Characterizations for classical finite hexagons

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Abstract

We characterize some classical finite hexagons as the only generalized hexagons containing ovoidal subspaces all of whose points are spanregular.

1 Introduction

A *generalized n-gon* $\Gamma = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ *of order* (s,t) is an incidence structure of points and lines with $s+1$ points incident with a line and $t+1$ lines incident with a point, $s,t \geq 1$, such that Γ has no ordinary k-gons for any $2 \leq k < n$, and any two elements are inside some ordinary $n-$gon. Distances are measured in the incidence graph.
If two points x, y are at distance 2, we call them *collinear* and write $x \sim y$.
If two points x, y are at distance 4 and $n > 4$, the unique point at distance 2 from x and at distance 2 from y is denoted by $x \bowtie y$.
If two elements u, v are at distance $k < n$, we denote the unique element at distance 1 from x and at distance $k-1$ from y by $\mathrm{proj}_x y$, and call this the *projection of* y *onto* x.
The set of all elements at distance i from an element u is denoted by $\Gamma_i(u)$.

The *trace* x^y, with x and y opposite elements (= at maximal distance n), is the set of all elements ($t+1$ if x is a point) at distance 2 from x and distance $n-2$ from y. The point x is said to be *regular*, if $\forall y, z$ opposite x, $|x^y \cap x^z| \geq 2 \Rightarrow x^y = x^z$. This implies that two traces x^{y_1} and x^{y_2} with x regular have $0, 1$ or $t+1$ points in common.

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The point x is said to be *spanregular*, if x is regular and for all points p, a, b with $d(x,p)=2,\ d(p,a)=n,\ d(p,b)=n : x \in p^a \cap p^b, |p^a \cap p^b| \geq 2 \Rightarrow p^a = p^b$. One could give the following interpretation: x is spanregular if x is regular and every point collinear with x behaves as a regular point in the neighbourhood of x.

Given some trace p^a with $u, v \in p^a$, we have the equivalent notations $p^a = (u \bowtie v)^a = \langle u, v\rangle_a$. The trace $\langle u, v\rangle_a$ through u and v and defined by a is called an *ideal line*, if every trace $\langle u, v\rangle_b$ through u and v coincides with $\langle u, v\rangle_a$. So we can use the notation $\langle u, v\rangle$ — independent of a — if this trace is an ideal line.

A sub-n-gon Γ' of order (s', t') of a generalized n-gon Γ of order (s, t) is a subgeometry of Γ which is itself a generalized n-gon of order (s', t'). If $s' = s$, Γ' is called *full*. If $t' = t$, Γ' is called *ideal*. A generalized n-gon of order (s, t) is called *thin*, whenever s or t is equal to 1, and is called *thick* whenever $s, t \geq 2$.

2 Definition of ovoidal subspace

An *ovoidal subspace* $\mathcal{A}$ of a generalized $2m$-gon $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a proper non-empty set of points $\mathcal{A} \subset \mathcal{P}$, with an induced set of lines $\mathcal{A}' = \{L \in \mathcal{L} \mid \Gamma_1(L) \subset \mathcal{A}\}$, such that all elements of Γ are at distance $\leq m$ from a certain point of $\mathcal{A}$, and such that for all elements of $\Gamma \backslash (\mathcal{A} \cup \mathcal{A}')$ at distance $< m$ from a certain point p of $\mathcal{A}$, this point p is unique.
The notion 'ovoidal' is inspired by the ovoids, being special cases of ovoidal subspaces.

To show the likeness between the definition of Γ itself and the definition of an ovoidal subspace of Γ, we define the distance between a point b and a point set $\mathcal{A}$ as $d(b, \mathcal{A}) = \min\{d(b,a) | a \in \mathcal{A}\}$. Then we can formally write their respective definitions as follows (disregarding the order (s,t)):

$\boxed{\Gamma}$ (1) Given a; $\max\{d(a,b) | b \text{ element of } \Gamma\} = 2m$
(2) Given a; $\forall b$ element of $\Gamma : d(a,b) < 2m$
$\Rightarrow \exists$ unique shortest path between a, b

$\boxed{\mathcal{A}}$ (1) Given $\mathcal{A}$; $\max\{d(\mathcal{A},b) | b \text{ element of } \Gamma\} = m$
(2) Given $\mathcal{A}$; $\forall b$ element of $\Gamma \backslash (\mathcal{A} \cup \mathcal{A}') : d(\mathcal{A}, b) < m$
$\Rightarrow \exists$ unique shortest path between $\mathcal{A}, b$

For Γ a generalized quadrangle of order (s,t), an ovoidal subspace is the same as a geometric hyperplane $\mathcal{A}$, which is defined as a set of points such that every line intersects $\mathcal{A}$ in exactly 1 or $s+1$ points. One can easily show that $\mathcal{A}$ is an ovoid $(\forall L : |L \cap \mathcal{A}| = 1)$, the point set of a subquadrangle of order (s, t'), $st' = t$ (called a grid if $t' = 1$), or the set of all points collinear with a given point.

For Γ a generalized quadrangle **of order** s (i.e. order (s,s)), it is known that all points are regular (and then Γ is known, i.e. Γ is the generalized quadrangle $W(s)$ arising from a symplectic polarity of $\mathrm{PG}(3,s)$) **iff** all points of a geometric hyperplane $\mathcal{A}$ are (span-)regular. For these proofs we refer to Payne & Thas [3]: 5.2.5 ($\mathcal{A}$ an ovoid), 5.2.6 ($\mathcal{A}$ the point set of a grid), 1.3.6(iv) ($\mathcal{A}$ the set of all points collinear with a given point x).

We extend this theorem to generalized hexagons.

