Characterizations by Automorphism Groups of some Rank 3 Buildings,

I. Some Properties of Half Strongly-Transitive Triangle Buildings.

H. Van Maldeghem*   K. Van Steen

Abstract

In a sequence of papers, we will show that the existence of a (half) strongly-transitive automorphism group acting on a locally finite triangle building Δ forces Δ to be one of the examples arising from $\mathbf{PSL}_3(\mathbb{K})$ for a locally finite local skew-field $\mathbb{K}$. Furthermore, we introduce some Moufang-like conditions in affine buildings of rank 3, and characterize those examples arising from algebraic, classical or mixed type groups over a local field. In particular, we characterize the p-adic-like affine rank 3 buildings by a certain p-adic Moufang condition, and show that such a condition has zero probability to survive in hyperbolic rank 3 buildings. This shows that a construction of hyperbolic buildings as analogues of p-adic affine buildings is very unlikely to exist.

1 Introduction, first definitions and statement of the Main Result

Let Δ be a building of irreducible type $\tilde{A}_2$, i.e., a building with a triangle as diagram. For this reason, Δ is also called a *triangle building*, see Van Maldeghem [13], Tits [12]. Moreover, Δ is an affine building and has rank 3. All irreducible affine buildings of rank > 3 are classified by Tits [11] and they all arise from algebraic, classical or mixed type groups over a local field. In this case, the corresponding building is sometimes referred to as the *Bruhat-Tits building over the corresponding group*, see Bruhat & Tits [5]. We will briefly say that Δ is a *Bruhat-Tits building*. Below we will describe the spherical building

*The first author is a Research Director of the Fund for Scientific Research – Flanders (Belgium).

Δ^∞ at infinity of Δ (assuming Δ has a maximal set of apartments, see TITS [11]); Δ^∞ is the building associated with a projective plane $\mathcal{P}$ and $\mathcal{P}$ is a Moufang plane precisely when Δ is a Bruhat-Tits building. Our aim is to characterize the class of Bruhat-Tits buildings of type $\tilde{A}_2$, i.e., we seek conditions on Δ that force Δ to be a Bruhat-Tits building. The idea is that the existence of a natural (B, N)-pair in the automorphism group of Δ should already be enough to ensure that, if Δ is locally finite (that means that the residues are finite projective planes, see below), then Δ is a Bruhat-Tits building. Such a result is highly non-trivial, since there are many buildings associated to a (B, N)-pair without being related to an algebraic, classical or mixed type group (or a Ree group in characteristic 2), see for instance TITS [10]. The existence of a natural (B, N)-pair in the automorphism group G of Δ is equivalent to G acting *strongly-transitively* on Δ, see BROWN [4] and RONAN [9]. We recall from the latter references that G acts strongly-transitive on Δ if the following two conditions are satisfied (where δ denotes the Weyl-distance in Δ, see e.g. RONAN [9]):

(ST1) If (C, C') and (D, D') are two pairs of chambers of Δ and $\delta(C, C') = \delta(D, D')$, then there exists $g \in G$, g type-preserving, such that $g(C) = D$ and $g(C') = D'$;

(ST2) If Σ is an apartment of Δ, then the stabilizer G_Σ of Σ of G in Δ induces in Σ a group N which contains the full Weyl group of Σ.

We remark that in (ST2) the group N can also contain non-type-preserving automorphisms. If it doesn't, then N coincides with the Weyl group. Now suppose that we have an automorphism group G of Δ which satisfies (ST1), but not necessarily (ST2). Then we say that G is a *half strongly-transitive automorphism group* of Δ. Also, if the residue of any vertex in Δ is a finite projective plane, then we say that Δ is *locally finite.*

We can now state our Main Result.

Main Result. *If Δ is a locally finite triangle building with a half strongly transitive automorphism group G, then Δ^∞ is associated to a desarguesian projective plane, and hence Δ is a Bruhat-Tits building and arises from a classical group $\mathbf{PSL}_3(\mathbb{K})$ over a locally finite local skewfield $\mathbb{K}$.*

The proof of the Main Result will occupy parts I and II of this paper. In part III, the second author will consider stronger conditions, but larger classes of buildings. In particular, characterization results for the class of all rank 3 Bruhat-Tits buildings will be proved. Several results and intermediate steps from the proof of the above Main Result will be useful there, and hence we have tried to collect those in Part I of this paper (although a few are due to the second author). Part II specializes to the locally finite case, and Part III specializes to the so-called *root-Moufang* case and *p-adic Moufang* case. In fact, the reader should be aware of the fact that, technically, some results of Parts II and III could still be unified and proved in Part I, but that would make Part I less “homogeneous”. Also, we will sometimes provide different proofs for such results. This should be useful for possible generalizations.

In the last part, Part IV, we apply a definition of p-adic Moufang building to the class of compact rank 3 hyperbolic buildings to obtain some non-existence results, which confirm in an elementary way results by Tits (unpublished).

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2 Tools and technique

In order to prove the Main Result, we use the geometries at distance n from any fixed vertex v, as defined by Van Maldeghem [14], and proved to be projective Hjelmslev planes of level n by Hanssens & Van Maldeghem [6]. We translate property (ST1) to these geometries and show that they have to satisfy the so-called *Moufang condition* (see below). This implies that they actually are desarguesian Hjelmslev planes (by Baker, Lane & Lorimer [2], see Part II for the details), and using Van Maldeghem [15], the result will follow. Unfortunately, in the course of the proof, we have to use the building Δ for some arguments and therefore we are not able to state a result on projective Hjelmslev planes. Also, our method uses induction on n. For $n = 1$, the result amounts to the classification of finite projective planes with an automorphism group acting transitively on (non-degenerate) triangles. For $n = 2$, the proof is somewhat different then in the general case, and so this has to be treated separately.

In the present paper, we are interested in some general properties of triangle buildings with a half strongly-transitive group. In some results, the induction step mentioned in the previous paragraph will be apparent. From now on, and throughout the paper (except in Section 3), we assume that Δ is a triangle building with a half strongly-transitive automorphism group G acting on it.

The *distance* $d(x, y)$ between two vertices x and y in Δ is the natural distance in the graph of vertices of Δ, where adjacent vertices belong to a common chamber of Δ. In order not to confuse with the Weyl-distance between chambers, we will always explicitly add the prefix "Weyl" to the latter. The convex closure $cl(v_1, v_2, \ldots, v_\ell)$ of some set $\{v_1, v_2, \ldots, v_\ell\}$ of vertices is the smallest subcomplex of Δ with the property that for any two vertices x, y of that complex, all vertices lying on every minimal path joining x and y belong to it. It may be of smaller rank than Δ itself.

We consider an arbitrarily chosen vertex O in Δ and associate geometries ${}^{n}H(O)$, $n \geq 1$, (or briefly ${}^{n}H$ if no confusion is possible) with it as in Van Maldeghem [14] and Hanssens-Van Maldeghem [6] as follows.

The residue of O is a projective plane, which we denote by ${}^{1}\!H(O)$. So the points of ${}^{1}\!H(O)$ are certain vertices of Δ adjacent to O, and similarly for the lines of ${}^{1}\!H(O)$. Let n be any natural number (distinct from 0). By definition, the points (respectively lines) of the geometry ${}^{n}\!H(O)$ are the sequences $(v_1, v_2, \ldots, v_n)$ of vertices of Δ, where v_1 is a point (respectively a line) of ${}^{1}\!H(O)$, and where $\{v_{i-1}, v_{i+1}\}$ represents a non-incident point-line pair in the residue of v_i, for all $i \in \{1, 2, 3, \ldots, n-1\}$ (the latter is independent of the choice of the names 'points' and 'lines' in the residue of v_i), with $v_0 = O$ by convention. A point of ${}^{n}\!H(O)$ represented by $(p_1, p_2, \ldots, p_n)$ is *incident* with a line $(l_1, l_2, \ldots, l_n)$ if all vertices $p_1, \ldots, p_n, l_1, \ldots, l_n$ are contained in a common apartment and $d(l_i, p_i) = i$. The corresponding geometry ${}^{n}\!H(O)$ is called a *projective Hjelmslev plane of level* n. Due to a result of Hanssens & Van Maldeghem [6], this definition is equivalent with the usual one, see for instance Artmann [1].

Usually, we denote ${}^{n}\!H(O)$ simply by ${}^{n}\!H$. The points of ${}^{n}\!H$ are denoted with upper case letters, the lines with lower case letters. Incidence is denoted by ${}^{n}I$. The point set of ${}^{n}\!H$ is ${}^{n}\mathcal{P}$, the line set is ${}^{n}\mathcal{L}$. We will sometimes identify a point or a line of ${}^{n}\!H$ with a vertex of Δ. This will always be the vertex x_n, where $(x_1, x_2, \ldots, x_n)$ is the point or line in question.

There is an obvious epimorphism ${}^{n}\pi$ from ${}^{m}\!H$ onto ${}^{n}\!H$, for $m \geq n$. Hence, one can consider the inverse limit of the sequence $({}^{n}\!H)$ with respect to these epimorphisms, see Artmann [1]. By a result of Van Maldeghem, [14], this is precisely one of the two mutual dual projective planes associated to the spherical building Δ^∞ at infinity of Δ as defined by Tits [11]. If this projective plane is a Moufang plane, then the building Δ is a Bruhat-Tits building, as follows from Tits [11], Section 14.

If for points P and Q of ${}^{n}\!H$ we have ${}^{i}\pi(P) = {}^{i}\pi(Q)$, $0 < i \leq n$, then we call P and Q *i-neighbouring*. For $i = 1$, we also simply talk about *neighbouring*; in symbols $P \sim Q$; not neighbouring is then denoted as $P \not\sim Q$. Similarly for lines. If P is a point of ${}^{n}\!H$ and l a line of ${}^{n}\!H$, then we say that P is *i-near* l, $0 < i \leq n$, if ${}^{i}\pi(P)\, {}^{i}I\ {}^{i}\pi(l)$. Also, 1-near will be simply called *near*.

Note that i-neighbouring is an equivalence relation and that it is determined by the geometry ${}^{n}\!H$. Indeed, neighbouring points are points which are incident with at least two common lines; 2-neighbouring points P and Q are neighbouring points that are incident with at least two common lines l and l' for which there exists at least one point R neighbouring both P and Q with R on l, but not on l', etc. Hence every collineation α of ${}^{n}\!H$ preserves the various neighbour relations, and so α induces in ${}^{i}\!H$, $0 < i \leq n$ a unique collineation $\alpha^{\star_i}$, which we call the $()^{\star_i}$-projection of α. We will use that notation throughout. Nevertheless, when acting on elements of ${}^{j}\!H$, $1 \leq j \leq n$, $\alpha^{\star_j}$ will be sometimes denoted by α, in order to simplify the notation (and there is no confusion possible).

An *elation* in ${}^{n}\!H$ with axis some line l and center some point P, where P is incident with l, is a collineation of ${}^{n}\!H$ fixing all points on l and fixing all lines through P. If the group of all elations with axis l and center P acts transitively on the points not near l incident with some line m (which is itself not neighbouring l, but which is incident with P), then we say that ${}^{n}H$ is (P,l)*-transitive*. If ${}^{n}H$ is (P,l)-transitive for all choices of such P and l, then we say that ${}^{n}H$ is a *Moufang Hjelmslev plane*, or that ${}^{n}H$ *satisfies the Moufang condition*.

It is easy to see that Δ is locally finite precisely when all Hjelmslev planes constructed in Δ as above, are finite. This is also equivalent to saying that ${}^{1}H(O)$ is a finite projective plane for at least one vertex O of Δ.

For more information on projective Hjelmslev planes and related geometries, we refer to Keppens [7]. For Hjelmslev planes of level n (which is the only class of Hjelmslev planes we are dealing with in this paper), the reader is directed to Artmann [1], and to Hanssens & Van Maldeghem [6].

Throughout, and for all $n \geq 1$, ${}^{n}\Psi(O)$ (or briefly ${}^{n}\Psi$ if no confusion is possible) will denote the collineation group acting on ${}^{n}H(O)$ that is induced by G (except in Section 3).

Our general goal is to show that all geometries ${}^{n}H$ are Moufang Hjelmslev planes, and that all possible elations are available in ${}^{n}\Psi$, $n \geq 1$. Using an inductive argument, this will be one of our main lemmas in Part I.

3 General results

We now collect some basic properties that we will need later. Since the proof of our Main Result will be an inductive one, and since the arguments for $n = 2$ will be somewhat different from the general one, we specialize from time to time some results to $n = 2$.

