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Generalized quadrangles and the Axiom of Veblen

J. A. Thas H. Van Maldeghem*

If x is a regular point of a generalized quadrangle $\mathcal{S} = (P, B, \mathrm{I})$ of order $(s, t), s \neq 1$, then x defines a dual net with $t + 1$ points on any line and s lines through every point. If $s \neq t, s > 1, t > 1$, then $\mathcal{S}$ is isomorphic to a $T_3(O)$ of Tits if and only if $\mathcal{S}$ has a coregular point x such that for each line L incident with x the corresponding dual net satisfies the Axiom of Veblen. As a corollary we obtain some elegant characterizations of the classical generalized quadrangles $Q(5, s)$. Further we consider the translation generalized quadrangles $\mathcal{S}^{(p)}$ of order $(s, s^2), s \neq 1$, with base point p for which the dual net defined by L, with $p \mathrm{I} L$, satisfies the Axiom of Veblen. Next there is a section on Property (G) and the Axiom of Veblen, and a section on flock generalized quadrangles and the Axiom of Veblen. This last section contains a characterization of the TGQ of Kantor in terms of the Axiom of Veblen. Finally, we prove that the dual net defined by a regular point of $\mathcal{S}$, where the order of $\mathcal{S}$ is (s, t) with $s \neq t$ and $s \neq 1 \neq t$, satisfies the Axiom of Veblen if and only if $\mathcal{S}$ admits a certain set of proper subquadrangles.

1 Introduction

For terminology, notation, and results concerning finite generalized quadrangles and not explicitly given here, see the monograph of Payne and Thas [11], which is henceforth denoted FGQ.

Let $\mathcal{S} = (P, B, \mathrm{I})$ be a (finite) generalized quadrangle (GQ) of order $(s, t), s \geq 1, t \geq 1$. So $\mathcal{S}$ has $v = |P| = (1 + s)(1 + st)$ points and $b = |B| = (1 + t)(1 + st)$ lines. If $s \neq 1 \neq t$, then $t \leq s^2$ and, dually, $s \leq t^2$; also $s + t$ divides $st(1 + s)(1 + t)$.

There is a point-line duality for GQ (of order (s, t)) for which in any definition or theorem the words "point" and "line" are interchanged and the parameters s and t are interchanged. Normally, we assume without further notice that the dual of a given theorem or definition has also been given.

Given two (not necessarily distinct) points x, x' of $\mathcal{S}$, we write $x \sim x'$ and say that x and x' are *collinear*, provided that there is some line L for which

*Senior Research Associate of the Belgian National Fund for Scientific Research

$x \mathrm{I} L \mathrm{I} x'$; hence $x \not\sim x'$ means that x and x' are not collinear. Dually, for $L, L' \in B$, we write $L \sim L'$ or $L \not\sim L'$ according as L and L' are concurrent or nonconcurrent. When $x \sim x'$ we also say that x is *orthogonal* or *perpendicular* to x', similarly for $L \sim L'$. The line incident with distinct collinear points x and x' is denoted xx', and the point incident with distinct concurrent lines L and L' is denoted $L \cap L'$.

For $x \in P$ put $x^\perp = \{x' \in P \parallel x \sim x'\}$, and note that $x \in x^\perp$. The trace of a pair $\{x, x'\}$ of distinct points is defined to be the set $x^\perp \cap x'^\perp$ and is denoted $\mathrm{tr}(x, x')$ or $\{x, x'\}^\perp$; then $|\{x, x'\}^\perp| = s+1$ or $t+1$ according as $x \sim x'$ or $x \not\sim x'$. More generally, if $A \subset P$, A "*perp*" is defined by $A^\perp = \cap\{x^\perp \parallel x \in A\}$. For $x \neq x'$, the *span* of the pair $\{x, x'\}$ is $\mathrm{sp}(x, x') = \{x, x'\}^{\perp\perp} = \{u \in P \parallel u \in z^\perp$ for all $z \in x^\perp \cap x'^\perp\}$. When $x \not\sim x'$, then $\{x, x'\}^{\perp\perp}$ is also called the *hyperbolic line* defined by x and x', and $|\{x, x'\}^{\perp\perp}| = s+1$ or $|\{x, x'\}^{\perp\perp}| \leq t+1$ according as $x \sim x'$ or $x \not\sim x'$.

2 Regularity

Let $\mathcal{S} = (P, B, \mathrm{I})$ be a finite GQ of order (s, t). If $x \sim x', x \neq x'$, or if $x \not\sim x'$ and $|\{x, x'\}^{\perp\perp}| = t+1$, where $x, x' \in P$, we say the pair $\{x, x'\}$ is *regular*. The point x is *regular* provided $\{x, x'\}$ is regular for all $x' \in P, x' \neq x$. Regularity for lines is defined dually.

A (finite) *net* of *order* $k\,(\geq 2)$ and *degree* $r\,(\geq 2)$ is an incidence structure $\mathcal{N} = (P, B, \mathrm{I})$ satisfying

(i) each point is incident with r lines and two distinct points are incident with at most one line;

(ii) each line is incident with k points and two distinct lines are incident with at most one point;

(iii) if x is a point and L is a line not incident with x, then there is a unique line M incident with x and not concurrent with L.

For a net of order k and degree r we have $|P| = k^2$ and $|B| = kr$.

Theorem 2.1 (1.3.1 of Payne and Thas [11]) . *Let x be a regular point of the* GQ $\mathcal{S} = (P, B, I)$ *of order $(s, t), s > 1$. Then the incidence structure with pointset $x^\perp - \{x\}$, with lineset the set of spans $\{y, z\}^{\perp\perp}$, where $y, z \in x^\perp - \{x\}, y \not\sim z$, and with the natural incidence, is the dual of a net of order s and degree $t+1$. If in particular $s = t > 1$, there arises a dual affine plane of order s. Also, in the case $s = t > 1$ the incidence structure π_x with pointset $x^\perp$, with lineset the set of spans $\{y, z\}^{\perp\perp}$, where $y, z \in x^\perp, y \neq z$, and with the natural incidence, is a projective plane of order s.*

3 Dual nets and the Axiom of Veblen

Now we introduce the *Axiom of Veblen* for dual nets $\mathcal{N}^* = (P, B, \mathrm{I})$.

