(K,\sigma)) \cong \mathrm{GSz}(K,\sigma).$$

The actions are equivalent to the natural actions of $\mathrm{PSL}_2K \rtimes K^\dagger/(K^2)^\times$ *(resp.* $\mathrm{GSz}(K,\sigma)$*) on the projective line over K (resp. the Suzuki-Tits ovoid).*

2 Preliminaries

We have already defined the notion of a generalized octagon in the introduction. Usually the Ree octagons are defined in the literature via the theory of BN-pairs. We spend a few words on that subject. Also, we will briefly give some generalities about the introduction of coordinates in generalized octagons. Finally, we spend a few words on a class of Moufang quadrangles, introduced by Tits [16], which are closely related to the Ree octagons.

Firstly we introduce some notation.

For any field K the set of elements of K distinct from 0 will be denoted by $K^\times$.

We always denote by $\Delta = (\mathcal{P}, \mathcal{L}, I)$ a generalized n-gon as defined in the introduction.

2.1 *Some further notation*

2.1.1 Paths and roots. As a general rule, we denote by L, M, L_0, etc. elements of L; by x, y, z, etc. elements of P; by v, v_0, v', etc. elements of $\mathcal{P}\cup\mathcal{L}$; by a, b, a', etc. elements of R_1 (see later); by k, k', l, etc. elements of R_2 (see later); and by e, e', etc. elements of $R_1\cup R_2$. The set of elements incident with a given element v will be denoted by $\Delta(v)$. As already stated in the introduction a *path* is a sequence of $d+1$ consecutively incident elements. If the first and the last element of a path coincide, then we have a *closed path*. A path is *non-stammering* if two consecutive points (resp. lines) never coincide. A closed non-stammering path of length $2n$ is called an *apartment*. A path of length 1 is called a *flag*. A non-stammering path of length n is called a *root*. The extremal elements of a root are called *opposite elements*. If two elements v, v' are not opposite, then by axiom (GP2), there is a unique chain $(v, v_0, \ldots, v')$ joining them. We call the element v_0 of that chain the *projection* of v' onto v.

Note that, if n is even, there are two kinds of roots: one kind has two points as extremities (and we call them *p-roots*) and the other one two lines (and we call them *l-roots*).

2.1.2 Root-elations and the Moufang property. A *root elation* with respect to the root $\phi = (v_0, v_1, v_2, \ldots, v_n)$, where $v_i \in \mathcal{P} \cup \mathcal{L}$, is a collineation of Δ fixing every flag containing v_1, $v_2, \ldots$ or v_{n-1}. It is easily seen that the group of all root elations with respect to a fixed root ϕ acts semi-regularly on the set of apartments containing ϕ. If this action is transitive, then we call the root ϕ *Moufang* and the corresponding group a *root group*. If every root of Δ is *Moufang*, then Δ itself is said to be *Moufang*. All Moufang generalized octagons have been determined by Tits [18]: they arise from Ree groups of twisted type $^2F_4$.

2.1.3 BN-pairs of Moufang polygons. Most of the theory of buildings goes along with the theory of BN-pairs, cp. Tits [15], Ronan [9]. The Moufang polygons are no exception. We will not get involved in this theory, nor will we even define the notion of a BN-pair. But, we will use a few very basic facts, that can be easily found in the literature, to give a little insight into the structure of the group that is generated by all root collineations; cp. also Tits [18], Section 2, p. 569.

Assume that Δ is a Moufang n-gon. We fix an apartment $\mathcal{A} = (v_0, v_1, \ldots, v_{2n} = v_0)$ with $v_i \in \mathcal{P} \cup \mathcal{L}$. Then A contains the roots $\phi_i = (v_i, \ldots, v_{i+n})$. Take indices modulo $2n$. The root group corresponding to ϕ_i will be denoted by $U^\Delta_{\phi_i}$. For an octagon we will later introduce a second notation for the root groups that is more convenient once the octagon is coordinatized. Set $G^\Delta = \langle U^\Delta_{\phi_i} \mid 0 \le i < 2n\rangle$ and let $B^\Delta = G^\Delta_{(v_{n-1}, v_n)}$ be the stabilizer of the flag (v_{n-1}, v_n) in the group G^Δ. For any $u \in U^\Delta_{\phi_i} \setminus \{1\}$ there are unique elements $u', u'' \in U^\Delta_{\phi_{i+n}}$ such that $m(u) = u'uu''$ stabilizes the apartment A. Let $u \in U^\Delta_{\phi_i}\setminus\{1\}$. Then $m(u)^{-1}U^\Delta_{\phi_j}m(u) = U^\Delta_{\phi_{2i+n-j}}$. Set $N^\Delta_{\phi_i} = \{m(u) \mid u \in U^\Delta_{\phi_i}\setminus\{1\}\}$, $N^\Delta = \langle N^\Delta_{\phi_i} \mid 0 \le i < 2n\rangle$ and $H^\Delta = B^\Delta \cap N^\Delta$. The pair (B^Δ, N^Δ) is well known to be a BN-pair for G^Δ, cp. Ronan [9], (6.16); especially we have $G^\Delta = \langle B^\Delta, N^\Delta\rangle$ and $B^\Delta = \langle U^\Delta_{\phi_0}, \ldots, U^\Delta_{\phi_{n-1}}\rangle H^\Delta$. Finally, $H^\Delta = \bigcap_i \mathcal{N}(U^\Delta_{\phi_i})$, where $\mathcal{N}(\cdot)$ denotes the normalizer in G^Δ, cp. Tits [18], 2.8. The action of H^Δ on the elements incident with an element of A is equivalent to the action of H^Δ on the root groups $U^\Delta_{\phi_i}$ by conjugation.

The structure of G^Δ does not depend on the choice of the apartment A, because G^Δ acts transitively on the set of all ordered apartments of Δ. This means that G^Δ is the group that is generated by *all* root elations.

2.1.4 Generalized homologies. Let $v, v' \in \mathcal{P} \cup \mathcal{L}$ be two opposite elements of the generalized polygon Δ. A *generalized homology* of Δ with *centers* v and v' is a collineation fixing all elements incident with v or v'. Let v^* be an arbitrary element incident with v. Then Δ is called (v, v')-*transitive* if the group of generalized homologies with centers v and v' acts transitively on the set of elements incident with v^*, different from v, and different from the projection of v' on v^*. This definition is independent from the element v^* on v or v'. Now let $v^{**} \neq v$ be the projection of v' on v^*. Then Δ is called (v, v')-*quasi-transitive* if the group of all generalized homologies of Δ with centers v, v' acts transitively on the set of elements incident with v^{**}, different from v^*, and different from the projection of v' on v^{**}. In Van Maldeghem [20], it is shown that a thick finite generalized octagon is Moufang and has order (q, q^2) if and only if it is (x, y)-transitive for every pair of opposite points (x, y), and (L, M)-quasi-transitive for every pair (L, M) of opposite lines. There is no obvious relationship between (v, v')-transitivity and (v, v')-quasi-transitivity; especially (v, v')-transitivity does not imply (v, v')-quasi-transitivity.

2.1.5 Subpolygons. A *subpolygon* $\Delta' = (\mathcal{P}', \mathcal{L}', I')$ of Δ is a generalized polygon, where $\mathcal{P}' \subset \mathcal{P}$, $\mathcal{L}' \subset \mathcal{L}$ and I' is the restriction of I to $\mathcal{P}' \times \mathcal{L}'$. A subpolygon is called *full*, or *ideal*, if for some $v \in \mathcal{P}' \cup \mathcal{L}'$ we have $\Delta(v) = \Delta'(v)$. In Van Maldeghem and Weiss [23] it is noted that it follows from a result of Thas [11] that no finite thick generalized octagon contains a full thick suboctagon.

We will also need the following well known result; a proof is indicated for the sake of completeness.