3.1 Affine planes in ${}^{n}H$

Suppose ${}^{i}\pi(Q)$ is a point of ${}^{i}H(O)$, $1 \leq i \leq n$, for some point $Q \in {}^{n}\mathcal{P}(O)$. Then the projective plane, viewed as a completed affine plane, associated with ${}^{i}\pi(Q)$ is denoted by ${}^{1}H({}^{i}\pi(Q))$ and defined as follows.

For $i = 1$, the vertex O, which can be defined as ${}^{0}\pi(Q)$, corresponds with the line at infinity of ${}^{1}H({}^{1}\pi(Q))$. Hence all vertices in Δ that are adjacent to O and to ${}^{1}\pi(Q)$, so all lines of ${}^{1}H$ that are incident in ${}^{1}H$ with ${}^{1}\pi(Q)$, correspond to points at infinity of ${}^{1}H({}^{1}\pi(Q))$. Finally, all points R of ${}^{2}H(O)$ with ${}^{1}\pi(R) = {}^{1}\pi(Q)$ correspond with the affine points of ${}^{1}H({}^{1}\pi(Q))$.

For $i > 1$, the point ${}^{i-1}\pi(Q)$ of ${}^{i-1}\mathcal{P}(O)$ corresponds with the line at infinity of ${}^{1}H({}^{i}\pi(Q))$. All vertices in Δ that are adjacent to both ${}^{i-1}\pi(Q)$ and ${}^{i}\pi(Q)$ correspond with points at infinity of ${}^{1}H({}^{i}\pi(Q))$. The points R of ${}^{i+1}H(O)$ with ${}^{i+1,i}\pi(R) = {}^{n,i}\pi(Q)$ correspond with the affine points of ${}^{1}H({}^{i}\pi(Q))$.

The projective Hjelmslev plane of level j associated with ${}^{i}\pi(Q)$, $1 \leq j$, is denoted by ${}^{j}H({}^{i}\pi(Q))$ and defined as the projective Hjelmslev plane of level j attached to the vertex ${}^{i}\pi(Q)$ of the triangle building Δ such that ${}^{j,1}\pi({}^{j}H({}^{i}\pi(Q))) = {}^{1}H({}^{i}\pi(Q))$.

Suppose ${}^{i}\pi(m)$ is a line of ${}^{i}H(O)$, $1 \leq i \leq n$, for some line $m \in {}^{n}\mathcal{L}(O)$. Then the projective plane, viewed as a completed dual affine plane, associated with ${}^{i}\pi(m)$ is denoted by ${}^{1}H({}^{i}\pi(m))$ and is defined in a similar way as ${}^{1}H({}^{i}\pi(Q))$. For $i = 1$, the vertex O corresponds with the point at infinity of ${}^{1}H({}^{1}\pi(m))$ and all vertices in Δ that are adjacent to both ${}^{1}\pi(m)$ and O correspond with lines at infinity of ${}^{1}H({}^{1}\pi(m))$. The line m of ${}^{2}H$ itself, as all lines u of ${}^{2}H(O)$ with ${}^{1}\pi(u) = {}^{1}\pi(m)$, correspond with the affine lines of ${}^{1}H({}^{1}\pi(m))$.

For $i > 1$, the line ${}^{i-1}\pi(m)$ of ${}^{i-1}\mathcal{L}(O)$ corresponds with the point at infinity of the dual projective plane ${}^{1}H({}^{i}\pi(m))$, and the vertices in Δ that are adjacent to both ${}^{i-1}\pi(m)$ and ${}^{i}\pi(m)$ correspond with the lines at infinity of ${}^{1}H({}^{i}\pi(m))$. Finally, all lines u of ${}^{i+1}H(O)$ with ${}^{i+1,i}\pi(u) = {}^{n,i}\pi(m)$ correspond with the affine lines of ${}^{1}H({}^{i}\pi(m))$.

The projective Hjelmslev plane of level j associated with ${}^{i}\pi(m)$, $1 \leq j$, is denoted by ${}^{j}H({}^{i}\pi(m))$ and defined as the projective Hjelmslev plane attached to ${}^{i}\pi(m)$ such that ${}^{j,1}\pi({}^{j}H({}^{i}\pi(m))) = {}^{1}H({}^{i}\pi(m))$.

3.2 Elations and quasi-elations

A crucial notion in the course of the proof of the Main Result, and also of Part III of this paper, is the notion of a quasi-elation. In the definition, the induction hypothesis is apparent. The general strategy of our proofs will be to construct a quasi-elation, then to show that there are many, this will imply the existence of at least one elation, and then one must show that there are many elations. In fact, it will then turn out (but this is irrelevant at that moment) that all quasi-elations are elations.

All results in this subsection hold true for arbitrary Hjelmslev planes of level n. No group with particular transitivity properties is hypothesized. Hence, in this section, ${}^{n}\Psi$ is just some automorphism group of ${}^{n}H$, $n \geq 1$.

Definition 1 A collineation δ of ${}^{n}H$, $n \geq 2$, is a *quasi-elation* if a point P and a line l of ${}^{n}H$ exist such that

(i) $(\delta)^{\star_{n-1}}$ is an elation with axis ${}^{n-1}\pi(l)$ and center ${}^{n-1}\pi(P)$, ${}^{n-1}\pi(P)\ {}^{n-1}I\ {}^{n-1}\pi(l)$;

(ii) all lines $(n-1)$-neighbouring l are fixed;

(iii) all points $(n-1)$-neighbouring P are fixed.

Every line m, ${}^{n-1}\pi(m) = {}^{n-1}\pi(l)$, that is incident with at least 3 two by two non-neighbouring fixed points is called a *quasi-axis* for δ. Every point Q, ${}^{n-1}\pi(Q) = {}^{n-1}\pi(P)$, that is incident with at least 3 two by two non-neighbouring fixed lines is a *quasi-center* for δ. We will prove below that every quasi-elation has at least one center and at least one axis (Remark 9). Also, we will show below that every elation is a quasi-elation (Lemma 5).

Now suppose α is a collineation in ${}^{2}\Psi$, fixing every point in ${}^{2}\mathcal{P}$ neighbouring some point P of ${}^{2}H$, and fixing every line in ${}^{2}\mathcal{L}$ neighbouring some line l of ${}^{2}H$, P near l. Then α induces in ${}^{1}H({}^{1}\pi(P))$ the identity, thus α also fixes every point at infinity of ${}^{1}H({}^{1}\pi(P))$. Hence α fixes every line of ${}^{1}H(O)$ that is incident with ${}^{1}\pi(P)$.

Dually, by using the same arguments as above, α fixes every point of ${}^{1}H(O)$ that is incident with ${}^{1}\pi(l)$. Hence, for $n = 2$, (i) in Definition 1 is a consequence of (i) and (iii) in Definition 1.

Before showing some general properties of quasi-elations in ${}^{n}H$, we prove two particular results on fixed point sets of collineations and elations in ${}^{2}H$. These can be easily generalized to ${}^{n}H$, but as we do not need that, we leave this to the reader.

Since for every point S of ${}^{2}H$, the geometry ${}^{1}H({}^{1}\pi(S))$ is determined by ${}^{2}H$, any collineation α of ${}^{2}H$ mapping S to a neighbouring point induces a unique collineation, which we denote by $\alpha_{/\,{}^{1}H({}^{1}\pi(S))}$, in ${}^{1}H({}^{1}\pi(S))$.

Lemma 2 *If α is a collineation in ${}^{2}\Psi$ with $(\alpha)^{\star_1} = 1$, fixing all points of ${}^{2}H$ neighbouring some point R of ${}^{2}H$, and such that $\alpha(S) = S$, for some point S of ${}^{2}H$, with $S \not\sim R$, then $\alpha_{/\,{}^{1}H({}^{1}\pi(S))} = 1$.*

Proof. Consider the line ${}^{1}\pi(m)$ determined by ${}^{1}\pi(R)$ and ${}^{1}\pi(S)$, and the associated projective plane ${}^{1}H({}^{1}\pi(m))$. Then α induces in ${}^{1}H({}^{1}\pi(m))$ a collineation fixing all lines at infinity of ${}^{1}H({}^{1}\pi(m))$ (since all points of ${}^{1}H(O)$ that are incident with ${}^{1}\pi(m)$ are fixed), fixing all points of ${}^{1}H({}^{1}\pi(m))$ that are incident with the line ${}^{1}\pi(R)$ at infinity of ${}^{1}H({}^{1}\pi(m))$ (since $\alpha_{/\,{}^{1}H({}^{1}\pi(R))} = 1$), and fixing at least one affine line of ${}^{1}H({}^{1}\pi(R))$ (i. e. the line of ${}^{2}H(O)$ determined by R and S). Hence α induces in ${}^{1}H({}^{1}\pi(m))$ the identity collineation. Combined with $(\alpha)^{\star_1} = 1$, this implies that α induces in ${}^{1}H({}^{1}\pi(S))$ an elation with axis at infinity, but at the same time fixes some affine point of ${}^{1}H({}^{1}\pi(S))$, namely S. Consequently $\alpha_{/\,{}^{1}H({}^{1}\pi(S))} = 1$. $\square$

A kind of 'converse' to this lemma is provided by the following result.

Lemma 3 *Suppose α is a collineation in ${}^{2}\Psi$, with $(\alpha)^{\star_1}$ a non-trivial elation with some axis ${}^{1}\pi(l)$, $l \in {}^{2}\mathcal{L}$, and some center ${}^{1}\pi(P)$, $P \in {}^{2}\mathcal{P}$. Then at most one point ${}^{1}\pi(R)$, $R \in {}^{2}\mathcal{P}$, exists such that all points R' of ${}^{2}H$ for which ${}^{1}\pi(R') = {}^{1}\pi(R)$ are fixed by α.*

Proof. Since $(\alpha)^{\star_1} \neq 1$, every fixed point for α in ${}^{2}\mathcal{P}$ is near l. Moreover, if $R \in {}^{2}\mathcal{P}$, then the points at infinity of ${}^{1}H({}^{1}\pi(R))$ correspond with the lines of ${}^{1}H$ that are incident with ${}^{1}\pi(R)$.

Suppose two points ${}^{1}\pi(S)$, $S \in {}^{2}\mathcal{P}$, and ${}^{1}\pi(T)$, $T \in {}^{2}\mathcal{P}$, incident with ${}^{1}\pi(l)$ exist such that all points $S' \sim S$ and all points $T' \sim T$ are fixed by α. Then α induces in both ${}^{1}H({}^{1}\pi(S))$ and ${}^{1}H({}^{1}\pi(T))$ the identity. So all points at infinity of ${}^{1}H({}^{1}\pi(S))$ and ${}^{1}H({}^{1}\pi(T))$ are fixed.
Hence, all lines in ${}^{1}\mathcal{L}$ incident with ${}^{1}\pi(S)$ and all lines in ${}^{1}\mathcal{L}$ incident with ${}^{1}\pi(T)$ are fixed by α. This would imply $(\alpha)^{\star_1} = 1$, a contradiction. $\square$

Lemma 4 *If δ is an elation in ${}^{2}\Psi$ with center P and axis l, $P \in {}^{2}\mathcal{P}$, $l \in {}^{2}\mathcal{L}$, such that $(\delta)^{\star_1} = 1$, then δ fixes all points of ${}^{2}H$ near l and all lines of ${}^{2}H$ near P.*

Proof. Let Q be some point of ${}^{2}H$ near l. Since, by earlier considerations, all lines of ${}^{1}H$ that are incident with ${}^{1}\pi(Q)$ (which are fixed by δ) correspond with points at infinity of ${}^{1}H({}^{1}\pi(Q))$, and since δ also fixes all points of ${}^{2}H$ that neighbour Q and are incident with l, δ induces in ${}^{1}H({}^{1}\pi(Q))$ a collineation with two axes. Consequently, δ induces the identity in ${}^{1}H({}^{1}\pi(Q))$. Hence all points of ${}^{2}H$ that neighbour Q are fixed points for δ, and so all points in ${}^{2}\mathcal{P}$ that are near l are fixed by δ.

Dually, all lines of ${}^{2}H$ that are near P are fixed by δ. $\square$

Lemma 5 *Every elation in ${}^{n}H$ is a quasi-elation.*

Proof. Let δ be an elation in ${}^{n}H$ with axis l and center P, with P on l. We have to show (ii) and (iii) of Definition 1. Since these are dual to each other, it suffices to show (iii). Clearly, δ fixes all points at infinity of ${}^{1}H({}^{n-1}\pi(P))$. But also all point on l in ${}^{1}H({}^{n-1}\pi(P))$ are fixed. This implies that δ, having two axes in ${}^{1}H({}^{n-1}\pi(P))$ fixes all points of ${}^{1}H({}^{n-1}\pi(P))$. $\square$

We now gather some properties of quasi-elations.

