Intersections of Hermitian and Ree Ovoids in the Generalized Hexagon H(q)

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ABSTRACT

For $q = 3^{2h+1}, h \geq 0$, we investigate the intersections of Hermitian and Ree ovoids of the generalized hexagon $H(q)$. © 1996 John Wiley & Sons, Inc.

1. INTRODUCTION

A finite generalized hexagon of order $(s, t), s, t \geq 1$ is a $1 - (v, s + 1, t + 1)$ design $S = (\mathcal{P}, \mathcal{B}, I)$ whose incidence graph has girth 12 and diameter 6, also denoted by $S(s, t)$. If $s = t$, S is said to have order s. Generalized hexagons (and more generally, generalized polygons) were introduced by Tits [12]. The only known (up to duality) finite generalized hexagons of order $s > 1$ arise from the Chevalley groups $G_2(q)$ and have order q, q power of a prime. They are due to Tits [12]. We denote the $G_2(q)$-hexagon by $H(q)$.

An *ovoid* of a generalized hexagon of order s is a set of $s^3 + 1$ points mutually at distance 6. A spread of a generalized hexagon of order s is defined dually. For example, consider the split-Cayley Moufang generalized hexagon $H(q)$ embedded in the nonsingular quadric $Q(6, q)$ (see Tits [12]). Let PG$(5, q)$ be a hyperplane of PG$(6, q)$ such that PG$(5, q) \cap Q$ is an elliptic quadric Q^-. Then the lines of $H(q)$ on Q^- constitute a spread of the generalized hexagon $H(q)$ ([9]). Further $\mathcal{O}$ is an ovoid of $H(q)$ if and only if $\mathcal{O}$ is an ovoid of the polar space $Q(6, q)$ ([10]). So $H(q)$ always has a spread.

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It has an ovoid if and only if $Q(6,q)$ has an ovoid. In particular $H(q)$, with q even, has no ovoid and $H(q)$, with $q = 3^h$, has an ovoid (see Thas [11]).

If $q = 3^{2h+1}$, $h \geq 0$ there are two kinds of ovoids known ([11]). Tits [13] showed that the generalized hexagon $H(q)$ admits a polarity if and only if q is an odd power of three (see also [14]). The set of absolute points of such a polarity is an ovoid of $H(q)$, namely the Ree ovoid $\mathcal{U}_R(q)$. The automorphism group in the group $G_2(q)$ of the Ree ovoid is the twisted Chevalley group ${}^2G_2(q)$ discovered by Ree ([7]) and also denoted by $R(q)$.

Applying such a polarity to a spread of $H(q)$ consisting of the lines of $H(q)$ on a $Q^-(5,q)$ gives an ovoid of $H(q)$. We will call these ovoids, the Hermitian ovoids.

These ovoids of $H(q)$ are also $2-(q^3+1, q+1, 1)$ designs (see Sec. 2.3), that is, unitals. In this article we will prove the following theorems:

- Let $q = 3^{2h+1}, h \geq 0$. The intersection-numbers of a Hermitian ovoid and a Ree ovoid of $H(q)$ are $q+1$, $q+\sqrt{3q}+1$ and $q-\sqrt{3q}+1$.
- The intersection sets D_r of order r of a Ree ovoid and a Hermitian ovoid of $H(q)$ form cyclic arcs in the corresponding Hermitian unital. For $r \neq q+1$, they are also cyclic arcs in the corresponding Ree unital.

Remark. Analogous results for the symplectic generalized quadrangle $W(q)$, $q = 2^{2h+1}$ and $h \geq 0$ are proved by Bagchi and Sastry [1]. Their proofs are based on group theory. Our proof uses rather elementary algebra. Bagchi and Sastry first determine the intersections of the groups and derive from this the intersections of the ovoids; we proceed in the opposite direction: we first determine the intersections of the ovoids and derive from this, with the help of a result of Kleidman [6], the intersections of the copies of ${}^2G_2(q)$ and $U_3(q):2$ inside $G_2(q)$ (see Theorem 2.3). Our methods can also be used to give an alternative proof of the results of Bagchi and Sastry (see the remark following Theorem 2.1).

2. INTERSECTION OF HERMITIAN OVOIDS AND REE OVOIDS IN $H(q)$

From now on we suppose that $q = 3^{2h+1}$, $h \geq 0$, and fix $H(q)$.

A. Description of the Ovoids with Coordinates

We will use the coordinatization of the finite Moufang hexagon $H(q)$ defined by the quadric $Q(6,q)$ with equation $X_0X_4 + X_1X_5 + X_2X_6 = X_3^2$ as described in [14]. We refer to [14] for the general coordinatization theory and for more details in the case of this specific example.