Proposition 2.1 *Every subpolygon of a Moufang polygon is itself Moufang.*

Proof: Let Δ' be a subpolygon of Δ and let A be an apartment in Δ'. Let ϕ be a root contained in A and let v, v' be its extremities; thus v is opposite to v'. Let B be an apartment of Δ' containing ϕ. Let u be the unique root elation with respect to ϕ mapping A to B (in Δ). Then $\Delta' \cap \Delta'^u$ contains B and all elements of $\Delta'(w)$, where w is an element of ϕ different from v and v'. It easily follows that $\Delta' \cap \Delta'^u$ is a subpolygon which must clearly coincide with Δ' (if Δ' is thick, this is almost immediate; if Δ' is not thick, then one should consider the corresponding generalized $\frac{n}{2}$-gon, where Δ' is a generalized n-gon, see Tits [12]). So u is a root elation in Δ'. The result follows. □

2.1.6 Projectivities. Assume that Δ is thick.

Choose a pair of opposite elements $v, w \in \mathcal{P} \cup \mathcal{L}$. For any $x \in \Delta(v)$ there is a unique y in $\Delta(w)$ which is nearest to x in the incidence graph of Δ. The element y is the projection of x onto w. Considering all elements incident with v we obtain a map

$$[v, w]\colon \Delta(v) \to \Delta(w),$$

the *perspectivity* from v to w. For a sequence of elements $v_1, \ldots, v_m \in \mathcal{P} \cup \mathcal{L}$, where v_i is opposite to v_{i+1}, the product

$$[v_1, \ldots, v_m] = [v_1, v_2] \cdots [v_{m-1}, v_m]$$

is called a *projectivity* from v_1 to v_m.

Fixing an element $v \in P \cup \mathcal{L}$ the set of all projectivities from v back to itself forms a group $\Pi(v)$ with composition as multiplication, the *group of projectivities* of v. The restriction to those projectivities which can be written as a product of an even number of perspectivities leads to a subgroup $\Pi_+(v)$, the group of *even projectivities*. It has index at most 2 in $\Pi(v)$.

Now take two lines $L, M \in \mathcal{L}$. Because of thickness there is some projectivity ρ from L to M, and the groups of projectivities of L and M are related by

$$\Pi(L) = \rho\Pi(M)\rho^{-1} \text{ and } \Pi_+(L) = \rho\Pi_+(M)\rho^{-1}.$$

This implies that the isomorphism types of $\Pi(L)$ and $\Pi_+(L)$ together with their actions on $\Delta(L)$ are independent of the choice of L. They are an invariant of Δ. Therefore we are allowed to refer to the group of (even) projectivities of an arbitrary line as *the* group of (even) projectivities of Δ; it will be denoted by $\Pi(\Delta)$ (resp. $\Pi_+(\Delta)$).

We will use the notation $\Pi^D(\Delta)$ (resp. $\Pi_+^D(\Delta)$) for the group of (even) projectivities of the dual Δ^D of Δ. When we have to deal with the projectivities of Δ^D we will often view a line of Δ^D as a point of Δ.

For n odd the groups $\Pi(\Delta)$, $\Pi_+(\Delta)$, $\Pi^D(\Delta)$, $\Pi_+^D(\Delta)$ coincide.

There are two results on the structure of the groups of projectivities of generalized polygons, both due to Knarr [5], (1.2) and (2.3), which are of crucial importance to this subject.

Proposition 2.2 *The group of even projectivities $\Pi_+(L)$ operates 2-transitively on the set of points on the line L.*

Proposition 2.3 *Let Δ be a Moufang polygon.*

Then the stabilizer of $v \in \mathcal{P} \cup \mathcal{L}$ in the group G^Δ that is generated by all root collineations of Δ induces the group $\Pi_+(v)$ on $\Delta(v)$.

Especially by the use of the latter proposition Knarr [5] determined the groups of projectivities of all finite Moufang polygons.

In the following we make use of the notation concerning BN-pairs that has been introduced in Section 2.1.3.

Corollary 2.4 *Let Δ be a Moufang polygon.*

Then the group $\langle U_{\phi_0}^\Delta, U_{\phi_n}^\Delta \rangle H^\Delta$ induces the group of even projectivities $\Pi_+(v_n)$ on $\Delta(v_n)$.

Proof: Of course, the group $\langle U_{\phi_0}^\Delta, U_{\phi_n}^\Delta \rangle$ fixes v_n and acts transitively on $\Delta(v_n)$; in fact it even acts 2-transitively. By (2.3), the assertion follows, if we can show that the stabilizer of the flag (v_{n-1}, v_n) in the group $\langle U_{\phi_0}^\Delta, U_{\phi_n}^\Delta \rangle H^\Delta$ induces the same group on $\Delta(v_n)$ as $G_{(v_{n-1},v_n)}^\Delta = B^\Delta = \langle U_{\phi_0}^\Delta, \ldots, U_{\phi_{n-1}}^\Delta \rangle H^\Delta$. But any element of $\Delta(v_n)$ is fixed by $U_{\phi_1}^\Delta, \ldots, U_{\phi_{n-1}}^\Delta$. □

Remark The intersection of $\langle U_{\phi_0}^\Delta, U_{\phi_n}^\Delta \rangle$ and H^Δ is exactly the stabilizer of v_{n-1} and v_{n+1} inside $\langle U_{\phi_0}^\Delta, U_{\phi_n}^\Delta \rangle$.

2.2 *Coordinatization of generalized octagons*

We will use the coordinatization method developed by Hanssens and Van Maldeghem [3] (for generalized quadrangles), generalized to polygons as in Van Maldeghem [20]. We will present here the weakest form of coordinatization. This means that we will not bother about *normalization*. See the remark at the end of this section.

So let $\Delta = (\mathcal{P}, \mathcal{L}, I)$ be a generalized octagon and let

$$\mathcal{A} = (x_0, L_0, x_1, L_2, x_3, L_4, x_5, L_6, x_7, L_7, x_6, L_5, x_4, L_3, x_2, L_1, x_0)$$

be a fixed ordered apartment with $x_i \in \mathcal{P}$, $L_i \in \mathcal{L}$. We choose sets R_1 and R_2 such that $R_1 \cap R_2 = \{0\}$, together with bijections

$$\begin{aligned} \pi_i: R_1 &\to \Delta(L_i)\backslash\{x_{i-1}\}, i \in \{1, 2, \ldots, 7\}, \\ \pi_0: R_1 &\to \Delta(L_0)\backslash\{x_0\}, \\ \lambda_i: R_2 &\to \Delta(x_i)\backslash\{L_{i-1}\}, i \in \{1, 2, \ldots, 7\}, \\ \lambda_0: R_2 &\to \Delta(x_0)\backslash\{L_0\}, \end{aligned}$$

satisfying

(Co1) $\pi_i(0) = x_{i+1}$, $i \in \{0, 1, \ldots, 6\}$, $\pi_7(0) = x_7$, $\lambda_i(0) = L_{i+1}$, $i \in \{0, 1, \ldots, 6\}$ and $\lambda_7(0) = L_7$,

(Co2) the projection of $\pi_i(a)$ onto L_{7-i} is exactly $\pi_{7-i}(a)$ and the projection of $\lambda_i(k)$ onto x_{7-i} is exactly $\lambda_{7-i}(k)$, $i \in \{0, 1, \ldots, 7\}$, $a \in R_1$ and $k \in R_2$.

If $e \in R_1 \cup R_2$, then we abbreviate $\underbrace{e, e, \ldots, e}_{i}$ by e^i. As a general rule, we will write coordinates of points in *round* parentheses and those of lines in *square* brackets. We label the point x_0 (resp. line L_0) by (∞) (resp. $[\infty]$), we label the point $\pi_i(a)$, $a \in R_1$ (resp. the line $\lambda_i(k)$, $k \in R_2$), $i \in \{0, 1, \ldots 6\}$, by $(0^i, a)$ (resp. $[0^i, k]$), and we label the point $\pi_7(a)$ by $(a, 0^6)$ (resp. the line $\lambda_7(k)$ by $[k, 0^6]$). Consider now a point x opposite to $x_0 = (\infty)$. Let $(x, M_1, y_1, M_2, y_2, M_3, y_3, [\infty])$ be a path connecting x with $[\infty]$. Suppose the point y_3 has been labelled (a), $a \in R_1$. Let $[0^6, l]$, $(0^5, a')$, $[0^4, l']$, $(0^3, a'')$, $[0^2, l'']$, $(0, a''')$ be the labels of the respective projections of $M_3, y_2, M_2, y_1, M_1, x$ on respectively x_6, L_5, x_4, L_3, x_2, L_1, then we label these elements as follows:

$$\begin{aligned} M_3 &\mapsto [a, l] \\ y_2 &\mapsto (a, l, a') \\ M_2 &\mapsto [a, l, a', l'] \\ y_1 &\mapsto (a, l, a', l', a'') \\ M_1 &\mapsto [a, l, a', l', a'', l''] \\ x &\mapsto (a, l, a', l', a'', l'', a'''). \end{aligned}$$

If we also apply the dual labelling, then we have labelled all points and lines of Δ in a unique way. We call this labelling a *coordinatization* of Δ. An element is said to have i coordinates, $i \in \{0, 1, \ldots, 7\}$ if it is labelled by an i-tuple (if $i \neq 0$), or if it is labelled (∞) or $[\infty]$ (if $i = 0$). A label of an element is also called *the coordinates* of that element. We will frequently identify an element with its coordinates without further notice.

