A Finite Generalized Hexagon Admitting a Group Acting Transitively on Ordered Heptagons is Classical

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December 15, 2010

Abstract

Let Γ be a thick finite generalized hexagon and let G be a group of automorphisms of Γ. If G acts transitively on the set of non-degenerate ordered heptagons, then Γ is one of the Moufang hexagons $H(q)$ or ${}^3H(q)$ associated to the Chevalley groups $G_2(q)$ or ${}^3D_4(q)$ respectively, or their duals; and G contains the corresponding Chevalley group. Moreover, we show that no thick generalized octagon admitting a group acting transitively on the set of ordered nonagons (enneagons) can exist. This completes the determination of all finite thick generalized n-gons, $n \geq 3$, with a group acting transitively on the set of ordered $(n+1)$-gons with elementary methods. Because we do not use the classification of the finite simple groups, from which these results also follow.

1 Introduction and Main Results

A finite generalized n-gon of order (s,t), $s,t \in \mathbf{N} \setminus \{0\}$, is an incidence geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ in which $\mathcal{P}$ and $\mathcal{L}$ are disjoint non-empty sets of objects called points and lines respectively, and for which I is a symmetric point-line incidence relation satisfying axioms (GP1), (GP2) and (GP3).

(GP1) Each point is incident with $1+t$ lines and two distinct points are incident with at most one line.

(GP2) Each line is incident with $1+s$ points and two distinct lines are incident with at most one point.

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(GP3) If the distance in the incidence graph between two elements (points and lines) v, w is — strictly — smaller than n, then there is a unique minimal (i.e. with a minimal number of elements) sequence of consecutive incident elements starting with v and ending in w.

For a generalized quadrangle (4-gon), hexagon (6-gon) and octagon (8-gon), we can write (GP3) respectively as follows:

(GQ3) For every non-incident pair $(x, L) \in \mathcal{P} \times \mathcal{L}$, there exists a unique pair $(y, M) \in \mathcal{P} \times \mathcal{L}$ for which $x\, I\, M\, I\, y\, I\, L$.

(GH3) For every non-incident pair $(x, L) \in \mathcal{P} \times \mathcal{L}$, there exists either a unique pair $(y, M) \in \mathcal{P} \times \mathcal{L}$ for which $x\, I\, M\, I\, y\, I\, L$, or a unique quadruple $(y, M, z, N) \in \mathcal{P} \times \mathcal{L} \times \mathcal{P} \times \mathcal{L}$ for which $x\, I\, N\, I\, z\, I\, M\, I\, y\, I\, L$.

(GO3) For every non-incident pair $(x, L) \in \mathcal{P} \times \mathcal{L}$, there exists either a unique pair $(y, M) \in \mathcal{P} \times \mathcal{L}$ for which $x\, I\, M\, I\, y\, I\, L$, or a unique quadruple $(y, M, z, N) \in \mathcal{P} \times \mathcal{L} \times \mathcal{P} \times \mathcal{L}$ for which $x\, I\, N\, I\, z\, I\, M\, I\, y\, I\, L$ or a a unique sixtuple $(y, M, z, N, u, X) \in \mathcal{P} \times \mathcal{L} \times \mathcal{P} \times \mathcal{L} \times \mathcal{P} \times \mathcal{L}$ for which $x\, I\, X\, I\, u\, I\, N\, I\, z\, I\, M\, I\, y\, I\, L$.

The following terminology will be used throughout. A finite generalized hexagon or octagon of order (s, t) is *thick* if $s, t \geq 2$ (the non-thick generalized hexagons and octagons are the flag complexes (or the *doubles*) of the projective planes and generalized quadrangles of order (s,s) respectively) and the dual of this (order $(1, s)$ and $(s, 1)$ respectively). A *heptagon* in a generalized hexagon is a subconfiguration consisting of seven distinct points and seven distinct lines such that each line (respectively point) is incident with exactly two points (respectively lines). An *ordered heptagon* is a heptagon in which the elements are ordered in such a way that two consecutive elements are incident. Similar definitions for *nonagons* and *ordered nonagons* in generalized octagons. A sub-n-gon of order $(1, 1)$ (a "usual" n-gon) in a generalized n-gon, $n = 6, 8$ is an *apartment*. A *skeleton* is a subconfiguration $\Omega = (\Sigma; L, p)$ where Σ is an apartment and L (respectively p) is a line (respectively point) not in Σ but incident with a point p_1 (respectively line L_1) of Σ, where $p_1\, I\, L_1$. We will always use upper case letters, such as L, M, N for lines and lower case ones, such as p, x, z, b for points. If in a generalized hexagon, a point x is collinear with two non-collinear points p_1 and p_2, then by axiom (GH3), x is unique with that property and we denote $x = p_1 * p_2$.

There are presently, up to duality, only two classes of thick finite generalized hexagons known and they are related to the Chevalley groups $G_2(q)$ and ${}^3D_4(q)$. We denote the first one by $H(q)$ (distinguishing it from its dual $H(q)^D$ by saying that $H(q)$ is naturally embedded in the quadric $Q_6(q)$, see Tits [16]) and the second one by ${}^3H(q)$ (distinguishing it by its dual ${}^3H(q)^D$ by telling that it has order (q, q^3)). From this, it follows that the dual of $H(q)$ is a subhexagon of ${}^3D_4(q)$ (see Tits [16] or Kantor [7]). We call the members of these 4 classes of finite generalized hexagons the finite *classical hexagons*. They were all discovered by Tits [16] (but the name "Tits hexagon" would cause confusion with the hexagons satisfying the Tits property, which was introduced by BUEKENHOUT & VAN MALDEGHEM [1]).