3 Main Result

We will use the following notations for the known finite generalized hexagons:

$H(q)$	the split Cayley hexagon (of order (q,q)) over the finite field GF(q), cfr. [6], par. 2.
$T(q,\sqrt[3]{q})$	the triality hexagon (of order $(q,\sqrt[3]{q})$) over the finite field $GF(q)$ with field automorphism $\sigma : x \mapsto x^3$.

Main Result
Let Γ be a generalized hexagon of order (s,t) having an ovoidal subspace $\mathcal{A}$, satisfying
($\star$) any 2 opposite points of Γ are contained in a thin ideal subhexagon $\mathcal{D}$,
then all points of $\mathcal{A}$ are spanregular $\Leftrightarrow \Gamma \cong H(q)$ or $T(q,\sqrt[3]{q})$.

From the proof, it will follow that the condition ($\star$) becomes superfluous in certain cases.

4 Preparations for the proof of the Main Result

4.1 Equivalent definition of ovoidal subspaces in generalized hexagons

Lemma 1 *Let $\Gamma = (\mathcal{P},\mathcal{L},\mathrm{I})$ be a generalized hexagon of order (s,t). An ovoidal subspace $\mathcal{A}$ is a set of points such that each point of the hexagon not in $\mathcal{A}$, is collinear with a unique point of $\mathcal{A}$.*

Proof. $\boxed{\Rightarrow}$ Take $x \in \Gamma\backslash\mathcal{A}$. As the distance between 2 points is even, x is at distance 2 from a certain point p of $\mathcal{A}$. By the second condition, this point p is unique.

$\boxed{\Leftarrow}$ Take $x \in \Gamma$. If $x \in \mathcal{A}$, it is at distance $0 \leq 3$ from a point of $\mathcal{A}$. If $x \notin \mathcal{A}$, it is at distance $2 < 3$ from a unique point of $\mathcal{A}$. ■

We will use the following properties of ovoidal subspaces of generalized hexagons frequently.

- Whenever a line meets $\mathcal{A}$ in 2 points, all points of the line belong to $\mathcal{A}$ — because they are collinear with two different points of $\mathcal{A}$.
- Whenever two points x, y at distance 4 belong to $\mathcal{A}$, $x \bowtie y$ belongs also to $\mathcal{A}$ (in the other case, $x \bowtie y$ would be collinear with 2 points of $\mathcal{A}$, $x \bowtie y$ being off $\mathcal{A}$).

4.2 Classification of ovoidal subspaces in generalized hexagons

Theorem 1 *An ovoidal subspace of a generalized hexagon of order (s,t) is either an ovoid, or the set of all points at distance 1 or 3 from a given line L, or the point set of a full generalized subhexagon of order $(s, \sqrt{\frac{t}{s}})$.*

Proof.

1. If every point, lying inside or outside $\mathcal{A}$, is collinear with exactly one point of $\mathcal{A}$, the subspace $\mathcal{A}$ is an ovoid — by definition.

2. Suppose there is a point in $\mathcal{A}$, collinear with a second point of $\mathcal{A}$; this means, suppose $\mathcal{A}$ contains a line L.

   (a) We show that for 2 points of $\mathcal{A}$, their distance $d_{\mathcal{A}}$ measured in $\mathcal{A}$ will be the same as their distance d_Γ measured in Γ, provided we add to $\mathcal{A}$ all lines N of Γ with $\Gamma_1(N) \subseteq \mathcal{A}$. Say $x, y \in \mathcal{A}$.
   If $d_\Gamma(x,y) < 6$, the unique path of length d_Γ between x and y also belongs to $\mathcal{A}$. It follows that $d_\Gamma(x,y) = d_{\mathcal{A}}(x,y)$.
   Suppose $d_\Gamma(x,y) = 6$. $\boxed{1}$ Suppose $d_\Gamma(x,L) = 5 = d_\Gamma(y,L)$. Draw the unique path $(x, xx_2, x_2, x_2x_3, x_3, L)$. As $d(x,x_3) = 4$ and $x, x_3 \in \mathcal{A}$, we know that all points of this path belong to $\mathcal{A}$. As $d_\Gamma(y, xx_2) = 5$, we can project y onto xx_2, and call this projection y'. As $d(y,y') = 4$ and $y, y' \in \mathcal{A}$, all points of the path between y and y' belong to $\mathcal{A}$. So we constructed a path in $\mathcal{A}$ of length 6 between x and y: $d_\Gamma(x,y) = d_{\mathcal{A}}(x,y)$.
   $\boxed{2}$ For $d_\Gamma(x,L) \neq 5$ or $d_\Gamma(y,L) \neq 5$, the proof is completely similar.

   (b) Now we claim that there are two points of $\mathcal{A}$ at distance 6 from each other. Take a point p of Γ, at distance 5 of L and denote the joining path by $(p, pp_2, p_2, p_2p_3, p_3, L)$. $\boxed{1}$ If $p \in \mathcal{A}$, one can find s pairs (p,u), $u \in L$, with u at distance 6 from p. $\boxed{2}$ If $p \notin \mathcal{A}$, p is collinear with a unique point x of $\mathcal{A}$. $\boxed{a}$ If $x = p_2$, then take a point q collinear with p, but not on pp_2. This point q does not belong to $\mathcal{A}$ (as p is collinear with just one point of $\mathcal{A}$), so is itself collinear with a unique point $y \in \mathcal{A}$. As $d(x,y) = 6, x, y, \in \mathcal{A}$, the claim follows. $\boxed{b}$ If $p_2 \neq x \in pp_2$, then $(x, xp_2, p_2, p_2p_3, p_3, L)$ belongs to $\mathcal{A}$, and so does p, a contradiction. $\boxed{c}$ If $x \notin pp_2$, then $d(x,p_3) = 6$.