- From [14] we recall the description of $H(q)$:

$$a, l, a', l', a'', k, b, k', b', k'' \in GF(q)$$

$$\begin{aligned} \text{points} &: (\infty), (a), (k,b), (a,l,a'), (k,b,k',b'), (a,l,a',l',a'') \\ \text{lines} &: [\infty], [k], [a,l], [k,b,k'], [a,l,a',l'], [k,b,k',b',k''] \end{aligned}$$

incidence : $(a,l,a',l',a'')I[a,l,a',l']I(a,l,a')I[a,l]I(a)I[\infty]I$
$(\infty)I[k]I(k,b)I[k,b,k']I(k,b,k',b')I[k,b,k',b',k'']$

$$(a,l,a',l',a'')I[k,b,k',b',k''] \Longleftrightarrow \begin{cases} a'' = ak + b \\ a' = a^2k + b' + 2ab \\ k'' = ka^3 + l \\ k' = k^2a^3 + l' - kl \end{cases}$$

- In that article the coordinates of a Ree ovoid are given:

$$U_R(q) = \{(\infty)\} \cup \{(a, a''^s - a^{3+s}, a', a^{3+2s} + a'^s + a^s a''^s, a'') \\ \| a, a', a'' \in GF(q)\}$$

with $q = 3^{2h+1}$ and $s = 3^{h+1}$.
The corresponding polarity maps (a,l,a',l',a'') to $[a^{3^{h+1}}, l^{3^h}, a'^{3^{h+1}}, l'^{3^h}, a''^{3^{h+1}}]$.

- A Hermitian ovoid of $H(q)$ is found by intersecting with a hyperplane Y: $x_1 + x_5 = 0$. The intersection of Y with the nonsingular quadric $Q(6,q)$ is an elliptic quadric $Q^-(5,q)$. So the lines of $Q^-(5,q)$ in $H(q)$ form a spread of $H(q)$. The image of this spread under the polarity fixing the Ree ovoid above is a Hermitian ovoid with the following coordinates:

$$\mathcal{U}_H(q) = \{(\infty)\} \cup \{(a, a''^3, a', -a^3, a'') \| a, a', a'' \in \mathrm{GF}(q)\}.$$

B. The Intersection

To investigate the intersection pattern of Ree ovoids and Hermitian ovoids, we fix a Ree ovoid, namely $\mathcal{U}_R(q)$ and look at the intersection with all Hermitian ovoids of $H(q)$. The interesection of $\mathcal{U}_R(q)$ with all Hermitian ovoids of $H(q)$ will be done in two steps. First we suppose they have at least one point in common, say (∞). In the second step, we will investigate if there exist Hermitian ovoids which are disjoint from $\mathcal{U}_R(q)$.

1. First Step. We denote by D the set of nonzero squares of the field $GF(q)$ (and we fix q). Take the Hermitian ovoid $\mathcal{U}_H(q)$ through (∞). The action of the group $G_2(q)_{(\infty)}$ on $\mathcal{U}_H(q)$ gives all the Hermitian ovoids of $H(q)$ through (∞). The group element taking (a,l,a',l',a'') to

$$(a, yl - ya^3K + L, ya' + ya^2K, y^2l' + L' + y^2a^3K^2 + y^2lK, ya'' + yaK)$$

with $K, L, L' \in GF(q)$ and $y \in D$ does not fix $\mathcal{U}_H(q)$. One can check this with the coordinates. All such elements form a subgroup G of order $\frac{q-1}{2} \cdot q^3$ and the latter is precisely the number of Hermitian ovoids through a fixed point [see (2-B.2)]. Hence G acts regularly on the set of Hermitian ovoids containing the point (∞).

So a general Hermitian ovoid of $H(q)$ through (∞) can be written in a unique way as $\mathcal{O}_{y,K,L,L'}$ and consists of the points $\{(\infty)\} \cup$

$$\{(a, yl^3 - ya^3K + L, ya' + ya^2K, -y^2a^3 + L' + y^2a^3K^2 + y^2l^3K, yl + yaK) \\ \| a, l, a' \in GF(q)\}.$$

Remark that $\mathcal{O}_{1,0,0,0} = \mathcal{U}_H(q)$. The intersection of $\mathcal{O}_{y,K,L,L'}$ and $\mathcal{U}_R(q)$ contains the following points: (∞) and

$$(a, yl^3 - ya^3K + L, a^2a^{3^{h+1}} - y^{2.3^h}a^{3^{h+1}} + L'^{3^h} + y^{2.3^h}a^{3^{h+1}}K^{2.3^h} + y^{2.3^h}l^{3^{h+1}}K^{3^h} - ay^{3^h}l^{3^{h+1}} + y^{3^h}a^{1+3^{h+1}}K^{3^h} - aL^{3^h}, -y^2a^3 + L' + y^2a^3K^2 + y^2l^3K, yl + yaK)$$

for all $(a, l) \in GF(q)^2$ with

$$a^{1+3^{h+1}} + y^{3^h}l^{3^{h+1}} - y^{3^h}a^{3^{h+1}}K^{3^h} + L^{3^h} = yl + yaK. \qquad (*)$$

So the order of the intersection of a Ree ovoid and a Hermitian ovoid of $H(q)$ through (∞) is determined by the number of solutions of the equation $(*)$.