Now we will be able to recover the generalized octagon Δ from the coordinatization only if we know the relations between the coordinates of incident elements. Obviously a point x with i coordinates, $i \in \{0, 1 \ldots, 7\}$, is incident with a line L with j coordinates, $j \in \{0, 1, \ldots, 7\}$, $(i, j) \neq (7, 7)$, if and only if one of the following holds

1. $x = (\infty)$ and $L = [k]$, $k \in R_2 \cup \{\infty\}$.
2. $L = [\infty]$ and $x = (a)$, $a \in R_1 \cup \{\infty\}$.
3. $x = (e_1, \ldots, e_i)$ and $L = [e_1, \ldots, e_i, e_{i+1}]$.
4. $L = [e_1, \ldots, e_j]$ and $x = (e_1, \ldots, e_j, e_{j+1})$.

Consider arbitrary elements $a, b, b', b'' \in R_1$ and $k, k', k'', k''' \in R_2$. We define "octanary" operations O_i, $i \in \{1, \ldots, 6\}$ as follows: let x be the point with coordinates $(a, l, a', l', a'', l'', a''')$ and let the projection L of $[k]$ on x have coordinates $[k, b, k', b', k'', b'', k''']$. Then

$$\begin{aligned} O_1(k, a, l, a', l', a'', l'', a''') &= k''' \\ O_2(k, a, l, a', l', a'', l'', a''') &= b'' \\ &\vdots \\ O_6(k, a, l, a', l', a'', l'', a''') &= b. \end{aligned}$$

Consequently, these equalities are satisfied if and only if x is incident with L. It is an elementary but tedious exercise to show that the sets R_1 and R_2 together with the operations $O_1, \ldots, O_6$, which we call an *octagonal octanary ring*, allow one to reconstruct in a unique way the generalized octagon Δ.

Remark The coordinatization of the points on the respective lines L_0, L_2, L_4 and L_6 has been done independently from each other. Hence in general, one cannot expect to get a unique octagonal octanary ring from this process. There are however ways to relate the coordinates on the lines mentioned. Then one assumes to have an element 1 in both R_1 and R_2 and one could require that the projection of the point (a), $a \in R_1$, on the line $[1, 0^6]$ coincides with the projection of $(0^6, a)$ on the same line (this relates the coordinates on L_0 to the coordinates on L_1, or equivalently, on L_6). If also the dual holds, then we say that the octagonal octanary ring is *normalized*. Further normalization can be obtained similarly by relating the coordinates of L_0 to those on L_2 and L_4. But as we do not want to establish a theory on coordinatization, and since we are only interested in one particular example, we do not consider this here.

2.3 *The Suzuki quadrangles*

The structure of the Ree octagons is intimately related to a class of exceptional Moufang quadrangles described in Tits [16]; see also Tits [15], Chapter 10. They arise as subquadrangles of the symplectic quadrangles over certain fields of characteristic 2.

Let K be a field of characteristic 2. Select a base e_0, e_1, e_2, e_3 of the vector space K^4. Then the set of totally isotropic one- or two-dimensional subspaces, with respect to the bilinear form β on K^4 defined by the matrix

$$\begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix},$$

with natural inclusion as incidence is a generalized quadrangle, the so-called *symplectic quadrangle* $W(K)$. Note that this symmetric bilinear form is also an alternating form because of char $K = 2$.

Set $v_{2i} = \langle e_i \rangle$ and $v_{2i+1} = \langle e_i, e_{i+1} \rangle$. Then $\mathcal{B}_0 = (v_0, v_1, \ldots, v_7, v_0)$ is an apartment of $W(K)$. Let $\phi_i = (v_i, \ldots, v_{i+4})$. The corresponding root groups are given by $U_{\phi_i}^{W(K)} = \{u_i(t) \mid t \in K\}$, where

$$u_0(t) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ t & & 1 & \\ & & & 1 \end{pmatrix}, u_1(t) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & t & 1 & \\ t & & & 1 \end{pmatrix},$$

$$u_2(t) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & t & & 1 \end{pmatrix}, u_3(t) = \begin{pmatrix} 1 & t & & \\ & 1 & & \\ & & 1 & \\ & & t & 1 \end{pmatrix},$$

and $u_{i+4}(t) = u_i(t)^{\mathrm{tr}}$. Altogether they generate the symplectic group $\mathrm{Sp}_4 K$. Note that, again because of char $K = 2$, this group is simple for $|K| > 2$, cp. Dieudonné [1], Section 5, p. 49.

Let K' be a subfield of K which contains the subfield of squares K^2. For any K'-vectorspace E which is contained in K and any K^2-vectorspace E' which is contained in K', the group which is generated by the respective restrictions of the root groups $u_{2i}(E')$ and $u_{2i+1}(E)$ fixes a subquadrangle $Q(K, K'; E, E')$. This can be derived immediately from the commutation relations for the root groups of the symplectic quadrangle. In Tits [16] the quadrangles $Q(K, K'; E, E')$ are characterized as the only Moufang quadrangles which have regular points and regular lines at the same time; cp. Payne and Thas [8], (1.3), for the definition of regularity.

According to Hanssens and Van Maldeghem [4], (1.5.1), a coordinatizing $*$-normalized quadratic quaternary ring for $Q(K, K'; E, E')$ is given by

$$Q_1^*(a, k, b, k') = ka + b \text{ and } Q_2^*(k, a, l, a') = a^2 k + l,$$

where $a, a', b \in E$ and $k, k', l \in E'$. Here we take the addition and the multiplication of the field K.

Here we are interested in a subclass of these quadrangles. From now on we always assume that K has a field endomorphism σ whose square is the Frobenius endomorphism $x \mapsto x^2$. Then we refer to the quadrangle $Q(K, K^\sigma; K, K^\sigma)$ as the *Suzuki quadrangle* $W(K, \sigma)$. The Suzuki quadrangles are self-dual. This can be deduced from the structure of its quadratic quaternary ring (see above). In fact, the Suzuki quadrangles are even self-polar, as it is proved in section 5.1, cp. also Tits [16]. If K is perfect, i.e. if the Frobenius endomorphism is surjective, we have $K^\sigma = K$ and $W(K, \sigma) = W(K)$. Note that this is always the case if K is finite.

If K is a finite field, then such an endomorphism σ exists if and only if $|K| = 2^{2r+1}$; moreover, in this case σ is uniquely determined, cp. [14], Section 4. Therefore σ will be omitted occasionally.

2.4 *The Suzuki-Tits ovoids and the Suzuki groups*

We use the same notation as in the previous section. We also keep the bilinear and alternating form β on the vector space K^4. The symplectic groups are meant to preserve β. Consider the set

$$\mathcal{O} = \{\langle e_0 + xe_1 + ye_2 + (xy + x^{\sigma+2} + y^\sigma)e_3\rangle \mid x, y \in K\} \cup \{\langle e_1\rangle\}$$

of points in the projective space $\mathrm{PG}_3 K$. This is an ovoid in $\mathrm{PG}_3 K$ as well as in the symplectic quadrangle $W(K)$; for a definition of the notion *ovoid* inside a generalized quadrangle see Payne and Thas [8], (1.8). Ovoids of this type are called *Suzuki-Tits ovoids*.

Define $\mathrm{GSz}(K, \sigma) = \{\gamma \in \mathrm{PGSp}_4 K \mid \mathcal{O}^\gamma = \mathcal{O}\}$ and $\mathrm{Sz}(K, \sigma) = \mathrm{GSz}(K, \sigma) \cap \mathrm{PSp}_4 K$, where $\mathrm{PGSp}_4 K$ is the group of symplectic similarities. The groups $\mathrm{Sz}(K, \sigma)$ are called *Suzuki groups*.