Axiom of Veblen. *If* $L_1 \mathrm{~I~} x \mathrm{~I~} L_2, L_1 \neq L_2, M_1 \not\mathrm{I}\, x \not\mathrm{I}\, M_2$, *and if* L_i *is concurrent with* M_j *for all* $i, j \in \{1, 2\}$, *then* M_1 *is concurrent with* M_2.

The only known dual net $\mathcal{N}^*$ which is not a dual affine plane and which satisfies the Axiom of Veblen is the dual net $H_q^n, n > 2$, which is constructed as follows : the points of H_q^n are the points of $\mathrm{PG}(n, q)$ not in a given subspace $\mathrm{PG}(n-2, q) \subset \mathrm{PG}(n, q)$, the lines of H_q^n are the lines of $\mathrm{PG}(n, q)$ which have no point in common with $\mathrm{PG}(n-2, q)$, the incidence in H_q^n is the natural one. By the following theorem these dual nets H_q^n are characterized by the Axiom of Veblen.

Theorem 3.1 (Thas and De Clerck [14]) *Let* $\mathcal{N}^*$ *be a dual net with* $s+1$ *points on any line and* $t+1$ *lines through any point, where* $t+1 > s$. *If* $\mathcal{N}^*$ *satisfies the Axiom of Veblen, then* $\mathcal{N}^* \cong H_q^n$ *with* $n > 2$ *(hence* $s = q$ *and* $t + 1 = q^{n-1}$*).*

4 Generalized quadrangles and the Axiom of Veblen

Consider a GQ $T_3(O)$ of Tits, with O an ovoid of $\mathrm{PG}(3, q)$; see 3.1.2 of FGQ. Here $s = q$ and $t = q^2$. Then the point (∞) is coregular, that is, each line incident with (∞) is regular. It is an easy exercise to check that for each line incident with (∞) the corresponding dual net is isomorphic to H_q^3. Hence for each line incident with the point (∞) the corresponding dual net satisfies the Axiom of Veblen. We now prove the converse.

Theorem 4.1 *Let* $\mathcal{S} = (P, B, I)$ *be a* GQ *of order* (s, t) *with* $s \neq t, s > 1$ *and* $t > 1$. *If* $\mathcal{S}$ *has a coregular point* x *and if for each line* L *incident with* x *the correponding dual net* $\mathcal{N}_L^*$ *satisfies the Axiom of Veblen, then* $\mathcal{S}$ *is isomorphic to a* $T_3(O)$ *of Tits.*

Proof Let L_1, L_2, L_3 be three lines no two of which are concurrent, let M_1, M_2, M_3 be three lines no two of which are concurrent, let $L_i \not\sim M_j$ if and only if $\{i, j\} = \{1, 2\}$ and assume that $x \mathrm{~I~} L_1$. By 5.3.8 of FGQ it is sufficient to prove that for any line $L_4 \in \{M_1, M_2\}^\perp$ with $L_4 \not\sim L_i, i = 1, 2, 3$, there exists a line M_4 concurrent with L_1, L_2, L_4.

So let $L_4 \in \{M_1, M_2\}^\perp$ with $L_4 \not\sim L_i, i = 1, 2, 3$. Consider the line R containing $L_2 \cap M_2$ and concurrent with L_1. Further, consider the line R' containing $M_2 \cap L_4$ and concurrent with L_1. By the regularity of L_1 there is a line $S \in \{M_1, M_3\}^{\perp\perp}$ through the point $L_3 \cap M_2$. Clearly the lines L_1 and S are concurrent. So the line L_1 is concurrent with the lines S, R, R'; also the line M_2 is concurrent with the lines S, R, R'. By the regularity of L_1 the line S belongs to the line $\{R, R'\}^{\perp\perp}$ of the dual net $\mathcal{N}^*_{L_1}$ defined by L_1. Hence the lines $\{R, R'\}^{\perp\perp}$ and $\{M_1, M_3\}^{\perp\perp}$ of $\mathcal{N}^*_{L_1}$ have the element S in common. By the Axiom of Veblen, also the lines $\{M_1, R'\}^{\perp\perp}$ and $\{M_3, R\}^{\perp\perp}$ of $\mathcal{N}^*_{L_1}$ have an element M_4 in common. Consequently M_4 is concurrent with L_1, L_2, L_4. Now from 5.3.8 of FGQ it follows that $\mathcal{S}$ is isomorphic to a $T_3(O)$ of Tits. $\square$

Corollary 4.2 *Let $\mathcal{S}$ be a* GQ *of order (s, t) with $s \neq t$, $s > 1$ and $t > 1$.*

(i) *If s is odd, then $\mathcal{S}$ is isomorphic to the classical* GQ $Q(5, s)$ *if and only if it has a coregular point x and if for each line L incident with x the corresponding dual net $\mathcal{N}^*_L$ satisfies the Axiom of Veblen.*

(ii) *If s is even, then $\mathcal{S}$ is isomorphic to the classical* GQ $Q(5, s)$ *if and only if all its lines are regular and if for at least one point x and all lines L incident with x the dual nets $\mathcal{N}^*_L$ satisfy the Axiom of Veblen.*

Proof Let (x, L) be an incident point-line pair of the GQ $Q(5, s)$. By 3.2.4 of FGQ there is an isomorphism of $Q(5, s)$ onto $T_3(O)$, with O an elliptic quadric of $\mathrm{PG}(3, s)$, which maps x onto the point (∞). It follows that $\mathcal{N}^*_L$ satisfies the Axiom of Veblen.