   (c) At this point, we know 2 points of $\mathcal{A}$ at distance 6 (in $\mathcal{A}$), say x and y. So $\mathcal{A}$ contains at least one path $(x, xx', x', M, y', y'y, y)$ between x and y (by (a)).
   If $\mathcal{A}$ contains an appartment, it is a full subhexagon of order (s,t'). By Thas [5], we know $st'^2 = t$. (Using the notations of the article mentioned, we know P' = point set of $\mathcal{A}$, $V = P$ and $W = \phi$. So $|W| = d = 0$, hence $t = st'^2$ if $s = s'$.) If Γ has order s, $\mathcal{A}$ will be of order $(s,1)$.
   If $\mathcal{A}$ does not contain any appartment, we show that $\mathcal{A} = \Gamma_1(M) \cup \Gamma_3(M)$.
   $\boxed{1}$ We show that every point of $\mathcal{A}$ is at distance ≤ 3 from M.
   Suppose $z \in \mathcal{A}, z \in \Gamma_5(M), \mathrm{proj}_M z = z'$. Without loss of generality, $z' \neq y'$, so $d(z, yy') = 5$. As $\mathrm{proj}_{yy'} z = y''$ belongs to $\mathcal{A}$, there are 2

paths of length 6 joining z and y'. This is an appartment, and hence a contradiction.

$\boxed{2}$ We show that every point of Γ at distance ≤ 3 from M belongs to $\mathcal{A}$. Suppose $u \notin \mathcal{A}, u \in \Gamma_3(M), \text{proj}_M u = u'$. Take a point z collinear with u, at distance 5 from M. As $z \notin \mathcal{A}$ (by the previous section), z is collinear with a unique point z' of $\mathcal{A}$. If $z' \in \Gamma_3(M)$, then there is a pentagon with edges $\{z', z, u, u', u' \bowtie z'\}$ (if $d(u', z') = 4$) or a quadrangle (if $d(u', z') = 2$). If $z' \in \Gamma_1(M)$, it's even worse: a quadrangle or a triangle arises.

∎

5 Proof of the Main Result

5.1 Organization

By the previous classification, we distinguish 3 different types of ovoidal subspaces in a generalized hexagon. We will consider each of them separately in the proof of the Main Result. Our proof is organized as follows:

1. To start with, we let $\mathcal{A}$ be an ovoid. As for all known finite generalized hexagons, it are only the ones with order $s = t$ which possibly possess an ovoid, we first consider this particular case. In fact, this proof is already known. The main idea is to count the thin ideal subhexagons $\mathcal{D}$ of the given hexagon Γ. This counting argument (1) can be written as follows:

   $$X \leq \beta \leq Y$$

   with

   X the number of pairs of opposite points through which there exists a $\mathcal{D}$ containing 2 points of $\mathcal{A}$;

   β the number of pairs of opposite points through which there exists a $\mathcal{D}$;

   Y the number of pairs of opposite points.

   Whenever $\boxed{1}$ $X = \beta$, each $\mathcal{D}$ contains 2 points of $\mathcal{A}$. Whenever $\boxed{2}$ $\beta = Y$, we know that through each $x, y \in \mathcal{P}$, there is a $\mathcal{D}$.

   For $\mathcal{A}$ being an ovoid in Γ of order s, condition $\boxed{1}$ as well as condition $\boxed{2}$ will be satisfied. Hence the Main Result holds without condition $(\star)$.

2. Then we consider $\mathcal{A} = \Gamma_1(L) \cup \Gamma_3(L)$, Γ of order s. In lemma 2 we do approximately the same counting as mentioned before, and — as $s = t$ — we conclude that $\boxed{1}$ and $\boxed{2}$ are satisfied. Hence the second part of the proof of the Main Result is completely similar to the first part. Here, too, the condition $(\star)$ is redundant.

3. Let $\mathcal{A}$ be the point set of a full subhexagon in the third part. Here we can prove that Γ should be of order s, while $\mathcal{A}$ has order $(s, 1)$. Indeed, if Γ of order (s, t) contains a subhexagon $\mathcal{A}$ of order (s, t'), we know $t' \leq s \leq t$ (see [6] 1.8.8). As Γ has spanregular points, we know $t \leq s$ (see [6] 1.9.5). So $t = s$, and $t' = 1$ ([6]).
   Unfortunately, we cannot use the same counting argument (1), as X is never equal to Y if $\mathcal{A}$ is a thin full subhexagon. Nevertheless, we are able to re-arrange the proof with only half of the countingargument: we assume that $\boxed{2}$ $\beta = Y$ (this is exactly condition $(\star)$), and we don't use the (wrong) assumption $\boxed{1}$ that $X = \beta$.

4. But by now, we can also re-arrange the proof in case of $\mathcal{A} = \Gamma_1(L) \cup \Gamma_3(L)$: we don't require s to be equal to t, but we assume condition $(\star)$. Only using condition $\boxed{2}$, we are still able to complete the proof.

5. At last, we can — technically — do the same for $\mathcal{A}$ being an ovoid. Suppose you don't know anything of the order (s, t) of Γ, then — assuming condition $(\star)$ — the Main Result is still true. (However, it is known $T(q, \sqrt[3]{q})$ does not have an ovoid.)

5.2 $\mathcal{A}$ an ovoid, Γ of order s

Theorem 2 *Let Γ be a finite generalized hexagon of order s containing an ovoid $\mathcal{A}$. Every point of $\mathcal{A}$ is spanregular $\Leftrightarrow$ Γ is isomorphic to $H(q), q = s$.*

Proof. This proof is given by V. De Smet and H. Van Maldeghem in [2]. ∎

5.3 $\mathcal{A} = \Gamma_1(L) \cup \Gamma_3(L)$, Γ of order s.