This equation can also be written as follows:

$$(a - y^{3^h}K^{3^h}) \cdot (a - y^{3^h}K^{3^h})^{3^{h+1}} - y^{3^h+1}K^{3^h+1} + L^{3^h} = yl - y^{3^h}l^{3^{h+1}}.$$

So we are looking for the number of solutions (x, l) of

$$xx^{3^{h+1}} + C = yl - y^{3^h}l^{3^{h+1}}$$

with $x = a - y^{3^h}K^{3^h}$, $C \in GF(q)$, and $y \in D$.

Suppose we know x.

We look for the number of $l \in GF(q)$ such that $yl - y^{3^h}l^{3^{h+1}} = c'$, (1)

with $c' = xx^{3^{h+1}} + C \in GF(q)$ and $y \in D$.

Put $l = y^{3^h} \cdot z$. Then eq. (1) is equal to

$$y \cdot y^{3^h} \cdot z - y^{3^h} \cdot y^{3^{2h+1}} \cdot z^{3^{h+1}} = c'. \qquad (2)$$

Since $y^{3^{2h+1}} = y$ in $GF(3^{2h+1})$, (2) is equivalent with

$$z - z^{3^{h+1}} = c'', \qquad (3)$$

with $c'' = c' \cdot y^{-1-3^h}$. The number of solutions in l of eq. (1) is equal to the number of solutions in z of eq. (3).

The map $z \mapsto z - z^{3^{h+1}}$ is an endomorphism of the additive group of the field. The kernel consist of those elements Z which satisfy $Z - Z^{3^{h+1}} = 0$. The latter is equivalent with $Z(1 - Z^{3^{h+1}-1}) = 0$. Since one easily computes that $(3^{2h+1} - 1, 3^{h+1} - 1) = 2$, we see that the kernel consists of the elements $0, 1, -1$. Hence the eq. (3) has either 3 or 0 solutions. It also follows that there are exactly 3^{2h} different values for c'' for which (3) has 3 solutions. On the other hand, there are exactly 3^{2h} elements a in $GF(3^{2h+1})$ with $\mathrm{Tr}(a) = 0$, where $\mathrm{Tr}(a) = \Sigma_{i=0}^{2h} a^{3^i}$. But noting that $\mathrm{Tr}(z - z^{3^{h+1}}) = 0$, we see that, if (3) has 3 solutions, then $\mathrm{Tr}(z'') = 0$. It follows that (3) has at least one—and hence exactly three—solution if and only if $\mathrm{Tr}(z'') = 0$.

How many x's?

We know that x must be such that $\mathrm{Tr}(c'') = 0$. Hence the condition on x is:

$$\mathrm{Tr}(y^{-1-3^h} \cdot (xx^{3^{h+1}} + C)) = 0$$

or

$$\mathrm{Tr}(y^{-1-3^h} \cdot xx^{3^{h+1}}) + \mathrm{Tr}(y^{-1-3^h} \cdot C) = 0.$$

where $\mathrm{Tr}(z) = \Sigma_{i=0}^{2h} z^{3^i}$ and C as defined before. Remark that $\mathrm{Tr}(z)$ is fixed by the automorphisms of $GF(q)$, so must lie in the prime field $GF(3)$. So the number of x's satisfying the condition is the number of x's such that

$$\begin{array}{lll} \text{either} & \mathrm{Tr}(y'xx^{3^{h+1}}) & \text{equals } 0 \\ \text{or} & \mathrm{Tr}(y'xx^{3^{h+1}}) & \text{equals } 1 \\ \text{or} & \mathrm{Tr}(y'xx^{3^{h+1}}) & \text{equals } -1,\ y' = y^{-1-3^h}, \end{array} \qquad (**)$$

depending on $\mathrm{Tr}(y^{-1-3^h} \cdot C)$.

We obtain three possible intersection numbers. To solve (**), we remark that, for all $x \in GF(q)$, $\mathrm{Tr}(y'xx^{3^{h+1}}) = 0$, 1 or -1. Let $X = \mathrm{Tr}(y'xx^{3^{h+1}})$, then $X.(X - 1).(X + 1) = 0$ (or equivalently $X^3 - X = 0$) has exactly q solutions in x.

Now,

$$X^3 - X = (x^{3^{2h+1}-1} - 1).y'.x^{3^h+1}.(x^2 + y'^{1-3^{h+1}})^{3^h}.$$

The greatest power of x in X, $X + 1$ and $X - 1$ is $3^{2h} + 3^h$. So $X = 0$, $X - 1 = 0$, and $X + 1 = 0$ have at most $3^{2h} + 3^h$ solutions in x.

We will show that $X = 0$ has exactly 3^{2h} solutions.