As for finite thick generalized octagons, the situation is even simpler. Only one such class is presently know (up to duality). It is also due to TITS [19] and it is related to the class of Ree groups of characteristic 2. These octagons have order (q, q^2) and we call them the *Ree-Tits octagons.*

It follows easily from the main result of VAN MALDEGHEM [20] that the finite classical hexagons admit an automorphism group G acting transitively on the set of skeletons. This in fact is equivalent with G acting transitively on the set of ordered heptagons, see below. The converse is also true. Suppose the finite generalized hexagon Γ admits a group G acting transitively on the set of ordered heptagons. Then G is a group with a (B, N)-pair of type G_2 and using the classification of the finite simple groups one can show that Γ must be classical (for an explicit proof, see BUEKENHOUT & VAN MALDEGHEM [1]). The aim of this paper is to give a proof of this result without using the classification of the finite simple groups. Once we have shown that the generalized hexagon must be classical, then a result of SEITZ [12] immediately implies that G must contain the simple group $G_2(q)$ $(q \geq 3)$, ${}^3D_4(q)$ or $G_2(2)' \cong U_3(3)$. In the latter case, the order of $H(2)$ is $(2, 2)$, the full automorphism group is $G_2(2)$ and the number of ordered heptagons is exactly equal to the order of $G_2(2)$ (which is $12,096$). So G must contain the corresponding Chevalley group.

Our first main result is:

THEOREM 1. *Let Γ be a finite thick generalized hexagon and let G be a group of automorphisms of Γ. Then G acts transitively on the set of ordered heptagons if and only if Γ is one of the classical generalized hexagons $H(q)$, ${}^3H(q)$, $H(q)^D$ or ${}^3H(q)^D$ and G contains the corresponding Chevalley group.*

A similar result is proved for finite generalized quadrangles by THAS & VAN MALDEGHEM [15]. Of course, for finite projective planes, an analogous result (transitivity on ordered quadrangles) follows immediately from the well-known theorem of OSTROM & WAGNER [8]. By a well known result of FEIT & HIGMAN [3], finite thick generalized n-gons exist only for $n = 2, 3, 4, 6, 8$. So we finally turn our attention to octagons. We will show as second main result:

THEOREM 2. *There does not exist a finite thick generalized octagon admitting a group of collineations acting transitively on the set of ordered nonagons.*

As a result, we have the following corollary:

COROLLARY. *A finite thick generalized n-gon Γ admitting a group G acting transitively on the set of all ordered $(n+1)$-gons is Moufang and a complete list of such pairs (Γ, G) is determined by elementary methods (i.e. without using the classification of the finite simple groups).*

We remark that the finiteness assumption cannot be dispensed with in the preceding results; indeed, this follows from a well-known construction method of Kegel and Schleiermacher as adapted by Tits [18].

We also remark that Theorem 1 improves on the main result of Van Maldeghem [20] in which all finite generalized hexagons with transitive apartments are classified. These hexagons satisfy automatically the hypothesis of Theorem 1.

The motivation for studying generalized polygons admitting an automorphsm group acting transitively on ordered circuits of a certain length stems from the need of a classification-free proof of the fact that all finite rank 2 Tits systems are known. In the case of quadrangles, Payne & Thas [9] have developed a geometric machinery which can be used to try to do so. A large part of that machinery must be used to show that all finite generalized quadrangles with a group acting transitively on ordered pentagons are known, see Thas & Van Maldeghem [15]. No such machinery is available for hexagons and octagons, but this paper wants to show that in spite of that, geometric reasonings can prove a lot. Also, by the geometric nature of our proof, certain substructures turn up (mainly affine planes), and a more systematic investigation of those must lead to a better understanding of the fact that so few finite hexagons are known. For octagons, the new idea of distance-i regularity (see Van Maldeghem [21]) is here succesfully applied.

In fact, it is the author's belief that a classification-free prove of the above mentioned fact is within reach, at least for the case of equal parameters (i.e. $s = t$). The geometry needed in the case of hexagons would not be much different from that turning up in the proof of our main result.

2 Proof of the Theorem 1

In this section, we denote by $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ a finite thick generalized hexagon of order (s, t) and by G a group of automorphisms of Γ acting transitively on the set of ordered heptagons. Note that we may assume $s, t \geq 3$ by Cohen & Tits [2].

Recall that the *distance* in Γ is the one inherited from the incidence graph. Elements at distance 6 (the maximal distance in Γ) are called *opposite*.

We notice that the proof is not longer than the one of the equivalent result for generalized quadrangles, despite the lack of a comparable machinery for finite generalized hexagons. With such a machinery (mainly on anti-regular points (see below) and on the *half Moufang condition*), our proof would be significantly shorter.

2.1 Some General Facts

2.1.1 Skeletons

Let $\chi = (p_1, L_1, \ldots, p_7, L_7)$, with $p_1 \, I \, L_1 \, I \, p_2 \, I \ldots I \, p_7 \, I \, L_7 \, I \, p_1$, be an ordered heptagon in Γ. Let p'_i be the point of L_{i+3} at distance 4 from p_i (where indices are taken modulo 7). Consider the skeleton $\Omega = (\Sigma; L_7, p'_5)$, where Σ is the apartment $(p_1, L_1, \ldots, p_4, L_4, p'_1, \ldots, p_1)$.

Then χ completely determines Ω and vice versa. Hence G acts transitively on the set of skeletons, and conversely, every group acting transitively on the set of skeletons also acts transitively on the set of ordered heptagons.