We give some properties of $\mathrm{GSz}(K, \sigma)$ and $\mathrm{Sz}(K, \sigma)$ which are due to Tits [13], [14]. The groups $\mathrm{Sz}(K, \sigma)$ and $\mathrm{GSz}(K, \sigma)$ operate 2-transitively on the Suzuki-Tits ovoid O, cp. [13], Theorem 4, and [14], Theorem 10.1.

Now assume that $|K| > 2$. Then $\mathrm{Sz}(K, \sigma)$ is simple; it is the commutator subgroup of $\mathrm{GSz}(K, \sigma)$, cp. [13], Theorem 8, and [14], Theorem 6.12. The stabilizer H of $\langle e_0\rangle$, $\langle e_1\rangle \in \mathcal{O}$ in $\mathrm{GSz}(K, \sigma)$ is the group

$$\{(x, y) \mapsto (bx, b^\sigma y) \mid b \in K^\times\},$$

where (x, y) is identified with $\langle e_0 + xe_1 + ye_2 + (xy + x^{\sigma+2} + y^\sigma)e_3\rangle \in \mathcal{O}$, cp. [13], Lem. 1. The intersection of H and $\mathrm{Sz}(K, \sigma)$ is given by

$$\{(x, y) \mapsto (bx, b^\sigma y) \mid b \in (K^\sigma)^\times\}.$$

The group Sz(2) is isomorphic as a permutation group to $\mathrm{AGL}_1(5)$ in its natural action, cp. [14], Thm. 6.12. Of course, the stabilizer of two points in the group $\mathrm{AGL}_1(5)$ is trivial.

3 Coordinatization of the Ree octagons

The Ree octagons arise naturally from the Ree groups of type ${}^2F_4$, see Tits [18]. For each field K of characteristic 2 admitting an endomorphism σ whose square is the Frobenius endomorphism $x \mapsto x^2$, there exists such a group ${}^2F_4(K, \sigma)$ and hence such a generalized octagon $O(K, \sigma)$. Our goal is now to describe these Ree octagons only using the field K and the endomorphism σ. Therefore we coordinatize $O(K, \sigma)$ using the commutation relations in Tits [18] as follows. We choose an apartment $\mathcal{A}_0$ (keeping this notation for the rest of the paper) in $O(K, \sigma)$ and label its elements (∞), $[\infty]$, (0), etc., as in Section 2.2. For each root ϕ in $\mathcal{A}_0$, there is a root group U_ϕ. According to Tits [18], half of these root groups, say those corresponding with l-roots, are parametrized by the elements of the field K (and the root group is in fact isomorphic to $(K, +)$); the other ones are parametrized by the pairs $(k_0, k_1) \in K \times K$ with operation law $(k_0, k_1) \oplus (l_0, l_1) = (k_0 + l_0, k_1 + l_1 + l_0 k_0^\sigma)$. Following Tits [18] we denote this group by $K_\sigma^{(2)}$. It is isomorphic to a regular normal subgroup of the point stabilizer in the Suzuki group $\mathrm{Sz}(K, \sigma)$.

Every root in $\mathcal{A}_0$ is determined by its middle element v; we denote the root by ϕ_v and the corresponding root group by U_v. The definition of the root groups in Section 2.1.3 is related to the new definition by e.g. $U_{(\infty)} = U_{\phi_{12}}^{O(K,\sigma)}$, $U_{[\infty]} = U_{\phi_{13}}^{O(K,\sigma)}$. The image of the point (0) (resp. line [0]) under the action of an element of the root group $U_{[0^3]}$ (resp. $U_{(0^3)}$) parametrized by $a \in K$ (resp. $(k_0, k_1) \in K_\sigma^{(2)}$) is given the coordinate (a) (resp. $[(k_0, k_1)]$). We will, however, abbreviate (0,0) by 0, to be consistent. The element of $U_{[0^3]}$ (resp. $U_{(0^3)}$)

mapping (0) to (a) (resp. [0] to $[(k_0, k_1)]$) will be called $u_{[0^3]}(a)$ (resp. $u_{(0^3)}(k_0, k_1)$). We also write $m_{[0^3]}(a) = m(u_{[0^3]}(a))$ (resp. $m_{(0^3)}(a) = m(u_{(0^3)}(k_0, k_1))$). In the same fashion, we give the coordinates $(0^2, a')$ (resp. $(0^4, a'')$, $(0^6, a''')$) to the image of the point (0^3) (resp. (0^5), (0^7)) under the action of an element of the root group $U_{[0]}$ (resp. $U_{[\infty]}$, $U_{[0^2]}$) parametrized by a' (resp. a'', a'''). Dually, we coordinatize lines. Applying rule (Co2), we obtain coordinates for every element incident with an element of $\mathcal{A}_0$. According to Section 2.2, this is enough to have coordinates for every element of $O(K, \sigma)$.

In order to have a complete description of $O(K, \sigma)$, we must find the octagonal octanary ring. So we have to find a necessary and sufficient condition for a point with seven coordinates and a line with seven coordinates to be incident. Let $x = (a, l, a', l', a'', l'', a''')$ be such a point and $L = [k, b, k', b', k'', b'', k''']$ be such a line, where $l = (l_0, l_1)$, $l' = (l'_0, l'_1), \ldots, k = (k_0, k_1)$, etc. Only in this section we further abbreviate $u_a = u_{[0^3]}(a)$, $u_l = u_{(0^2)}(l)$, $u_{a'} = u_{[0]}(a')$, etc. Similarly we define (in a dual way) the root elations u_k, u_b, $u_{k'}$, etc. Put

$$u_x = u_{a'''} u_{l''} u_{a''} u_{l'} u_{a'} u_l u_a \tag{1}$$

and

$$u_L = u_{k'''} u_{b''} u_{k''} u_{b'} u_{k'} u_b u_k. \tag{2}$$

Then it is clear that $x = (0^7)^{u_x}$ and $L = [0^7]^{u_L}$. Also remark that the automorphism u_x does not change the first coordinate of any line. This implies that $x\ I\ L$ if and only if $(0^7)\ I\ L^{u_x^{-1}}$ (the latter must then be $[k, 0^6]$ by the previous remark, hence:) if and only if $L^{u_x^{-1}} = [0^7]^{u_k}$ if and only if $[0^7]^{u_L u_x^{-1} u_k^{-1}} = [0^7]$. Since $[0^7]^{u_a} = [0^7]$, the latter is equivalent with $[0^7]^{u_a u_L u_x^{-1} u_k^{-1}} = [0^7]$. Dually, $x\ I\ L$ if and only if $(0^7)^{u_k u_x u_L^{-1} u_a^{-1}} = (0^7)$, which can be rewritten as $(0^7)^{u_a u_L u_x^{-1} u_k^{-1}} = (0^7)$. Since the group generated by all root elations fixing the flag $\mathcal{F} = ((\infty), [\infty])$ acts regularly on the set of flags opposite $\mathcal{F}$ (opposite flags are flags whose respective points and lines are opposite) (see Tits [18], condition (P2)), we have that $x\ I\ L$ if and only if $u_a u_L = u_k u_x$. Using the commutation relations given in *loc. cit.*, one can actually compute this condition and the result is the following (the computations take approximately five handwritten pages; we have checked afterwards our result with a computer and found no mistakes).