The zero-solution appears $3^h + 1$ times in $X^3 - X$, and it can only appear in the factor $X = 0$. So $X = 0$ has at most $3^{2h} + 3^h - 3^h = 3^{2h}$ different solutions in x.

$$\text{Define } \theta : \quad GF(q) \longrightarrow GF(3)$$
$$x \longrightarrow \mathrm{Tr}(x).$$

Then

$$|\ker\theta| = \frac{q}{3}.$$

One of the elements of ker θ is zero and the other elements come in pairs $\{z, -z\}$. Since -1 is not a square, one element of such a pair is a square and the other is not. Now we count the number of x's such that $\mathrm{Tr}(xx^{3^{h+1}}.y') = 0$. This is in other words the number of x's such that $y'.x.x^{3^{h+1}} = z$ with $\mathrm{Tr}(z) = 0$.

$$\text{If } z = 0, \quad \text{then } x = 0.$$

If $z \neq 0$, there are $\frac{\frac{q}{3}-1}{2}$ pairs $\{z, -z\}$, each containing exactly one square. Since $x.x^{3^{h+1}}$ is a square, exactly one $u \in \{z, -z\}$ satisfies $x.x^{3^{h+1}} = \frac{u}{y'}$ and u is a square if and only if y' is. But with u, there correspond two opposite values for x.

So $X = 0$ has exactly $1 + \frac{q}{3} - 1 = 3^{2h}$ solutions in x, which was the maximum possible. Since

$$\gcd(X - 1, X + 1) = 2 \text{ or } 1,$$
$$\gcd(X, X - 1) = 1,$$
$$\gcd(X, X + 1) = 1$$

and since the factor $(x^2 + y'^{1-3^{h+1}})$ is irreducible, the complete power of this factor, $(x^2 + y'^{1-3^{h+1}})^{3^h}$, must appear in one of the factors X, $X + 1$ or $X - 1$. The factor $X = 0$ can be excluded by the previous observation.

Thus, $X = 0$ has 3^{2h} solutions in x and one of $X - 1 = 0$ or $X + 1 = 0$ has at most $3^{2h} + 3^h - 2.3^h$ solutions in x and the other has at most $3^{2h} + 3^h$ solutions in x. But the total number of solutions must be $q = 3^{2h+1}$ so the *at most* becomes *exactly*.

We proved that the intersection numbers of a Hermitian ovoid and a Ree ovoid, which have at least one point in common, are

- $3^{2h}.3 + 1 = q + 1$
- $(3^{2h} - 3^h).3 + 1 = q - \sqrt{3q} + 1$
- $(3^{2h} + 3^h).3 + 1 = q + \sqrt{3q} + 1$

Remark. Note that the number of Hermitian ovoids through (∞) meeting $\mathcal{U}_R(q)$ in $q + 1$ points equals the number of such ovoids meeting $\mathcal{U}_R(q)$ in $q - \sqrt{3q} + 1$ resp. in $q + \sqrt{3q} + 1$ points [since $\mathrm{Tr}(x) = 0$, 1 or -1 has equally many solutions].

2. Second Step. We will show that all Hermitian ovoids have at least one point in common with a Ree ovoid.

The total number of Hermitian ovoids in $H(q)$ is

$$\frac{|G_2(q)|}{|U_3(q){:}\,2|} = \frac{q^6.(q^6-1).(q^2-1)}{2.q^3.(q^3+1).(q^2-1)} = q^3 \cdot \frac{q^3-1}{2}$$

The total number of Hermitian ovoids in $H(q)$ through a fixed point is

$$\frac{q^6 \cdot (q-1)(q^2-1)}{2.q^3(q^2-1)} = q^3 \cdot \frac{q-1}{2}$$

From the last remark we know that there are $\frac{q^3}{3} \cdot \frac{q-1}{2}$ Hermitian ovoids through a fixed point of $\mathcal{U}_R(q)$ intersecting the Ree ovoid $\mathcal{U}_R(q)$ in $q+1$ points, the same number of Hermitian ovoids through a fixed point of $\mathcal{U}_R(q)$ intersecting in $q+\sqrt{3q}+1$ points and $q-\sqrt{3q}+1$ points.

In order to count the number of Hermitian ovoids, K_i, which intersect $\mathcal{U}_R(q)$ in $i = q+1$, $q+\sqrt{3q}+1$ or $q-\sqrt{3q}+1$ points, we count in two ways, the couples (p,K) with $p \in \mathcal{U}_R(q), K$ a Hermitian ovoid of $H(q)$ which intersect $\mathcal{U}_R(q)$ in, respectively, $q+1$, $q+\sqrt{3q}+1$, and $q-\sqrt{3q}+1$ points and $p\mathrm{I}K$. We obtain:

1. $(q^3+1) \cdot \frac{q^3}{3} \cdot \frac{q-1}{2} = K_{q+1}.(q+1)$

The number of Hermitian ovoids of $H(q)$ which intersect $\mathcal{U}_R(q)$ in $q+1$ points is

$$\frac{q^3}{3} \cdot \frac{q-1}{2} \cdot (q^2-q+1)$$

2. $(q^3+1) \cdot \frac{q^3}{3} \cdot \frac{(q-1)}{2} = K_{q+\sqrt{3q}+1}.(q+\sqrt{3q}+1)$