Lemma 6 *Suppose $n \geq 3$. If α is a quasi-elation in ${}^{n}\Psi$, $(\alpha)^{\star_1} \neq 1$, with $(\alpha)^{\star_{n-1}}$ axial with some axis ${}^{n-1}\pi(l)$, $l \in {}^{n}\mathcal{L}$, and central with some center ${}^{n-1}\pi(P)$, $P \in {}^{n}\mathcal{P}$, $P \; {}^{n}I \; l$, then α fixes all points of ${}^{2}H$ that are 1-near ${}^{2}\pi(l)$ and neighbour ${}^{2}\pi(P)$, $\cdots$, all points of ${}^{k}H$ that are $(k-1)$-near ${}^{k}\pi(l)$ and neighbour ${}^{k}\pi(P)$, $\cdots$, all points of ${}^{n-1}H$ that are $(n-2)$-near ${}^{n-1}\pi(l)$ and neighbour ${}^{n-1}\pi(P)$.*

Proof. We give an inductive proof. Suppose α is a quasi-elation in ${}^{3}\Psi$, $(\alpha)^{\star_1} \neq 1$, with $(\alpha)^{\star_2}$ an elation with some axis ${}^{2}\pi(l)$, $l \in {}^{2}\mathcal{L}$, and some center ${}^{2}\pi(P)$, $P \in {}^{2}\mathcal{P}$, $P \; {}^{3}I \; l$. Then, by Lemma 5, $(\alpha)^{\star_2}$ is a quasi-elation in ${}^{2}\Psi$ and all points of ${}^{2}H$ that are near l and neighbour ${}^{2}\pi(P)$ are fixed, by definition.

Now suppose α is a quasi-elation in ${}^{n}\Psi$ $(n > 3)$, $(\alpha)^{\star_1} \neq 1$, with $(\alpha)^{\star_{n-1}}$ an elation with some axis ${}^{n-1}\pi(l)$, $l \in {}^{n}\mathcal{L}$, and some center ${}^{n-1}\pi(P)$, $P \in {}^{n}\mathcal{P}$, $P \ {}^{n}I \ l$. Then obviously, $(\alpha)^{\star_{n-1}}$ is a quasi-elation in ${}^{n-1}\Psi$. By the induction hypothesis, $(\alpha)^{\star_{n-1}}$ (hence α) fixes all points of ${}^{2}H$ that are near ${}^{2}\pi(l)$ and neighbour ${}^{2}\pi(P)$, $\cdots$, all points of ${}^{k}H$ that are $(k-1)$-near ${}^{k}\pi(l)$ and neighbour ${}^{k}\pi(P)$, $\cdots$, all points of ${}^{n-2}H$ that are $(n-3)$-near ${}^{n-2}\pi(l)$ and neighbour ${}^{n-2}\pi(P)$.

Next consider an arbitrary point Q of ${}^{n}H$ such that ${}^{n-1}\pi(Q)$ is $(n-2)$-near ${}^{n-1}\pi(l)$, and such that ${}^{n-1}\pi(Q)$ neighbours ${}^{n-1}\pi(P)$. Since all points of ${}^{n-2}H$ that are $(n-3)$-near ${}^{n-2}\pi(l)$ and neighbour ${}^{n-2}\pi(P)$ are fixed by α, α fixes all affine points of ${}^{1}H({}^{n-3}\pi(Q))$. It follows that α induces an axial collineation in ${}^{1}H({}^{n-2}\pi(Q))$ with axis the line at infinity of ${}^{1}H({}^{n-2}\pi(Q))$. Since $(\alpha)^{\star_{n-1}}$ is axial with axis ${}^{n-1}\pi(l)$, α consequently induces the identity collineation in ${}^{1}H({}^{n-2}\pi(Q))$. Thus $\alpha({}^{n-1}\pi(Q)) = {}^{n-1}\pi(Q)$.

The lemma follows. $\square$

Lemma 7 *If α is a quasi-elation in ${}^{n}\Psi$ with non-trivial $()^{\star_1}$-projection, such that $(\alpha)^{\star_{n-1}}$ is an elation with some axis ${}^{n-1}\pi(l)$, $l \in {}^{n}\mathcal{L}$, and some center ${}^{n-1}\pi(P)$, $P \ {}^{n}I \ l$, then all fixed lines for α that do not neighbour l are $(n-1)$-near P.*

Proof. Suppose m is some line of ${}^{n}H$ with ${}^{1}\pi(m) \neq {}^{1}\pi(l)$, with $\alpha(m) = m$ and such that m is not $(n-1)$-near P. Since $(\alpha)^{\star_1} \neq 1$, it is clear that m is near P. Let $m' \in {}^{n}\mathcal{L}$ be some line $(n-1)$-near P such that ${}^{1}\pi(m') = {}^{1}\pi(m)$. Then a point ${}^{1}\pi(R)$ of ${}^{1}H$ exists, $R \in {}^{n}\mathcal{P}$, ${}^{1}\pi(R)$ not incident with ${}^{1}\pi(l)$ and R incident with both m and m', such that $(\alpha)^{\star_1}({}^{1}\pi(R)) = {}^{1}\pi(R)$. Hence $(\alpha)^{\star_1} = 1$, a contradiction. $\square$

Lemma 8 *If α is a quasi-elation in ${}^{n}\Psi$, $(\alpha)^{\star_1} \neq 1$, with $(\alpha)^{\star_{n-1}}$ axial with some axis ${}^{n-1}\pi(l)$, $l \in {}^{n}\mathcal{L}$, and central with some center ${}^{n-1}\pi(P)$, $P \in {}^{n}\mathcal{P}$, $P \ {}^{n}I \ l$, then α induces, for every point $T \ {}^{n}I \ l$, ${}^{1}\pi(T) \neq {}^{1}\pi(P)$, a non-trivial elation in ${}^{1}H({}^{n-1}\pi(T))$ with center at infinity and with an affine axis.*

Proof. Let l' be an arbitrary line of ${}^{n}H$ satisfying ${}^{n-1}\pi(l') = {}^{n-1}\pi(l)$. Since $\alpha(l') = l'$, α fixes $cl(l', {}^{n-1}\pi(T))$ and in particular $l' \cap {}^{1}H({}^{n-1}\pi(T))$, where $l' \cap {}^{1}H({}^{n-1}\pi(T))$ is the formal notation for the unique vertex in Δ that is adjacent to both T and ${}^{n-1}\pi(T)$ and at distance $n-1$ from l'. Now notice that $l' \cap {}^{1}H({}^{n-1}\pi(T))$ corresponds with an affine line of ${}^{1}H({}^{n-1}\pi(T))$ that is incident with the point at infinity ${}^{n-1}\pi(l) \cap {}^{1}H({}^{n-2}\pi(T))$ of ${}^{1}H({}^{n-1}\pi(T))$.

Since the points at infinity of ${}^{1}H({}^{n-1}\pi(T))$ correspond with lines of ${}^{1}H({}^{n-2}\pi(T))$ that are incident with ${}^{n-1}\pi(T)$, and since $(\alpha)^{\star_1}$ is assumed to be non-trivial, only one point at infinity of ${}^{1}H({}^{n-1}\pi(T))$ can be fixed by α. Hence α induces a non-trivial elation with center at infinity and an affine axis in ${}^{1}H({}^{n-1}\pi(T))$. $\square$

Remark 9 It follows from Lemma 8 that every quasi-elation α in ${}^{n}\Psi$, with $(\alpha)^{\star_1} \neq 1$ and $(\alpha)^{\star_{n-1}}$ an elation with some axis ${}^{n-1}\pi(l)$, $l \in {}^{n}\mathcal{L}$, some center ${}^{n-1}\pi(P)$, $P \in {}^{n}\mathcal{P}$, ${}^{n-1}\pi(P)$ ${}^{n}I$ ${}^{n-1}\pi(l)$, has at least one quasi-axis $(n-1)$-neighbouring l and at least one quasi-center $(n-1)$-neighbouring P.

Lemma 10 *If α is a quasi-elation in ${}^{n}\Psi$ with non-trivial $()^{\star_1}$-projection, such that $(\alpha)^{\star_{n-1}}$ has some axis ${}^{n-1}\pi(l)$, $l \in {}^{n}\mathcal{L}$, and some center ${}^{n-1}\pi(P)$, $P \,{}^{n}I\, l$, if m is a fixed line for α, ${}^{1}\pi(m) \neq {}^{1}\pi(l)$, and T the point for which $m \,{}^{n}I\, T \,{}^{n}I\, l$, then every line of ${}^{n}H$ that $(n-1)$-neighbours m and that is incident with T is fixed by α.*

Proof. This is an immediate consequence of the dual of Lemma 8. □

Lemma 11 *If α is a quasi-elation in ${}^{n}\Psi$, $(\alpha)^{\star_1} \neq 1$, with $(\alpha)^{\star_{n-1}}$ axial with some axis ${}^{n-1}\pi(l)$, $l \in {}^{n}\mathcal{L}$, and central with some center ${}^{n-1}\pi(P)$, $P \,{}^{n}I\, l$, then α induces, for every point $T \,{}^{n}I\, l$, ${}^{1}\pi(T) = {}^{1}\pi(P)$, an elation in ${}^{1}H({}^{n-1}\pi(T))$ with a center at infinity of ${}^{1}H({}^{n-1}\pi(T))$, and with axis the line at infinity of ${}^{1}H({}^{n-1}\pi(T))$.*

Proof. Suppose α is a quasi-elation in ${}^{n}\Psi$, $(\alpha)^{\star_1} \neq 1$, as in the statement of the lemma. Then α fixes every point in ${}^{n-1}\mathcal{P}$ that is $(n-2)$-near ${}^{n-1}\pi(l)$ and neighbours ${}^{n-1}\pi(P)$, using Lemma 6. Hence α induces in ${}^{1}H({}^{n-1}\pi(T))$, for all points T of ${}^{n}H$ incident with l and ${}^{1}\pi(T) = {}^{1}\pi(P)$, an axial collineation with axis the line at infinity of ${}^{1}H({}^{n-1}\pi(T))$. Since α fixes every line in ${}^{n}\mathcal{L}$ that $(n-1)$-neighbours l, α also induces a central collineation in ${}^{1}H({}^{n-1}\pi(T))$ (with center at infinity of ${}^{1}H({}^{n-1}\pi(T))$).

This proves the lemma. □

3.3 ${}^{k}h$-collineations and generalized 1-homologies

Definition 12 For all k, $1 \leq k \leq n-1$, a ${}^{k}h^{l}_{P}$-collineation in ${}^{n}\Psi$ is an elation in ${}^{n}\Psi$ with axis $l \in {}^{n}\mathcal{L}$ and center $P \in {}^{n}\mathcal{P}$, and $()^{\star_{n-k}}$-projection trivial.

If the knowledge of an axis and/or center is not relevant, we drop the superscript l and/or the subscript P.

The set of all ${}^{k}h$-collineations (respectively with center R, with axis l, with center R and with axis l) in ${}^{n}\Psi$ is denoted by ${}^{k}h\mathcal{C}$ (respectively ${}^{k}h\mathcal{C}_{(R)}$, ${}^{k}h\mathcal{C}^{l}$, ${}^{k}h\mathcal{C}^{l}_{(R)}$). Obviously ${}^{k}h\mathcal{C} \subseteq {}^{k+1}h\mathcal{C}$, for $1 \leq k < n-1$. An elation with axis $l \in {}^{n}\mathcal{L}$, center $P \in {}^{n}\mathcal{P}$ and $()^{\star_1}$-projection not trivial, will sometimes be called an ${}^{n}h$-collineation. The identity might sporadically be called a ${}^{0}h$-collineation.

Lemma 13 *Suppose α is a ${}^{k}h$-collineation in ${}^{n}\Psi$, $1 \leq k \leq n-1$. Then $(\alpha)^{\star_{n-1}}$ is a ${}^{k-1}h$-collineation in ${}^{n-1}\Psi$.*

Proof. This is readily verified. $\square$

Lemma 14 *Suppose α is a ${}^{k}h$-collineation in ${}^{n}\Psi$, $1 \leq k \leq n-1$, with some axis $l \in {}^{n}\mathcal{L}$ and some center $P \in {}^{n}\mathcal{P}$. Then α fixes all points (respectively lines) of ${}^{n}H(O)$ that are k-near l (respectively P), all points (respectively lines) of ${}^{n-1}H(O)$ that are $(k-1)$-near ${}^{n-1}\pi(l)$ (respectively ${}^{n-1}\pi(P)$), $\cdots$, all points (respectively lines) of ${}^{n-k+1}H(O)$ that are near ${}^{n-k+1}\pi(l)$ (respectively ${}^{n-k+1}\pi(P)$).*

Proof. The statement holds for ${}^{2}\Psi$, using Lemma 4. Suppose the statement holds for ${}^{n-1}\Psi$, $n \geq 3$.