For $k = (k_0, k_1)$, set $tr(k) = k_0^{\sigma+1} + k_1$ (the *trace* of k) and set $N(k) = k_0^{\sigma+2} + k_0 k_1 + k_1^\sigma$ (the *norm* of k); let $K^\dagger$ be the subgroup of the multiplicative group $K^\times$ which is generated by all norm elements. Define a multiplication $a \otimes k = a \otimes (k_0, k_1) = (ak_0, a^{\sigma+1} k_1)$ for $a \in K$ and $k \in K_\sigma^{(2)}$. Also write $(k_0, k_1)^\sigma$ for (k_0^σ, k_1^σ). Then the point $(a, l, a', l', a'', l'', a''')$ is incident with the line $[k, b, k', b', k'', b'', k''']$ if and only if the following six equations hold:

$$(k_0''', k_1''') = (l_0, l_1) \oplus a \otimes (k_0, k_1) \oplus (0, al'_0 + a^\sigma l''_0) \tag{3}$$

$$\begin{aligned} b'' &= a' + a^{\sigma+1} N(k) + k_0(al'_0 + a^\sigma l''_0 + \mathrm{tr}(l)) \\ &\quad + a^\sigma (a''' + l_0 k_1) + a l_0''^{\sigma} + l_0 l'_0 \end{aligned} \tag{4}$$

$$\begin{aligned} (k''_0, k''_1) &= a^\sigma \otimes (k_1, \mathrm{tr}(k) N(k)) \oplus k_0 \otimes (l_0, l_1)^\sigma \oplus (0, \mathrm{tr}(k) N(l) \\ &\quad + a^{\sigma+1} l_0 N(k)^\sigma + \mathrm{tr}(k)(aa' + a^\sigma l_0 l''_0 + a^{\sigma+1} a''') \\ &\quad + \mathrm{tr}(l)(k_1^\sigma a + a''') + k_1^\sigma a^{\sigma+1} l''_0 + k_0^{\sigma+1} a^2 l_0''^{\sigma} \\ &\quad + k_0(a' + a l_0''^{\sigma} + k_1 a^\sigma l_0 + a^\sigma a''')^\sigma \\ &\quad + k_0^\sigma l_0 (a' + a l_0''^{\sigma} + k_1 a^\sigma l_0 + a^\sigma a''') + a(l''_1 + a'''^{\sigma} l_0 + a''' l'_0) \\ &\quad + l''_0(a' + a^\sigma a''') + a'' l_0 + l_0 l'_0 l''_0) \oplus (l'_0, l'_1) \end{aligned} \tag{5}$$

$$b' = a'' + a^{\sigma+1}N(k)^\sigma + a(k_0 l_0'' + l_0 k_1 + a''')^\sigma + \mathrm{tr}(k)(l_1 + a^\sigma l_0'') + k_0^\sigma(a' + a^\sigma a''') + l_0' l_0'' + l_0^\sigma a''' \tag{6}$$

$$(k_0', k_1') = (l_0'', l_1'') \oplus a \otimes (\mathrm{tr}(k), k_0 N(k)^\sigma) \oplus l_0 \otimes (k_0, k_1)^\sigma \oplus (0, N(k)(a^\sigma l_0'' + l_1) + k_0(a'' + l_0' l_0'' + a a'''^\sigma + l_0^\sigma a''') + k_1(k_1 l_0 a^\sigma + a' + a l_0''^\sigma + a^\sigma a''') + k_0 k_1^\sigma a l_0^\sigma + a'''^\sigma l_0 + a''' l_0') \tag{7}$$

$$b = a''' + aN(k) + l_0 k_1 + k_0 l_0'' \tag{8}$$

Note that these equalities define respectively the operations $O_1, O_2, \ldots, O_6$. Also remark that this octagonal octanary ring is normalized. There is also a set of *dual* operations, a kind of *inverse formulae*, giving as output the $l, a', l', \ldots, a'''$ when $k, b, \ldots k'''$ and a is given as input. But we will not need them here explicitly.

4 Some immediate applications

4.1 Generalized homologies

In Van Maldeghem [21], it is stated that the finite Moufang octagons possess "a lot of" generalized homologies. We can here extend this result to all Moufang octagons $O(K, \sigma)$, giving the explicit form of these automorphisms in terms of the coordinates. Indeed, it is an elementary exercise to check that the following map $\eta(A, B)$ preserves the incidence relation (made explicit by the octanary operations above):

$$\begin{aligned} &(a, l, a', l', a'', l'', a''') \\ &\mapsto (Aa, (AB) \otimes l, A^{\sigma+1}B^{\sigma+2}a', (A^\sigma B^{\sigma+1}) \otimes l', A^{\sigma+1}B^{2\sigma+2}a'', (AB^{\sigma+1}) \otimes l'', \\ &\quad AB^{\sigma+2}a'''), [k, b, k', b', k'', b'', k'''] \\ &\mapsto [B \otimes k, AB^{\sigma+2}b, (AB^{\sigma+1}) \otimes k', A^{\sigma+1}B^{2\sigma+2}b', (A^\sigma B^{\sigma+1}) \otimes k'', A^{\sigma+1}B^{\sigma+2}b'', \\ &\quad (AB) \otimes k'''], \end{aligned}$$

where $A, B \in K^\times$ and where the action on elements with less than seven coordinates is defined by "restricting" the above action to the appropriate number of coordinates. Putting $B = 1$, one sees that the corresponding group $\mathcal{H}(*, 1)$ of automorphisms $\eta(A, 1)$ fixes every line through on (∞) and acts transitively (even regularly) on the set of points incident with $[\infty]$, different from (∞) and (0). Now we remark that the map $a \mapsto a^{\sigma+2}$ is always injective; moreover it is bijective if K is perfect. Indeed, suppose $a^{\sigma+2} = b^{\sigma+2}$, then $c^{\sigma+2} = 1$, for $c = a/b$. Applying σ to both sides, we obtain $c^{2\sigma+2} = 1 = c^{\sigma+2}$, implying $c^\sigma = 1$, hence $c = 1$. If K is perfect, then $b = a^{1-\sigma^{-1}}$ satisfies $b^{\sigma+2} = a$, proving the remark. Now put $A = 1$ in the above formulae, then we see that the corresponding group $\mathcal{H}(1, *)$ of automorphisms of type $\eta(1, B)$ fixes every point on $[\infty]$ and acts semi-regularly on the points of $[0]$ different from (∞) and $(0, 0)$; this action is regular if K is perfect. Hence we have shown:

Theorem A *Every Ree octagon $O(K, \sigma)$ is (x, y)-transitive for all pairs of opposite points (x, y), and it is (L, M)-quasi-transitive for all pairs of opposite lines (L, M) if and only if K is perfect.*

Proof: The only thing left is to show that $O(K,\sigma)$ is not (x,y)-quasi-transitive if K is not perfect. This follows directly from Corollary 4.4 below, proven independently. □

We will show in the next subsection that all generalized homologies are conjugate to the ones just described by an element of $G^{O(K,\sigma)}$.

The two following results give some information about the relationship between generalized homologies and root elations of the Ree octagons. We will need this information later to determine the groups of projectivities. In fact, (4.2) is a reformulation of 1.10 (vii) in Tits [18]. Remark that (4.4) which is used in the proof of (4.2) is proved independently.

Lemma 4.1 *For $u_1, u_2 \in U_{\phi_i}^{O(K,\sigma)} \backslash \{1\}$ the collineation $m(u_1)m(u_2)$ is a generalized homology with centers v_{i+4}, v_{i-4}.*

Proof: Omit all upper indices $O(K,\sigma)$. Clearly $m(u_1)m(u_2) \in H = \bigcap_j \mathcal{N}(U_{\phi_j})$. There are unique $u_1', u_1'', u_2', u_2'' \in U_{\phi_{i+n}}$ with $m(u_k) = u_k' u_k u_k''$. From the commutation relations of Tits [18], 1.7.1 (1), we infer that $[U_{\phi_j}, U_{\phi_{j+4}}] = 1$ for arbitrary j. This implies that $m(u_k)$ centralizes $U_{\phi_{i+4}}$ and $U_{\phi_{i-4}}$. Therefore the claim. □

Proposition 4.2 $H^{O(K,\sigma)} = \langle \mathcal{H}(K^\dagger, 1), \mathcal{H}(1, K^\times) \rangle$.

Proof: Again omit all upper indices $O(K,\sigma)$. Following the proof of [18], 1.10 (vii), we have $H = \langle m_{(\infty)} m'_{(\infty)}, m_{[\infty]} m'_{[\infty]} \mid m_v, m'_v \in N_v \rangle$. From [18], 1.8.1 (12), we know that (see also Sarli [10], Table II, p. 6)

$$[k]^{m_{[\infty]}(a) m_{[\infty]}(b)} = [(a^{\sigma-1} b^{1-\sigma}) \otimes k],$$

$a, b \in K^\times, k \in K_\sigma^{(2)}$. By (4.1) the collineation $m_{[\infty]}(a) m_{[\infty]}(b)$ is a generalized homology with centers $[\infty], [0^7]$. We infer $m_{[\infty]}(a) m_{[\infty]}(b) = \eta(1, a^{\sigma-1} b^{1-\sigma})$ by (4.4).