2.1.2 The Moufang condition

Consider an apartment $\Sigma = (p_1, L_1, \ldots, p_6, L_6, p_1)$ (where consecutive elements are incident). If the group of collineations of Γ fixing every point incident with either L_1 or L_2, and also fixing every line incident with either p_1, p_2 or p_3 acts transitively on the set of apartments containing $p_1, p_2, p_3, L_6, L_1, L_2$ and L_3, then we say that Γ is $(p_1, L_1, p_2, L_2, p_3)$-transitive (and these collinations are called $(p_1, L_1, p_2, L_2, p_3)$-elations). Dually, one defines $(L_1, p_2, L_2, p_3, L_3,)$-transitivity. If Γ is $(p_1, L_1, p_2, L_2, p_3)$-transitive for all possible choices of the points p_1, p_2, p_3 and the lines L_1 and L_2 in Γ, and if moreover, also the dual transitivity property holds for every possible choice, then one calls Γ *Moufang*. From a theorem of FONG & SEITZ [4, 5] follows that finite Moufang generalized hexagons are classical. The converse is also true, see TITS [17]. We will use that characterization in the proof of Theorem 1.

2.1.3 Half regular and anti-regular points

Let x be a point of Γ and consider the set of points $\Gamma(x)$ collinear with x. We define blocks in this set as follows: for every point y opposite x, the block x^y is the set of points collinear with x and at distance 4 from y.

1. If this geometry is a semi-linear space (i. e. two points in $\Gamma(x)$ determine at most one such block), then we say that x is *half regular*. This notion is introduced by VAN MALDEGHEM & BLOEMEN [22], where the authors remark that from a theorem of RONAN [10] follows that, if every point of a generalized hexagon is half regular, then it is Moufang. An alternative (and actually better) name for half regular is *distance-2 regular*, see VAN MALDEGHEM [21] (it is better since one can generalize this to arbitrary distance from x). Anyway, we will also use this characterization of the classical hexagons in the finite case (Ronan's result is indeed also valid in the infinite case).

2. If in this geometry two blocks meet in at most 2 points, and if every 3 pairwise non-collinear points in $\Gamma(x)$ are contained in a least (and hence exactly) one block, then we call x *anti-regular*. This definition is modelled on the same situation in generalized quadrangles (see PAYNE & THAS [9]). We will meet this property in our proof thus giving a motivation to study anti-regularity in generalized hexagons separately without assuming a group.

The geometry obtained in this paragraph will be denoted by Γ_x

2.2 Reduction to two main cases

Our proof is inspired by Ronan [11], although we cannot use his results (because, as we will see, we will have the intersection set property, but not the regulus property).

For two points x, y at distance 4 from each other, we denote by $x * y$ the unique point at distance 2 from both x and y.

Consider two points x, y at distance 4 from each other via the chain $x \, I \, L_x \, I \, (x * y) \, I \, L_y \, I \, y$. Let p be a point collinear with x but not incident with L_x. Let p_1 and p_2 be two points collinear with y, not incident with L_y and at distance 4 from p. There are two possibilities.

1. Suppose $x^{p_1} = x^{p_2}$. By the transitivity property, there is a collineation fixing $x, x*y, y$ and p_1 and mapping the line yp_2 to any desired line L through y, $yp_1 \neq L \neq L_y$. The set x^{p_1} is preserved and p_2 is mapped to a point p_3 on L. Obviously, $x^{p_3} = x^{p_2} = x^{p_1}$. Since L was arbitrary, every point z collinear with y, opposite x and at distance 4 from p has the property $x^z = x^{p_1}$. By the transitivity, we can now let p vary over the set of all points collinear with x but not on L_x, and hence we obtain the property that whenever z_1 and z_2 are points collinear with y and opposite x, then either $x^{z_1} = x^{z_2}$, or $x^{z_1} \cap x^{z_2} = \{x * y\}$. By transitivity, this holds for every such pair (x, y). Following Ronan [11], we say that all intersection sets (w.r.t. points) of Γ have order 1.

2. Suppose now $x^{p_1} \neq x^{p_2}$. Then there exists a line L through x incident with two distinct points a_1 and a_2 at distance 4 from p_1 and p_2 respectively. By the transitivity property, there is a collineation θ fixing x, y, p_1, a_1 and yp_2 and mapping a_2 to any desired point on L distinct from x and a_1. The point p_2 will be mapped onto any point p_2^θ incident with yp_2, except for y and z, where z has distance 4 from a_1. Obviously, $x^{p_1} \cap x^{p_2^\theta}$ contains p^θ and $x^{p_1} \cap x^z$ contains a_1. Since p_2^θ was essentially arbitrary, and since by transitivity also the line yp_2 is arbitrary and for the same reason p_1 as well, we conclude that whenever z_1 and z_2 are points at distance 4, collinear with y and opposite x, then $|x^{z_1} \cap x^{z_2}| \geq 2$. Reversing the roles of x and y, we also see that whenever u_1 and u_2 are two non-collinear points in $\Gamma(x)$ opposite y, then there exists a point z collinear to y and opposite x such that $\{u_1, u_2\} \subseteq x^z$.

   Let us get back to the above situation involving a_1, a_2, p_1 and p_2. There exist also collineations fixing x, y, p_1, a_1 and p and mapping a_2 to any point u of L different from x and a_1. Of course, such mappings do not preserve the line yp_2, and in fact every other choice for u gives another image of yp_2, hence $s - 1 \leq t - 1$, implying $s \leq t$.

   The properties obtained in this paragraph are also valid for every choice of such a pair (x, y), by transitivity. Following Ronan [11], we say that all intersection sets of Γ have order ≥ 2.

