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NOTE

A Note on Finite Self-Polar Generalized Hexagons and Partial Quadrangles

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In 1976, Cameron *et al.* [2] proved that, if a finite generalized hexagon of order s admits a polarity, then either s or $3s$ is a perfect square. Their proof used standard eigenvalue arguments. Later on, Ott [3] showed that for a self polar finite thick generalized hexagon of order s, $3s$ always has to be a perfect square. His proof used Hecke algebras. It was surprising that one had to use this more complicated method to achieve this result. The goal of the present note is to prove it with the original elementary eigenvalue technique.

Suppose that Γ is a finite generalized hexagon of order s admitting a polarity θ. We number the points of Γ as $p_1, p_2, ..., p_v$, $v=(1+s)(1+s^2+s^4)$. Let $A=(a_{ij})$ be the $(v\times v)$-matrix with $a_{ij}=1$ if $p_i^\theta \; I \; p_j$, and $p_{ij}=0$ otherwise. It is shown in Cameron *et al.* [2] that A has eigenvalues $1+s$, 0, $\pm\sqrt{s}$, and $\pm\sqrt{3s}$. It is also shown there that $\mathrm{tr}(A)=1+s^3$ and that the multiplicity of the eigenvalue $1+s$ is 1. If we denote the multiplicity of $\epsilon\sqrt{s}$ by k_ϵ and the multiplicity of $\epsilon\sqrt{3s}$ by l_ϵ, $\epsilon\in\{+,-\}$, and if we put $k=k_+-k_-$, $l=l_+-l_-$, then one has the equation

$$(1+s)+k\sqrt{s}+l\sqrt{3s}=1+s^3. \tag{1}$$

This equation allowed Cameron *et al.* [2] to conclude that either $\sqrt{s}$ or $\sqrt{3s}$ is an integer. Now we look at the matrix $A^3=:(b_{ij})$. Noting that A^2 is nothing else than an adjacency matrix of the collinearity graph plus $1+s$ times the identity matrix, one easily sees that, if $p_i \; I \; p_i^\theta$, then the element b_{ii} is equal to $1+s$ plus the number of points p_j collinear in Γ with p_i and is such that $p_i \; I \; p_j^\theta$. So in this case $b_{ii}=2s+1$. Otherwise, b_{ii} equals the

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number of points p_j which are collinear with p_i and such that $p_i \; I \; p_j^\theta$. If $b_{ii} \neq 0$, it follows that $p_j \; I \; p_j^\theta$ and hence $b_{ii} = 1$, and this happens if and only if p_i is incident with some line $p_j^\theta \; I \; p_j$ for $j \neq i$. Note that there are $s(1+s^3)$ such points. Hence $\operatorname{tr}(A^3) = (1+s^3)(3s+1)$, and we have

$$(1+s)^3 + ks\sqrt{s} + 3ls\sqrt{3s} = (1+s^3)(3s+1). \tag{2}$$

Solving Eqs. (1) and (2), we obtain $k = 0$ and $l = (s^3 - s)/\sqrt{3s}$. Since l is an integer, we show Ott's result:

THEOREM (Ott [3]). *If a finite thick generalized hexagon of order s is self-polar, then $\sqrt{3s}$ is an integer.*

Note that the classical generalized hexagons of order 3^{2h+1} are the only known finite thick self-polar generalized hexagons.

We can now apply the same technique to self-polar partial quadrangles. So consider a self-polar partial quadrangle with $s+1$ points on a line, $\alpha + 1$ points collinear with two non-collinear points, $\alpha < s$ and with m absolute points (with respect to the given polarity). Without making the calculations here explicitly, one obtains integers k, l such that

$$\begin{cases} (1+s) + k\sqrt{3s-\alpha+\sqrt{4s^2+(s-\alpha)^2}} \\ \qquad + l\sqrt{3s-\alpha-\sqrt{4s^2+(s-\alpha)^2}} = m, \\ (1+s)^3 + k\sqrt{3s-\alpha+\sqrt{4s^2+(s-\alpha)^2}}^{\,3} \\ \qquad + l\sqrt{3s-\alpha-\sqrt{4s^2+(s-\alpha)^2}}^{\,3} = (3s+1)\,m, \end{cases}$$

from which it easily follows that $\sqrt{3s-\alpha+\sqrt{4s^2+(s-\alpha)^2}}$ or $\sqrt{3s-\alpha-\sqrt{4s^2+(s-\alpha)^2}}$ is an integer. Note that $\sqrt{4s^2+(s-\alpha)^2}$ is an integer; see Cameron [1]. No finite self-polar partial quadrangles with $s > 1$ and $\alpha < s$ are known. In fact, no partial quadrangles of order $s = t > \alpha$ are known (where $s+1$ is the number of points on a line and $t+1$ the number lines through a point); see again Cameron [1].

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