For this part of the proof, we will use a similar counting argument (lemma 2) as used in [2].

Lemma 2 *Let Γ be a finite generalized hexagon of order (s, t), which contains a set $\mathcal{A} = \Gamma_1(M) \cup \Gamma_3(M)$ for which all points are spanregular. Then every thin ideal subhexagon of Γ contains 2 collinear points of $\mathcal{A}$* **if and only if** *$s = t$.*

Proof.

1. First we count the thin ideal subhexagons containing M. There are $\frac{(s+1)s^3t^2}{2}$ sets $\{u, v\}$ of opposite points in $\mathcal{A}$. As u is spanregular, there is a thin ideal subhexagon through u and v, named $\Gamma(u, v)$, containing M (see [6] 1.9.10). But in every ideal subhexagon $\Gamma(u, v)$, one can find t^2 sets $\{u', v'\}$ of opposite points in $\mathcal{A}$. So there are $\frac{s^3(s+1)}{2}$ thin ideal subhexagons containing M — and hence containing $2 + 2t$ points of $\mathcal{A}$.

2. Now we count the thin ideal subhexagons $\mathcal{D}$ containing two collinear points u, v of $\Gamma_3(M)$. Hence M is not a line of $\mathcal{D}$, as there are only 2 points on the line uv in $\mathcal{D}$. We count in 2 different ways the couples $(\{u, v\}, \mathcal{D})$, with $\{u, v\}$

a set of collinear points in $\Gamma_3(M)$, and $\mathcal{D}$ a thin ideal subhexagon containing u and v (as u is spanregular, there will be an ideal subhexagon through u). Denoting the number of $\mathcal{D}$'s by X, it follows that

$$\frac{(s+1)s(s-1)t}{2} \cdot s^2 = 1 \cdot X$$

3. Now we compare these 2 quantities with the total number of thin ideal subhexagons in Γ. We count the pairs $(\{u,v\},\mathcal{D})$ with $\{u,v\}$ a set of opposite points in Γ, and $\mathcal{D}$ a thin ideal subhexagon containing u and v. Denoting the total number of $\mathcal{D}$'s by α, and noting that for each set $\{u,v\}$ there is at most 1 subhexagon $\mathcal{D}$, we know

$$\frac{(1+s)(1+st+s^2t^2)s^3t^2}{2} \cdot 1 \geq \frac{2(1+t+t^2)t^2}{2} \cdot \alpha$$

The total number of thin ideal subhexagons containing 2 (collinear) points of $\mathcal{A}$ will be less than or equal to α:

$$\frac{(s+1)s^3}{2} + \frac{(s+1)s^3t(s-1)}{2} \leq \alpha \leq \frac{(1+s)(1+st+s^2t^2)s^3}{2(1+t+t^2)} \qquad (1)$$

Equality in both cases is satisfied if and only if $t^2(t-s)(s-1) = 0$.

For $s = t$, we can conclude two things: the equality between the first and second quantity expresses that every $\mathcal{D}$ contains 2 collinear points of $\mathcal{A}$; while the second equality expresses that through every 2 points of Γ, there is a thin ideal subhexagon $\mathcal{D}$. ∎

Corollary

Let Γ be a finite generalized hexagon of order (s,t), which contains a set $\mathcal{A} = \Gamma_1(M) \cup \Gamma_3(M)$ for which all points are spanregular. Then, through every 2 points at distance 6, there exists 1 thin ideal subhexagon; through every 2 points at distance 4, there are s thin ideal subhexagons; through every 2 points at distance 2, there are s^2 thin ideal subhexagons; through every point, there are s^3 thin ideal subhexagons.

Theorem 3 *Let* $\Gamma = (\mathcal{P},\mathcal{L},\mathrm{I})$ *be a finite generalized hexagon of order* s*. Consider the set* $\mathcal{A}$ *consisting of all points at distance* 1 *or* 3 *of a certain line. Every point of* $\mathcal{A}$ *is spanregular* $\Leftrightarrow$ Γ *is isomorphic to* $H(q), q = s$*.*

Proof.

$\boxed{\Leftarrow}$ This follows from Ronan [4].

$\boxed{\Rightarrow}$ Due to Ronan [4] we have to prove all traces of Γ are ideal lines. So, for 2 points $x, y \in \mathcal{P}$ with $d(x,y) = 4, z = x \bowtie y$ we must prove that z^w, $w \in \Gamma_4(x) \cap \Gamma_4(y) \cap \Gamma_6(z)$, is independent of w.

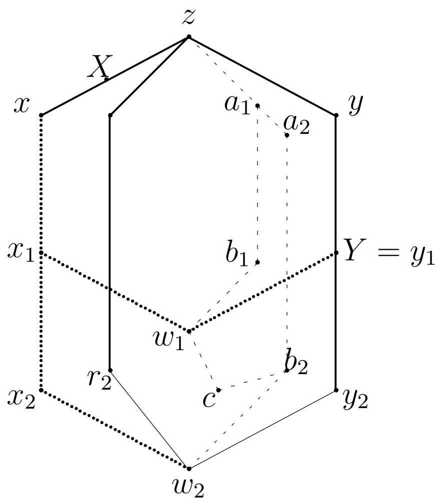

Figure 1: $X \neq Y, X = Z$

From the corollary, it follows that there are s thin ideal subhexagons $\mathcal{D}_i$ containing x and y. They can be obtained by choosing a point y_i on a line through y at distance 5 from x and they all contain 2 collinear points of $\mathcal{A}$. Since there is only one trace z^w in $\mathcal{D}_i$ (there are only 2 points on a line), $z^w = z^{w'}$, $\forall w, w' \in \Gamma_4(x) \cap \Gamma_4(y) \cap \mathcal{D}_i$. So we have to prove that $z^{w_1} = \ldots = z^{w_s}$ with $w_i \in \mathcal{D}_i$.
If $x \notin \mathcal{A}$, we denote the unique point of $\mathcal{A}$ collinear with x by a capital letter X, possibly with some index i (depending on the thin ideal subhexagon $\mathcal{D}_i$ where this point belongs to). The same for $y \sim Y$ and $z \sim Z$. We denote the line yY by L.

