On the dimensions of the root groups of full subquadrangles of Moufang quadrangles arising from algebraic groups

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Abstract

Let Γ be a Moufang quadrangle arising from an algebraic group in the sense of Tits [4]. We call the pair (s,t) the dimensions of Γ if the root group of point-elations (respectively line-elations) has dimension s (respectively t). If (s,t) are the dimensions of Γ, and if (s,t') are the dimensions of a subquadrangle Γ', then we show that $s+t' \leq t$. This generalizes a result of Thas [3] for finite quadrangles, and of Kramer and Van Maldeghem (in preparation) for compact topological quadrangles. The proof we present is a geometric one using the notion of a subtended ovoid.

1 Introduction

By the recent classification of Moufang quadrangles by Tits and Weiss (book in preparation), the Moufang quadrangles fall into two classes: one class consists of those quadrangles which *arise from classical or algebraic groups* in the sense of [4]; another class consists of those quadrangles *of algebraic origin* which are related to groups of so-called "mixed type" (the type B_n corresponds to the mixed quadrangles ("quadrangles indifférents" in [6]); the type F_4 corresponds to the exceptional Moufang quadrangles of type F_4 recently discovered by Richard Weiss, and proved to be of algebraic origin by Mühlherr and Van Maldeghem [1]). This note is concerned

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only with the first type of Moufang quadrangles, and more exactly, only with those quadrangles arising from algebraic groups. For short, we will call a member of this class a *rational Moufang quadrangle*, because it arises from an algebraic group over an algebraically closed field by taking the “rational points” with respect to a subfield (the name *classical Moufang quadrangle* will be used for a Moufang quadrangle arising from a classical group; note that only “infinite dimensional” classical Moufang quadrangles are not rational — and with ”dimension” we here understand the dimension of the vector space over the center of the skew field involved). Note that every Moufang quadrangle in characteristic $\neq 2$ is a classical or rational Moufang quadrangle (and the characteristic of a Moufang quadrangle, or in general, Moufang polygon, can be defined as the order of any central or axial elation).

Let Γ be a rational Moufang quadrangle, and suppose that Γ' is a rational Moufang subquadrangle of Γ. Moreover, we assume that Γ' is a *full* subquadrangle, i.e., every point of Γ on any line of Γ' belongs to Γ' (the dual notion of a full subquadrangle will be called an *ideal* subquadrangle). Fix an apartment Σ of Γ and let x_i, $i = 1, 3, 5, 7 \mod 8$, be the points of Σ, with $x_i \perp x_{i+2}$, $i = 1, 3, 5, 7 \mod 8$, where $a \perp b$ means that a is collinear with b. Put $x_{2i} = x_{2i-1}x_{2i+1}$. An (x_i, x_{i+1}, x_{i+2})-*elation*, or briefly an *elation*, is a collineation of Γ fixing all elements incident with one of x_i, x_{i+1} and x_{i+2}. If x_i is a point, then such an elation is called a *point-elation*, otherwise a *line-elation*. The group of all (x_i, x_{i+1}, x_{i+2})-elations acts semi-regularly on the set of elements of Γ incident with x_{i-1}, but different from x_i. The Moufang condition states precisely that this action is transitive, hence regular, for all i. We denote the group of all (x_i, x_{i+1}, x_{i+2})-elations by U_i. By [5], there exists $i \in \mathbb{Z} \mod 8$ such that U_{i+2n} is commutative for all $n \in \mathbb{Z} \mod 8$, and U_{i+1+2n} is nilpotent of class ≤ 2. Moreover, putting $V_{i+1+2n} = [U_{i+2n}, U_{i+2+2n}]$, one has the following commutation relations:

(CR0) $[U_j, U_{j+k}] \leq U_{j+1} \cdots U_{j+k-1}$, for all $j \in \mathbb{Z} \mod 8$, $k = 1, 2, 3$. In particular, $[U_j, U_{j+1}]$ is always trivial.