The number of Hermitian ovoids of $H(q)$ which intersect $\mathcal{U}_R(q)$ in $q+\sqrt{3q}+1$ is

$$\frac{q^3}{3} \cdot \frac{q-1}{2} \cdot (q+1) \cdot (q-\sqrt{3q}+1)$$

3. $(q^3+1) \cdot \frac{q3}{3} \cdot \frac{(q-1)}{2} = K_{q-\sqrt{3q}+1} \cdot (q=\sqrt{3q}+1)$

The number of Hermitian ovoids of $H(q)$ which intersect $\mathcal{U}_R(q)$ in $q-\sqrt{3q}+1$ is

$$\frac{q^3}{3} \cdot \frac{q-1}{2} \cdot (q+1) \cdot (q+\sqrt{3q}+1)$$

So we have $\frac{q^3}{3} \cdot \frac{q-1}{2} \cdot [q^2-q+1+2(q+1)\cdot(q+1)] = \frac{q^3}{2} \cdot (q^3-1)$ Hermitian ovoids which have at least one point in common with $\mathcal{U}_R(q)$. But this number is exactly the total number of Hermitian ovoids of $H(q)$.

So all Hermitian ovoids intersect $\mathcal{U}_R(q)$ in $q+1$, $q+\sqrt{3q}+1$ or $q-\sqrt{3q}+1$ points. This proves the first theorem:

Theorem 2.1. *Let $q = 3^{2h+1}$, $h \geq 0$. The intersection-numbers of a Hermitian ovoid and a Ree ovoid of $H(q)$ are $q+1$, $q+\sqrt{3q}+1$, and $q-\sqrt{3q}+1$.*

Remark. We can apply the same method to obtain the intersection numbers of an elliptic ovoid and a Suzuki ovoid of the generalized quadrangle $\mathcal{W}(q)$ with $q = 2^{2h+1}$, $h \geq 0$.

We mention here some major steps of the proof, using the coordinatization introduced in [4]

- First step

  1. The Suzuki ovoid: $\{(\infty)\} \cup \{(a, a.a^{2^h} + a'^{2^h}, a') \| a, a' \in GF(q)\}$
     Elliptic ovoids through (∞):
     $\{(\infty)\} \cup \{(a, ya + ya' + aK + L, y^2a' + aK^2) \| a, a' \in GF(q)\}$

with

$$y \in GF(q)^*, L \in GF(q)$$

and $K \in D \subset GF(q)$ with D a set of cardinality $\frac{q}{2}$ satisfying: if $x \in D$ then $x + 1 \notin D$.
The intersection contains besides (∞), those points of the Hermitian ovoid for which

$$aa^{2h} + y^{2^{h+1}} \cdot a'^{2^h} + a^{2^h} \cdot K^{2^{h+1}} = ya + ya' + aK + L,$$

or,

$$a''^{2^h+1} + ya'' = ya' + (y^2a')^{2h} + C \quad \text{with } a'' = a + K^{2^{h+1}}.$$

2. Via a system of linear equations over GF(2), we find two solutions for a' for every fixed a'' for which $\mathrm{Tr}(y^{-2^{h+1}-1} \cdot (C + a''^{2^h+1} + ya'')) = 0$. In this way we find two intersection numbers for a Suzuki ovoid and an elliptic ovoid through a fixed point of the Suzuki ovoid, namely

$$q + \sqrt{2q} + 1 \quad \text{and} \quad q - \sqrt{2q} + 1 \text{ with } q = 2^{2h+1}.$$

- Second step
By an analogous counting argument, we find that a Suzuki ovoid always meets an elliptic ovoid nontrivially. So the two mentioned intersection numbers are the only ones.

C. Study of the Intersection Sets

With the Ree and Hermitian ovoids of $H(q)$, there correspond $2 - (q^3 + 1, q + 1, 1)$ designs, that is, unitals. For this, we have to define the *blocks* of the design.

- A block through two points x and y of a Hermitian ovoid is the point regulus $R(x, y)$ ([8]). This is the set of $q + 1$ points at distance 3 from the lines which are at distance 3 from x and y. The points of a Hermitian ovoid $H(q)$ and the corresponding point reguli form a $2 - (q^3 + 1, q + 1, 1)$ design, isomorphic to the Hermitian unital.
- A block through two points x and y of a Ree ovoid $H(q)$ is defined as follows: Let L_y (resp. L_x) denote the unique line through y (resp. x) at distance 3 from the absolute line through x (resp. y). Then every line of the regulus $R(L_x, L_y)$ contains a unique point of the Ree ovoid. This can be checked immediately: Take the Ree ovoid $\mathcal{U}_R(q)$ as defined in the previous section. Since $R(q)$ is 2-transitive, we only have to construct the block through $p = (\infty)$ and $q = (0,0,0,0,0)$. The line $[0]$ through (∞) is at distance 3 from $(0,0,0,0,0)$ and the line $[0,0,0,0]$ through $(0,0,0,0,0)$ is at distance 3 from $[\infty]$. The regulus through $[0]$ and $[0,0,0,0]$ contains the lines $[0,b,0,0]$. These lines contain a unique point of $\mathcal{U}_R(q)$ namely $(0,b,0,0,b^{3^h})$. Remark that such a set of $q + 1$ points is independent of the choice of any two points in it.