From this, we derive two possibilities:

Case (i) In either Γ or its dual, all intersection sets have order 1;

Case (ii) In both Γ and its dual, all intersection sets have order ≥ 2.

Remark that Case (ii) implies $s = t$.

2.3 Case (i)

In fact, Ronan [11] proves that, if Γ has also the *regulus* condition, then it is Moufang. We are not in a position to show directly the regulus condition, but a little weaker version will do the trick here. We may assume, by duality, that all intersection sets w.r.t. points have order 1 in Γ.

We consider an apartment $(p_1, L_1, \ldots, p_6, L_6, p_1)$ as in 2.1.2. Denote by $\mathcal{S}$ the set of points of the form $x*y$, with $x\, I\, L_2$, $y\, I\, L_5$ and x at distance 4 from y. Using the transitivity property as above, one shows completely similar to the argument in the previous paragraphs, that either $p_1^u = p_1^{p_4}$, for all $u \in \mathcal{S} \setminus \{\sqrt{\infty}\}$, or every point on every line L through p_1, $L_1 \neq L \neq L_6$, is at distance 4 from exactly 1 element of $\mathcal{S}$. We call these cases Subcase (a) and Subcase (b) respectively. If Subcase (a) holds for one apartment and one choice of p_1 in that apartment, then Subcase (a) holds for all apartments and choices of p_1, by the transitivity. Similarly for Subcase (b).

2.3.1 Subcase (a)

We assume in this paragraph that Subcase (a) (respectively Subcase (b)) holds. Let p be any point of Γ. We show that p is half regular — or distance-2 regular— (respectively anti-regular). Therefore, let a and b be two non-collinear points collinear with p. Let x and y be two points opposite p both at distance 4 from both a and b. We must show $p^x = p^y$ (respectively $p^x = p^y$ or $p^x \cap p^y = \{a, b\}$; by the Subcase (b) assumption, we already have at least one block through every three pairwise non-collinear points in $\Gamma(p)$). We may suppose that either $x * a$ is not collinear with $y * a$, or $x * b$ is not collinear with $y * b$, otherwise the result follows from the Subcase (a) (respectively (b)) assumption. So suppose $x * b$ is not collinear with $y * b$. Let L be the line joining b and $y * b$ and let M be the line joining $a * x$ and a. Let u and v be the points collinear with $y * b$ and $x * a$ respectively, and at distance 3 from M and L respectively.

By the Case (i) assumption, $p^x = p^v$ and $p^y = p^u$, and by the Subcase (a) assumption, $p^v = p^u$ (respectively $p^v = p^u$,or $p^v \cap p^u = \{a, b\}$), implying $p^x = p^y$ (respectively $p^x = p^y$, or $p^x \cap p^y = \{a, b\}$). This shows that p is half regular (respectively anti-regular). But this means that all points are half regular and Γ is classical.

2.3.2 Subcase (b)

We show the result in four steps.

STEP 1

In this step, we make the additional assumption that G acts regularly on the set of ordered heptagons in Γ. Fix an apartment $\Sigma = (p_1, L_1, \ldots, p_6, L_6, p_1)$, where consecutive elements are incident. Let L be any line incident with p_1, $L_1 \neq L \neq L_6$. The group H_1 fixing $p_1, p_2, p_3, L, L_6, L_1, L_2$ and L_3 has order $s(s-1)$ and acts sharply doubly transitively on the set V of points incident with L_6 but different from p_1. Hence it acts on that set as a Frobenius group and so H_1 had a unique normal regular subgroup N_1 of order s, which is elementary abelian. So $s = \pi_1^{n_1}$ with π_1 a prime and $n_1 \in \mathbf{N}_0$. Similarly $t = \pi_2^{n_2}$, π_2 prime and $n_2 \in \mathbf{N}_0$.

Now let p be a point incident with L_1, $p_1 \neq p \neq p_2$. The subgroup of H_1 fixing p acts regularly on the set V (see above), hence this subgroup is N_1. Since p was essentially arbitrary, N_1 fixes L, L_6 and L_3, fixes every point on L_1, every point on L_2 and acts regularly on V. Let N_1' be the subgroup of H_1 fixing p_6. Suppose an element $\theta \in N_1'$ fixes some point x on L_2, $p_2 \neq x \neq p_3$. Let x' be the point collinear with x and at distance 3 from L_5. By the assumption of Subcase (b), the unique point at distance 4 from x' on L is different from the unique point at distance 4 from p_4 on L, but both these points are fixed by θ. Hence θ fixes a skeleton, which implies that θ is the identity. This shows that N_1' acts regularly on the set V' of points incident with L_2 but distinct from p_2 and p_3. Hence H_1 acts transitively on V' and $N_1 \trianglelefteq H_1$ partitions V' in equal orbits. But $|V'| = s-1$ is relatively prime to π_1. Since N_1 is a π_1-group, this implies that N_1 fixes all elements of L_2.

Hence N_1 fixes L, L_3, L_6 and every point on L_1 and on L_2. Similarly, the dual result holds.

We now forget about the above notation to derive a geometric property. Consider a point p in Γ. Consider the geometry Γ_p. Fix a block K, a point x on K and a point y off K, with x and y non-collinear in Γ. Remember that p is anti-regular, so every 3 points of Γ_p which are non-collinear in Γ define a unique block. Hence, the number of blocks through y and x meeting K in exactly 2 points is $t-1$. On the other hand, there are in total s blocks through x and y, at least one of which meets K in exactly $\{x\}$ (if $K = p^u$, then there is a point w on the line joining u and $x * u$ at distance 4 from y; p^u and p^w have only x in their intersection, otherwise a pentagon arises), so at most $s-1$ blocks through x and y meet K in a second point. This implies $t \leq s$.