(a) If $x \in \mathcal{A}$ or $y \in \mathcal{A}$ then it is immediate that z^w is ideal.

(b) If $X = Y = z$ then it is immediate that z^w is ideal.

(c) Suppose $X \neq Y, X = Z$.
With every point $y_i \in L \backslash \{y\}$ (with $y_1 = Y$, without loss of generality) , there corresponds a thin ideal subhexagon $\mathcal{D}_i$ through x, y and y_i. First we look at $\mathcal{D}_1$ and the hyperbolic line $\langle x, y\rangle_1$ in $\mathcal{D}_1$. We will show that the hyperbolic lines $\langle x, y\rangle_i$ in the other $\mathcal{D}_i$'s are the same.
Let y_2 be a point of $L \backslash \{y, y_1\}$ and let $\mathcal{D}_2$ be the thin ideal subhexagon through x, y and y_2. By lemma 2, each $\mathcal{D}_i$ contains 2 collinear points of $\mathcal{A}$, say r_i and s_i. If $d(r_is_i, z) = 5$ and $\mathrm{proj}_{r_is_i} z = r_i$, then $d(r_i, z) = 4$. If $d(r_is_i, z) = 3$ and $\mathrm{proj}_{r_is_i} z = r_i$, then $d(s_i, z) = 4$. If $d(r_is_i, z) = 1$, then $r_i = s_i$ or $r_i = z$, a contradiction. So z is at distance 4 from one of these 2 points; say at distance 4 from r_i. Let $\mathcal{D}_i = \mathcal{D}_2$. Since r_2 and L are in $\mathcal{D}_2$, also the shortest path between them lies in $\mathcal{D}_2$. So the projection of r_2 onto L should be y_2 (as $d(r_2, y) = 6$), and we denote $r_2 \bowtie y_2$ by w_2. As $d(w_2, xz) = 5$, also the path between w_2 and x belongs to $\mathcal{D}_2$. Say $x_2 := w_2 \bowtie x$.
Denote $\mathrm{proj}_{xx_2} y_1$ by x_1, and $x_1 \bowtie y_1$ by w_1. Suppose that $\langle x, y\rangle_2 = z^{w_2}$ is

different from $\langle x,y\rangle_1 = z^{w_1}$. So there is a line N through z on which the point a_1 at distance 4 from w_1 is different from the point a_2 at distance 4 from w_2. Denote $a_i \bowtie w_i$ by b_i. One can show (see [6] 1.9.9) that whenever a trace contains a spanregular point, this trace is an ideal line. As y_1 and r_2 are spanregular, we have ideal lines $\langle x_1,y_1\rangle$ and $\langle x_2,y_2\rangle$. So $w_1^z = w_1^{w_2}$ and $w_2^z = w_2^{w_1}$. As $b_2 \in w_2^z = w_2^{w_1}, d(b_2,w_1) = 4$. Denote $b_2 \bowtie w_1$ by c. As $c \in w_1^{w_2} = w_1^z$, $d(c,z) = 4$. But $d(c,z) = 6$ as one supposed that $(z, za_2, a_2, a_2b_2, b_2, b_2c, c)$ is a path of length 6. So this is a contradiction. To solve this, a_1 should be a_2, and hence $b_1 = c$, and a_1, b_1, b_2 are collinear.

(d) Suppose $X \neq Y, Y = Z$. Similar to the previous case.

(e) Suppose $X \neq Y \neq Z \neq X$. If $Z \in z^w$ for some $w \in \Gamma_4(x) \cap \Gamma_4(y) \cap \Gamma_6(z)$ then $\langle x,y\rangle_w$ is ideal since it contains the spanregular point Z. If not, take a point $w \in \Gamma_4(x) \cap \Gamma_4(y) \cap \Gamma_6(z)$ and put $\mathrm{proj}_{zZ} w = t$. By case (c) (with x replaced by t, and with X replaced by $T = Z$), we have that $\langle t,y\rangle_w$ is ideal, so $\langle x,y\rangle_w$ is ideal. ■

5.4 $\mathcal{A}$ a full subhexagon, and conditon ($\star$) is satisfied

Theorem 4 *Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a finite generalized hexagon of order (s,t). Consider a proper full subhexagon $\mathcal{A}$ of Γ, and suppose there is a thin ideal subhexagon $\mathcal{D}$ through any 2 points of Γ. Then every point of $\mathcal{A}$ is spanregular $\Leftrightarrow$ Γ is isomorphic to $H(q), q = s = t$, with q a power of 3.*

Proof.

$\boxed{\Leftarrow}$ This follows from Ronan [4].

$\boxed{\Rightarrow}$ By the preliminary remark in lemma 2, we know that Γ has order s, and $\mathcal{A}$ is thin.

If $s = 2$, the result is trivially true by Cohen and Tits [1]. Hence we may assume $s > 2$.

Due to Ronan [4] we have to prove that Γ has ideal lines. So, for 2 points $x, y \in \mathcal{P}$ with $d(x,y) = 4, z = x \bowtie y$ we must prove that $\langle x,y\rangle_w = z^w$, $w \in \Gamma_4(x) \cap \Gamma_4(y) \cap \Gamma_6(z)$, is independent of w.