Conversely, assume that the GQ $\mathcal{S}$ of order (s, t), with s odd, $s \neq t, s > 1$ and $t > 1$, has a coregular point x such that for each line L incident with x the dual net $\mathcal{N}^*_L$ satisfies the Axiom of Veblen. Then by Theorem 4.1 the GQ $\mathcal{S}$ is isomorphic to $T_3(O)$. By Barlotti [2] and Panella [9] each ovoid O of $\mathrm{PG}(3, s)$, with s odd, is an elliptic quadric. Now by 3.2.4 of FGQ we have $\mathcal{S} \cong T_3(O) \cong Q(5, s)$.

Finally, assume that for the GQ $\mathcal{S}$ of order (s, t), with s even, $s \neq t, t > 1$, all lines are regular and that for at least one point x and all lines L incident with x the dual nets $\mathcal{N}^*_L$ satisfy the Axiom of Veblen. Then by Theorem 4.1 the GQ $\mathcal{S}$ is isomorphic to $T_3(O)$. Since all lines of $\mathcal{S} \cong T_3(O)$ are regular, by 3.3.3(iii) of FGQ we finally have $\mathcal{S} \cong T_3(O) \cong Q(5, s)$. $\square$

5 Translation generalized quadrangles and the Axiom of Veblen

Let $\mathcal{S} = (P, B, \mathrm{I})$ be a GQ of order $(s, t), s \neq 1, t \neq 1$. A collineation θ of $\mathcal{S}$ is an *elation* about the point p if $\theta = \mathrm{id}$ or if θ fixes all lines incident with p

and fixes no point of $P - p^\perp$. If there is a group H of elations about p acting regularly on $P - p^\perp$, we say $\mathcal{S}$ is an *elation generalized quadrangle* (EGQ) with *elation group* H and *base point* p. Briefly, we say that $(\mathcal{S}^{(p)}, H)$ or $\mathcal{S}^{(p)}$ is an EGQ. If the group H is abelian, then we say that the EGQ $(\mathcal{S}^{(p)}, H)$ is a *translation generalized quadrangle.* For any TGQ $\mathcal{S}^{(p)}$ the point p is coregular so that the parameters s and t satisfy $s \leq t$; see 8.2 of FGQ. Also, by 8.5.2 of FGQ, for any TGQ with $s \neq t$ we have $s = q^a$ and $t = q^{a+1}$, with q a prime power and a an odd integer; if s (or t) is even then by 8.6.1(iv) of FGQ either $s = t$ or $s^2 = t$.

In $\mathrm{PG}(2n+m-1, q)$ consider a set $O(n, m, q)$ of q^m+1 $(n-1)$-dimensional subspaces $\mathrm{PG}^{(0)}(n-1, q)$,$\mathrm{PG}^{(1)}(n-1, q), \ldots$, $\mathrm{PG}^{(q^m)}(n-1, q)$, every three of which generate a $\mathrm{PG}(3n-1, q)$ and such that each element $\mathrm{PG}^{(i)}(n-1, q)$ of $O(n, m, q)$ is contained in a $\mathrm{PG}^{(i)}(n+m-1, q)$ having no point in common with any $\mathrm{PG}^{(j)}(n-1, q)$ for $j \neq i$. It is easy to check that $\mathrm{PG}^{(i)}(n+m-1, q)$ is uniquely determined, $i = 0, 1, \ldots, q^m$. The space $\mathrm{PG}^{(i)}(n+m-1, q)$ is called the *tangent space* of $O(n, m, q)$ at $\mathrm{PG}^{(i)}(n-1, q)$. For $n = m$ such a set $O(n, n, q)$ is called a *generalized oval* or an $[n-1]$-*oval* of $\mathrm{PG}(3n-1, q)$; a generalized oval of $\mathrm{PG}(2, q)$ is just an oval of $\mathrm{PG}(2, q)$. For $n \neq m$ such a set $O(n, m, q)$ is called a *generalized ovoid* or an $[n-1]$-*ovoid* or an *egg* of $\mathrm{PG}(2n+m-1, q)$; a $[0]$-ovoid of $\mathrm{PG}(3, q)$ is just an ovoid of $\mathrm{PG}(3, q)$.

Now embed $\mathrm{PG}(2n+m-1, q)$ in a $\mathrm{PG}(2n+m, q)$, and construct a point-line geometry $T(n, m, q)$ as follows.

Points are of three types :

(i) the points of $\mathrm{PG}(2n+m, q)$–$\mathrm{PG}(2n+m-1, q)$;

(ii) the $(n+m)$-dimensional subspaces of $\mathrm{PG}(2n+m, q)$ which intersect $\mathrm{PG}(2n+m-1, q)$ in one of the $\mathrm{PG}^{(i)}(n+m-1, q)$;

(iii) the symbol (∞).

Lines are of two types :

(a) the n-dimensional subspaces of $\mathrm{PG}(2n+m, q)$ which intersect $\mathrm{PG}(2n+m-1, q)$ in a $\mathrm{PG}^{(i)}(n-1, q)$;

(b) the elements of $O(n, m, q)$.

Incidence in $T(n, m, q)$ is defined as follows. A point of type (i) is incident only with lines of type (a); here the incidence is that of $\mathrm{PG}(2n+m, q)$. A point of type (ii) is incident with all lines of type (a) contained in it and with the unique element of $O(n, m, q)$ contained in it. The point (∞) is incident with no line of type (a) and with all lines of type (b).

Theorem 5.1 (8.7.1 of Payne and Thas [11]) *$T(n,m,q)$ is a* TGQ *of order (q^n,q^m) with base point (∞). Conversely, every* TGQ *is isomorphic to a $T(n,m,q)$. It follows that the theory of the* TGQ *is equivalent to the theory of the sets $O(n,m,q)$.*

Corollary 5.2 *The following hold for any $O(n,m,q)$:*

(i) *$n=m$ or $n(c+1)=mc$ with c odd;*

(ii) *if q is even, then $n=m$ or $m=2n$.*

Let $O(n,2n,q)$ be an egg of $\mathrm{PG}(4n-1,q)$. We say that $O(n,2n,q)$ is *good at the element* $\mathrm{PG}^{(i)}(n-1,q)$ of $O(n,2n,q)$ if any $\mathrm{PG}(3n-1,q)$ containing $\mathrm{PG}^{(i)}(n-1,q)$ and at least two other elements of $O(n,2n,q)$, contains exactly q^n+1 elements of $O(n,2n,q)$.