STEP 2

In this paragraph, we keep our assumption about the regularity of G on the set of ordered heptagons, or equivalently, on the set of skeletons of Γ. But we handle the case $t < s$, which will be assumed throughout this step. We consider again the notation of the first two paragraphs of Step 1. In particular the subgroup N_1 fixes L_6, L_3 and L, and it fixes L_1 and L_2 pointwise. In fact, L was essentially arbitrary. So we can define a regular group N_2 for a different line M through p_1, $L_6 \neq M \neq L_1$ in the same way as N_1 was defined for L. By the transitivity on V there exists for every $\theta \in N_1$ an element $\theta' \in N_2$ such that $(p_6^\theta)^{\theta'} = p_6$. So $\theta\theta'$ fixes Σ and all points on L_1 or L_2, hence $\theta\theta'$ fixes a subhexagon of order (s, t'), implying $s \leq t$ or $t = t'$ by THAS [13]. By our assumption, $t = t'$, so $\theta' = \theta^{-1}$ and so θ fixes both L and M. Since M was essentially arbitrary, θ fixes every line through p_1. A similar argument shows that θ also fixes every line through p_3. Now consider a line X through p_2, $L_1 \neq X \neq L_2$. The group H_2 fixing Σ and X has order $s - 1$ and acts transitively on the set of points incident with L_1 and different from p_1 and p_2. Suppose any element $\varphi \in H_2$ fixes a point x on L_6, $p_1 \neq x \neq p_6$. By the Subcase (b) assumption, the point u on X, at distance 4 from the point w, which is defined by: w is collinear with x and at distance 3 from L_3, is different from the point u' at distance 4 from p_5 and incident with X. Hence φ fixes the skeleton determined by the apartment containing p_1, p_2, u', p_5 and p_6 and furthermore consisting of the line L_2 and the point u. By the regularity of the action of G on the set of skeletons, φ must be the identity. Hence H_2 acts regularly on the set of points of L_6 different from p_1 and p_6. Since $s > 2$, this group is non-trivial, and letting p_6 now vary over V, we obtain a group H_3 of order $s(s-1)$ acting sharply doubly transitively on V and fixing p_1, p_2, p_3 and L_6, L_1, L_2 and X. A similar argument as above shows that in fact N_1 is a subgroup of H_3 and hence we conclude that Γ is $(p_1, L_1, p_2, L_2, p_3)$-transitive.

By transitivity, Γ is also $(p_2, L_2, p_3, L_3, p_4)$-transitive with corresponding group N_3 (so every element of N_3 fixes all elements incident with one of the points p_2, p_3, p_4 or with one of the lines L_2, L_3). Suppose the commutator $[N_1, N_3]$ is trivial. It is easy to see that this implies that every element of N_1 fixes every line through every point of L_1 respectively L_2. This implies that, with dual notation, $L_1^{L_4} = L_1^{M_4}$, where M_4 is a line opposite L_1, meeting L_3 and at distance 4 from L_6. This means that the dual Γ^D of Γ satisfies the assumption of Case (i), and hence we may assume that Γ^D also satisfies the assumption of Subcase (b). So $s \leq t$, a contradiction. Hence $[N_1, N_3]$ is non-trivial. But it is easily seen that any element θ of $[H_1, N_2]$ fixes every element incident with one of L_1, p_2, L_2, p_3, L_3. By conjugating θ with the subgroup of G fixing Σ, we see that Γ is $(L_1, p_2, L_2, p_3, L_3)$-transitive. Hence Γ is Moufang. But this is again a contradiction, because no Moufang hexagon satisfies the assumption of Subcase (b), see e.g. RONAN [10],(5.9). Hence this situation cannot occur.

STEP 3

In this step, we still assume that G acts regularly on the set of skeletons of Γ, but by the last paragraph, we necessarily have $s = t$. We consider the notation of the first two paragraphs of Step 1 again, in particular the group N_1 fixing L_6, L_3 and L, and fixing L_1 and L_2 pointwise. Note that this group must fix at least one other line L' through p_2, $L_1 \neq L' \neq L_2$, and one other line L'' through p_3, $L_2 \neq L'' \neq L_3$ (because N_1 is a π-group for some prime π and t is a power of π). Similarly, there is a group N_4 of order $s = t$ fixing p_1, p_2, p_3 and p_4, all lines through p_2 and p_3, a point p on L_1, p' on L_2 and p'' on L_3, p, p', p'' not in Σ.

We now interrupt our proof for a moment to get back to the notation of the last paragraph of Step 1, in order to derive some more geometric properties. In the geometry Γ_p we fix a point $p * p'$ with p' a point of Γ at distance 4 from p. Let L be the line joining p' and

