Flat Lax and Weak Lax Embeddings of Finite Generalized Hexagons

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Abstract

In this paper we study laxly embedded generalized hexagons in finite projective spaces (a generalized hexagon is laxly embedded in $\mathbf{PG}(d,q)$ if it is a spanning subgeometry of the natural point-line geometry associated to $\mathbf{PG}(d,q)$), satisfying the following additional assumption: for any point x of the hexagon, the set of points collinear in the hexagon with x is contained in some plane of $\mathbf{PG}(d,q)$. In particular, we show that $d \leq 7$, and if $d = 7$, we completely classify all such embeddings. A classification is also carried out for $d = 5, 6$ under some additional hypotheses. Finally, laxly embedded generalized hexagons satisfying other additional assumptions are considered, and also here classifications are obtained.

1 Introduction

This paper presents improvements of the results obtained by the same authors in Thas & Van Maldeghem [1996b], and can be seen as a continuation of the latter paper, where the reader is referred to for additional information on embeddings of generalized hexagons.

Definition 1 A *(finite) generalized hexagon* $\mathcal{H}$ *of order* (s,t), $s,t \geq 1$, is a non-empty point-line incidence geometry satisfying the following axioms (we denote the symmetric incidence relation by I).

(i) Every line contains $s+1$ points and two lines are incident with at most one point.

(ii) Every point is on $t+1$ lines and two points are incident with at most one line.

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(*iii*) Given two distinct elements v, w (points and/or lines), there always exists a minimal path $v = v_0 \mathrm{I}\, v_1 \mathrm{I}\, v_2 \mathrm{I} \ldots \mathrm{I}\, v_k = w$ with $k \leq 6$, and if $k < 6$, then this minimal path is unique.

If a generalized hexagon $\mathcal{H}$ has point set P and line set B, and if we denote incidence with I, then we write $\mathcal{H} = (P, B, \mathrm{I})$. If $s, t > 1$, then we call $\mathcal{H}$ *thick*. A geometry $\mathcal{H}' = (P', B', \mathrm{I}')$ is a *subhexagon of* $\mathcal{H}$ if $P' \subseteq P$, $B' \subseteq B$, I' is the restriction of I to P' and B', and $\mathcal{H}'$ is a generalized hexagon.

Definition 2 Let us view the lines of a given generalized hexagon as subsets of the set of points. This is possible by axiom (*i*) above. Likewise, we will view the lines of any projective space $\mathbf{PG}(d, q)$ as sets of points of $\mathbf{PG}(d, q)$. Now let $\mathcal{H} = (P, B, \mathrm{I})$ be a generalized hexagon. Then we say that $\mathcal{H}$ is *(laxly) embedded in* $\mathbf{PG}(d, q)$ if P is a set of points of $\mathbf{PG}(d, q)$ generating $\mathbf{PG}(d, q)$, if every line $L \in B$ is a subset of a line L' of $\mathbf{PG}(d, q)$, and if different lines of B are not subsets of a common line of $\mathbf{PG}(d, q)$. If a hexagon $\mathcal{H}$ is embedded in a projective space $\mathbf{PG}(d, q)$, and L is some line of $\mathcal{H}$, then we will always denote by L' the corresponding line in $\mathbf{PG}(d, q)$. If $L = L'$ for all lines of $\mathcal{H}$, or equivalently, if $\mathcal{H}$ has order (q, t), then we call the lax embedding a *full embedding.* On the other hand, we call the lax embedding *flat* if the following condition is satisfied:

(F) For every point x of $\mathcal{H}$, the set $x^\perp$ of points of $\mathcal{H}$ collinear with x in $\mathcal{H}$ is contained in a plane of $\mathbf{PG}(d, q)$.

In order to make a distinction between collinearity in $\mathbf{PG}(d, q)$ and that in $\mathcal{H}$, we will call two points of $\mathcal{H}$ which are collinear in $\mathcal{H}$ *polycollinear* (as a shorthand for *collinear* in the *poly*gon), as in Thas & Van Maldeghem [1996b]. Also, two elements in $\mathcal{H}$ will be called *opposite* if there exist at least two minimal paths between them. The *distance* $d(v, w)$ of two elements of $\mathcal{H}$ is the length of a minimal path between v and w. Two elements of $\mathcal{H}$ are called *opposite* if and only if their distance is 6.

Now we will say that a lax embedding of $\mathcal{H}$ in $\mathbf{PG}(d, q)$ is *weak*, if the following condition is satisfied:

(W) For every point x of $\mathcal{H}$, the set $x^{\perp\!\!\perp}$ of points of $\mathcal{H}$ not opposite x is contained in a hyperplane of $\mathbf{PG}(d, q)$.

A lax embedding which is both flat and weak will be called *regular* or *ideal.*

Generalized hexagons were introduced by Tits [1959]. All presently known finite generalized hexagons of order (s, t) with $s, t \geq 3$ are described in *loc. cit.*, up to duality, as full embedded hexagons in $\mathbf{PG}(7, s)$ or $\mathbf{PG}(6, s)$. The first class of hexagons is related to the triality group ${}^3D_4(q)$, $q = t$, and the corresponding hexagon is denoted by $T(q^3, q)$ and has order (q^3, q). It is contained in the triality quadric $Q^+(7, q^3)$, and we call this full embedding the *natural embedding* of $T(q^3, q)$. A second class of finite hexagons is related to Dickson's group $G_2(q)$, $q = s$, and the corresponding hexagon is denoted by $H(q)$. It has order (q, q) and it has a *natural (full) embedding* in $\mathbf{PG}(6, q)$, in which case $H(q)$ is contained in some non-degenerate quadric $Q(6, q)$. In fact all points of $Q(6, q)$ are points of $H(q)$, but the set of lines of $H(q)$ is only a subset of the set of lines of $Q(6, q)$. If q is even, one can project the natural embedding of $H(q)$ from the nucleus of the quadric $Q(6, q)$ onto any hyperplane not containing the nucleus and obtain a full embedding in $\mathbf{PG}(5, q)$. In this case, the points of $H(q)$ are all points of $\mathbf{PG}(5, q)$, and the lines are some lines of $\mathbf{PG}(5, q)$, which are totally isotropic with respect to a symplectic polarity. We call this full embedding also a *natural embedding* of $H(q)$. The hexagons $T(q^3, q)$ and $H(q)$ are called the *classical hexagons*. All natural embeddings we mentioned are full and regular.