With these blocks and the points of a Ree ovoid of $H(q)$ we have again a $2 - (q^3 + 1, q + 1, 1)$ design. Indeed, the points $(0,b,0,0,b^{3h})$ and (∞) are fixed pointwise by the following involution σ preserving $\mathcal{U}_R(q)$:

$$\sigma : (a, l, a', l', a'') \mapsto (-a, l, -a', -l', a'').$$

Hence, by definition, the unital we defined geometrically is exactly the usual Ree unital ([13]).

Lemma 2.2. *Two Hermitian unitals of $H(q)$ intersect in either one point or in a block.*

Proof. Take two dual Hermitian unitals (so two $Q^-(5,q)$'s) in $Q(6,q)$. If they intersect in at least two lines, then the regulus through those two lines lies in the intersection. Suppose they intersect in more than a block. Then we have 3 lines of a $Q^-(5,q)$ not on one regulus. Since they span a 5-dimensional space, they span the whole $Q^-(5,q)$ and the two unitals coincide. So two Hermitian unitals intersect in zero, one or $q+1$ points.

We will now show that every two Hermitian unitals intersect. Every two different 5-dimensional spaces of PG$(6,q)$ intersect in a 4-dimensional subspace. If we can prove that every 4-dimensional subspace of a 5-dimensional subspace which intersect $Q(6,q)$ in a $Q^-(5,q)$ contains at least one line of a spread of $Q^-(5,q)$ then we are done.

Fix such a 5-dimensional subspace Π_5 and call the spread S. If the 4-space is a tangent hyperplane of Π_5 then it contains exactly one line of S. A nontangent 4-subspace Π_4 contains $|Q(4,q)| = (q^2+1)\cdot(q+1)$ points of $Q(6,q)$. Let x be the number of lines of S which lie in Π_4 and y the number of other lines of S, so these intersect Π_4 in a point. We have $x+y = q^3+1$. On the other hand, by counting in two ways the number of pairs (p,R) with $p \in Q(4,q)$, $R \in S$ and pIR, we get $(q^2+1)\cdot(q+1) = x\cdot(q+1)+y$. This implies that $x = q+1$. □

Theorem 2.3. *If $r \neq q+1$, then the intersections of order r of a Hermitian and Ree unital are orbits under a cyclic subgroup of order r of the intersection D of the corresponding automorphism groups $R(q)$ and $U_3(q):2$. The group D itself is isomorphic to $r:6$ (Atlas notation). If $r = q+1$, then the intersections of order r of a Hermitian and Ree unital are orbits under the fourgroup normalizer $(2^2 \times D_{\frac{q+1}{2}}):3$.*

Proof. Take a copy of $R(q)$ and denote the corresponding Ree unital by $\mathcal{U}_R(q)$. Denote by D_r the intersection of order r of $\mathcal{U}_R(q)$ with a suitable Hermitian ovoid $\mathcal{U}_H(q)$. In the previous section we proved that r is either $q+1$, $q+\sqrt{3q}+1$ or $q-\sqrt{2q}+1$.

In this paragraph we suppose that $(q,r) \neq (3,4)$.

From the coordinates of the points in D_r we conclude that D_r is not a subset of a block of $\mathcal{U}_H(q)$. Hence every element of $G_2(q)$ stabilizing D_r also preserves $\mathcal{U}_H(q)$. Therefore the group D stabilizing D_r coincides with the subgroup of $R(q)$ stabilizing D_r. Denote by K_r all copies of $\mathcal{U}_H(q)$ intersecting $\mathcal{U}_R(q)$ in r points. What is the order of the automorphism group of D_r (denoted by $|Aut(D_r)|$) in $R(q)$?

The order of $R(q)$ is $(q^3+1).q^3.(q-1)$. Suppose the set K_r is divided in k orbits under the action of $R(q)$. If all these orbits have the same size, then $|K_r| = k.\frac{(q^3+1)\cdot q^3\cdot(q-1)}{|Aut(D_r)|}$. From the previous section we know the values of $|K_r|$. We find that $|Aut(D_r)| = k.6.r$. If not all those orbits have the same size then there exists at least one D_r with $|Aut(D_r)| \geq k.6.r$. We consider such a D_r.