$p * p'$. Let x be a point collinear with p' and opposite p. By the Case (i) assumption, there are exactly s points y opposite p and collinear with p' for which $p^x = p^y$. Evidently, every other point u collinear with p' and opposite p gives rise to a different block $p^u \neq p^x$ of Γ_p and moreover $p^u \cap p^x = \{p * p'\}$. So p' defines exactly s blocks of Γ_p which meet two by two in $p * p'$. Varying p' over L (keeping it at distance 4 from p of course), we see that the set of points opposite p and at distance 3 from L define at most s^2 blocks in Γ_p ("at most" since some of them could coincide). Now consider two arbitrary lines L_1 and L_2 through p, not incident with $p * p'$, and consider arbitrary points p_1 and p_2 on L_1 and L_2 respectively, $p_1 \neq p \neq p_2$. Let p'_1 be a point not incident with L but collinear with p' and at distance 4 from p_1. Then p'_1 defines a block in $\mathcal{S}$ through p_1, and hence by the Subcase (b) assumption, there is a point x at distance 3 from L such that $\{p_1, p_2\} \subseteq p^x$. Varying p_1 and p_2 over L_1 respectively L_2, we see that at least s^2 blocks through $p * p'$ are defined by points at distance three from L, hence exactly s^2. It is now easy to see that the incidence structure $\Pi(p, L)$ with point set the set of points collinear with p but not collinear with $p * p'$, and line set the set of blocks of the form p^x with x at distance 3 from L and opposite p, together with the ordinary lines through p, forms an affine plane (with the obvious incidence relation). Every point on L different from $p * p'$ symbolizes a point at infinity of $\Pi(p, L)$ and also p is a point at infinity of $\Pi(p, L)$ in an obvious way.

We now get back to our previous situation (first paragraph of this Step 3). Let θ be any non-trivial element of N_1. This collineation induces in $\Pi(p_2, L_6)$ a non-trivial axial collineation (indeed, all points of the line L_2 are fixed), hence θ is central. Since θ fixes the lines L', L_2 and the line at infinity of $\Pi(p_2, L_6)$, the center must be the point p_2 at infinity, which is incident with all three fixed lines mentioned. Hence θ fixes all lines through p_2. Similarly θ induces an axial non-trivial collineation in $\Pi(p_1, L_2)$ (all points at infinity of $\Pi(p_1, L_2)$ are fixed), and hence θ is central, but as already three lines through the point p_1 at infinity are fixed (the line at infinity, L and L_1), p_1 must be the center, hence θ fixes all lines through p_1 and similarly, also all lines through p_3. We conclude that Γ is $(p_1, L_1, p_2, L_2, p_3)$-transitive. Considering again the commutator $[N_1, N_3]$ as in the second paragraph of Step 2, we obtain that either Γ^D has anti-regular points, in which case Γ is dually $(L_1, p_2, L_2, p_3, L_3)$-transitive and hence Moufang; or $[N_1, N_3]$ is non-trivial and Γ is Moufang again. But as before, this is a contradiction since a Moufang hexagon cannot satisfy the assumption of Subcase (b).

So we conclude that Case (i), Subcase (b) cannot occur if G acts regularly on the set of ordered heptagons.

STEP 4

We now drop every extra assumption on Γ and G. Consider a certain fixed heptagon in Γ and take the intersection of all subhexagons containing this heptagon. This is again a thick generalized hexagon Γ' which does not contain strictly any thick subhexagon. Clearly G induces in Γ' a group of collineations acting transitively on the set of ordered heptagons, but since Γ' does not contain strictly any subhexagon, this action must be regular. It is also clear that Γ' satisfies the assumptions of Case (i) and Subcase (b), so by the previous steps, Γ' cannot exist. Hence Γ cannot exist.

This completes the proof of Case (i).

2.4 Case (ii)

Here we assume that all intersection sets have order ≥ 2 in both Γ and Γ^D, and that $s = t$. As in the previous case, it suffices to show that this situation cannot occur if G acts regularly on the set of ordered heptagons.

Note that Γ does not even contain any subhexagon of order $(1, s)$ or $(s, 1)$ since this would imply that Γ satisfies the condition of Case (i). Indeed, if Γ' is a subhexagon of order $(1, s)$ containing two points p, p' at distance 4 from each other, then all points u of Γ' collinear with p' and opposite p determine the same set p^u as this set consists of *all* points of Γ' collinear with p.

Consider an apartment $\Sigma = (p_1, L_1, \ldots, p_6, L_6, p_1)$ as before, then we again obtain a sharply doubly transitive permutation group and a group N_1 fixing L_6, L_3, fixing L_1 and L_2 pointwise, and fixing some lines L, L', L'' through p_1, p_2, p_3 respectively and not contained in Σ; N_1 acts regularly on the set of points incident with L_6 but different from p_1. By the transitivity property of G, we can choose either L or L' or L'' (but we obtain possibly a different group N_1^*) arbitrarily (but with the same restrictions). With every element θ of N_1 corresponds an element θ' of N_1^* such that $\theta\theta'$ fixes Σ elementwise. But it also fixes every point on L_1 and on L_2, hence it fixes a subhexagon of order (s, t'). By Thas [13], $t' = 1$ or $t' = s$. We already ruled out $t' = 1$, hence $t' = s$ and so $\theta = \theta'$. We conclude that N_1 also fixes every line through p_1, every line through p_2 and every line through p_3. So Γ is $(p_1, L_1, p_2, L_2, p_3)$-transitive. Also the dual transitivity holds here and so Γ is Moufang. But this is impossible since every Moufang hexagon of order (s, s) has a subhexagon of order $(1, s)$ or $(s, 1)$, see e.g. Ronan [10],(6.11).

This completes the proof of the Theorem 1.

3 Proof of Theorem 2

Now we suppose that Γ is a finite thick generalized octagon of order (s, t) admitting a group G acting transitively on the set of all ordered nonagons of Γ, or equivalently (as for hexagons), on the set of all skeletons of Γ. Upon taking the intersection of all suboctagons containing a fixed nonagon, we may suppose that G acts regularly on the above sets (because if we show that this *regular* situation cannot occur, then also the more general *transitive* situation is impossible). By the fact that $2st$ must be a perfect square, see Feit & Higman [3], we may assume that $s < t$.