(CR1) $[U_{i+1+2n}, U_{i+1+2n}] \leq V_{i+1+2n}$;

(CR2) $[U_{i-1+2n}, V_{i+1+2n}]$ and $[V_{i-1+2n}, U_{i+1+2n}]$ are trivial;

(CR3) $[U_{i+2n}, V_{i+3+2n}] \leq V_{i+1+2n}U_{i+2+2n}$.

Note that (CR0) easily implies that V_{i+1+2n} is a normal subgroup of U_{i+1+2n}. We put $W_{i+1+2n} = U_{i+1+2n}/V_{i+1+2n}$. Also, the group U_+ generated by U_1, U_2, U_3, U_4 has a unique decomposition into the product $U_1U_2U_3U_4$. The subset $U_1U_2U_3$ is a subgroup and it acts regularly on the set of points of Γ *opposite* x_3, i.e., on the set of points of Γ not collinear with x_3 (this is easily shown, see also [7](5.2.4)). Moreover, the group $U_1U_2U_3$ only depends on x_3 (and not on Σ), hence we call this group the *whorl group* about x_3 (because all its elements are whorls about x_3 in the sense of [2]).

Now let $\mathbb{K}$ be the field over which Γ is defined, i.e., the elements of the corresponding algebraic group are the $\mathbb{K}$-rational points of an algebraic group over the algebraic closure of $\mathbb{K}$. Then the groups U_{i+2n} can be turned into a vector space over $\mathbb{K}$. The addition of vectors coincides with the product in U_{i+2n}, the multiplication with a scalar is given by conjugation with a certain element of the *torus*, i.e., a certain collineation stabilizing Σ. The set of all such elements forms a subgroup of the torus isomorphic to the direct product of two copies of the multiplicative group of $\mathbb{K}$. Geometrically, one copy fixes all points on a certain line of Σ, the other fixes every line through a certain point of Σ. It is now clear that each of these act on suitable U_{i+2n} by conjugation. The dimension of U_{i+2n} over $\mathbb{K}$ is briefly called the *dimension of* U_{i+2n}. Similarly, V_{i+1+2n} and W_{i+1+2n} can be turned into vector spaces over $\mathbb{K}$, and we call the sum of their dimensions briefly the *dimension of* U_{i+1+2n}, and this is equivalent to the usual definition of dimension of a root group as an algebraic variety (Tits, personal communication through lectures given at Collège de France, 1995-1996). If s and t are the dimensions of the root groups containing point-elations and line-elations, respectively, then we call (s,t) the dimensions of Γ. In the finite case, Γ has then order (q^s, q^t) for some prime power q. In the compact topological case, Γ has then topological parameters (s,t), except in the cases where the field $\mathbb{K}$ is isomorphic to the field $\mathbb{C}$ of complex numbers: in this case the topological parameters are $(2s, 2t)$. This only happens when Γ is isomorphic or anti-isomorphic to the symplectic quadrangle over $\mathbb{C}$.

Now put $X_{i+1} = [U_{i+1}, U_{i+1}]$ for short. Since multiplication with a scalar is in fact an automorphism of U_{i+1}, the commutative group X_{i+1} is a vector space over $\mathbb{K}$, and it is a subspace of V_{i+1}, by (CR1). Similarly, the commutative group U_{i+1}/X_{i+1} is a vector space (over $\mathbb{K}$), and V_{i+1}/X_{i+1} is a subspace of it. Moreover, the group morphism $U_{i+1}/X_{i+1} \longrightarrow W_{i+1} : uX_{i+1} \mapsto uV_{i+1}$ clearly preserves scalar multiplication, hence it is a vector space morphism and so $\dim U_{i+1}/X_{i+1} = \dim W_{i+1} + \dim V_{i+1}/X_{i+1}$. Similarly, $\dim V_{i+1}/X_{i+1} = \dim V_{i+1} - \dim X_{i+1}$. This now implies that we may define the dimension of U_{i+1} as $\dim U_{i+1}/[U_{i+1}, U_{i+1}] + \dim[U_{i+1}, U_{i+1}]$.