The statement is true for $k = 1$. Indeed, suppose α is a ${}^{1}h$-collineation in ${}^{n}\Psi$, with some axis $l \in {}^{n}\mathcal{L}$ and some center $P \in {}^{n}\mathcal{P}$. Then by definition it fixes all points in ${}^{n}\mathcal{P}(O)$ that are incident with l and all lines in ${}^{n}\mathcal{L}(O)$ that are incident with P. Moreover, its $()^{\star_{n-1}}$-projection is trivial. Let Q be some point near l, $Q \,{}^{n}\!\!\not{I}\, l$. Consider any line m in ${}^{n}\mathcal{L}(O)$ with $m \,{}^{n}I\, Q$, ${}^{1}\pi(m) \neq {}^{1}\pi(l)$. And call m' the line in ${}^{n}\mathcal{L}(O)$ such that $\alpha(m) = m'$. Then $m \cap m' \cap l \neq \emptyset$ (the intersection is to be concieved as the intersection of the sets of points on the respective lines). Let us denote by $R \in {}^{n}\mathcal{L}$ the common point of m, m' and l. Notice that R is uniquely determined since ${}^{1}\pi(m) \neq {}^{1}\pi(l)$ and ${}^{1}\pi(R) = {}^{1}\pi(Q)$. Since ${}^{n-1}\pi(m) = {}^{n-1}\pi(m')$, the points of ${}^{n-1}H({}^{1}\pi(R))$ in ${}^{n}\mathcal{P}(O)$ that are incident with m are also incident with m'.

So $m \cap {}^{n-1}H({}^{1}\pi(R))$ (where $m \cap {}^{n-1}H({}^{1}\pi(R))$ is the formal notation for the unique vertex in Δ that is adjacent to m and at distance $n-1$ from ${}^{1}\pi(R)$ and R) is fixed by α. By the arbitrary choice of m, Q must be fixed by α. Therefore we have proved that all points Q that are near l are fixed by α. Dually, we conclude that every line of ${}^{n}H$ near P is fixed by α.

Suppose the statement is true for $k-1$, $1 \leq k-1 < n-1$. Let β be a ${}^{k}h$-collineation in ${}^{n}\Psi$ with some axis $l \in {}^{n}\mathcal{L}$ and some center $P \in {}^{n}\mathcal{P}$. Then $(\beta)^{\star_{n-1}}$ is a ${}^{k-1}h$-collineation in ${}^{n-1}\Psi$ with axis ${}^{n-1}\pi(l)$ and center ${}^{n-1}\pi(P)$ (Lemma 13). By the induction hypothesis, $(\beta)^{\star_{n-1}}$ fixes all points (lines) of ${}^{n-1}H(O)$ that are $(k-1)$-near ${}^{n-1}\pi(l)$ (${}^{n-1}\pi(P)$), $\cdots$, all points (lines) of ${}^{n-k+1}H(O)$ that are near ${}^{n-1}\pi(l)$ (${}^{n-1}\pi(P)$). Consequently, β fixes all points (lines) of ${}^{n-1}H(O)$ that are $(k-1)$-near l (P), $\cdots$, all points (lines) of ${}^{n-k+1}H(O)$ that are near l (P). Now we still have to prove that β fixes all points (lines) of ${}^{n}H(O)$ that are k-near l (P). For this purpose we consider a point Q of ${}^{n}H(O)$ that is k-near l, $Q \,{}^{n}\!\!\not{I}\, l$. Then consider any line m in ${}^{n}\mathcal{L}(O)$, ${}^{1}\pi(m) \neq {}^{1}\pi(l)$, $Q \,{}^{n}I\, m$. Denote by m' the image of m under β. Thus ${}^{n-k}\pi(m) = {}^{n-k}\pi(m')$ and $m \cap m' \cap l \neq \emptyset$ (the intersection again to be interpreted as the intersection of the sets of points incident with the lines m, m', l). Let $S \in {}^{n}\mathcal{L}$ be the unique common point of m, m' and l. Then all points of ${}^{n-k}H({}^{k}\pi(S))$ in ${}^{n}\mathcal{P}(O)$ that are incident with m, are also incident with m'. Hence $m \cap {}^{n-k}\!H({}^{k}\!\pi(S))$ (where $m \cap {}^{n-k}\!H({}^{k}\!\pi(R))$ is the formal notation for the unique vertex in Δ that is at distance $n-k$ from both ${}^{k}\!\pi(R)$ and R, and at distance k from m) is fixed by α. By the arbitrary choice of m, Q is a fixed point for β. Therefore also the last step in the proof, namely proving that all points (dually: lines) of ${}^{n}\!H(O)$ that are k-near l (dually: P) are fixed by β, is completed. $\square$

Definition 15 A *generalized* 1*-homology in* ${}^{n}\Psi$ is a non-trivial collineation of ${}^{n}\!H$ with $()^{\star_{n-1}}$-projection trivial, and with an axis $l \in {}^{n}\mathcal{L}$ and a center $P \in {}^{n}\mathcal{P}$, with l not near P.

Lemma 16 *Let δ be a non-trivial collineation of ${}^{n}\!H$ with an axis l and with $()^{\star_{n-1}}$-projection trivial. Then*

(i) δ fixes all points near l;

(ii) δ is either a ${}^{1}h$-collineation, or a generalized 1-homology.

In particular,

(iii) every generalized 1-homology of ${}^{n}\!H$ with axis l and center P (not near l) fixes all lines near P and all points near l;

(iv) no ${}^{1}h$-collineation of ${}^{n}\!H$ fixes a point not near an axis of it.

Also,

(v) no generalized 1-homology fixes a point not near an axis and not neighbouring a center.

Proof. We first prove (i). Let Q be any point near l. Let m be any line not neighbouring l, but incident with Q. Then $\delta(m)$ is $(n-1)$-neighbouring m, and since the intersection point $m \cap l$ is fixed by δ, the set of points on m neigbouring Q is stabilized by δ. Since m was essentially arbitrary, it follows that δ fixes Q (it suffices to consider m' through Q not neighbouring l and not neighbouring m). Whence (i). By this and the dual, (iii) follows.

We now show (ii), and the rest along our way. Suppose δ fixes a point P not near l. Then clearly P is a center for δ and δ is a generalized 1-homology. Suppose now that there is a second center Q not neighbouring P. Every point not near the line through P and Q is the intersection of two fixed lines (one through P and one through Q), hence is fixed itself, hence is a center if it is not near l. We easily deduce that δ is the identity. In particular, (v) follows, and also δ is not an ${}^{1}h$-collineation. This proves (iv). Suppose now that δ does not fix any point not near l. We show that δ has a center near l. Let Q be any point not near l, and let m be any line through Q and $\delta(Q)$. Clearly, m is not neighbouring l, and since δ

fixes the intersection point R of l and m, δ also fixes m. Now let m' be any line through R. We show that δ fixes m'. We may assume that m' does not neighbour l. Let Q' be incident with m' and not near l. Every line through Q' and $\delta(Q')$ is fixed by δ, as above. If such a line is not neigbouring m', then it meets m in a point not near l, which must be fixed by δ, a contradiction. Hence all lines through Q' and $\delta(Q')$ are neighbouring m', which implies that m' is one of these. Hence m' is fixed by δ. This shows that R is a center, and (ii) follows.

This proves the lemma. $\square$

4 Properties of $\overset{n}{H}$ induced by the half strongly-transitivity

In this section, we assume that $\overset{n}{\Psi}$ is induced by the group G acting half strongly-transitively on Δ.

4.1 Well-formed triangles

A *well-formed triangle* in the projective Hjelmslev plane $\overset{n}{H}$ is a set of three pairwise non-neighbouring points $\{P, Q, S\}$ such that $\overset{1}{\pi}(P), \overset{1}{\pi}(Q), \overset{1}{\pi}(S)$ are not collinear in $\overset{1}{H}$.

Property 17 (transitivity on the well-formed triangles of $\overset{n}{H}$) *Suppose $\{P_1, P_2, P_3\}$ and $\{Q_1, Q_2, Q_3\}$ are well-formed triangles of $\overset{n}{H}$. Then a collineation α in $\overset{n}{\Psi}$ exists such that $\alpha(P_i) = Q_i$, for all $i \in \{1, 2, 3\}$.*

Proof. For every $w \in W$, where W is the Coxeter group corresponding with Δ, G acts transitively on the ordered pairs of chambers (x, y) where $\delta(x, y) = w$ (this is due to the half strongly-transitive action).

Suppose $\{P_1, P_2, P_3\}$ and $\{Q_1, Q_2, Q_3\}$ are two well-formed triangles of $\overset{n}{H}$. Then an apartment Σ exists such that $\{O, P_1, P_2, P_3\} \subseteq \Sigma$ and an apartment Σ' for which $\{O, Q_1, Q_2, Q_3\} \subseteq \Sigma'$. Notice that possibly $\Sigma' = \Sigma$. We can identify P_1 with a unique vertex p_1 in Δ at distance n from O in the obvious way (see the introduction). Similarly for p_2, p_3 and q_1, q_2 and q_3. Let v_i, $i = 1, 2$, be the unique vertex of Σ at distance n from p_i and at distance $2n$ from both O and p_3. Also, let w_i, $i = 1, 2$, be the unique vertex in Σ' at distance n from q_i and at distance $2n$ from both O and q_3. It is easily seen that there is a unique chamber C_i in the convex closure $cl(v_1, v_2)$ containing v_i, $i = 1, 2$. Also, it is easily seen that the convex closure $cl(C_1, C_2)$ contains p_1, p_2, p_3 and O. Similarly, we obtain chambers D_1, D_2 containing w_1 and w_2 respectively, and such that $cl(D_1, D_2)$ contains q_1, q_2, q_3 and O. Moreover, by our particular construction, we have $\delta(C_1, C_2) = \delta(D_1, D_2)$. Hence there is an automorphism g in G mapping C_i to D_i, $i = 1, 2$. One easily sees that O must be fixed by g, and hence

g induces some collineation α in ${}^{n}\Psi$. By construction of g, (and using the fact that g acts type-preserving), α maps P_j to Q_j, $j = 1, 2, 3$, as required. $\square$

In the course of the proof of the Main Results in Parts II and III of this paper, we will need to construct a non-trivial ${}^{1}h$-collineation. In the next subsection, we will prove three useful lemmas to that end. Afterwards, we will show that, if such a ${}^{1}h$-collineation exists in ${}^{n}\Psi$, and if ${}^{n-1}H$ satisfies the Moufang property, $n > 2$ (and all elations belong to ${}^{n-1}\Psi$), then also ${}^{n}H$ satisfies the Moufang property and all elation belong to ${}^{n}\Psi$ (in fact, both the assumption and the conclusion are required for all vertices O of Δ). For $n = 2$, one must additionally ask that there exists a quasi-elation with non-trivial $()^{\star_1}$-projection. This provides a general strategy for any type of assumption on the automorphism group of Δ to show that Δ is a Bruhat-Tits building. We will apply this strategy in Parts II and III, thus giving two concrete examples where this can be used. This way of presenting our proof was suggested to us by one of the referees.

4.2 Three more lemmas

Lemma 18 *Suppose ${}^{k}H(v)$ is a Moufang Hjelmslev plane, for all vertices v of Δ, and for all positive integers $k \leq n-1$, with $n > 2$, and suppose that all elations in ${}^{k}H(v)$ are inherited from G. Then at least one quasi-elation γ in ${}^{n}\Psi$ exists with non-trivial $()^{\star_1}$-projection.*

Proof. Suppose l is some line of ${}^{n}H(O)$ and P some point of ${}^{n}H(O)$, $P\ {}^{n}I\ l$. Since ${}^{k}H$ satisfies the Moufang condition for all $k < n$, there exist non-trivial ${}^{n-2}h$-collineations β in ${}^{n-1}\Psi({}^{1}\pi(l))$, with axis the line l of ${}^{n-1}H({}^{1}\pi(l))$, with center P' corresponding with the vertex in $cl(P, {}^{1}\pi(l))$ at distance 1 from both P and ${}^{n-1}\pi(P)$ and at distance $n-1$ from ${}^{1}\pi(l)$, and with two by two different and non-trivial actions in ${}^{2}H({}^{1}\pi(l))$.