To prove the dual it is necessary to embed the field K in its perfect closure; cp. [18], 1.8.2 (b). By virtue of [18], 1.8.1 (10), the result

$$(a)^{m_{(\infty)}(k) m_{(\infty)}(l)} = (N(k)^{1-\sigma} N(l)^{\sigma-1} a),$$

$a \in K$, $k, l \in K_\sigma^{(2)} \backslash \{(0,0)\}$, can be read off again from [18], 1.8.1 (12). We have $m_{(\infty)}(k) m_{(\infty)}(l) = \eta(N(k)^{1-\sigma} N(l)^{\sigma-1}, 1)$ by (4.4).

Now the claim follows because $\sigma - 1$ and $1 - \sigma$ are bijections from $K^\times$ onto $K^\times$ where $(K^\dagger)^{\sigma-1} = (K^\dagger)^{1-\sigma} = K^\dagger$. □

4.2 Suboctagons

Let $\Delta = O(K,\sigma)$ be a Moufang octagon coordinatized as in Section 2.2. Let K' be a subfield of K such that $K'^\sigma \subseteq K'$. Then by restricting the coordinates to K' and $K'^{(2)}_\sigma$ respectively, one obtains in a natural and standard way a suboctagon $\Delta' \cong O(K', \sigma)$ of Δ. This suboctagon Δ', or any isomorphic image under an element $\eta(A, B)$ of Δ', will be called *a standard suboctagon.* The main result of this section is that no other thick suboctagons containing $\mathcal{A}_0$ exist.

Proposition 4.3 *No Ree octagon possesses a proper thick full suboctagon.*

Proof: We use the commutation relations (1.7.1) of Tits [18]. So let Δ' be a full suboctagon of $\Delta = O(K, \sigma)$. First suppose that $\Delta(L) = \Delta'(L)$ for every line L of Δ' and suppose that Δ' is thick. We may assume that $\mathcal{A}_0$ is in Δ'. By proposition 2.1 and its proof, we have that $U_L \leq G'$ for every line L of $\mathcal{A}_0$ (where G' is the collineation group of Δ'). By the thickness assumption, $U_x \cap G' \neq 1$, for all points x of $\mathcal{A}_0$. By the relations (2) and (3) of (1.7.1) in [18], $U_x' \leq G'$ (where the former is the subgroup of U_x with trivial first component when U_x is identified with $K_\sigma^{(2)}$). By (6) of (1.7.1) in [18], $(U_x'')^{-1} \subseteq G'$ (where U_x'' is the subset of U_x consisting of elements with trivial second component when U_x is identified with $K_\sigma^{(2)}$). It now follows easily that $U_x \leq G'$ for all points x in $\mathcal{A}_0$, and hence $\Delta' = \Delta$.

Using relation (8) of (1.7.1) in [18], one shows similarly that whenever $\Delta(x) = \Delta'(x)$ for every point x in $\mathcal{A}_0$ (and Δ' is not necessarily thick), then $\Delta = \Delta'$. □

Note that we also proved that no non-thick full suboctagon Δ' exists with $\Delta(x) = \Delta'(x)$, for all points x of Δ'. Later on, we shall see that there is a unique (up to an isomorphism of Δ) full non-thick suboctagon Δ' with $\Delta(L) = \Delta'(L)$, for all lines L of Δ'.

Theorem B *Every thick suboctagon of any Ree octagon $O(K, \sigma)$ arises in a standard way from a subfield $K' \leq K$ closed under σ.*

Proof: We may assume that the apartment $\mathcal{A}_0$ belongs to a suboctagon Δ' of $\Delta = O(K, \sigma)$. As in the previous proof, we may also assume, by the Moufang property of Δ', that $U'_{(0,0,0)} \cap G' \neq 1$ (where G' is the collineation group of Δ' and $U'_{(0,0,0)}$ is as in the previous proof) and hence the line $[(0, k_1)]$ is in Δ' for some $k_1 \in K^\times$. By applying suitable generalized homologies, we can assume that Δ' contains the point (1) and the line $[(0, 1)]$. Let S be the set of all $x \in K$ such that (x) is a point of Δ' and let $a \in S$. Then the generalized homology $\eta(a, 1)$ maps Δ' to a suboctagon which intersects Δ' in a full suboctagon, hence by the previous proposition, $\Delta'^{\eta(a,1)} = \Delta'$ and so, S is closed under multiplication. Similarly, S is closed under taking inverses. By the Moufang property, S is also closed under addition. Hence S is a subfield of K. We now claim that S is closed under σ, i.e. $S^\sigma \subseteq S$. For any $b \in S$, the octagon $\Delta'^{\eta(b,b^{-1})}$ coincides with Δ (it has all points on $[\infty]$ in common and also all lines through (0); this follows from the explicit form of $\eta(b, b^{-1})$ in 4.1). Since $[(k_0, k_1)]$ is mapped by $\eta(b, b^{-1})$ to $[b^{-1} \otimes k]$, it follows that whenever $[k]$ is in Δ', the line $c \otimes k$ is also in Δ', for all $c \in S$. This in turn has as consequence that $\Delta'^{\eta(d^{-\sigma-2},d)}$ coincides with Δ' (since all points on [0] are fixed by $\eta(d^{-\sigma-1}, d)$), whenever $d \in S$, which means that $d^{-\sigma-2} \in S$ (look at the image of the point (1)). So we have $d^\sigma \in S$ and S is closed under σ.

We conclude that there is a Ree suboctagon $\Delta'' = O(S, \sigma)$ naturally embedded in Δ via the restriction of the coordinates to S. Assume that $\Delta' \neq \Delta''$. Then the intersection $\Delta' \cap \Delta''$ is a proper thick full suboctagon in at least one of Δ' and Δ'' if $\Delta' \neq \Delta''$. This contradicts the previous proposition and hence the result follows. □

As an application we show the following

Corollary 4.4 *The only generalized homologies of a Ree octagon $O(K, \sigma)$ in its full automorphism group are, up to conjugation, the generalized homologies $\eta(A, 1)$ and $\eta(1, B)$, $A, B \in K$.*

Proof: Let η be a generalized homology with centers (∞) and (0^7). Suppose η maps (1) to (a), $a \in K$. Then $\eta(a^{-1}, 1)$ fixes a thick full suboctagon. This must be $O(K, \sigma)$ itself by the proposition, hence $\eta = \eta(a, 1)$.

As for the dual, we remark that each root elation $u_{(0^2)}(0, k_1)$ with $k_1 \in K$ has order 2, while each root elation $u_{(0^2)}(k_0, k_1)$, $k_0 \neq 0$, has order 4. Any collineation α preserving $\mathcal{A}_0$ and mapping $[(0, k_1')]$ to $[(k_0, k_1)]$, $k_0, k_1, k_1' \in K$, $k_0 \neq 0$, would give rise to the contradiction $u_{(0^2)}(0, k_1')^\alpha = u_{(0^2)}(k_0, k_1)$. Hence every generalized homology η with centers $[\infty]$ and $[0^7]$ maps the line $[(0, 1)]$ to a line $[(0, k_1)]$. As before, this must be the generalized homology $\eta(1, k_1)$. This completes the proof of the corollary. □