On the other hand, we can calculate the stabilizer of a point of the automorphism group of D_r:

Let $(q,r) \neq (3,1)$ and suppose the intersection D_r contains besides (∞) also the points $(a,(l-C)^3,-a^s-a^{2+s}-a.l,-a^3,l)$ with $s = 3^{h+1}$ and $a,l \in GF(q)$ such that $a^{s+1} = l-l^s+C^s$ (according to Tr(C) we have a $q+1$, $q+\sqrt{3q}+1$ or $q-\sqrt{3q}+1$ intersection). The stabilizer of (∞) in the automorphism group of $\mathcal{U}_R(q)$ transforms the other points in $(xa+A, x^{3+s}(l-C)^3+A''^s-A^sx^3a^3-A^{3+s}, -x^{2+s}(a^s+a^{2+s}+al)+A'-A''xa+A^2x^2a^2-AA'', -x^{3+2s}a^3+A'^s+$

$A^{2s}x^3a^3 + A^sx^{2+s}(l - C)^3 + A^{2s+3},x^{1+s}l + A'' + A^sxa)$ with $x \in GF(q)^*$ and $A, A', A'' \in GF(q)$.

Expressing that these points must belong to D_r gives the conditions $A = A' = 0$, $A'' \in GF(3)$, and $x = 1$ or -1. So we have a group $D_{(\infty)}$ of order 6.

Let l be the length of the orbit of (∞) under $Aut(D_r)$. Then $l = \frac{|Aut(D_r)|}{|Aut(D_r)_{(\infty)}|} \geq k.r$. From $l \leq r$ it follows that $k = 1$, $|Aut(D_r)| = 6.r$ and the automorphism group of $\mathcal{U}_R(q)$ is transitive on D_r. Hence all orbits in K_r considered above have equal length and the argument above shows that $|Aut(D_r)| = 6.r$ in every case.

Remark that the automorphism group of D_r does not fix any point of $\mathcal{U}_R(q)$ since $D_{(\infty)}$ only fixes the point (∞).

Now D_r must be a subgroup of at least one maximal subgroup M of the automorphism group of $\mathcal{U}_R(q)$. For $q \neq 3$, there are only 6 possibilities ([6]):

1. $\underline{M \text{ fixes a point.}}$
   Excluded by the remark above.
2. $\underline{M \text{ fixes a block of the unital.}}$

   - Suppose the fixed block lies in $\mathcal{U}_R(q)D_r$. Take an element β of order 3 of $Aut(D_r)$ fixing a point of D_r. Then $q + 1$ must be divided in orbits of length 1 or 3. So β has at least two fixpoints which is impossible.
   - Suppose the fixed block lies partly in D_r. Taking again such an element of order 3, we know that it must fix an element of D_r. Since $Aut(D_r)$ is transitive, we can choose this fixpoint outside the block and an analogous reasoning leads to a contradiction.
   - Suppose the fixed block lies entirely in D_r. In the Theorem 2.5 we will show that this is never the case (note that in the proof of Theorem 2.5 we use the argument that D is transitive on D_r, which we have already proved).

3. $\underline{M = R(\sqrt[n]{q}) \text{ with } n \text{ a prime.}}$
   In this case, the $Aut(D_r)$ must be a subgroup of at least one maximal subgroup of $R(\sqrt[n]{q})$. It cannot be the whole group because the highest power of 3 in $6.r$ is 1 but 27 divides the order of $R(q)$, for all $q = 3^{2h+1}$, $h \geq 0$. So again we can examine the 6 cases. By induction we can exclude the first three cases and cases 4, 5, and 6 are excluded because their orders are too small.
   We are left with the last three possibilities:
4. $\underline{M = \left(2^2 \times D_{\frac{q+1}{2}}\right) : 3.}$
5. $\underline{M = Z_{(q+\sqrt{3q}+1)} : 6.}$
6. $\underline{M = Z_{(q-\sqrt{3q}+1)} : 6.}$

So for $r \neq q + 1$, we have $Aut(D_r) = r : 6$.
For $r = q + 1$, $Aut(D_r) = (2^2 \times D_{\frac{q+1}{2}}) : 3$.

Now let $q = 3$. Cases 2 and 4 coincide, case 3 can not occur and case 6 is not maximal. If $r = 4$, then since D_r is a block of the unital $\mathcal{U}_H(q), D \cong 2^3 : 3$ (cases 2,4). If $r = 7$, we have case 5 so $D \cong 7 : 6$. Finally if $r = 1$, a direct computation shows that the corresponding point stabilizers share a cyclic group of order 6. □

Corollary 2.4. *The intersections D_r with $r = q + \sqrt{3q} + 1$ and $q - \sqrt{3q} + 1$ are arcs in the corresponding Hermitian unital and can be extended to maximal $(q^2 - q + 1)$-arcs of the unital.*