Theorem 5.3 *Let $\mathcal{S}^{(p)}$ be a* TGQ *of order $(s,s^2), s\neq 1$, with base point p. Then the dual net $\mathcal{N}_L^*$ defined by the regular line L, with $p\ I\ L$, satisfies the Axiom of Veblen if and only if the egg $O(n,2n,q)$ which corresponds to $\mathcal{S}^{(p)}$ is good at its element $PG^{(i)}(n-1,q)$ which corresponds to L.*

Proof Assume that the dual net $\mathcal{N}_L^*$ satisfies the Axiom of Veblen. Let the egg $O(n,2n,q)$ correspond to $\mathcal{S}^{(p)}$ and let $\mathrm{PG}^{(i)}(n-1,q)$ correspond to L. We have $s=q^n$. The dual net has q^n+1 points on a line and q^{2n} lines through a point. By Theorem 3.1 the dual net $\mathcal{N}_L^*$ is isomorphic to $H^3_{q^n}$. Consider the TGQ $T(n,2n,q)\cong\mathcal{S}^{(p)}$ and let $\mathrm{PG}(3n,q)$ be a subspace skew to $\mathrm{PG}^{(i)}(n-1,q)$ in the projective space $\mathrm{PG}(4n,q)$ in which $T(n,2n,q)$ is defined. Let $O(n,2n,q)=\{\mathrm{PG}^{(0)}(n-1,q),\mathrm{PG}^{(1)}(n-1,q),\ldots,\ \mathrm{PG}^{(q^{2n})}(n-1,q)\}$, let $\langle\mathrm{PG}^{(i)}(n-1,q),\mathrm{PG}^{(j)}(n-1,q)\rangle\cap\mathrm{PG}(3n,q)=\pi_j$ for all $j\neq i$ (π_j is $(n-1)$-dimensional), let $\mathrm{PG}(4n-1,q)\cap\ \mathrm{PG}(3n,q)=\mathrm{PG}(3n-1,q)$ with $\mathrm{PG}(4n-1,q)$ the space of $O(n,2n,q)$, and let $\mathrm{PG}^{(i)}(3n-1,q)\cap\mathrm{PG}(3n,q)=\mathrm{PG}(2n-1,q)$ with $\mathrm{PG}^{(i)}(3n-1,q)$ the tangent space of $O(n,2n,q)$ at $\mathrm{PG}^{(i)}(n-1,q)$. Then the dual net $\mathcal{N}_L^*$ is isomorphic to the following dual net $\mathcal{N}^*$: points of $\mathcal{N}^*$ are the q^{2n} spaces $\pi_j, j\neq i$, and the q^{3n} points of $\mathrm{PG}(3n,q)-\mathrm{PG}(3n-1,q)$, lines of $\mathcal{N}^*$ are the q^{4n} n-dimensional subspaces of $\mathrm{PG}(3n,q)$ which are not contained in $\mathrm{PG}(3n-1,q)$ and contain an element $\pi_j, j\neq i$, and incidence is the natural one. Clearly the points $\pi_j, j\neq i$, of $\mathcal{N}^*$ form a parallel class of points. Let M be a line of $\mathcal{N}^*$ incident with π_j and let $\pi_k\neq\pi_j, k\neq i\neq j$. As $\mathcal{N}^*\cong H^3_{q^n}$ the elements π_k and M of $\mathcal{N}^*$ generate a dual affine plane $\mathcal{A}^*$ in $\mathcal{N}^*$, and the plane $\mathcal{A}^*$ contains q^n points $\pi_l, l\neq i$. Clearly the points of $\mathcal{A}^*$ not of type π_l are the q^{2n} points of the subspace $\langle\pi_k,M\rangle$ of $\mathrm{PG}(3n,q)$ which are not contained in $\mathrm{PG}(3n-1,q)$. Hence the q^n points of $\mathcal{A}^*$ of type π_l are contained in $\langle\pi_k,M\rangle\cap$ $\mathrm{PG}(3n-1,q)$. It follows that these q^n elements π_l are contained in a $(2n-1)$-dimensional space $\mathrm{PG}'(2n-1,q)$; also, they form a partition of $\mathrm{PG}'(2n-1,q) - \mathrm{PG}(2n-1,q)$. Consequently for any two elements $\pi_l, \pi_{l'}, l \neq i \neq l'$, the space $\langle \pi_l, \pi_{l'} \rangle$ contains exactly q^n elements $\pi_r, r \neq i$. Hence for any two spaces $\mathrm{PG}^{(l)}(n-1,q)$ and $\mathrm{PG}^{(l')}(n-1,q)$ of $O(n,2n,q) - \{\mathrm{PG}^{(i)}(n-1,q)\}$, the $(3n-1)$-dimensional space $\langle \mathrm{PG}^{(i)}(n-1,q), \mathrm{PG}^{(l)}(n-1,q), \mathrm{PG}^{(l')}(n-1,q) \rangle$ contains exactly q^n+1 elements of $O(n,2n,q)$. We conclude that $O(n,2n,q)$ is good at $\mathrm{PG}^{(i)}(n-1,q)$.