As we supposed that any 2 opposite points are contained in a thin ideal subhexagon $\mathcal{D}$, there are s $\mathcal{D}_i$'s containing x and y. They can be obtained by choosing a point $y_i \neq y$ on a fixed line through y at distance 5 from x. Since there is only one trace z^w in $\mathcal{D}_i$, $z^w = z^{w'}$ $\forall w, w' \in \Gamma_4(x) \cap \Gamma_4(y) \cap \Gamma_6(z) \cap \mathcal{D}_i$. So we have to prove that $z^{w_1} = \ldots = z^{w_s}$ with $w_i \in \mathcal{D}_i$.

If $x \notin \mathcal{A}$, we denote the unique point of $\mathcal{A}$ collinear with x by a capital letter X and some index i (depending on the thin ideal subhexagon $\mathcal{D}_i$ where this point belongs to). The same for $y \sim Y$ and $z \sim Z$.

1. If $x \in \mathcal{A}$ or $y \in \mathcal{A}$ then it is immediate that z^w is ideal.

2. If $X = Y = z$ then it is immediate that z^w is ideal.

3. Suppose $X \neq Y \neq Z \neq X$ and $d(X,Y) = 6$, so $\mathcal{D}_1 = \Gamma(y,X) \neq \Gamma(x,Y) = \mathcal{D}_2$. Attaching indices, we get $X = X_1, Y = Y_2$. We denote $\mathrm{proj}_{xX_1}Y_2$ by x_2, $\mathrm{proj}_{yY_2}X_1$ by y_1, and $w_1 := X_1 \bowtie y_1, w_2 := x_2 \bowtie Y_2$.
Take a point $w_3 \in \Gamma_3(xX_1) \cap \Gamma_3(yY_2)$, $w_3 \neq z$, and suppose $\Gamma(z,w_3) = \mathcal{D}_3$ not equal to $\mathcal{D}_1$ or $\mathcal{D}_2$. We show that $z^{w_1} = z^{w_2} = z^{w_3}$. As $X_1 \in w_1^z$ and $Y_2 \in w_2^z$, these traces are ideal lines. So $w_1^z = w_1^{w_2} = w_1^{w_3}$ and $w_2^z = w_2^{w_1} = w_2^{w_3}$. Using the same arguments (and notations) as in the proof of theorem 3 case (c), we know that $z^{w_1} = z^{w_2}$ and also $w_3^{w_1} = w_3^{w_2}$. Using this knowledge, we show that $z^{w_1} = z^{w_3}$.
Suppose $z^{w_1} \neq z^{w_3}$; this means there is a line N through z on which the point $a_1 = a_2$ at distance 4 from w_1 (and w_2) is different from the point a_3 at distance 4 from w_3. Denote $a_i \bowtie w_i$ by b_i. In the proof of theorem 3, we showed already that a_1, b_1 and b_2 are collinear. As $b_1 \in w_1^z = w_1^{w_3}, d(b_1,w_3) = 4$. Similarly $d(b_2,w_3) = 4$. But then we have a pentagon, a quadrangle or a triangle, unless $w_3 \bowtie b_2 = w_3 \bowtie b_1$ and $w_3 \bowtie b_i \sim b_i$, $i = 1,2$. Conclusion: $d(b_1b_2,w_3) = 3$ and $a_1 = a_2 = a_3$.

4. Suppose $X \neq Y \neq Z \neq X$ with $d(X,Y) = 4$, and suppose $s \geq 4$. So the path between $X = X_1$ and $Y = Y_1$ belongs also to $\mathcal{A}$ and we can denote $X_1 \bowtie Y_1$ by the capital letter W_1. Take a point $w_3 \in (\Gamma_3(xX_1) \cap \Gamma_3(yY_1)) \backslash \mathcal{D}_1$ and say $\mathrm{proj}_{yY_1}w_3 = y_3$, $\mathrm{proj}_{xX_1}w_3 = x_3$. Take a line through z, different from zx, zy or zZ, and project w_3 onto this line. The projection is the point u. As $u \notin \mathcal{A}$ (otherwise $z \notin \mathcal{A}$ would be collinear with 2 points of the ovoidal subspace), u is collinear with a unique point U of $\mathcal{A}$. Suppose this spanregular point is also at distance 4 from X_1 and Y_1. Then we take another line through z, we project w_3 onto this line, denoting the projection and its unique collinear point of $\mathcal{A}$ by v and V, respectively. Now we show that V is at distance 6 from at least one of the three points X_1, Y_1 or U. The points X_1, Y_1, U define an ordinary sixgon in the thin full subhexagon $\mathcal{A}$. Suppose $d(U,V) = 4$ and $T := U \bowtie V$. As there are only 2 lines through one point in $\mathcal{A}$, T should be on the line $\langle U, U \bowtie Y_1 \rangle$ or on the line $\langle U, U \bowtie X_1 \rangle$. Say $T \in \langle U, U \bowtie Y_1 \rangle$. If $T \neq U \bowtie Y_1, d(V,Y_1) = 6$. If $T = U \bowtie Y_1$, V should be on the line $\langle U \bowtie Y_1, Y_1 \rangle$ (as there are only 2 lines through a point in $\mathcal{A}$), hence $d(V,X_1) = 6$. So in this situation one can find a spanregular point $V = V'$ at distance 6 from X_1, Y_1 or U. Suppose $d(V',X_1) = 6$. We now use case (3.) of this proof, for $X_1 \neq V' \neq Z \neq X_1$.
First suppose $d(w_3,V') = 6$, and see figure 2. Put $\mathrm{proj}_{xX_1}V' = x_2$, $\mathrm{proj}_{vV'}X_1 = v_1'$, $\mathrm{proj}_{vV'}x_3 = v_3'$, $w_3 \bowtie v = v_3$, $x_3 \bowtie v_3' = w_3'$, $X_1 \bowtie v_1' = w_1'$, $x_2 \bowtie V' = w_2'$. By case (3.) of the proof, $z^{w_1'} = z^{w_2'} = z^{w_3'} = z^a$, for all $a \in \Gamma_3(xx_3) \cap \Gamma_3(vv_3) \cap \Gamma_6(z)$. As w_3 and w_3' are in the same thin ideal subhexagon $\mathcal{D}_3 = \Gamma(x_3,v)$, we know that $\langle x,v \rangle_{w_3} = \langle x,v \rangle_{w_3'}$. As $x,u,v,y \in \Gamma_2(z) \cap \mathcal{D}_3$, $\langle x,v \rangle_{w_3} = \langle x,y \rangle_{w_3}$. So $\langle x,y \rangle_{w_3} = \langle x,v \rangle_{w_3} = \langle x,v \rangle_{w_3'} = \langle x,v \rangle_a$, for all $a \in \Gamma_3(xx_3) \cap \Gamma_3(vv_3) \cap \Gamma_6(z)$. This finishes the proof if $d(w_3,V') = 6$.
Suppose $d(w_3,V') = 4$. Then $\langle x,v \rangle_{w_2'} = \langle x,v \rangle_{w_3}$ by case (3.) of this proof.