We claim that every such a collineation β gives rise to a collineation α acting on ${}^{n}H(O)$ such that ${}^{n-1}\pi(l)$ is an axis for $(\alpha)^{\star_{n-1}}$. Indeed, it is obvious that, since β acts trivially on ${}^{1}H({}^{1}\pi(l))$ and thus fixes O, β can be 'extended' to a collineation α in ${}^{n}\Psi(O)$. Suppose ${}^{n-1}\pi(Q)$, $Q \in {}^{n}\mathcal{P}(O)$, is an arbitrary point in ${}^{n-1}\mathcal{P}(O)$ incident with ${}^{n-1}\pi(l)$. Then ${}^{n-1}\pi(Q)$, O, and ${}^{n-1}\pi(l)$ determine some "segment" $cl({}^{n-1}\pi(Q), O, {}^{n-1}\pi(l))$ with top vertex O and $cl({}^{n-1}\pi(Q), O)$ and $cl({}^{n-1}\pi(l), O)$ as "boundary sides". A unique vertex X exists at distance $n-2$ from ${}^{1}\pi(l)$ such that the segment $cl({}^{n-1}\pi(l), {}^{1}\pi(l), X)$ with $cl({}^{n-1}\pi(l), {}^{1}\pi(l))$ and $cl(X, {}^{1}\pi(l))$ as boundary sides is a subsegment of $cl({}^{n-1}\pi(Q), O, {}^{n-1}\pi(l))$. It is clear that X can be identified with a point in ${}^{n-2}\mathcal{P}({}^{1}\pi(l))$ that is incident with the line ${}^{n-1}\pi(l)$ in ${}^{n-2}\mathcal{L}({}^{1}\pi(l))$, and that ${}^{n-1}\pi(Q)$ is an affine line of ${}^{1}H(X)$.
Since β is an ${}^{n-2}h$-collineation in ${}^{n-1}\Psi({}^{1}\pi(l))$, and using Lemma 14, β fixes all points in ${}^{n-1}\mathcal{P}({}^{1}\pi(l))$ that are $(n-2)$-near l (with respect to ${}^{1}\pi(l)$). So all affine points of ${}^{1}H(X)$ are fixed by β. Hence β induces the identity in ${}^{1}H(X)$. This also implies that $\beta({}^{n-1}\pi(Q)) = {}^{n-1}\pi(Q)$, or $\alpha({}^{n-1}\pi(Q)) = {}^{n-1}\pi(Q)$. Hence the claim.

Next we claim that α fixes all lines of ${}^{n}H(O)$ that are $(n-1)$-neighbouring l. Indeed, by Lemma 14 all lines of ${}^{n-1}H({}^{1}\pi(l))$ that are $(n-2)$-near a center for β (with respect to ${}^{1}\pi(l)$ again) are fixed by β. Hence all lines of ${}^{n}H(O)$ that $(n-1)$-neighbour $l \in {}^{n}\mathcal{L}(O)$ are fixed by α.

We now show that $(\alpha)^{\star_1}$ is a non-trivial elation with axis ${}^{1}\pi(l)$ and center ${}^{1}\pi(P)$. Since $(\beta)^{\star_1} = 1$, in other words β induces the identity in ${}^{1}H({}^{1}\pi(l))$, α fixes all points in ${}^{1}\mathcal{P}(O)$ that are incident with ${}^{1}\pi(l)$. Now suppose ${}^{1}\pi(U)$, $U \in {}^{n}\mathcal{P}(O)$, is some point of ${}^{1}H(O)$, ${}^{1}\pi(U)\ {}^{1}\!\!\not{I}\ {}^{1}\pi(l)$. Then ${}^{1}\pi(U)$ can be regarded as a point of ${}^{2}H({}^{1}\pi(l))$ not near $l \in {}^{n-1}\mathcal{L}({}^{1}\pi(l))$. Since β is an ${}^{n-2}h$-collineation in ${}^{n-1}\Psi({}^{1}\pi(l))$ with axis l of ${}^{n-1}H({}^{1}\pi(l))$ and with a non-trivial action in ${}^{2}H({}^{1}\pi(l))$, $\beta({}^{1}\pi(U)) \neq {}^{1}\pi(U)$. Hence $(\alpha)^{\star_1}$ must be a non-trivial elation with axis ${}^{1}\pi(l)$. What about the center of $(\alpha)^{\star_1}$? All lines of ${}^{n-1}H({}^{1}\pi(l))$ that are incident (incidence in ${}^{n-1}H({}^{1}\pi(l))$) with P' are fixed by β. Consequently, by taking convex closures with O, every line of ${}^{1}H(O)$ that is incident (incidence in ${}^{1}H(O)$) with ${}^{1}\pi(P)$ is fixed by β, and thus by α. We conclude that $(\alpha)^{\star_1}$ is central with center ${}^{1}\pi(P)$.

By using dual arguments, some collineation δ in ${}^{n}\Psi(O)$ exists such that ${}^{n-1}\pi(R)$, for some point R of ${}^{n}H$ incident with l, ${}^{1}\pi(R) \neq {}^{1}\pi(P)$, is a center for $(\delta)^{\star_{n-1}}$, such that δ fixes all points of ${}^{n}H(O)$ that $(n-1)$-neighbour R, and for which $(\delta)^{\star_1}$ is a non-trivial elation with center ${}^{1}\pi(R)$ and some axis ${}^{1}\pi(r)$, $r \in {}^{n}\mathcal{L}(O)$, ${}^{1}\pi(r) \neq {}^{1}\pi(l)$.

Then $[\alpha, \delta]$ fixes all lines of ${}^{n}H(O)$ that $(n-1)$-neighbour l, and all points of ${}^{n}H(O)$ that $(n-1)$-neighbour R. Moreover, $([\alpha,\delta])^{\star_{n-1}}$ is an elation with axis ${}^{n-1}\pi(l)$ and center ${}^{n-1}\pi(R)$, and $([\alpha,\delta])^{\star_1}$ is a non-trivial elation with axis ${}^{1}\pi(l)$ and center ${}^{1}\pi(R)$. By definition, $[\alpha,\delta]$ is a quasi-elation in ${}^{n}\Psi(O)$, having a non-trivial $()^{\star_1}$-projection. $\square$

Lemma 19 *Suppose ${}^{k}H(v)$ is a Moufang Hjelmslev plane, for all vertices v of Δ, and for all positive integers $k \leq n-1$, $n > 2$, and suppose that all elations in ${}^{k}H(v)$ are inherited from G. Suppose also that there is a non-trivial collineation α of ${}^{n}H$ in ${}^{n}\Psi$ fixing all points $(n-1)$-near some line $l \in {}^{n}\mathcal{L}$, and with $(\alpha)^{\star_{n-1}} = 1$, then a non-trivial ${}^{1}h$-collineation exists in ${}^{n}\Psi$.*

Proof. By Lemma 16, we may assume that α is a generalized 1-homology with some center T (not near l). Using Lemma 18, we can consider a quasi-elation β in ${}^{n}\Psi$ with $(\beta)^{\star_1} \neq 1$ such that $(\beta)^{\star_{n-1}}$ is axial with axis ${}^{n-1}\pi(l)$ and central with some center ${}^{n-1}\pi(P)$, $P\ {}^{n}I\ l$.

We first claim that $[\beta,\alpha]$ fixes every line in ${}^{n}\mathcal{L} \setminus \{m \in {}^{n}\mathcal{L} \,|\, m \sim l\}$ that is $(n-1)$-near P. Indeed, using the dual of Lemma 8, β induces in ${}^{1}H({}^{n-1}\pi(m))$, for every line m of ${}^{n}H$ incident with P, ${}^{1}\pi(m) \neq {}^{1}\pi(l)$, a non-trivial elation with axis at infinity and with an affine center. Suppose m' is some line of ${}^{n}H$, ${}^{n-1}\pi(m')\ {}^{n-1}I\ {}^{n-1}\pi(P)$, ${}^{1}\pi(m') \neq {}^{1}\pi(l)$, satisfying $\beta(m') = m'$. Let us denote the common point in ${}^{n}\mathcal{P}$ of m' and l by U. Then $U\ {}^{n}I\ \alpha(m')$ since $\alpha(U) = U$ (U is $(n-1)$-near l). Using the fact that $(\alpha)^{\star_{n-1}} = 1$, m' and $\alpha(m')$ are $(n-1)$-neighbouring lines of ${}^{n}H$. Hence, by Lemma 10, $\beta(\alpha(m')) = \alpha(m')$. Consequently, $[\beta,\alpha](m') = m'$. Using

Lemma 10 again and since $([\beta,\alpha])^{\star_1} = 1$, $[\beta,\alpha]$ induces in ${}^{1}H({}^{n-1}\pi(m'))$ a collineation with at least two centers, one of which is the point at infinity of ${}^{1}H({}^{n-1}\pi(m'))$. It follows that $[\beta,\alpha]$ induces the identity collineation in ${}^{1}H({}^{n-1}\pi(m'))$. By the arbitrary choice of m' as a fixed line for β in ${}^{n}\mathcal{L} \setminus \{m \in {}^{n}\mathcal{L} \,|\, m \sim l\}$, ${}^{n-1}\pi(m') \; {}^{n-1}I \; {}^{n-1}\pi(P)$, the claim follows.

We next claim that $[\beta,\alpha]$ fixes every point of ${}^{n}H$ that is near l. Indeed, β stabilizes the set of points near l, and α fixes all elements of this set by Lemma 16. Hence the claim.

As a consequence of the previous paragraphs, $[\beta,\alpha]$ is a collineation in ${}^{n}\Psi$ with axis l and center P, and with $()^{\star_{n-1}}$ -projection trivial. Moreover, $[\beta,\alpha]$ is non-trivial. For if $[\beta,\alpha](T) = T$, then $\beta^{-1}\alpha^{-1}\beta(T) = T$, hence $\alpha^{-1}(\beta(T)) = \beta(T)$. However, since ${}^{1}\pi(\beta(T)) \neq {}^{1}\pi(T)$, this implies that the generalized 1-homology α (with axis l) fixes at least two non-neighbouring points of ${}^{n}H$ that are not near l. Hence $\alpha = 1$ by Lemma 16, a contradiction.

Hence $[\beta,\alpha]$ is a non-trivial ${}^{1}h$-collineation. $\square$

Lemma 20 *Suppose ${}^{n}H$ satisfies the Moufang condition, $n \geq 1$, and all elations belong to ${}^{n}\Psi$. Suppose $\{P_1, P_2, P_3\}$ is a well-formed triangle of ${}^{n}H$. Let l be the line of ${}^{n}H$ determined by P_1 and P_2. Then a subgroup of ${}^{n}\Psi$ exists, fixing P_1, P_2 and P_3, acting transitively on the points in ${}^{1}\mathcal{P} \setminus \{{}^{1}\pi(P_1), {}^{1}\pi(P_2)\}$ that are incident with ${}^{1}\pi(l)$.*

Proof. Since ${}^{1}H$ satisfies the Moufang condition and ${}^{1}\Psi$ contains all elations, a subgroup Υ of ${}^{1}\Psi$ exists, containing all collineations fixing ${}^{1}\pi(P_1)$, ${}^{1}\pi(P_2)$ and ${}^{1}\pi(P_3)$, and acting transitively on the points of ${}^{1}\mathcal{P} \setminus \{{}^{1}\pi(P_1), {}^{1}\pi(P_2)\}$ that are incident with ${}^{1}\pi(l)$ (this follows from Pickert [8](7.3.13)).

Suppose $\beta \in \Upsilon$. Then a collineation $\alpha \in {}^{n}\Psi$ exists for which $(\alpha)^{\star_1} = \beta$. Applying ${}^{n-1}h$-collineations in ${}^{n-1}\Psi$ (which are easily seen to exist by the Moufang property), the existence of a collineation in ${}^{n-1}\Psi$ with $(\delta)^{\star_1} = (\alpha)^{\star_1}$ and satisfying $\delta(P_i) = P_i$ $(i = 1, 2, 3)$ follows.