Conversely, assume that $O(n,2n,q)$ is good at the element $\mathrm{PG}^{(i)}(n-1,q)$ which corresponds to L. As in the first part of the proof we project onto a $\mathrm{PG}(3n,q)$ and we use the same notations. Since $O(n,2n,q)$ is good at $\mathrm{PG}^{(i)}(n-1,q)$, for any two elements $\pi_l, \pi_{l'}, l \neq i \neq l'$, the space $\langle \pi_l, \pi_{l'} \rangle$ contains exactly q^n elements $\pi_r, r \neq i$; these q^n elements form a partition of the points of $\langle \pi_l, \pi_{l'} \rangle$ which are not contained in $\mathrm{PG}(2n-1,q)$. If M, M' are distinct concurrent lines of $\mathcal{N}^*$, then it is easily checked that M and M' generate a dual affine plane $\mathcal{A}^*$ of order q^n in $\mathcal{N}^*$. As $\mathcal{A}^*$ satisfies the Axiom of Veblen, also $\mathcal{N}^*$ satisfies the Axiom of Veblen. $\square$

Let $O = O(n,2n,q)$ be an egg in $\mathrm{PG}(4n-1,q)$. By 8.7.2 of FGQ the $q^{2n}+1$ tangent spaces of O form an $O^* = O^*(n,2n,q)$ in the dual space of $\mathrm{PG}(4n-1,q)$. So in addition to $T(n,2n,q) = T(O)$ there arises a TGQ $T(O^*)$ with the same parameters. The TGQ $T(O^*)$ is called the *translation dual* of the TGQ $T(O)$. Examples are known for which $T(O) \cong T(O^*)$, and examples are known for which $T(O) \ncong T(O^*)$; see Thas [13].

6 Property (G) and the Axiom of Veblen

Let $\mathcal{S} = (P, B, \mathrm{I})$ be a GQ of order $(s, s^2), s \neq 1$. Let x_1, y_1 be distinct collinear points. We say that the pair $\{x_1, y_1\}$ has *Property* (G), or that $\mathcal{S}$ has *Property* (G) *at* $\{x_1, y_1\}$, if every triple $\{x_1, x_2, x_3\}$ of points, with x_1, x_2, x_3 pairwise noncollinear and $y_1 \in \{x_1, x_2, x_3\}^\perp$, is 3-regular; for the definition of 3-regularity see 1.3 of FGQ. The GQ $\mathcal{S}$ has *Property* (G) *at the line* L, or the line L has *Property* (G), if each pair of points $\{x, y\}, x \neq y$ and $x \mathrm{\ I\ } L \mathrm{\ I\ } y$, has Property (G). If (x, L) is a flag, that is, if $x \mathrm{\ I\ } L$, then we say that $\mathcal{S}$ has *Property* (G) *at* (x, L), or that (x, L) has *Property* (G), if every pair $\{x, y\}, x \neq y$ and $y \mathrm{\ I\ } L$, has Property (G). Property (G) was introduced in Payne [10] in connection with generalized quadrangles of order (q^2, q) arising from flocks of quadratic cones in $\mathrm{PG}(3, q)$.

Theorem 6.1 *Let $\mathcal{S} = (P, B, I)$ be a* GQ *of order (s^2, s), s even, satisfying Property (G) at the point x. Then x is regular in $\mathcal{S}$ and the dual net $\mathcal{N}_x^*$ satisfies the Axiom of Veblen. Consequently $\mathcal{N}_x^* \cong H_s^3$.*

Proof Let $\mathcal{S} = (P, B, \mathrm{I})$ be a GQ of order (s^2, s), s even, satisfying Property (G) at the point x. By 3.2.1 of [13] the point x is regular. Let y be a point of the dual net $\mathcal{N}_x^*$, let A_1 and A_2 be distinct lines of $\mathcal{N}_x^*$ containing y, let B_1 and B_2 be distinct lines of $\mathcal{N}_x^*$ not containing y, and let $A_i \cap B_j \neq \emptyset$ for all $i, j \in \{1, 2\}$. Let $\{z\} = A_1 \cap B_1$ and let $z \mathrm{I} M$, with $x \not\mathrm{I} M$. Further, let $x \mathrm{I} L$, with $z \not\mathrm{I} L$, let u be the point of A_1 on L, and let v be the point of B_1 on L. The line of $\mathcal{S}$ incident with u resp. v and concurrent with M is denoted by C resp. D; the line incident with z and x is denoted by N. Since $\mathcal{S}$ satisfies Property (G) at x, the triple $\{C, D, N\}$ is 3-regular. By 2.6.2 of TGQ the lines of $\mathcal{S}$ concurrent with at least two lines of $\{C, D, N\}^\perp \cup \{C, D, N\}^{\perp\perp}$ are the lineset of a subquadrangle $\mathcal{S}'$ of order (s, s) of $\mathcal{S}$. As x is regular for $\mathcal{S}$ it is also regular for $\mathcal{S}'$. By Theorem 2.1 the point x defines a projective plane π_x of order s. Clearly A_1, A_2, B_1, B_2 are lines of the projective plane π_x. Hence B_1 and B_2 intersect in π_x. Consequently $\mathcal{N}_x^*$ satisfies the Axiom of Veblen, and so $\mathcal{N}_x^* \cong H_s^3$. □

Theorem 6.2 (Thas [13]) *A* TGQ $T(n, 2n, q)$ *satisfies Property* (G) *at the pair* $\{(\infty), \bar{\zeta}\}$*, with* $\bar{\zeta}$ *a point of type* (ii) *incident with the line* ζ *of type* (b) *(or, equivalently, at the flag* $((\infty), \zeta)$*) if and only if, for any two elements* ζ_i, ζ_j $(i \neq j)$ *of* $O(n, 2n, q) - \{\zeta\}$*, the* $(n-1)$*-dimensional space* $PG(n-1, q) = \tau \cap \tau_i \cap \tau_j$*, with* τ, τ_i, τ_j *the respective tangent spaces of* $O(n, 2n, q)$ *at* ζ, ζ_i, ζ_j*, is contained in exactly* $q^n + 1$ *tangent spaces of* $O(n, 2n, q)$*.*

Theorem 6.3 *Let* $\mathcal{S}^{(p)}$ *be a* TGQ *of order* $(s, s^2), s \neq 1$*, with base point* p*. Then the dual net* $\mathcal{N}_L^*$ *defined by the regular line* L*, with* $p\, I\, L$*, satisfies the Axiom of Veblen if and only if the translation dual* $\mathcal{S}'^{(p')}$ *of* $\mathcal{S}^{(p)}$ *satisfies Property* (G) *at the flag* (p', L')*, where* L' *corresponds to* L*; in the even case,* $\mathcal{N}_L^*$ *satisfies the Axiom of Veblen if and only if* $\mathcal{S}^{(p)}$ *satisfies Property* (G) *at the flag* (p, L)*.*