Let x be a point of Γ not contained in Γ'. Let $\mathcal{O}$ be the set of points of Γ' collinear with x. It is easy to see that every line of Γ' is incident with precisely one point of $\mathcal{O}$. Hence $\mathcal{O}$ is an *ovoid* of Γ', which we call a *subtended ovoid* (subtended by Γ and x).

Finally, we introduce some more terminology. For a set of points S of Γ, the set $S^\perp$ is the set of points x such that $x \perp y$, for every $y \in S$. We write $S^{\perp\perp}$ for $(S^\perp)^\perp$. A point x of Γ is called *regular* if for each point y not collinear with x, the set $\{x,y\}^\perp \cup \{x,y\}^{\perp\perp}$ is the point set of an ideal subquadrangle of Γ. Dually, one defines a regular line.

2 Main Result and some consequences

Main Result. *Let Γ be a rational Moufang quadrangle with dimensions (s,t) over the field $\mathbb{K}$; let Γ' be a rational Moufang full subquadrangle with dimensions (s,t') over $\mathbb{K}$; let $\mathcal{O}$ be an ovoid in Γ' subtended by Γ and x; let p be any point of $\mathcal{O}$ and let U_p be the subgroup of the whorl group about p stabilizing $\mathcal{O}$. Then we have:*

(*i*) *U_p acts transitively and hence regularly on $\mathcal{O} \setminus \{p\}$. In particular, the group generated by all U_a, $a \in \mathcal{O}$, acts 2-transitively on $\mathcal{O}$. Moreover, the corresponding permutation representation is equivalent to the one arising from the subgroup of the stabilizer of x in the group generated by all elations of Γ fixing x, acting on the set of lines through x that contain a point of $\mathcal{O}$;*

(*ii*) *U_p is nilpotent of class at most 2. Moreover, both $[U_p, U_p]$ and $U_p/[U_p, U_p]$ can be given the structure of a vector space over $\mathbb{K}$, and the sum of the dimensions of these vector spaces is equal to $s + t'$;*

(*iii*) *$s + t' \leq t$ and equality holds if and only if every line of Γ contains at least one point of Γ';*

(*iv*) *the ovoid $\mathcal{O}$ contains all elements of $\{a, b\}^{\perp\perp}$ which belong to Γ', for all $a, b \in \mathcal{O}$, $a \neq b$.*

Before proving the Main Result, let us look at some elementary applications.

Let us consider the exceptional Moufang quadrangles Γ of type E_i, $i = 6, 7, 8$, over some field $\mathbb{K}$ (depending on i), see [6]. Up to duality, these quadrangles have dimensions $(6, 9)$, $(8, 17)$ and $(12, 33)$, respectively. These quadrangles have rational Moufang full subquadrangles Γ' which are dual to quadrangles arising from a quadric of Witt index 2 in the projective space $\mathrm{PG}(d, \mathbb{K})$, with $d = 9, 11, 15$, respectively, and having (dual) dimensions $(6, 1)$, $(8, 1)$ and $(12, 1)$, respectively. Hence, in each case there exist lines in the exceptional Moufang quadrangle which do not meet the corresponding full subquadrangle. Also, these exceptional quadrangles do not contain regular points, since a regular point gives rise to an ideal subquadrangle of dimensions $(0, t)$, $t = 9, 17, 33$, respectively, and this would imply that $t \leq s$, a contradiction. Similarly regular lines are ruled out in the full subquadrangles, and hence they can neither exist in the exceptional Moufang quadrangles. This implies for instance that neither the exceptional Moufang quadrangles nor their duals can be weakly embedded of degree 2 in projective space (for definitions, see Steinbach, these proceedings), because intersecting the embedded quadrangle with suitable 3-space, one sees that all lines must be regular.