This proves the lemma. $\square$

Lemma 20 will be very useful in constructing "many" (quasi-)elations from one (quasi-)elation. For example, we have the following result (and we use the previous lemma only for $n = 1$):

Corollary 21 *Suppose ${}^{1}H$ satisfies the Moufang condition (and every elation is induced by G, which is still to be assumed to act half strongly-transitively on Δ). If, for $n \geq 2$, there exists a quasi-elation in ${}^{n}H$ with $()^{\star_1}$ -projection non-trivial, then for every line l of ${}^{n}H$, every point P of ${}^{n}H$ incident with l, every line m through P not neighbouring l, and every pair of points Q and Q', both not neighbouring P, but incident with m, there exists a quasi-elation in ${}^{n}H$ with quasi-axis neighbouring l and quasi-center neighbouring P mapping Q to a point neighbouring Q'.*

Proof. If Q is neighbouring Q', then we can take the identity. So suppose that Q does not neighbour Q'. By Property 17 and Remark 9, there exists a quasi-elation δ with quasi-axis l and quasi-center neighbouring P, and with $()^{\star_1}$-projection non-trivial. The result now follows by conjugating with an automorphisms of ${}^{n}H$ for which the $()^{\star_1}$-projection fixes ${}^{1}\pi(l)$ and ${}^{1}\pi(Q)$ and maps $\delta(Q)$ to some point neighbouring Q' (and existing by Lemma 20). $\square$

4.3 The case $n = 2$

In this subsection, we assume that, for every vertex v of Δ, ${}^{1}H(v)$ is a Moufang plane (with all elations inherited from G), that there exists at least one ${}^{1}h$-collineation in ${}^{2}H(v)$, and that there exists at least one quasi-elation in ${}^{2}H(v)$ with non-trivial $()^{\star_1}$-projection. In order not to be forced to repeat these conditions each time, we will abbreviate this to saying that (Δ, G) *satisfies the radius* 2 *induction hypothesis.*

Lemma 22 *If (Δ, G) satisfies the radius 2 induction hypothesis, then at least one axial collineation in ${}^{2}\Psi$ exists for which the $()^{\star_1}$-projection is a non-trivial elation.*

Proof. In ${}^{1}H({}^{1}\pi(l))$, l acts as an affine line , O acts as the point at infinity, and every vertex that is adjacent to both ${}^{1}\pi(l)$ and O (this is a point of ${}^{1}H(O)$ incident with ${}^{1}\pi(l)$) plays the role of a line at infinity. Moreover, for every vertex ${}^{1}\pi(P) \, {}^{1}I \, {}^{1}\pi(l)$, l induces in ${}^{1}H({}^{1}\pi(P))$ an affine line. These affine lines of ${}^{1}H({}^{1}\pi(P))$ correspond with affine points of ${}^{1}H({}^{1}\pi(l))$ that are incident (in ${}^{1}H({}^{1}\pi(l))$) with the affine line l of ${}^{1}H({}^{1}\pi(l))$.

Fix ${}^{1}\pi(P)$ such that ${}^{1}\pi(P) \, {}^{1}I \, {}^{1}\pi(l)$. By assumption and by Property 17, there is a non-trivial ${}^{1}h$-collineation $\beta \in {}^{1}h\mathcal{C}({}^{1}\pi(l))$ (where we use the notation ${}^{1}h\mathcal{C}({}^{1}\pi(l))$ instead of ${}^{1}h\mathcal{C}$ to indicate that the base vertex is now ${}^{1}\pi(l)$ and not O) which fixes every point of ${}^{2}H({}^{1}\pi(l))$ the projection onto ${}^{1}H({}^{1}\pi(l))$ of which is incident with l, and for which the set of centers is equal to the set of points which are projected onto the intersection of ${}^{1}\pi(P)$ and l, both viewed as lines of ${}^{1}H({}^{1}\pi(l))$.

Since O is fixed by β, β can be extended to a collineation acting on ${}^{2}H(O)$. The extended collineation, say β', fixes all points that are incident with l. Since the ${}^{1}h$-collineation β is not trivial in ${}^{2}H({}^{1}\pi(l))$, the only points in ${}^{1}\mathcal{P}(O)$ that are fixed by β' are incident with ${}^{1}\pi(l)$. Moreover, since β fixes every line of ${}^{2}H({}^{1}\pi(l))$ 'coming' from the line ${}^{1}\pi(P)$ of ${}^{1}H({}^{1}\pi(l))$, $(\beta')^{\star_1}$ is central with center ${}^{1}\pi(P) \in {}^{1}\mathcal{P}(O)$. Hence β' is an axial collineation of ${}^{2}\Psi$ with the extra property that $(\beta')^{\star_1}$ is a non-trivial elation. $\square$

Theorem 23 *If (Δ, G) satisfies the radius 2 induction hypothesis, then at least one elation in ${}^{2}\Psi$ exists with non-trivial $()^{\star_1}$-projection.*

Proof. Using Lemma 22 and its dual form, in combination with Property 17, a collineation α exists with some axis $l \in {}^{2}\mathcal{L}$, with $(\alpha)^{\star_1}$ a non-trivial elation with axis ${}^{1}\pi(l)$ and some center ${}^{1}\pi(P)$, $P \in {}^{2}\mathcal{P}$, and a collineation β exists with some center $Q \in {}^{2}\mathcal{P}$, $Q \,{}^{2}\mathrm{I}\, l$, ${}^{1}\pi(Q) \neq {}^{1}\pi(P)$, such that $(\beta)^{\star_1}$ is a non-trivial elation with some axis ${}^{1}\pi(m)$, $m \in {}^{2}\mathcal{L}$, ${}^{1}\pi(m) \neq {}^{1}\pi(l)$, and center ${}^{1}\pi(Q)$.

Obviously, $[\beta, \alpha]$ is an elation with axis l and center Q. It is also straightforward that $([\beta, \alpha])^{\star_1} \neq 1$. $\square$

Remark that if the order of ${}^{1}H$ is 2 (as a projective plane, so ${}^{2}H$ has 7 points and 7 lines), then quasi-axes which are not axes and quasi-centers which are not centers do not exist in ${}^{2}H$. Hence in this case, the radius 2 induction hypothesis trivially implies the existence of an elation in ${}^{2}H$ with non-trivial $()^{\star_1}$-projection (and induced by G; indeed, otherwise this is true for every Hjelmslev plane of level 2 with canonical image of order 2 since all such planes are Moufang (there are only 2 isomorphism classes)).

Lemma 24 *Suppose (Δ, G) satisfies the radius 2 induction hypothesis. Let l be some line and let P be some point of ${}^{2}H$, $P \,{}^{2}\mathrm{I}\, l$. Suppose R is some point of ${}^{2}H$ for which ${}^{1}\pi(R) \,{}^{1}\!\!\not\mathrm{I}\, {}^{1}\pi(l)$. Then the set of ${}^{1}h$-collineations in ${}^{2}\Psi$ with axis l and center P acts transitively on the points of ${}^{1}H({}^{1}\pi(R))$ in ${}^{2}\mathcal{P}(O)$ that are incident with the line m of ${}^{2}H$ determined by P and R.*

Proof. By assumption, at least one non-trivial ${}^{1}h$-collineation α in ${}^{2}\Psi$ exists with axis l and center P (using the transitivity on the well-formed triangles (Property 17)). Consider some point R of ${}^{2}H$ not near l, and suppose m is the line of ${}^{2}H$ determined by P and R. Let us denote $\alpha(R)$ by S and let T be an arbitrary point of ${}^{1}H({}^{1}\pi(R))$ in ${}^{2}\mathcal{P}(O)$, $T \neq R$, that is incident with m.

By the radius 2 induction hypothesis, ${}^{1}\Psi(O)$ contains all possible elations acting on ${}^{1}H(O)$. As a consequence, and replacing the base-vertex O by ${}^{1}\pi(R)$, the group ${}^{1}\Psi({}^{1}\pi(R))$ (the by Ψ induced group in the projective plane ${}^{1}H({}^{1}\pi(R))$) contains a group fixing a well-formed triangle $\{P_1, P_2, P_3\}$ of ${}^{1}H({}^{1}\pi(R))$ and acting transitively on the points of the set $\{P \in {}^{1}\mathcal{P}({}^{1}\pi(R)) \,|\, P \neq P_1,\, P \neq P_2, P$ is incident with the line determined by P_1 and $P_2\}$ (using Lemma 20).

Therefore it is possible to consider a collineation β in ${}^{1}\Psi({}^{1}\pi(R))$ fixing R, $m \cap {}^{1}H({}^{1}\pi(R))$, the line at infinity of ${}^{1}H({}^{1}\pi(R))$, mapping T to S. Since the line at infinity of ${}^{1}H({}^{1}\pi(R))$ corresponds with the base-vertex O in Δ, and thus $\beta(O) = O$, β can be 'extended' to some collineation acting on ${}^{2}H(O)$. We use the same notation β for this 'extended' collineation.

Thus $\beta^{-1}\alpha\beta$ is again a ${}^{1}h$-collineation in ${}^{2}\Psi$, now with some axis $l' \in {}^{2}\mathcal{L}(O)$ and some center $P' \in {}^{2}\mathcal{P}(O)$, $P' \,{}^{2}\mathrm{I}\, l'$, ${}^{1}\pi(P') \,{}^{1}\mathrm{I}\, {}^{1}\pi(m)$, and such that $\beta^{-1}\alpha\beta(R) = T$.

Case 1: ${}^{1}\pi(l) = {}^{1}\pi(l')$.

Then $\beta^{-1}\alpha\beta$ is axial with axis l and central with center P.

Case 2: ${}^{1}\pi(l) \neq {}^{1}\pi(l')$.

Then ${}^{1}\pi(l)$ and ${}^{1}\pi(l')$ have a unique common intersection point in ${}^{1}\mathcal{P}(O)$, say ${}^{1}\pi(Q)$ for some point Q in ${}^{2}\mathcal{P}(O)$. By assumption and Corollary 21, a quasi-elation γ in ${}^{2}\Psi$ exists with quasi-center neighbouring R and quasi-axis neighbouring the line r determined by R and Q, that maps ${}^{1}\pi(l)$ to ${}^{1}\pi(l')$. Hence $\gamma^{-1}(\beta^{-1}\alpha\beta)\gamma$ is a ${}^{1}h$-collineation in ${}^{2}\Psi$ with axis l and center P, that additionally maps R to T, as required. $\square$

Lemma 25 *If (Δ, G) satisfies the radius 2 induction hypothesis, then the group of all elations in ${}^{2}\Psi$ with prechosen axis l and center P acts sharply transitively on the points that are incident with a fixed line m through P, m not neighbouring l, and that do not neighbour P.*

Proof. Theorem 23 provides at least one elation with $()^{\star_1}$-projection not trivial. Property 17 implies that there is such an elation with axis l and center P. Completely similar to the end of the proof of Corollary 21, one shows that for every pair of points Q, Q' in ${}^{2}H$, with Q and Q' on m, and with P, Q, Q' pairwise non neighbouring, there exists an elation with axis l and center P mapping Q to some point neighbouring Q'. Composing with a suitable ${}^{1}h$-collineation, guarenteed to exist by Lemma 24, we obtain an elation with axis l and center P mapping Q to Q'.

If Q and Q' are neighbouring, then the existence of an elation with axis l and center P mapping Q to Q' follows from Lemma 24.

The semi-regularity follows from the semi-regularity in ${}^{1}H$ and Lemma 16(iv). The lemma is proved. $\square$

Theorem 26 *If (Δ, G) satisfies the radius 2 induction hypothesis, then ${}^{2}H$ satisfies the Moufang condition and ${}^{2}\Psi$ contains all elation of ${}^{2}H$.*

Proof. This follows immediately from the previous lemma. $\square$

4.4 The case $n \geq 3$

In this subsection, we introduce another induction hypothesis. We assume that $n \geq 3$, that ${}^{k}H(v)$ is a Moufang Hjelmslev plane of level k, for every $k \leq n-1$ (and all elations are induced by G) and for all vertices v of Δ, and that for every vertex v of Δ, there exists some non-trivial ${}^{1}h$-collineation in ${}^{n}H(v)$ (and induced by G). Let us call these conditions the *radius n induction hypothesis* on (Δ, G). Note that by Lemma 18, there exists a quasi-elation in ${}^{n}H$ with non-trivial $()^{\star_1}$-projection. Hence the radius n induction hypothesis is a natural generalization of the radius 2 induction hypothesis considered in the previous subsection.

Our aim is to show that ${}^{n}H$ satisfies the Moufang condition.

Lemma 27 *Suppose (Δ, G) satisfies the radius n induction hypothesis and let k be arbitrary in $\{1, \cdots, n-1\}$. Then a collineation α in ${}^{n}\Psi$ exists with some axis $l \in {}^{n}\mathcal{L}$, fixing some line m of ${}^{n}H$, ${}^{1}\pi(m) \neq {}^{1}\pi(l)$, with $(\alpha)^{\star_k} = 1$, $(\alpha)^{\star_{k+1}} \neq 1$, and such that none of the points ${}^{k+1}\pi(R)$ of ${}^{k+1}H$, $R \in {}^{n}\mathcal{P}$, that are incident with ${}^{k+1}\pi(m)$ and for which ${}^{1}\pi(R) \; {}^{1}\!\!\not{I} \; {}^{1}\pi(l)$, is fixed by α (here, $(\alpha)^{\star_0} = 1$ must be read as an identity).*

Proof. By assumption, a non-trivial ${}^{1}h$-collineation α in ${}^{n}\Psi$ exists with some axis $l \in {}^{n}\mathcal{L}$, fixing some line m of ${}^{n}H$, ${}^{1}\pi(m) \neq {}^{1}\pi(l)$ (taking m for instance through the center of α). By definition of a non-trivial ${}^{1}h$-collineation, α is an elation with $(\alpha)^{\star_{n-1}} = 1$ and $(\alpha)^{\star_n} = \alpha \neq 1$. It is clear that none of the points in ${}^{n}\mathcal{P}$ that are incident with m and that are not near l is fixed by α, see Lemma 16(iv). Hence we verified the lemma for $k = n-1$.