Proof By Theorem 5.3 the dual net $\mathcal{N}_L^*$ satisfies the Axiom of Veblen if and only if $O(n, 2n, q)$ is good at the element $\mathrm{PG}^{(i)}(n-1, q)$ which corresponds to L. By Theorem 6.1 the egg $O(n, 2n, q) = O$ is good at $\mathrm{PG}^{(i)}(n-1, q)$ if and only if $T(O^*)$ satisfies Property (G) at the flag $((\infty), \mathrm{PG}^{(i)}(3n-1, q))$, with $\mathrm{PG}^{(i)}(3n-1, q)$ the tangent space of O at $PG^{(i)}(n-1, q)$; by Theorem 4.3.2 of [13], for q even, $T(O^*)$ satisfies Property (G) at the flag $((\infty), \mathrm{PG}^{(i)}(3n-1, q))$ if and only if $T(O)$ satisfies Property (G) at the flag $((\infty), \mathrm{PG}^{(i)}(n-1, q))$. □

Theorem 6.4 *Let* $\mathcal{S}^{(p)}$ *be a* TGQ *of order* $(s, s^2), s$ *odd and* $s \neq 1$*, with base point* p*. If the dual net* $\mathcal{N}_L^*$ *defined by some regular line* L*, with* $p\, I\, L$*, satisfies the Axiom of Veblen, then* $\mathcal{S}^{(p)}$ *contains at least* $s^3 + s^2$ *classical subquadrangles* $Q(4, s)$*.*

Proof This follows immediately from the preceding theorem and Theorem 4.3.4 of Thas [13]. $\square$

Theorem 6.5 *Let $\mathcal{S}^{(p)}$ be a* TGQ *of order (s, s^2), s odd and $s \neq 1$, with base point p. If $p \, \mathrm{I} \, L$ and if the dual net $\mathcal{N}_L^*$ satisfies the Axiom of Veblen, then all lines concurrent with L are regular.*

Proof Let N be concurrent with $L, p \not\mathrm{I}\, N$, and let the line M of $\mathcal{S}^{(p)}$ be nonconcurrent with N. By Theorem 4.3.4 of Thas [13] the lines N, M are lines of a subquadrangle of $\mathcal{S}^{(p)}$ isomorphic to $Q(4, q^n)$. Hence $\{N, M\}$ is a regular pair of lines. We conclude that the line N is regular in $\mathcal{S}^{(p)}$. $\square$

7 Flock generalized quadrangles and the Axiom of Veblen

Let F be a flock of the quadratic cone K with vertex x of $\mathrm{PG}(3, q)$, that is, a partition of $K - \{x\}$ into q disjoint irreducible conics. Then, by Thas [12], with F there corresponds a GQ $\mathcal{S}(F)$ of order (q^2, q). In Payne [10] it was shown that $\mathcal{S}(F)$ satisfies Property (G) at its point (∞).

Let $F = \{C_1, C_2, \ldots, C_q\}$ be a flock of the quadratic cone K with vertex x_0 of $\mathrm{PG}(3, q)$, with q odd. The plane of C_i is denoted by $\pi_i, i = 1, 2, \ldots, q$. Let K be embedded in the nonsingular quadric Q of $\mathrm{PG}(4, q)$. The polar line of π_i with respect to Q is denoted by L_i; let $L_i \cap Q = \{x_0, x_i\}, i = 1, 2, \ldots, q$. Then no point of Q is collinear with all three of $x_0, x_i, x_j, 1 \leq i < j \leq q$. In [1] it is proved that it is also true that no point of Q is collinear with all three of $x_i, x_j, x_k, 0 \leq i < j < k \leq q$. Such a set U of $q + 1$ points of Q will be called a *BLT-set* in Q, following a suggestion of Kantor [7]. Since the GQ $Q(4, q)$ arising from Q is isomorphic to the dual of the GQ $W(q)$ arising from a symplectic polarity in $\mathrm{PG}(3, q)$, to a BLT-set in Q corresponds a set V of $q + 1$ lines of $W(q)$ with the property that no line of $W(q)$ is concurrent with three distinct lines of V; such a set V will also be called a *BLT-set*.

To F corresponds a GQ $\mathcal{S}(F)$ of order (q^2, q). Knarr [8] proves that $\mathcal{S}(F)$ is isomorphic to the following incidence structure.

Start with a symplectic polarity θ of $PG(5, q)$. Let $(\infty) \in \mathrm{PG}(5, q)$ and let $\mathrm{PG}(3, q)$ be a 3-dimensional subspace of $\mathrm{PG}(5, q)$ for which $(\infty) \notin \mathrm{PG}(3, q) \subset (\infty)^\theta$. In $\mathrm{PG}(3, q)$ θ induces a symplectic polarity θ', and hence a GQ $W(q)$. Let V be the BLT-set defined by F of the GQ $W(q)$ and construct a geometry $\mathcal{S} = (P, B, \mathrm{I})$ as follows.

Points : (i) (∞); (ii) lines of $\mathrm{PG}(5, q)$ not containing (∞) but contained in one of the planes $\pi_t = (\infty)L_t$, with L_t a line of the BLT-set V; (iii) points of $\mathrm{PG}(5, q)$ not in $(\infty)^\theta$.

Lines : (a) planes $\pi_t = (\infty)L_t$, with $L_t \in V$; (b) totally isotropic planes of θ not contained in $(\infty)^\theta$ and meeting some π_t in a line (not through (∞)).

The incidence relation I is the natural incidence inherited from PG$(5, q)$.

Then Knarr [8] proves that $\mathcal{S}$ is a GQ of order (q^2, q) isomorphic to the GQ $\mathcal{S}(F)$ arising from the flock F defining V.