We adopt the following notation throughout: Σ is the apartment $(p_1, L_1, p_2, \ldots, p_8, L_8, p_1)$, where consecutive elements are incident. The *distance* is again the one inherited from the incidence graph and elements at distance 8 are called *opposite*.

The reader can easily generalize the definition of Moufang condition to octagons and by Tits [19], all Moufang octagons are Ree-Tits octagons.

We again consider some steps.

STEP A.

In this step, we show the claim that G acts regularly on configurations of the form (Σ, p, L), where L is a line incident with p_1, p is a point incident with L_2 and neither L nor p is in Σ. The subgroup H_1 of G fixing Σ elementwise has order $(s-1)(t-1)$ and acts transitively on the set of lines through p_1 different from L_8 and L_1. The stabilizer H_2 of L in H_1 has therefore order $(s-1)$. We have to show that H_2 acts transitively on the set of $s-1$ points on L_2 different from p_2 and p_3. Suppose this is not the case, then there is a collineation θ in H_2 fixing some point p on L_2, $p_2 \neq p \neq p_3$. There is a unique line M at distance 4 from L and at distance 3 from p_5; there is a unique point x on M at distance 6 from p; there is a unque point y on L_7 (different from p_7 and p_8) at distance 6 from x and there is a unique point z on L_3 at distance 6 from y. Obviously, θ fixes all these elements. Dually, there is a line Z through p_8, $L_7 \neq Z \neq L_8$, fixed by θ. But now θ fixes a skeleton $(\Sigma; Z, y)$, hence θ is the identity and our claim follows.

STEP B

We show half of the Moufang condition. As for generalized hexagons (Step 1 of 2.3.2), we have a sharply doubly transitive group H_3 acting on the lines through p_8 different from L_8 fixing $p_8, p_1, \cdots, p_4, p$, where p is a certain arbitrarily chosen point on L_8 (p not in Σ). As in 2.3.2 and using Step A above, one shows again that H_3 has a regular normal subgroup N_3 fixing every line through p_1, p_2 and p_3. Since we have now $t > s$, and by Thas [14], Γ does not contain a suboctagon of order (s', t) unless $s = s'$, we can dualize the argument of Step 2 of 2.3.2 (using also Step A above) to obtain that Γ is $(L_8, p_1, \ldots, L_3)$-transitive.

Note that, dually, we have a group N_1 of order s fixing L_8, L_4 and a certain arbitrary line L through p_1, $L_8 \neq L \neq L_1$, and fixing every point on L_1, L_2 and L_3. The group N_1 acts regularly on the set of points incident with L_8 but different from p_1.

In particular we deduce that both s and t are powers of a prime (not necessarily the same one, but since $2st$ is a square, at least one of these primes equals 2).

STEP C

In this step, we determine two geometric properties that Γ must have, if it were not Moufang.

(R1) If Γ is not Moufang, then the commutator $[N_3, N_4]$ must be trivial (N_4 is the group of all $(L_1, p_2, \ldots, L_4)$-elations). As in Step 2 of 2.3.2, this means that, whenever x is a point collinear with p_6 (or p_4) and at distance 6 from p_2 (or p_8), then $p_1^x = p_1^{p_5}$ (where y^z is the set of points collinear with y and at distance 6 from z; y and z must be opposite).

(R2) Let p be any point on L_1, $p_1 \neq p \neq p_2$. Define the elements $p\,I\,M_1\,I\,x_1\,I\,M_2\,I\,x_2\,I\,M_3\,I\,x_3\,I\,L_5$. Let M be any line through p_4, $L_3 \neq M \neq L_4$, and let θ (respectively θ') be the unique $(L_8, p_1, \ldots, L_3)$-elation (respectively $(M_1, p, L_1, \ldots, L_3)$-elation) mapping L_4 onto M.

First assume that the unique point x on M at distance 6 from p_8 is opposite x_1. Then the collineation $\theta'\theta^{-1}$ fixes all points of L_1, L_2 and L_3, it fixes all lines through p_2 and p_3 and it does not fix all points on L_4. By composing with a suitable $(L_1, p_2, \ldots, L_4)$-elation, we obtain a collineation φ fixing L_8 and L_4, every point on L_1, L_2 and L_3 and every line through p_2 and p_3. By conjugation, the group N_5 of such collineations acts transitively (hence regularly) on the points incident with L_4 different from p_4. If at least one element φ of N_5 fixes at least one line L through p_1, $L_8 \neq L \neq L_1$, then by conjugation with the subgroup H_2 of G fixing Σ and L, every element of N_5 must fix L. By conjugation with the subgroup H_1 fixing Σ (and acting transitively on the set of lines through p_1 different from L_8 and L_1), we see that N_5 must fix every line through p_1. By a similar argument, N_5 fixes all lines through p_4 if it does not act semi-regularly on the set of lines through p_4 different from L_3 and L_4. So Γ is $(p_1, L_1, \ldots, p_4)$-transitive unless N_5 acts semi-regularly on the set of $t-1$ lines through p_1 different from L_8 and L_1, or on the set of $t-1$ lines through p_4 different from L_3 and L_4. If this happens, then s divides $t-1$.