Proof. The groups of these intersections are also subgroups of a maximal subgroup of $U_3(q) : 2$. Since $q + \sqrt{3q} + 1$ and $q - \sqrt{3q} + 1$ are divisors of $q^2 - q + 1$, the corresponding intersection groups are subgroups of the maximal subgroups of $U_3(q) : 2$ of order $6(q^2 - q + 1)$. Indeed, consider a prime p dividing $r = q + \sqrt{3q} + 1$ and let P be the (unique) Sylow p-subgroup in $\mathrm{Aut}(D_r)$. Clearly P is also a Sylow p-subgroup of $U_3(q) : 2$: and hence also of a certain "Singer cycle" of order $q^2 - q + 1$, by which it is centralized. But also $\mathrm{Aut}(D_r)$ centralizes P, hence the result. In [5], [3], and [2] it is proved that these maximal subgroups are related to maximal arcs. □

Theorem 2.5. *The intersection sets D_r of order r of a Ree ovoid and a Hermitian ovoid of $H(q)$ form an arc in the corresponding Hermitian unital. They are arcs in the corresponding Ree unital if and only if $r \neq q + 1$.*

Proof: Since there exists a transitive group on every intersection set D_r, it is an arc if the blocks through one point of D_r contain at most one other point of D_r. If we take the Ree ovoid $\mathcal{U}_R(q)$ from the previous section and the corresponding suitable Hermitian ovoid, we have the following intersection set D_r:
(∞) and $(a, (l - C)^3, -a^s - a^{2+s} - a.l, -a^3, l)$ with $s = 3^{h+1}$ and $a, l \in GF(q)$ such that $a^{s+1} = l - l^s + C^s$ (according to $\mathrm{Tr}(c)$ we have a $q + 1$, $q + \sqrt{3q} + 1$ or $q - \sqrt{3q} + 1$ intersection).

The blocks through (∞) in the unital of the Hermitian ovoid are the sets:

$$\{(\infty)\} \cup \{(A, A''^s - A^{s+3}, a', A^{3+2s} + a'^s + A^s A''^s, A'') \| a' \in GF(q)\}.$$

It is obvious that there is at most one other point of D_r on such a block.

The blocks through (∞) in the unital of the Ree ovoid are the sets:

$$\{(\infty)\} \cup \{(A, a''^s - A^{s+3}, A', A^{3+2s} + A'^s + A^s a''^s, a'') \| a'' \in GF(q)\}.$$

If $A \neq 0$ then there is at most one other point of D_r on such a block.

From the condition of A of the intersection sets, it follows that points with $A = 0$ belong to D_r only when $\mathrm{Tr}(C) = 0$, so only when $r = q + 1$. In this case, the 3 points of D_{q+1} with $A = 0$ lie on the same block through (∞). Applying the transitive group on D_{q+1}, the $q + 1$ points are divided in 4-subsets of $\frac{q+1}{4}$ blocks of the unital. □

Remark. These intersection-arcs are not complete in the unital (see the previous corollary for the Hermitian unitals; for the Ree unitals, this was checked by computer using CAYLEY for small values of q).

REFERENCES

[1] B. Bagchi and N. S. N. Sastry, *Intersection pattern of the classical ovoids in symplectic 3-space of even order,* J. Algebra **126** (1989), 147–160.

[2] E. Boros and T. Szönyi, *On the sharpness of a theorem of B. Segre.* Combinatorica, **6** (1986), 261–268.

[3] J. C. Fisher, J. W. P. Hirschfeld, and Thas J. A., *Complete arcs in planes of square order.* Ann. Discrete Math., **30** (1986), 243–250.

[4] G. Hanssens and H. Van Maldeghem, *A new look at the classical generalized quadrangles.* Ars Combin., **24** (1987), 199–210.

[5] B.C. Kestenband, *A family of complete arcs in finite projective planes.* Colloq. Math., **57** (1989), 59–67.

[6] P.B. Kleidman, *The maximal subgroups of the Chevalley groups $G_2(q)$ with q odd, of the Ree groups $^2G_2(q)$, and of their automorphism groups.* J. Algebra, **117** (1988), 30–71.

[7] R. Ree, *A family of simple groups associated with the simple Lie-algebra of type (G_2).* Amer. J. Math., **83** (1961), 432–462.

[8] M.A. Ronan, *A geometric characterization of Moufang hexagons.* Invent. Math., **57** (1980), 227–262.

[9] J.A. Thas, *Polar spaces, generalized hexagons and perfect codes.* J. Comb. Theory, **29** (1980), 87–93.

[10] ——, *Ovoids and spreads of finite classical polar spaces.* Geom. Dedicata, **10** (1981), 135–144.

[11] ——, *Old and new results on spreads and ovoids of finite classical polar spaces.* Combinatorics, (1990), 529–544.

[12] J. Tits, *Sur la trialité et certains groupes qui s'en déduisent.* Inst. Hautes Etudes Sci. Publ. Math., **2** (1959), 14–60.

[13] ——, *Les groupes simples de Suzuki et de Ree.* Séminaire Bourbaki, **13**(210) (1960/61), 1–18.

[14] H. Van Maldeghem and V. De Smet, *The finite Moufang hexagons coordinatized.* Contr. Alg. Geom., **34** (1993), 217–232.

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