Suppose the lemma is true for some k, $1 < k \leq n-1$. Then we prove that the statement holds for $k-1$.

So we may assume that a collineation α in ${}^{n}\Psi(O)$ exists with some axis $l \in {}^{n}\mathcal{L}(O)$, fixing some line m of ${}^{n}H(O)$, ${}^{1}\pi(m) \neq {}^{1}\pi(l)$, with $(\alpha)^{\star_k} = 1$, $(\alpha)^{\star_{k+1}} \neq 1$, and such that none of the points ${}^{k+1}\pi(R) \; {}^{k+1}I \; {}^{k+1}\pi(m)$, $R \in {}^{n}\mathcal{P}$, that are not near l is fixed by α. Let us identify the points and lines of ${}^{n}H$ with the corresponding vertices at distance n from O. Then, taking ${}^{1}\pi(l)$ as a base-vertex, there is also a collineation β acting on ${}^{n}H({}^{1}\pi(l))$ with some axis l', fixing some line u' of ${}^{n}H({}^{1}\pi(l))$ not neighbouring this axis, such that $l \in cl(l', {}^{1}\pi(l))$ and $u' \in cl(m, {}^{1}\pi(l))$, with a trivial action in ${}^{k}H({}^{1}\pi(l))$ but a non-trivial action in ${}^{k+1}H({}^{1}\pi(l))$, and such that none of the points of ${}^{k+1}H({}^{1}\pi(l))$ that are incident (incidence in ${}^{k+1}H({}^{1}\pi(l))$) with ${}^{k+1}\pi(u')$ and does not neighbour l' (neighbour relation in ${}^{k+1}H({}^{1}\pi(l))$) is fixed by β (we have denoted by ${}^{k+1}\pi(u')$ the unique vertex in $cl(u', {}^{1}\pi(l))$ at distance $k+1$ from ${}^{1}\pi(l)$). Since $k \geq 1$, $\beta(O) = O$. Consequently β can be 'extended' to some collineation δ in ${}^{n}\Psi(O)$. Which properties does δ have?

Consider an arbitrary point V of ${}^{n}H(O)$ that is incident with $l \in {}^{n}\mathcal{L}(O)$. Then the convex closure of the axis l' for β in ${}^{n}\mathcal{L}({}^{1}\pi(l))$ with V determines a point V' of ${}^{n}H({}^{1}\pi(l))$ that is incident (incidence in ${}^{n}H({}^{1}\pi(l))$) with l'. Hence $\beta(V') = V'$. As a consequence, since $\beta(cl(V', O)) = cl(V', O)$, $\beta(V) = V$ or $\delta(V) = V$. This implies that $\delta \in {}^{n}\Psi(O)$ is axial with axis l of ${}^{n}H(O)$. Using $(\beta)^{\star_k} = 1$, it follows that $(\delta)^{\star_{k-1}} = 1$ (for $k = 1$, this is trivial). Since u' is a fixed line in ${}^{n}H({}^{1}\pi(l))$ for β, and $\beta(O) = O$, also $\beta(cl(u', O)) = cl(u', O)$. The vertex in $cl(u', O)$ at distance 1 from u' and $n-1$ from O corresponds with a line, say u, of ${}^{n-1}H(O)$. Hence, $\beta(u) = u$ or $\delta(u) = u$.

Every point ${}^{k}\pi(S)$ of ${}^{k}H(O)$ that is incident (in ${}^{k}H(O)$) with ${}^{k}\pi(u)$, corresponds with a vertex in Δ at distance $k+1$ from ${}^{1}\pi(l)$ such that $O \in cl({}^{1}\pi(l), {}^{k}\pi(S))$. Let u'' be the vertex in $cl(u', {}^{1}\pi(l))$ at distance $k+1$ from ${}^{1}\pi(l)$. Then $cl(u'', {}^{1}\pi(l))$ and $cl({}^{k}\pi(S), {}^{1}\pi(l))$ determine a segment $cl(u'', {}^{1}\pi(l), {}^{k}\pi(S))$ containing $cl({}^{k}\pi(u), O)$.

Hence $\beta({}^{k}\pi(S)) \neq {}^{k}\pi(S)$ or $\delta({}^{k}\pi(S)) \neq {}^{k}\pi(S)$. Let T be an arbitrary point of ${}^{n}H(O)$ satisfying ${}^{1}\pi(T) \; {}^{1}\!\!\not{I} \; {}^{1}\pi(l)$. Suppose $\delta({}^{k}\pi(T)) = {}^{k}\pi(T)$. Then any line of ${}^{k}H(O)$ that is incident with

${}^{k}\pi(T)$ and does not neighbour ${}^{k}\pi(u)$ intersects ${}^{k}\pi(u)$ in a fixed point in ${}^{k}\mathcal{P}(O)$ for δ. This contradicts earlier mentioned arguments. Now notice that any line of ${}^{n}H$ incident with T and $\delta(T)$ is a fixed line for δ, since δ is an axial collineation with axis l.

This proves the lemma, by induction. □

It is convenient to rephrase the previous lemma for $k = 0$ as follows:

Corollary 28 *If (Δ, G) satisfies the radius n induction hypothesis, then a collineation α in ${}^{n}\Psi$ exists with some axis $l \in {}^{n}\mathcal{L}$, fixing some line m of ${}^{n}H$, ${}^{1}\pi(m) \neq {}^{1}\pi(l)$, with $(\alpha)^{\star_1}$ a non-trivial elation with axis ${}^{1}\pi(l)$ and center the intersection point in ${}^{n}\mathcal{P}(O)$ of ${}^{1}\pi(m)$ and ${}^{1}\pi(l)$.*

Proof. This is an immediate consequence of the proof of Lemma 27. □

Lemma 29 *Suppose (Δ, G) satisfies the radius n induction hypothesis. Then, for all k, $1 \leq k \leq n-1$, at least one ${}^{k}h$-collineation exists in ${}^{n}\Psi$ with non-trivial $()^{\star_{n-k+1}}$-projection.*

Proof. There exists a non-trivial ${}^{1}h$-collineation in ${}^{n}\Psi$ by the radius n induction hypothesis.

Suppose a ${}^{k}h$-collineation in ${}^{n}\Psi$ exists with a non-trivial $()^{\star_{n-k+1}}$-projection, for some k, $1 \leq k < n-1$. Then we prove that a ${}^{k+1}h$-collineation in ${}^{n}\Psi$ exists, with $()^{\star_{n-k}}$-projection not trivial.

Suppose l is some line of ${}^{n}H(O)$ and consider a ${}^{k}h$-collineation in ${}^{n}\Psi({}^{1}\pi(l))$ with a non-trivial action in ${}^{n-k+1}H({}^{1}\pi(l))$, and with some axis l' of ${}^{1}H(l)$, $l \in cl(l', {}^{1}\pi(l))$. Since $n-k > 1$ (and consequently O is fixed), and using similar arguments as in the inductive proof of Lemma 27, some collineation α in ${}^{n}\Psi(O)$ exists with axis l, with $(\alpha)^{\star_{n-k-1}} = 1$, that fixes some line m of ${}^{n}H(O)$, ${}^{1}\pi(m) \neq {}^{1}\pi(l)$, and such that none of the points ${}^{n-k}\pi(U)\ {}^{n-k}I\ {}^{n-k}\pi(m)$, $U \in {}^{n}\mathcal{P}(O)$, that are not near l is fixed. Let us denote the intersection point of m and l in ${}^{n}\mathcal{P}(O)$ by Q. Using the dual of Corollary 28 and applying the transitivity of ${}^{n}\Psi(O)$ on the set of well-formed triangles of ${}^{n}H(O)$, a collineation β in ${}^{n}\Psi(O)$ exists with some center $R\ {}^{n}I\ l$, ${}^{1}\pi(R) \neq {}^{1}\pi(Q)$, fixing some point $S\ {}^{n}I\ m$, ${}^{1}\pi(S) \neq {}^{1}\pi(Q)$, with $(\beta)^{\star_1}$ a non-trivial elation with center ${}^{1}\pi(R)$ and axis the line of ${}^{1}H(O)$ determined by ${}^{1}\pi(S)$ and ${}^{1}\pi(R)$.

As a consequence, $[\alpha, \beta]$ fixes all points incident with l and all lines incident with R. Since $(\alpha)^{\star_{n-k-1}} = 1$, $([\alpha, \beta])^{\star_{n-k-1}} = 1$. Moreover, $\beta(S) = S$ and $\alpha(S)$ is some point $T\ {}^{n}I\ m$ with ${}^{n-k}\pi(T) \neq {}^{n-k}\pi(S)$, ${}^{n-k-1}\pi(T) = {}^{n-k-1}\pi(S)$. Since m is mapped by β^{-1} to some line m' of ${}^{n}H$ incident with S and not neighbouring m, ${}^{n-k}\pi(T)$ cannot be fixed by β^{-1}. Hence $[\alpha, \beta]({}^{n-k}\pi(S)) \neq {}^{n-k}\pi(S)$. Thus $([\alpha, \beta])^{\star_{n-k-1}} = 1$, but $([\alpha, \beta])^{\star_{n-k}} \neq 1$.

We conclude that $[\alpha, \beta]$ is a ${}^{k+1}h$-collineation in ${}^{n}\Psi$ with non-trivial $()^{\star_{n-k}}$-projection. □

Lemma 30 *If (Δ, G) satisfies the radius n induction hypothesis, then at least one elation in ${}^{n}\Psi$ exists with $()^{\star_1}$ -projection not trivial.*

Proof. Applying Corollary 28, a collineation α in ${}^{n}\Psi$ exists with some axis l of ${}^{n}H$ and $(\alpha)^{\star_1}$ a non-trivial elation with axis ${}^{1}\pi(l)$ and some center ${}^{1}\pi(P)$, $P \,{}^{n}\mathrm{I}\, l$.

Dually, there is a collineation β in ${}^{n}\Psi$ with some center Q of ${}^{n}H$ incident with l, ${}^{1}\pi(P) \neq {}^{1}\pi(Q)$, and $(\beta)^{\star_1}$ a non-trivial elation with center ${}^{1}\pi(Q)$ and some axis ${}^{1}\pi(u)$, $u \,{}^{n}\mathrm{I}\, Q$, ${}^{1}\pi(u) \neq {}^{1}\pi(l)$ (using Property 17 if necessary).

Then $[\alpha, \beta]$ fixes all points of ${}^{n}H$ that are incident with l and all lines of ${}^{n}H$ that are incident with Q. Obviously, $([\alpha, \beta])^{\star_1} \neq 1$. Hence $[\alpha, \beta]$ is an elation as required. $\square$

Theorem 31 *If (Δ, G) satisfies the radius n induction hyposthesis, then for every k, $1 \leq k \leq n-1$, the set of ${}^{k}h$-collineations in ${}^{n}\Psi$ with some chosen axis l of ${}^{n}H$ and some chosen center P of ${}^{n}H$, $P \,{}^{n}\mathrm{I}\, l$, acts transitively on the points of ${}^{1}H({}^{n-k}\pi(R))$ in ${}^{n-k+1}\mathcal{P}(O)$ that are incident with ${}^{n-k+1}\pi(m)$, for some arbitrary line m of ${}^{n}H$, $P \,{}^{n}\mathrm{I}\, m$, ${}^{1}\pi(m) \neq {}^{1}\pi(l)$, and some point R of ${}^{n}H$, $R \,{}^{n}\mathrm{I}\, m$, ${}^{1}\pi(R) \,{}^{1}\!\!\not\mathrm{I}\, {}^{1}\pi(l)$.*

Proof. Let P, R, l and m be as in the statement of the theorem.

We first claim that the set of ${}^{1}h$-collineations in ${}^{n}\Psi$ with axis l and center P acts transitively on the points of ${}^{1}H({}^{n-1}\pi(R))$ in ${}^{n}\mathcal{P}(O)$ that are incident with m. By assumption and using Property 17, at least one non-trivial ${}^{1}h$-collineation α in ${}^{n}\Psi$ exists with axis l and center P. The image of R under α is some point S incident with m, ${}^{n-1}\pi(S) = {}^{n-1}\pi(R)$.