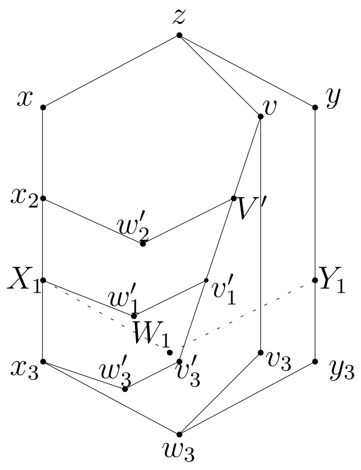

Figure 2: $d(X_1, Y_1) = 4, d(w_3, V') = 6$

Using $\langle x, y\rangle_{w_3} = \langle x, v\rangle_{w_3}$, we have the same result as before.
Suppose $d(w_3, V') = 2$. Then $V' = v_3$. As in case (c) of the proof of theorem 3, one shows that $\langle x, v\rangle_{w_3} = \langle x, v\rangle_{w_1'}$.

4.bis Suppose $s = t = 3$.
As we assumed the existence of five lines through a point in the previous section, we now investigate the case $s = t = 3$, for $X \neq Y \neq Z \neq X$ and $d(X, Y) = 4$. So X_1, Y_1 are in the same $\mathcal{D}_1$, and $W_1 := X_1 \bowtie Y_1$. Take w_3 at distance 3 from xX_1 and yY_1, and define $x_3 := \text{proj}_{xX_1} w_3$ and $y_3 := \text{proj}_{yY_1} w_3$. As we must prove $\langle x, y\rangle_{W_1}$ to be equal to $\langle x, y\rangle_{w_3}$, we suppose $Z \notin \langle x, y\rangle_{w_3}$ (otherwise the proof is done). As Z is spanregular, x, Z define an ideal line $\langle x, Z\rangle$. If y would be in $\langle x, Z\rangle$, this would imply Z to be in $\langle x, y\rangle_{w_3}$ — a contradiction. For the same reason, $x \notin \langle y, Z\rangle$.
Now we look at the fourth line through z, let's call it L. As $\langle x, Z\rangle$and $\langle y, Z\rangle$ are different ideal lines, their intersection only contains the point Z. So their respective intersection points with L are different — and by this named t_x and t_y, respectively.
Now we consider again the traces $\langle x, y\rangle_{W_1}$ and $\langle x, y\rangle_{w_3}$. If t_x would be in $\langle x, y\rangle_{w_3}$, the trace $\langle x, y\rangle_{w_3}$ contains 2 points (x and t_x) of the ideal line $\langle x, Z\rangle$, and hence $\langle x, y\rangle_{w_3} = \langle x, Z\rangle$. This is of course a contradiction. For the same reason, $t_y \notin \langle x, y\rangle_{w_3}$. We can conclude that $|\langle x, y\rangle_{w_3} \cap L \cap \langle x, y\rangle_{W_1}| = 1$, and we call this intersection point t. We put $a_1 := t \bowtie W_1$ and $a_3 := t \bowtie w_3$.
As W_1^z contains spanregular points X_1 and Y_1, this trace is ideal . As $W_1^{w_3}$ intersects W_1^z in at least 2 points, $W_1^{w_3}$ should be equal to W_1^z. So $a_1 \in W_1^{w_3}$, which means $d(a_1, w_3) = 4$. If a_1 is not on the line ta_3, there arises an ordinary pentagon with edges $t, a_1, a_1 \bowtie w_3, w_3, a_3$. So a_1 is on ta_3
Now we construct $s_1 := \text{proj}_{zZ} W_1$; $s_3 := \text{proj}_{zZ} w_3$; $b_1 := s_1 \bowtie W_1$; $b_3 := s_3 \bowtie w_3$. By a previous argument, neither s_1 nor s_3 coincide with Z (because $\langle x, Z\rangle$ is ideal and doesn't contain y). We know that $b_1 \in W_1^z = W_1^{w_3}$, so $d(b_1, w_3) = 4$.

As there are only 4 lines through w_3, and the lines w_3x_3, w_3a_3, w_3y_3 already correspond to the respective points $X_1, a_1, Y_1 \in W_1^{w_3}$, we know that $b_1 \bowtie w_3$ is on b_3w_3. But this results in an ordinary pentagon $b_1, s_1, s_3, b_3, b_3 \bowtie b_1$ if b_1 is not on s_3b_3. Conclusion: b_1 is on s_3b_3 and $s_1 = s_3$. So $z^{W_1} = z^{w_3}$, and this part of the proof is completed.