We now show that also $s-1$ must divide $t-1$. Let $\theta \in N_5$ be non-trivial. Let $\theta' \in N_1$ such that $\theta'\theta^{-1}$ fixes Σ. Since θ does not fix L, $\theta'\theta^{-1}$ is non-trivial and fixes every point on L_1, L_2 and L_3, hence it fixes a suboctagon of order $(s, 1)$. This suboctagon is easily seen to be Moufang (by the presence of the group N_1 or N_5), and by transitivity, it is also self-dual. Hence it is the double of a symplectic quadrangle of characteristic 2 and, if $s \neq 2$, the elations generate the symlectic group $PSp_4(s)$ which contains a subgroup K (of "generalized homologies") of order $s-1$ which fixes Σ and every point on L_1, and which acts transitively on the remaining points of L_8 (indeed, this follows from the fact that the symplectic group $PSp_4(s)$ is simple in this case). No non-trivial element of K can fix an additional line through p_1 since otherwise a non-trivial thick suboctagon is fixed. Hence the claim for $s \neq 2$. But if $s = 2$, then $t = 4$ and the result follows.

Hence $s(s-1)$ divides $t-1$. But this implies that either $s(s-1) = t-1$, or $2s(s-1) \leq t-1$. Note that certainly t is odd, and hence t is a square. But if $s(s-1) = t-1$, then $s^2 - s + 1 = t$ and so $(s-1)^2 < t < s^2$, a contradiction. Hence $2s(s-1) \leq t-1 \leq s^2-1$ (by the inequality of Higman [6]), implying $s = 1$. This shows that our assumption is false. Hence x is at distance 6 from x_1, for every choice of M and p.

From this we derive the following property of Γ:

(RR) *If x and y are opposite points of Γ, L is a line at distance 3 from y and 5 from x, and z is opposite x and at distance 3 from L, then either $x^y = x^z$, or $|x^y \cap x^z| = 1$.*

Indeed, suppose $|x^y \cap x^z| \geq 2$. Let a be the unique point collinear with x and at distance 3 from L, and let b be a second point in $x^y \cap x^z$. Let L_y and L_z be the unique lines through b at distance 5 from y and z respectively and let p_y and p_z be the points on L collinear with y and z respectively. If $p_y = p_z$ or $L_y = L_z$, then $x^y = x^z$ by (R1) and (R2) respectively. So suppose $p_y \neq p_z$ and $L_y \neq L_z$. Let u be the unique point collinear with z and at distance 5 from L_y. By (R1), $x^z = x^u$ and by (R2), $x^y = x^u$. Hence (RR) follows.

The property (RR) expresses a kind of distance-2 regularity. In the next and last step, we will show that this is impossible for any generalized octagon. The proof is completely the same as the one ruling out distance-2 regular octagons in Van Maldeghem [21], since in fact (RR) is the only thing used in the proof in that paper. For the convenience of the reader, we repeat this proof here.

STEP D

Let $(p_1, L_1, p_2, \ldots, p_8, L_8, p_1)$ be as above. We assume that property (RR) holds for every pair of opposite points x, y (which we may by transitivity). Let p_1' be incident with L_8 but different from both p_1 and p_8 (Γ is thick). Construct the sequence $(p_1', p_1'p_2', p_2', p_2'p_3', p_3', p_3'p_4', p_4')$ such that p_4' is incident with L_4. Let $(p_2, p_2p_3'', p_3'', p_3''p_4'', p_4'', p_4''p_5'', p_5'', p_5''p_6, p_6)$ be a sequence with $p_3 \neq p_3'' \neq p_1$ (again possible by the thickness assumption). Since $\{p_1, p_3\} \subseteq p_2^{p_6} \cap p_2^{p_3'}$, property (RR) implies that p_3' is at distance 6 from p_3'' and so we can define a sequence $(p_3', p_3'x_1, x_1, x_1x_2, x_2, x_2p_3'', p_3'')$. Clearly x_1 is incident with neither $p_2'p_3'$, nor $p_3'p_4'$.

Suppose first that x_2 is not incident with $p_3''p_4''$. Let x_3 be the unique point on $p_3''p_4''$ at distance 6 from p_4'. Clearly $p_3'' \neq x_3 \neq p_4''$. But $\{x_2, p_2\} \subseteq (p_3'')^{p_2'} \cap (p_3'')^{p_4'}$, hence the distance between p_2' and x_3 is 6, so p_2' and p_4'' are opposite. Now $p_6^{p_2}$ and $p_6^{p_2'}$ share the points p_5 and p_7, and so there is a sequence $(p_2', p_2'y_1, y_1, y_1y_2, y_2, y_2p_5'', p_5'')$. Clearly y_1 is incident with neither $p_1'p_2'$ nor $p_2'p_3'$. And if y_2 were incident with $p_4''p_5''$, then the distance between p_2' and p_4'' would be 6, contradicting the fact that they are opposite. Also, $p_2'x_1 \neq p_2'y_1$ (otherwise a cycle of length 14 or 12 via x_3 and p_5'' arises). Clearly p_3'' and y_1 are opposite, but this contradicts $\{p_1', p_3', y_1\} \subseteq (p_2')^{p_6}$ and $\{p_1', p_3'\} \subseteq (p_2')^{p_3''} \cap (p_2')^{p_3}$ and property (RR). We conclude that x_2 must be incident with $p_3''p_4''$.

So we may suppose that x_2 is incident with $p_3''p_4''$. By symmetry, y_2 (defined as in the previous paragraph) must be incident with $p_4''p_5''$. But then $(p_2', y_1, y_2, p_4'', x_2, x_1, p_3', p_2')$ forms a cycle of length 14 in Γ, a contradiction.

This completes the proof of Theorem 2.

ACKNOWLEDGEMENT

I would like to thank J. Tits who, after reading a preliminary version of the present paper, drew my attention on the construction method of Kegel and Schleiermacher as adapted by himself to show that the finiteness condition in my results is essential.

I would also like to thank L. Storme for requiring a missing argument in Step 1 of 2.3.2.

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