Consider the spread $\mathcal{S}$ in the dual of Γ' subtended by Γ and a point of Γ not in Γ'. By (*iv*), for any two elements L and M of this spread, all lines of the regulus defined by L and M are contained in $\mathcal{S}$. Moreover, this spread is stabilized by a 2-transitive group G. This group can probably be viewed as an algebraic group of relative rank 1 of dimension $7, 9, 13$, respectively. Anyway, from the proof below, it follows that G contains a normal subgroup isomorphic to $\mathbb{K}, +$ (this is U_2' in the proof), and the corresponding quotient is isomorphic to the (commutative) root group of both Γ and Γ' consisting of point-elations (this is U_3 in the proof below). This follows immediately from the structure of the root groups of the exceptional Moufang quadrangles given by Tits (unpublished), see also [7](5.5.5).

Another consequence of the Main Result is that no rational Moufang quadrangle Γ can have both full and ideal (thick) rational Moufang subquadrangles. Indeed, if the dimensions of Γ are (s, t), then this would imply $s < t$ and $t < s$, respectively, a contradiction.

Applied to the finite case, a subquadrangle of order $(q^s, q^{t'})$ of some Moufang quadrangle of order (q^s, q^t) satisfies $q^s q^{t'} \leq q^t$, which is exactly the inequality in [3]. Also the condition in (*iii*) for equality is exactly the same as the one in [3].

3 Proof of the Main Result

Let G_p be the whorl group in Γ with respect to p. Let p_1 and p_2 be two points of $\mathcal{O}$ distinct from p. Since there is in Γ exactly one whorl about p mapping p_1 to p_2, we have to show that it stabilizes $\mathcal{O}$. Let Σ be any apartment in Γ' through p and p_1, and label its elements as in the introduction with $p = x_3$ and consequently $p_1 = x_7$. We also will use the notation U_i as in the introduction.

Let U_p be the subgroup of G_p stabilizing $\mathcal{O}$. Let $u \in G_p$ be the unique element mapping p_1 onto p_2. Since u stabilizes all lines through p, hence also the line px, u fixes x. Also, u stabilizes Γ', since Γ'^u shares with Γ' the apartment Σ^u, all points on the line x_2, and all lines through the point p (and so $\Gamma' \cap \Gamma'^u$ coincides with both Γ' and Γ'^u, cfr. [7](1.8.1;1.8.2)). Consequently, u stabilizes $\mathcal{O}$ and so $u \in U_p$. Also, by [8](Lemma 1), u fixes all points on the line px. Now let U_x be the group of (px, x, x_2)-elations. We define the map $\theta : U_p \longrightarrow U_x : u \mapsto u'$, where u' is defined by $N^u = N^{u'}$, for all lines N through x (and this is well defined by [8](Lemma 1)). Clearly θ is a monomorphism, and hence U_p is isomorphic to a subgroup of U_x. This proves (i). Since U_x is nilpotent of class ≤ 2, also U_p is. Notice that, if U_x is commutative, then also U_p is. Now, for every $u_1 \in U_1$, there exists $u \in U_p$ such that $u = u_1u_2u_3$, with $u_i \in U_i$, $i = 1, 2, 3$ (indeed, this follows easily from the fact that every point on x_4 is collinear with some point of $\mathcal{O}$). Hence we see, using obvious notation for $u, u' \in U_p$, that $1 = [u, u'] = [u_1u_2u_3, u_1'u_2'u_3'] \in [u_1, u_1']U_2[u_3, u_3']$. We conclude that U_1 is commutative. On the other hand, if U_x is non-commutative, then U_1 is automatically commutative by [5]. Hence U_1 and U_3 are always commutative (hence the corollary below).