Now suppose T is an arbitrary point in ${}^{n}\mathcal{P} \setminus \{R\}$, $T \,{}^{n}\mathrm{I}\, m$, ${}^{n-1}\pi(T) = {}^{n-1}\pi(R)$. Then by Lemma 20 (applied to ${}^{n-1}H({}^{n-1}\pi(R))$), it is possible to consider a collineation β in ${}^{n}\Psi$ fixing R and mapping T to S. We distinguish two cases.

Case 1: ${}^{1}\pi(\beta^{-1}(l)) \neq {}^{1}\pi(l)$.

Then l and $\beta^{-1}(l)$ determine a unique intersection point of ${}^{n}H$, say U. Next we consider a quasi-elation δ in ${}^{n}\Psi$ with $\delta({}^{1}\pi(l)) = {}^{1}\pi(\beta^{-1}(l))$, and with $(\delta)^{\star_{n-1}}$ an elation with axis the line defined by ${}^{n-1}\pi(R)$ and ${}^{n-1}\pi(U)$ and center ${}^{n-1}\pi(R)$, and which exists by Corollary 21. Then $\delta^{-1}(\beta^{-1}\alpha\beta)\delta$ is a ${}^{1}h$-collineation of ${}^{n}\Psi$ with axis l and center P. Moreover, R is mapped to T (note that δ fixes every point of ${}^{n}H$ that $(n-1)$-neighbours R).

Case 2: ${}^{1}\pi(\beta^{-1}(l)) = {}^{1}\pi(l)$.

Then $\beta^{-1}\alpha\beta$ is a ${}^{1}h$-collineation in ${}^{n}\Psi$ with axis l and center P, such that $\beta^{-1}\alpha\beta(R) = T$.

Hence the claim.

We now proceed by induction. To do this, we have to show the more general result that the statement is true for O varying over the set of vertices of Δ.

Suppose the set of ${}^{k}h$-collineations in ${}^{n}\Psi(O)$, $1 \le k < n-1$, with axis l and center P, acts transitively on the points of ${}^{1}H({}^{n-k}\pi(R))$ in ${}^{n-k+1}\mathcal{P}(O)$ that are incident with ${}^{n-k+1}\pi(m)$. Then the set of ${}^{k}h$-collineations α in ${}^{n}\Psi({}^{1}\pi(l))$ with some axis l' of ${}^{1}H(l)$ for which $l \in cl(l', {}^{1}\pi(l))$, some center $P' \in {}^{n}\mathcal{P}({}^{1}\pi(l))$, P' incident with l' (in ${}^{n}H({}^{1}\pi(l))$) and $P \in cl(P', O)$, acts transitively on the points of ${}^{1}H({}^{n-k}\pi(R'))$ (${}^{n-k}\pi(R')$ refers to the unique vertex in $cl(R', {}^{1}\pi(l))$ at distance $n-k$ from ${}^{1}\pi(l)$) that are incident (in ${}^{n-k+1}H({}^{1}\pi(l))$) with ${}^{n-k+1}\pi(m')$ (where ${}^{n-k+1}\pi(m')$ is the unique vertex in $cl(m', {}^{1}\pi(l))$ at distance $n-k+1$ from the vertex ${}^{1}\pi(l)$ in Δ), m' is the line of ${}^{n}\mathcal{L}({}^{1}\pi(l))$ incident with P' such that the vertex in Δ corresponding with m' is in $cl(m, {}^{1}\pi(l))$, and where R' is the point of ${}^{n}\mathcal{P}({}^{1}\pi(l))$ incident with m', $R' \in cl(O, R)$.

Since all collineation are induced by G (by the proofs of their existence), these collineations can be'extended' to collineations α' acting on ${}^{n}H(O)$. They are axial with axis l, fix the lines ${}^{n-1}\pi(r)$ of ${}^{n-1}H(O)$, ${}^{1}\pi(r) \neq {}^{1}\pi(l)$, that are incident with ${}^{n-1}\pi(P)$ and have a trivial action in ${}^{n-k-1}H(O)$. Moreover, the set Υ consisting of the 'extensions' α' acts transitively on the points of ${}^{1}H({}^{n-k}\pi(R'))$ that are incident (incidence in ${}^{n-k}H(O)$) with ${}^{n-k}\pi(m)$. Let, using Lemma 30, β be some elation in ${}^{n}\Psi$ with some center V of ${}^{n}H$ incident with l, ${}^{1}\pi(V)$ ${}^{1}\!\!\not{\!I}$ ${}^{1}\pi(m)$, fixing some point S of ${}^{n}H$, $S \; {}^{n}I \; m$, ${}^{1}\pi(S)$ ${}^{1}\!\!\not{\!I}$ ${}^{1}\pi(l)$, such that $(\beta)^{\star_1}$ is a non-trivial elation in ${}^{1}\Psi$. Then the set of commutators $\Upsilon' = \{[\alpha', \beta] \mid \alpha' \in \Upsilon\}$, is a set of elations with axis l and center V, that all have a trivial action in ${}^{n-k-1}H(O)$ and two by two different and non-trivial actions in ${}^{n-k}H(O)$.

It is now readily seen that the transitivity property of Υ translates into the desired transitivity property for Υ'. $\square$

Lemma 32 *If (Δ, G) satisfies the radius n induction hypothesis, then the set of elations with some axis $l \in {}^{n}\mathcal{L}$ and some center $P \in {}^{n}\mathcal{P}$, $P \; {}^{n}I \; l$, acts transitively on the points in ${}^{1}\mathcal{P} \setminus \{{}^{1}\pi(P)\}$ that are incident with some line ${}^{1}\pi(m)$, $m \, {}^{n}I \, P$, ${}^{1}\pi(m) \neq {}^{1}\pi(l)$.*

Proof. Using Lemma 20, this is similar to the proof of Corollary 21. $\square$

Lemma 33 *If (Δ, G) satisfies the radius n induction hypothesis, then the set of elations with some axis $l \in {}^{n}\mathcal{L}$, and some center $P \in {}^{n}\mathcal{P}$, $P \, {}^{n}I \, l$, acts transitively on the points in ${}^{n}\mathcal{P} \setminus \{Q \in {}^{n}\mathcal{P} \mid {}^{1}\pi(Q) = {}^{1}\pi(P)\}$ that are incident with some line m of ${}^{n}H$ incident with P, ${}^{1}\pi(m) \neq {}^{1}\pi(l)$.*

Proof. By composing suitable ${}^{k}h$-collineations and elations, this is an immediate consequence of Theorem 31 and Lemma 32. $\square$

Theorem 34 *If (Δ, G) satisfies the radius n induction hypothesis, then ${}^{n}\!H$ satisfies the Moufang condition and all elations belong to ${}^{n}\Psi$.*

Proof. Applying Lemma 33, all possible elations occur in ${}^{n}\Psi$. Therefore ${}^{n}\!H$ is a Moufang projective Hjelmslev plane. $\square$

Finally, we show the following analogue of a well-konwn result for projective planes.

Theorem 35 *If ${}^{n}\!H$ is Moufang, with all elations in ${}^{n}\Psi$, then the set of elations with fixed axis $l \in {}^{n}\mathcal{L}$ forms an abelian subgroup of ${}^{n}\Psi$.*

Proof. We first prove that the set of elations is a group. Suppose l is some line of ${}^{n}\!H$. Suppose α and β are elations in ${}^{n}\Psi$ with axis l. Then $\beta^{-1}\alpha$ is a collineation with axis l. If $\beta(R) = \alpha(R)$ for all points of ${}^{n}\!H$ not mear l, then $\alpha = \beta$ and $\beta^{-1}\alpha = 1$. Hence there is some point R of ${}^{n}\!H$ not near l with $\beta^{-1}\alpha(R) = S \neq R$. Every line m through R and S is fixed by $\beta^{-1}\alpha$. Let P be the intersection of such line m with l. As at the end of the proof of Lemma 16, one shows that every line incident with P and not neighbouring l is fixed by $\beta^{-1}\alpha$. The question is whether every line of ${}^{n}\!H$ that is incident with P and neighbours l is also fixed by $\beta^{-1}\alpha$. Let R and S be as above. Then, by assumption, an elation δ in ${}^{n}\Psi$ exists with axis l and center P mapping S to R. So $\delta\beta^{-1}\alpha(R) = R$. Hence R is a center for $\delta\beta^{-1}\alpha$. This implies easily that $\delta\beta^{-1}\alpha$ fixes all points not near l and not near m (because these points are the unique interesection of two lines — one through R, one through P — fixed by $\delta\beta^{-1}\alpha$), forcing $\delta\beta^{-1}\alpha = 1$. It follows that $\beta^{-1}\alpha$ is an elation in ${}^{n}\Psi$ with axis l and center P (see also Baker, Lane & Lorimer [3]).

Next we prove that two elations with axis l commute. So let α and β be such elations with center P and Q, respectively (both P and Q are incident with l).

Suppose first that ${}^{1}\pi(P) \neq {}^{1}\pi(Q)$. Then $[\beta, \alpha]$ fixes every line incident with P and every line that is incident with Q. Hence $[\beta, \alpha] = 1$.

Now suppose that P and Q are neighbouring. Let U be incident with l and not neighbouring P. Let δ be an elation with axis l and center U, and with non-trivial $()^{\star_1}$-projection. From the previous paragraph, we know that $[\alpha, \delta] = 1$ and that $[\beta, \delta] = 1$. Since the center of $\alpha\delta$ cannot neighbour P (by looking at $(\alpha\delta)^{\star_1}$), and using the previous paragraph again, it follows that $[(\alpha\delta), \beta] = 1$. So $(\beta\alpha)\delta = \beta(\alpha\delta) = (\alpha\delta)\beta = \alpha(\delta\beta) = \alpha(\beta\delta) = (\alpha\beta)\delta$. Hence $[\beta, \alpha] = 1$. $\square$

References

[1] Artmann B., Existenz und projektive Limiten von Hjelmslev-Ebenen n-ter Stufe, *Atti del Convegno di Geometria Combinatoria e sue Applicazioni,* Perugia (1971), 27 – 41.

[2] BAKER C., LANE N. D. and LORIMER J. W., The Artin-Zorn theorem for finite punctually cohesive projective Klingenberg Planes, *Ars Combin.* **29** (1990), 143 – 149.

[3] BAKER C., LANE N. D. and LORIMER J. W., An affine characterization of Moufang projective Klingenberg planes, *Res. in Math.* **17** (1990).

[4] BROWN K. S., *Buildings*, Springer, Berlin-New York (1989).

[5] BRUHAT F. and TITS J., Groupes réductifs sur un corps local. I. Données radicielles valuées, *Inst. Hautes Études Sci. Publ. Math.* **41** (1972), 5 – 252.

[6] HANSSENS G. and VAN MALDEGHEM H., On projective Hjelmslev planes of level n, *Glasgow Math. J.* (1989), 257 – 261.

[7] KEPPENS D., *Klingenberg-incidentiestructuren, een bijdrage tot de studie van ringmeetkunden,* Ph.D.-thesis, University of Ghent, 1987.

[8] PICKERT G., *Projektive Ebenen*, Springer, Berlin-New York (1975), second edition.

[9] RONAN M. A., Lectures on buildings, Academic Press, *Persp. Math.* **7** (1989).

[10] TITS J., Endliche Spiegelungsgruppen, die als Weylgruppen auftreten, *Inventiones Math.* **43** (1977), 283 – 295.

[11] TITS J., Immeubles de type affine, **in** *Buildings and the Geometry of Diagrams*, *Springer Lecture Notes in Mathematics* **1181** (L. Rosati, ed.), Springer, Berlin (1986), 159 – 190.

[12] TITS J., Spheres of radius 2 in triangle buildings. I., **in** *Finite Geometries, Buildings, and related topics,* Pap. Conf. Pingree Park/CO (USA) (1990), 17 – 28.

[13] VAN MALDEGHEM H, Non-classical triangle buildings, *Geom. Dedicata* **24** (1987), 123 – 206.

[14] VAN MALDEGHEM H., Valuations on PTR's induced by triangle buildings, *Geom. Dedicata* **26** (1988), 29 – 84.

[15] VAN MALDEGHEM H., Automorphisms of non-classical triangle buildings, *Tijdsch. Belg. Wisk. Gen.* **42** (1990), 201 – 237.

Address of the authors:
Hendrik Van Maldeghem and Kristel Van Steen,
Department of Pure Mathematics and Computer Algebra
University of Ghent,
Galglaan 2,
9000 Gent
BELGIUM
email: hvm@@cage.rug.ac.be and kvs@@cage.rug.ac.be