5 Some remarkable configurations

5.1 The Suzuki subquadrangles

We keep our notation $\Delta = O(K, \sigma)$ and the apartment $\mathcal{A}_0$ in it. We have already remarked that there is no full non-thick suboctagon Δ' for which $\Delta(x) = \Delta'(x)$, for all points x of Δ'. However, if we replace R_2 by $\{0\}$ and keep R_1, then all octanary operations (1) up to (6) are well defined, and they become either trivial or

$$b'' = a' + a^\sigma a''' \tag{S1}$$

$$b' = a'' + a a'''^\sigma \tag{S2}$$

$$b = a''' \tag{S3}$$

This coordinatizes a non-thick full suboctagon Δ' which is the "double" of a thick generalized quadrangle Δ_* as follows. The points of Δ_* are the lines of Δ' having 0, 3, 4 or 7 coordinates; the lines of Δ_* are the lines of Δ' having 1, 2, 5 or 6 coordinates; incidence is defined by the existence of a common point. We now coordinatize Δ_*. As there are no elements of R_2 involved anymore, we drop our general notational assumption about $k, l \ldots \in R_2$ and replace it by $k, l, \ldots \in R_{2*}$, where the subscript "$*$" means "with respect to Δ_*". Now put $R_{1*} = R_{2*} = K$. Put $(\infty)_* = [\infty]$, $[\infty]_* = [0]$, $(a)_* = [0, a, 0]$, $[k]_* = [k, 0]$, $(0, b)_* = [0^2, b, 0]$, $[0, l]_* = [0^3, l, 0]$, $(0^3)_* = (0^7)$ and $[0^3]_* = [0^6]$. This determines a coordinatization of Δ_* in a unique way. It is easy to check that we have the following correspondences:

$$\begin{aligned} (\infty)_* &= [\infty] & [\infty]_* &= [0] \\ (a)_* &= [0, a, 0] & [k]_* &= [k, 0] \\ (k, b)_* &= [k, 0, b, 0] & [a, l]_* &= [0, a, 0, l, 0] \\ (a, l, a')_* &= [0, a, 0, l, 0, a', 0] & [k, b, k']_* &= [k, 0, b, 0, k', 0] \end{aligned}$$

The quaternary operations expressing incidence between points and lines with three coordinates can be deduced immediately from (S1), (S2) and (S3) above :

$$(a, l, a')_* \, I \, [k, b, k']_* \iff \begin{cases} a' = k^\sigma a + b \\ k' = a^\sigma k + l \end{cases} \tag{S$_*$}$$

Remark (1) If we recoordinatize by simply putting R_{2*} equal to K^σ (and thus formally substituting $[k^\sigma]_*$ for $[k]_*$, etc.), then the new quadratic operations become

$$(a, l, a')_* \, I \, [k, b, k']_* \iff \begin{cases} a' = ka + b \\ k' = a^2k + l \end{cases}$$

and hence we see that Δ_* is isomorphic to the Suzuki quadrangle $W(K, \sigma)$. But we will stick to the operations (S_*).
(2) The existence of these full non-thick suboctagons in the Ree octagons is well known; cp. e.g. Sarli [10], (6.1.3); see also Tits [18], 3.17.

Note that the flags of Δ_* are exactly the points of Δ'. We will call Δ_*, by abuse of language, a Suzuki *subquadrangle* of Δ. Occasionally we will also use the notation $W(K, \sigma) \leq O(K, \sigma)$.

Clearly, Δ_* is self-polar and a polarity is given by swapping the round parentheses with the square brackets. The set of absolute points (resp. lines) with respect to that polarity consists of the point $(\infty)_*$ (resp. line $[\infty]_*$) and the points with coordinates $(a, a' + a^{\sigma+1}, a')_*$ (resp. lines with coordinates $[k, k' + k^{\sigma+1}, k']_*$), with $a, a', k, k' \in K$. So a flag consisting of two absolute elements looks like $((a, a' + a^{\sigma+1}, a')_*, [a, a' + a^{\sigma+1}, a']_*)$ or $((\infty)_*, [\infty]_*)$. Translated to a point of Δ', this is $(a, 0, a' + a^{\sigma+1}, 0, a', 0, a)$ resp. (∞). We denote the set of these points by ST, a *Suzuki-Tits set* of points in Δ', and hence in Δ.

Now, the set of absolute points of this polarity in a Suzuki quadrangle is a Suzuki-Tits ovoid on which the Suzuki group $\mathrm{Sz}(K, \sigma)$ acts as a doubly transitive group. Also, the set of lines through a point in Δ can be thought of as a Suzuki-Tits ovoid. Indeed, the group $\mathrm{Sz}(K, \sigma)$ is contained in ${}^2F_4(K, \sigma)$ as the group which is generated by two opposite p-root groups and it acts on that set in the same way as on the ovoid; see Section 6. The question is: is there a geometric connection between these two Suzuki-Tits ovoids? First, we remark:

Proposition 5.1 *No non-trivial automorphism of $O(K, \sigma)$ fixes the Suzuki quadrangle pointwise.*

Proof: Such a collineation would be a generalized homology, but clearly, no non-trivial generalized homology fixes all points on Δ'. This can be established by simply looking at the coordinates. □

From this proposition, it immediately follows that the Suzuki group acting on the Suzuki-Tits ovoid in Δ_*, and hence in Δ', acts in a unique way on the Suzuki-Tits set ST as an automorphism group of Δ. Now let x be any point at distance 4 from both (∞) and (0^7) (i.e. x lies in the middle of a root with extremities (∞) and (0^7)). By the coordinatization, it follows easily that either $x = (k, 0^3)$ or $x = (0^3)$. If we now require that x has also distance 4 from each other element of ST, then from the octanary operation (6), it follows $N(k) = 1$, and from the operation (5), one deduces $\mathrm{tr}(k) = 0$. This implies $k = (1, 1)$. Now the octanary operations (4), (5) and (6) show that the point $((1, 1), 0, 0, 0)$ does indeed meet our requirement. We call this point the *nucleus* x_{ST} of the set ST. Since this point is unique with respect to a geometric property, it is fixed by the group stabilizing the set ST. The projection of the set ST on x_{ST} is the set of lines $[(1, 1), 0, 0, 0, (a_0^\sigma, a_1^\sigma)]$ together with $[(1, 1), 0, 0]$. The Suzuki group acts on this set in a natural way. Hence it also acts on R_2. Note that the map $ST \mapsto x_{ST}$ is not bijective since obviously the stabilizer of x_{ST} can move ST around. But it is surjective of course. Note also that the action on a pencil of lines of the Suzuki group which is induced by the action of the Suzuki group on ST just described is in fact on R_2^σ and not on R_2 itself. We do not have a geometric interpretation of that fact.

From all this it follows also that no other point or line in Δ is fixed by the Suzuki group stabilizing ST. Indeed, such a point must be fixed by every generalized homology $\eta(A, 1)$, $A \in K$ (since this stabilizes ST). It follows that all its coordinates must be 0 except if it has an even number of coordinates and then only the first one can differ from 0. It is now an elementary exercise to show that only x_{ST} satisfies the assumption.

Now consider the root ϕ with middle element x_{ST} and extremities (∞) and (0^7). Consider with respect to this root, the root elation u mapping the line $[\infty]$ to $[0]$. This root elation must preserve Δ' (since Δ'^u contains $[0]$ and $[0^6]$, and this completely defines Δ'). Hence u maps $[0]$ back to $[\infty]$ and so it is an involution. It preserves the set ST, and it fixes every line through x_{ST}, hence, since every point of ST uniquely defines a line through x_{ST}, u fixes pointwise the set ST. But u induces in Δ_* a polarity, and the set of "absolute flags" amounts exactly to ST; hence we have shown the following theorem.

Theorem C *Let ST be a set of points of the Ree octagon Δ for which there exists a Suzuki subquadrangle Δ' in which ST corresponds to the set of flags of a Suzuki-Tits ovoid in the way described above.*
Then there exists a unique point x_{ST} of Δ such that
(i) *x_{ST} is at distance 4 (within the octagon Δ) from each point of ST,*
(ii) *x_{ST} is fixed by the full group of automomorphisms of Δ stabilizing ST,*
(iii) *x_{ST} is the middle element of the root with respect to which there is a(n involutory) root elation in Δ inducing a polarity in Δ' defining the Suzuki-Tits ovoid corresponding to ST.*

The point x_{ST} is uniquely determined by each of the properties (i), (ii) *or* (iii) □

5.2 *The grid configuration*

Consider an apartment in $\Delta = O(K, \sigma)$ for which we can take without loss of generality $\mathcal{A}_0$. Take the two opposite l-roots $\phi_{[\infty]}$, $\phi_{[0^7]}$ in $\mathcal{A}_0$. Both of them have the extremities $[0^3]$ and $[0^4]$. Let $\phi_{[0^3,b,0^3]}$ be the root with same extremities and with the middle element $[0^3, b, 0^3]$, $b \in K^\times$. Then the set of middle elements of roots with extremities $[\infty]$ and $[0^7]$ coincides with the set of middle elements of the roots $[\infty]$ and $[0^3, b', 0^3]$. This can be checked by use of the coordinates; it also follows immediately from the regularity of every element in the Suzuki quadrangles. We call this the *grid configuration* on $\phi_{[\infty]}$, $\phi_{[0^7]}$ and $\phi_{[0^3,b,0^3]}$.