Now we consider an apartment Σ' through x, p, x_1 and $p_1 = x_7$ (see above). Then $\Sigma' = \Sigma^v$, where v is an (x_0, x_1, x_2)-elation in Γ. Let τ be an arbitrary element of the torus with respect to Σ defining by conjugation a scalar multiplication in $[U_2', U_2']$, where U_2' is the intersection of U_2 with the collineation group of Γ'. Then τ also defines a scalar multiplication in $[U_2', U_2']$. Also, τ^v defines a scalar multiplication in $(U_2')^v$. But τ^v has the same action on the set of lines through x_1 as τ, hence τ^v preserves Γ'. Let G_1 be the subgroup of U_p fixing the line x_2 pointwise. Then the preceding argument shows precisely that the map $\phi : G_1 \longrightarrow U_2'$ defined by "$u^\phi u^{-1}$ fixes x_0" is not only an isomorphism of groups, but it preserves scalar multiplication, and also that θ (see above) restricted to G_1 preserves scalar multiplication. Notice that $G_1 \trianglelefteq U_p$. Similarly, one shows that there is an isomorphism from U_p/G_1 to U_3 preserving scalar multiplication (and also the isomorphism from U_p/G_1 to U_x^θ/G_1^θ induced by θ preserves scalar multiplication).

Now we calculate the natural number $d = \dim_{\mathbb{K}}[U_p, U_p] + \dim_{\mathbb{K}} U_p/[U_p, U_p]$. In view of the definition of dimension of root groups above, d can be considered as the dimension of U_p. Since U_3 is commutative, we see that $[U_p, U_p] \leq G_1 \cong U_2'$. Hence the group $[U_p, U_p]/[G_1, G_1]$ is a subspace of $G_1/[G_1, G_1]$ (because the relevant action of the torus stabilizes U_p, and hence also $[U_p, U_p]$), and $G_1/[U_p, U_p]$ is a subspace of $U_p/[U_p, U_p]$. Noting that $t' = \dim U_2'/[U_2', U_2'] + \dim[U_2', U_2']$ (all dimensions are over

$\mathbb{K}$), we obtain:

$$\begin{aligned}\dim U_2/[U_2,U_2] &= \dim G_1/[G_1,G_1]\\ &= \dim[U_p,U_p]/[G_1,G_1]+\dim G_1/[U_p,U_p]\\ &= \dim[U_p,U_p]-\dim[G_1,G_1]+\dim U_p/[U_p,U_p]-\dim U_p/G_1\\ &= d-\dim[U_2,U_2]-\dim U_3\\ &= d-s-\dim[U_2,U_2],\end{aligned}$$

imlying $d = s + t'$, and proving (ii). Notice that, in the previous calculation, we used some isomorphisms between (commutative) groups. It is however easily checked (using the torus as above) that all isomorphisms are also isomorphisms of vector spaces over $\mathbb{K}$.

A similar calculation inside U_x and using the monomorphism θ defined above, shows that $d \leq t$. Indeed, identifying $(U_p)^\theta$ with U_p, we have:

$$\begin{aligned} d &= \dim U_p/[U_p,U_p]+\dim[U_p,U_p]\\ &\leq \dim U_p[U_x,U_x]/[U_p,U_p]+\dim[U_p,U_p]\\ &\leq \dim U_p[u_x,U_x]/[U_x,U_x]+\dim[U_x,U_x]/[U_p,U_p]+\dim[u_p,U_p]\\ &\leq \dim U_x/[U_x,U_x]+\dim[U_x,U_x]\\ &\leq t,\end{aligned}$$

and equality holds if and only if

$$U_x = U_p[U_x,U_x] = U_p,$$

hence if and only if every line through x meets Γ'. This shows (iii). Now (iv) follows immediately from the fact that $x \in \{a,b\}^\perp$. The proof of the Main Result is complete.

There is an interesting corollary to this proof:

Corollary. *If a Moufang quadrangle Γ with root groups U_1 (point-elations) and U_2 (line-elations) has a full subquadrangle Γ' with root groups U_1 (point-elations) and U_2' (line-elations), then the common root group U_1 is commutative.*

This corollary also holds for classical Moufang quadrangles and Moufang quadrangles of algebraic origin, because the proof only uses the Moufang condition and the subtended ovoid in Γ'. Let me point out that all Moufang quadrangles related to groups of mixed type only have commutative root groups. This could now be explained by the previous corollary and the fact that these quadrangles have lots of full and ideal subquadrangles, see Tits and Weiss (book in preparation), or [7].

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