Theorem 7.1 *For any* GQ $\mathcal{S}(F)$ *of order* (q^2, q) *arising from a flock* F*, the point* (∞) *is regular.*

Proof The GQ $\mathcal{S}(F)$ satisfies Property (G) at its point (∞). Then for q even, by 3.2.1 of Thas [13], the point (∞) is regular. Now let q be odd, and consider the construction of Knarr. If the point y is not collinear with (∞), that is, if y is a point of PG$(5, q)$ not in $(\infty)^\theta$, then $\{(\infty), y\}^{\perp\perp}$ consists of the $q + 1$ points of the line $(\infty)y$ of PG$(5, q)$. As $|\{(\infty), y\}^{\perp\perp}| = q + 1$ the point (∞) is regular. □

Let K be the quadratic cone with equation $X_0X_1 = X_2^2$ of PG$(3, q), q$ odd. Then the q planes π_t with equation $tX_0 - mt^\sigma X_1 + X_3 = 0, t \in GF(q), m$ a given nonsquare of $GF(q)$, and σ a given automorphism of $GF(q)$, define a flock F of K; see Thas [12]. The corresponding GQ $\mathcal{S}(F)$ were first discovered by Kantor [6], and so these flocks F will be called *Kantor flocks*. Any such GQ $\mathcal{S}(F)$ is a TGQ for some base line, and so the point-line dual of $\mathcal{S}(F)$ is isomorphic to some $T(O)$, with O an $[n - 1]$-ovoid. Also, in Payne [10] it is proved that $T(O)$ is isomorphic to its translation dual $T(O^*)$; there is an isomorphism of $T(O)$ onto $T(O^*)$ conserving types of points and lines and mapping the line ζ of type (b) of $T(O)$ onto the line τ of type (b) of $T(O^*)$, where τ is the tangent space of O at ζ.

Theorem 7.2 *Consider the* GQ $\mathcal{S}(F)$ *of order* (q^2, q) *arising from the flock* F*. If* q *is even, then the dual net* $\mathcal{N}^*_{(\infty)}$ *always satisfies the Axiom of Veblen and so* $\mathcal{N}^*_{(\infty)} \cong H^3_q$*. If* q *is odd, then the dual net* $\mathcal{N}^*_{(\infty)}$ *satisfies the Axiom of Veblen if and only if* F *is a Kantor flock.*

Proof Consider the GQ $\mathcal{S}(F)$ of order (q^2, q) arising from the flock F. Then $\mathcal{S}(F)$ satisfies Property (G) at the point (∞).

First, let q be even. Then by Theorem 6.1 the dual net $\mathcal{N}^*_{(\infty)}$ satisfies the Axiom of Veblen, and so $\mathcal{N}^*_{(\infty)} \cong H^3_q$.

Next, let q be odd. Suppose that F is a Kantor flock. Then the point-line dual of $\mathcal{S}(F)$ is isomorphic to some $T(O)$, and by [10] $T(O) \cong T(O^*)$. The point (∞) of $\mathcal{S}(F)$ corresponds to some line ζ of type (b) of $T(O)$. Hence $T(O)$ satisfies Property (G) at ζ. By Theorem 6.3 the dual net $\mathcal{N}^*_\tau$ which corresponds with the regular line τ of $T(O^*)$, where τ is the tangent space of O at ζ, satisfies the Axiom of Veblen. Hence also the dual net $\mathcal{N}^*_\zeta$ which corresponds with the regular line ζ of $T(O)$ satisfies the Axiom of Veblen. It follows that the dual net $\mathcal{N}^*_{(\infty)}$ satisfies the Axiom of Veblen. Conversely, suppose that the dual net $\mathcal{N}^*_{(\infty)}$ satisfies the Axiom of Veblen. Hence $\mathcal{N}^*_{(\infty)} \cong H^3_q$. In the representation of Knarr, this dual net looks as follows : points of $\mathcal{N}^*_{(\infty)}$ are the lines of $\mathrm{PG}(5,q)$ not containing (∞) but contained in one of the planes π_t, lines of $\mathcal{N}^*_{(\infty)}$ can be identified with the threedimensional subspaces of $(\infty)^\theta$ not containing (∞), and incidence is inclusion. By point-hyperplane duality in $(\infty)^\theta$, the net $\mathcal{N}_{(\infty)}$, which is the point-line dual of $\mathcal{N}^*_{(\infty)}$, is isomorphic to the following incidence structure : points of $\mathcal{N}_{(\infty)}$ are the points of $(\infty)^\theta - \mathrm{PG}(3,q)$, lines of $\mathcal{N}_{(\infty)}$ are the planes of $(\infty)^\theta$ not contained in $\mathrm{PG}(3,q)$ but containing one of the lines of the BLT-set V in $\mathrm{PG}(3,q)$, and incidence is the natural one. As the net $\mathcal{N}_\infty$ is isomorphic to the dual of H^3_q, it is easily seen to be derivable; see e.g. De Clerck and Johnson [4]. In $W(q)$ the lineset $S = \{L_0, L_1\}^{\perp\perp} \cup \{L_0, L_2\}^{\perp\perp} \cup \ldots \cup \{L_0, L_q\}^{\perp\perp}$ is a linespread containing V; see e.g. [12]. As $\mathcal{N}_{(\infty)}$ is derivable, by [3] there are two distinct lines in $\mathrm{PG}(3,q)$, but not in $\{L_0, L_1\}^\perp \cup \{L_0, L_2\}^\perp \cup \ldots \cup \{L_0, L_q\}^\perp$, intersecting the same $q+1$ lines of S. Then by Johnson and Lunardon [5], the flock F is a Kantor flock. $\square$

Corollary 7.3 *Suppose that the* TGQ $T(O)$, *with* $O = O(n, 2n, q)$ *and* q *odd, is the point-line dual of a flock* GQ $\mathcal{S}(F)$ *where the point* (∞) *of* $\mathcal{S}(F)$ *corresponds to the line* ζ *of type* (b) *of* $T(O)$. *Then* $T(O)$ *is good at the element* ζ *if and only if* F *is a Kantor flock.*