5. Suppose $X \neq Y = Z$.
   Take $w_3 \in \Gamma_3(xX) \cap \Gamma_4(y)$, and say $\mathrm{proj}_{xX} w_3 = x_3$. Take a line N through z, different from zx or zy, and say $\mathrm{proj}_N w_3 = v_3$. As $v_3 \notin \mathcal{A}, v_3$ is collinear with a unique point $V \in \mathcal{A}$, $V \notin v_3z$. At this point, we can use parts (3.) and (4.) of the proof to conclude that $\langle x, v_3\rangle_{w_3} = \langle x, v_3\rangle_{w_i}$, $i = 1, 2, 3$. As $x, y, v_3 \in \Gamma_2(z) \cap \mathcal{D}_3$, we know $\langle x, y\rangle_{w_3} = \langle x, v_3\rangle_{w_3}$, so $\langle x, y\rangle_{w_3}$ is ideal.

By now, we know $\Gamma \cong H(q)$. As Γ contains a full as well as ideal subhexagons, q must be a power of 3 by [6] 3.5.7.

5.5 $\mathcal{A} = \Gamma_1(L) \cup \Gamma_3(L)$, Γ of order (s, t), and condition ($\star$) is satisfied

Theorem 5 *Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a finite generalized hexagon of order (s, t). Consider the set $\mathcal{A}$ consisting of all points at distance 1 or 3 from a certain line L, and suppose there is a thin ideal subhexagon $\mathcal{D}$ through any 2 points of Γ. Then every point of $\mathcal{A}$ is spanregular $\Leftrightarrow$ Γ is isomorphic to $H(s)$ or to $T(s, \sqrt[3]{s})$.*

Proof. If $s \neq t$, we cannot use lemma 2. But by assuming $\beta = Y$ (see 5.1.1), we can re-arrange the (provisional) proof of theorem 3 in the same way as in proof 4: the new proof only uses the second equality in (1).

Where possible, we refer to the proof of theorem 4.

$\boxed{\Leftarrow}$ This follows from Ronan [4].

$\boxed{\Rightarrow}$ Due to Ronan [4] we have to prove that Γ has ideal lines.

For $z^w = z^{w'}$ $\forall w, w' \in \Gamma_4(x) \cap \Gamma_4(y) \cap \mathcal{D}_i$: cfr. theorem 4.
For $z^{w_1} = \ldots = z^{w_s}$ with $w_i \in \mathcal{D}_i$: cfr. below.

1. cfr. theorem 4 (1.)
2. cfr. theorem 4 (2.)
3. cfr. theorem 4 (3.)
4. cfr. theorem 4 (4.)*:* Suppose $X \neq Y \neq Z \neq X$ and $d(X, Y) = 4$. So the path between $X = X_1$ and $Y = Y_1$ belongs also to $\mathcal{A}$ and we can denote $X_1 \bowtie Y_1$ by the capital letter W_1. Take a point $w_3 \in (\Gamma_3(xX_1) \cap \Gamma_3(yY_1)) \backslash \mathcal{D}_1$ and say $\mathrm{proj}_{yY_1} w_3 = y_3$, $\mathrm{proj}_{xX_1} w_3 = x_3$. Take a line through z, different from zx, zy or zZ, and project w_3 onto this line. The projection is the point u^1. As $u^1 \notin \mathcal{A}$ (otherwise $z \notin \mathcal{A}$ would be collinear with 2 point of the ovoidal subspace), u^1 is collinear with a unique point U^1 of $\mathcal{A}$.

New for this proof:
We can do the same for the remaining lines through z, to obtain the points $U^1, \ldots, U^{t-2}$.
($\bullet$) If we suppose that none of these points U^j is at distance 6 from X_1 or at distance 6 from Y_1, then they should all be at distance 4 from X_1 and Y_1, and hence at distance 2 from W_1 (as $\mathcal{A}$ contains no appartment). So W_1 is a point of the 'central' line L of $\mathcal{A}$. None of the t lines W_1X_1, W_1Y_1, W_1U^j is equal to L. Indeed, suppose $W_1U^1 = L$. We know $Z \in \mathcal{A} = \Gamma_1(L) \cup \Gamma_3(L)$, so $d(Z, W_1U^1) = 3$ (as Z doesn't belong to W_1U^1). But this results in an ordinary pentagon. Conclusion: the line L is the projection of Z onto W_1, and this completes the linepencil $\Gamma(W_1)$. So $d(W_1, Z) = 4$. This means: $Z \in z^{W_1} = \langle x, y\rangle_{W_1}$. By this, $\langle x, y\rangle_{W_1}$ contains a spanregular point and hence is ideal.
If on the other hand the assumption ($\bullet$) is false, i.e. if there is a point U^j at distance 6 from X_1 or Y_1, then we refer to *theorem* 4 (4.) for the remaining part of the proof.

5. cfr. theorem 4 (5.) ■

5.6 $\mathcal{A}$ an ovoid, Γ of order (s,t), and condition ($\star$) is satisfied

Theorem 6 *Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a finite generalized hexagon of order (s,t) containing an ovoid $\mathcal{A}$. Suppose there is a thin ideal subhexagon $\mathcal{D}$ through any 2 points of Γ. Then every point of $\mathcal{A}$ is spanregular $\Leftrightarrow$ Γ is isomorphic to $H(q), q = s$.*

Proof. In a completely similar way as in the proof of theorem 4 — noting that all points of $\mathcal{A}$ are at distance 6 from each other (and hence case (4.) of the proof of 4 cannot occur) —, we prove that Γ is classical. As it is known that $T(q, \sqrt[3]{q})$ does not have an ovoid (see [6] 7.2.4), Γ is isomorphic to $H(q)$, $s = t = q$. ■

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