One can ask if there exist three p-roots carrying a (dual) grid configuration. To answer this, we can consider without loss of generality the roots $\phi_{(\infty)}$, $\phi_{(0^7)}$ and $\phi_{(0^3,l',0^3)}$ all having the points (0^3) and (0^4) as extremities, and having respectively the point (∞), (0^7) and $(0^3, l', 0^3)$ as middle element. The set of middle elements of the roots with extremities (∞) and (0^7) is $\{(k, 0^3) \mid k \in R_2\} \cup \{(0^3)\}$ (this follows immediately from the coordinatization process). From the octanary operations (4), (5) and (6) it follows that the projection of the line $[k]$ on $(0^3, l', 0^3)$ is the line $[k, 0^3, l', 0^2]$. Hence we have a grid configuration on any three different roots with common extremities.

This yields an interesting consequence for the perspectivities between the middle elements of the three roots forming a grid configuration. We have

$$[[\infty], [0^7]] = [[\infty], [0^3, b, 0^3], [0^7]]$$

and

$$[(\infty), (0^7)] = [(\infty), (0^3, l, 0^3), (0^7)].$$

5.3 The Löwe configuration

Löwe [7] shows that a generalized quadrangle arises from a finite Ree octagon $\Delta = O(2^{2r+1})$ as soon as Δ does not contain the following configuration $\mathcal{C}$. Consider a non-degenerate 9-gon and denote its lines in cyclic order by L_i, $i = 1, 2, \dots, 9$. Let x be a point of Δ and suppose that there are paths of length 5 connecting x with L_3, resp. L_6, L_9. Then $\mathcal{C}$ consists of the 9-gon and the three paths mentioned. Löwe [7] does not actually prove or disprove the existence of such a configuration. We will show here that such a configuration is contained in every Ree octagon, finite or infinite.

We start by remarking that every Ree octagon has a finite suboctagon isomorphic to $O(2)$, since $GF(2)$ is a subfield of every field of characteristic 2 and obviously, it is fixed by every field endomorphism and hence also by σ. So if we establish a configuration isomorphic to $\mathcal{C}$ above in $O(2)$, then it exists in every Ree octagon.

Consider the 9-gon

$(0,0,0,0,0,1)$	I	$[0,0,0,0,0,1,0]$	I	$(0,0,1,0,0,0,0)$	I
$[(1,0),0,0,1,0,1,0]$	I	$((1,0),0,0,1,0,1)$	I	$[(1,0),0,0,1,0]$	I
$((1,0),0,0,1,0,0)$	I	$[(1,0),0,0,1,0,0,0]$	I	$(0,0,0,0,1,(0,1),0)$	I
$[0,0,0,0,1,(0,1)]$	I	$(0,0,0,0,1)$	I	$[0,0,0,0]$	I
$(0,0,0,0,0)$	I	$[0,0,0,0,0,0]$	I	$(0,0,0,0,0,0,0)$	I
$[0,0,0,0,0,0,0]$	I	$(0,0,0,0,0,0)$	I	$[0,0,0,0,0]$.	

Then the point (∞) has obviously chains of length 5 connecting it to the respective lines $[(1,0),0,0,1,0]$, $[0,0,0,0,0]$ and $[0,0,0,0]$. This shows our assertion.

6 Projectivities

6.1 The groups of projectivities of the Suzuki quadrangles

We will now compute the groups of projectivities of the Suzuki quadrangles $W(K,\sigma)$.

For any subgroup $G \leq K^\times$ that contains all squares $(K^2)^\times$ let $\mathrm{PSL}_2K \rtimes G/(K^2)^\times$ denote the 2-transitive subgroup of PGL_2K whose stabilizer of 0 and ∞ on the projective line over K is given by the set $\{x \mapsto bx \mid b \in G\}$.

In Section 2.3 the Suzuki quadrangles have been introduced as subquadrangles of the symplectic quadrangles $W(K)$. So the groups of projectivities of a Suzuki quadrangle are subgroups of the respective groups of projectivities of the corresponding symplectic quadrangle. Knarr [5] has proved that $\Pi(W(K)) = \Pi_+(W(K)) \cong \mathrm{PGL}_2K$ and $\Pi^D(W(K)) = \Pi_+^D(W(K)) \cong \mathrm{PSL}_2K$. Although the result is stated only for finite fields, for the symplectic quadrangles it holds in the infinite case as well without any change in the proof being necessary. The proof of the next theorem is also similar to that result and hence we omit the proof. Note that $\mathrm{PSL}_2K^\sigma \rtimes (K^2)^\times/(K^{2\sigma})^\times$ is contained in PSL_2K.

Theorem E

$$\Pi(W(K,\sigma)) = \Pi_+(W(K,\sigma)) \cong \mathrm{PSL}_2K \rtimes (K^\sigma)^\times/(K^2)^\times \cong \Pi_+^D(W(K,\sigma)) = \Pi^D(W(K,\sigma)).$$

The action is equivalent to the natural action of $\mathrm{PSL}_2K \rtimes (K^\sigma)^\times/(K^2)^\times$ *on the projective line over* K.

6.2 The groups of projectivities of the Ree octagons

For the finite Ree octagons $O(2^{2r+1})$ the groups of projectivities have been determined by Knarr [5].

Theorem F

$$\Pi(O(K,\sigma)) = \Pi_+(O(K,\sigma)) \cong \mathrm{PSL}_2K \rtimes K^\dagger/(K^2)^\times$$

and

$$\Pi^D(O(K,\sigma)) = \Pi_+^D(O(K,\sigma)) \cong \mathrm{GSz}(K,\sigma).$$

The actions are equivalent to the natural actions of $\mathrm{PSL}_2K \rtimes K^\dagger/(K^2)^\times$ (*resp.* $\mathrm{GSz}(K,\sigma)$) *on the projective line over* K (*resp. the Suzuki-Tits ovoid*).

Proof: We have $\Pi = \Pi_+$; this follows from the existence of the grid configurations, cp. Section 5.2; it also follows from the regularity of the lines in the Suzuki quadrangles by virtue of $W(K,\sigma) \le O(K,\sigma)$. Dually $\Pi^D = \Pi_+^D$ because of the existence of the dual grid configurations, cp. Section 5.2.

According to Tits [13], p. 5, and [18], 1.8.2, we have $\langle U_{[0^3]}, U_{[0^4]}\rangle \cong \mathrm{SL}_2K \cong \mathrm{PSL}_2K$ acting on the points of the line $[\infty]$ as PSL_2K acts naturally on the projective line, and dually $\langle U_{(0^3)}, U_{(0^4)}\rangle \cong \mathrm{Sz}(K,\sigma)$ acting on the pencil of lines through (∞) as on the Suzuki-Tits ovoid.

The action of $H^{O(K,\sigma)}$ has been determined in (4.2). So we infer

$$H^{O(K,\sigma)}|_{O(K,\sigma)([\infty])} = \mathcal{H}(K^\dagger, 1)|_{O(K,\sigma)([\infty])} = \{(a) \mapsto (Aa) \mid A \in K^\dagger\},$$
$$H^{O(K,\sigma)}|_{O(K,\sigma)((\infty))} = \mathcal{H}(1, K^\times)|_{O(K,\sigma)((\infty))} = \{[k] \mapsto [B \otimes k] \mid B \in K^\times\}.$$

The stabilizer of (0) and (∞) inside the group $\langle U_{[0^3]}, U_{[0^4]}\rangle$ acting on the points of the line $[\infty]$ equals $\{(a) \mapsto (b^2a) \mid b \in K^\times\}$. The dual is stated in Section 2.4. □

Remark Because of $W(K,\sigma) \le O(K,\sigma)$ we have immediately $\mathrm{PSL}_2K \rtimes (K^\sigma)^\times/(K^2)^\times \le \Pi$. For perfect K (e.g. K finite) the result is $\Pi = \Pi_+ \cong \mathrm{PGL}_2K \cong \mathrm{PSL}_2K$ and $\Pi^D = \Pi_+^D \cong \mathrm{GSz}(K,\sigma) = \mathrm{Sz}(K,\sigma)$.

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