Proof This follows immediately from Theorems 5.3 and 7.2. $\square$

8 Subquadrangles and the Axiom of Veblen

Theorem 8.1 *Let* $\mathcal{S} = (P, B, I)$ *be a* GQ *of order* $(s,t), s \neq 1 \neq t$, *having a regular point* x. *If* x *together with any two points* y, z, *with* $y \not\sim x$ *and* $x \sim z \not\sim y$, *is contained in a proper subquadrangle* $\mathcal{S}'$ *of* $\mathcal{S}$ *of order* (s', t), *with* $s' \neq 1$, *then* $s' = t = \sqrt{s}$ *and the dual net* $\mathcal{N}^*_x$ *satisfies the Axiom of Veblen. It follows that* s *and* t *are prime powers, and that for each subquadrangle* $\mathcal{S}'$ *the projective plane* π_x *of order* t *defined by the regular point* x *of* $\mathcal{S}'$ *is desarguesian. Conversely, if the dual net* $\mathcal{N}^*_x$ *satisfies the Axiom of Veblen, then either* (a) $s = t$, *or* (b) $s = t^2$, s *and* t *are prime powers,* x *and any two points* y, z *with* $y \not\sim x$ *and* $x \sim z \not\sim y$ *are contained in a subquadrangle* $\mathcal{S}'$ *of* $\mathcal{S}$ *of order* (t,t), *and the projective plane* π_x *of order* t *defined by the regular point* x *of* $\mathcal{S}'$ *is desarguesian.*

Proof Let $\mathcal{S} = (P, B, \mathrm{I})$ be a GQ of order $(s,t), s \neq 1 \neq t$, having a regular point x.

First, assume that x together with any two points y, z with $y \not\sim x$ and $x \sim z \not\sim y$ is contained in a proper subquadrangle $\mathcal{S}'$ of $\mathcal{S}$ of order (s', t), with $s' \neq 1$. As x is also regular for $\mathcal{S}'$, the GQ $\mathcal{S}'$ contains subquadrangles of order $(1, t)$. Then, by 2.2.2 of FGQ, we have $s' = t = \sqrt{s}$. By Theorem 2.1 the dual net $\mathcal{N}_x'^*$ arising from the regular point x of $\mathcal{S}'$, is a dual affine plane of order s. Hence $\mathcal{N}_x'^*$ satisfies the Axiom of Veblen. Now consider distinct lines A_1, A_2, B_1, B_2 of the dual net $\mathcal{N}_x^*$, where $A_1 \cap A_2 = \{z\}, z \notin B_1, z \notin B_2$, and $A_i \cap B_j \neq \emptyset$ for all $i, j \in \{1, 2\}$. Let $A_1 \cap B_1 = \{u\}, A_2 \cap B_2 = \{w\}$, and let $y \in \{u, w\}^\perp - \{x\}$. Let $\mathcal{S}'$ be a subquadrangle of order t containing the points x, y, z of $\mathcal{S}$. Then A_1, A_2, B_1, B_2 are lines of the dual net $\mathcal{N}_x'^*$. As $\mathcal{N}_x'^*$ satisfies the Axiom of Veblen, we have $B_1 \cap B_2 \neq \emptyset$. It follows that the dual net $\mathcal{N}_x'^*$ satisfies the Axiom of Veblen. Consequently $\mathcal{N}_x'^* \cong H_t^3$, and so s and t are prime powers. For any subquadrangle $\mathcal{S}'$ the dual net $\mathcal{N}_x'^*$ is a dual affine plane of order t, which is isomorphic to a dual affine plane of order t in H_t^3. Hence the dual net $\mathcal{N}_x'^*$, and consequently also the corresponding projective plane π_x, are desarguesian.

Conversely, assume that the dual net $\mathcal{N}_x^*$ satisfies the Axiom of Veblen. Also, suppose that $s \neq t$, that is, $s > t$ by 1.3.6 of FGQ. Then, by Theorem 3.1, we have $\mathcal{N}_x^* \cong H_q^n$ with q a prime power and $n > 2$. As $s = q^{n-1}, t = q$ and $s \leq t^2$ (by the inequality of Higman, see 1.2.3 of FGQ), we necessarily have $n = 3$. Hence $s = t^2, t = q$, and $\mathcal{N}_x^* \cong H_q^3$. Now consider any two points y, z, with $y \not\sim x, x \sim z \not\sim y$. As $\mathcal{N}_x^* \cong H_q^3$ it is easily seen that z and $\{x, y\}^\perp$ generate a dual affine plane $\mathcal{A}$ of order q in $\mathcal{N}_x^*$. Let $A_1, A_2, \ldots, A_{q^2}$ be the lines of $\mathcal{A}$. Further, let P' be the pointset of $\mathcal{S}$ consisting of the points of $A_1^\perp \cup A_2^\perp \cup \ldots \cup A_{q^2}^\perp$ and the points of $\mathcal{A}$. Clearly P' contains z and y, and $|P'| = q^3 + q^2 + q + 1$. Further, any line of $\mathcal{S}$ incident with at least one point of P' either contains x or a point of $\mathcal{A}$; the set of all these lines is denoted by B'. Also, any point incident with two distinct lines of B' belongs to P'. Then, by 2.3.1 of FGQ, $\mathcal{S}' = (P', B', \mathrm{I}')$ with I' the restriction of I to $(P' \times B') \cup (B' \times P')$ is a subquadrangle of $\mathcal{S}$ of order q. As in the first part of the proof one now shows that for any such subquadrangle $\mathcal{S}'$ the projective plane π_x defined by x is desarguesian. $\square$

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J.A. Thas Department of Pure Mathematics and Computer Algebra, University of Ghent, Krijgslaan 281, B-9000 Gent, Belgium
e-mail : jat@cage.rug.ac.be

H. Van Maldeghem Department of Pure Mathematics and Computer Algebra, University of Ghent, Krijgslaan 281, B-9000 Gent, Belgium
e-mail : hvm@cage.rug.ac.be