We can now state the main result of our paper.

Main Result

(*i*) *If $\mathcal{H}$ is a thick generalized hexagon of order (s, t) regularly lax embedded in $\mathbf{PG}(d, q)$, then $d \in \{5, 6, 7\}$, $\mathcal{H}$ is a classical generalized hexagon, and there exists a subspace $\mathbf{PG}(d, s)$ over the subfield $\mathbf{GF}(s)$ of $\mathbf{GF}(q)$ such that $\mathcal{H}$ is naturally embedded in $\mathbf{PG}(d, s)$.*

(*ii*) *If the thick generalized hexagon $\mathcal{H}$ of order (s, t) is flatly and fully embedded in $\mathbf{PG}(d, s)$, then $d \in \{4, 5, 6, 7\}$ and $t \leq s$. Also, if $d = 7$, then $\mathcal{H} \cong T(s, \sqrt[3]{s})$ and the embedding is natural. If $d = 6$ and $t^5 > s^3$, then $\mathcal{H} \cong H(s)$ and the embedding is the natural one. If $d = 5$ and $s = t$, then $\mathcal{H} \cong H(s)$, with s even, and the embedding is the natural one.*

(*iii*) *If the thick generalized hexagon $\mathcal{H}$ of order (s, t) is flatly lax embedded in $\mathbf{PG}(d, q)$, then $d \leq 7$. Also, if $d = 7$, then $\mathcal{H}$ is regularly embedded, and hence we can apply (i). If $d = 6$, and if $\mathcal{H}$ is classical or dual classical with $s \neq t^3$, then $\mathcal{H} \cong H(s)$ and the embedding is regular, and hence we can apply (i) again.*

(*iv*) *If the thick generalized hexagon $\mathcal{H}$ of order (s, t) is weakly lax embedded in $\mathbf{PG}(d, q)$, then $d \geq 5$. Also, if $d = 5$, then $\mathcal{H}$ is a regular lax embedding of $H(s)$, s even, and hence we can apply (i). If $d = 6$, if the embedding is full and if q is odd, then $\mathcal{H}$ is a natural embedding of $H(q)$ in $\mathbf{PG}(6, q)$.*

Before we embark on the proof of the Main Result, we introduce some more terminology. A subhexagon $\mathcal{H}'$ of order (s', t') of a hexagon $\mathcal{H}$ of order (s, t) is called an *ideal* subhexagon if $t = t'$ (this was called a full subhexagon in Thas & Van Maldeghem [1996b], but here we follow the terminology of VAN MALDEGHEM [19**]). Note that, if also $s = s'$, then $\mathcal{H} = \mathcal{H}'$. A circuit consisting of six different points and six different lines will be called an *apartment*. If Σ is an apartment, and if L, M are two opposite lines of Σ, then we call the set $\Sigma \cup \{x, N, y, R, z\}$, where $L \mathrm{~I~} x \mathrm{~I~} N \mathrm{~I~} y \mathrm{~I~} R \mathrm{~I~} z \mathrm{~I~} M$, with $x \notin \Sigma$, a *window* of $\mathcal{H}$. Let x, y be two opposite points of $\mathcal{H}$. We denote $x^y := x^\perp \cap y^{\perp\!\!\perp}$, where $y^{\perp\!\!\perp}$ denotes the set of points not opposite y. If for all opposite pairs x, y, the point set x^y is uniquely defined by any two of its elements, then it follows from RONAN [1980] that $\mathcal{H}$ is a classical hexagon. Also, it follows from THAS & VAN MALDEGHEM [1996b] that every regular full embedding of a generalized hexagon $\mathcal{H}$ is a natural embedding of some classical generalized hexagon.

We now prove the Main Result in a sequence of lemmas and theorems.

2 Regular lax embeddings

In this section, we assume that the thick generalized hexagon $\mathcal{H} = (P, B, \mathrm{I})$ of order (s, t) is regularly lax embedded in $\mathbf{PG}(d, q)$, except if explicitly stated otherwise, as for instance in the first lemma.

Lemma 2.1 *Let $\mathcal{H}$ be flatly lax embedded in $\mathbf{PG}(d, q)$. Let U be a subspace of $\mathbf{PG}(d, q)$ containing an apartment of $\mathcal{H}$. Then all points of $\mathcal{H}$ contained in U and incident with at least two lines of $\mathcal{H}$ in U, together with the lines of $\mathcal{H}$ through these points (which automatically lie in U) and the natural incidence, form an ideal subhexagon $\mathcal{H}'$ of $\mathcal{H}$.*

Proof. See Lemma 1 and Remark 2 in THAS & VAN MALDEGHEM [1996b]. $\square$

If U and $\mathcal{H}'$ are as in the above lemma, then we say that $\mathcal{H}'$ is *induced* by U.

Consider any $x \in P$, with P the point set of $\mathcal{H}$. The points not opposite x span a subspace which we denote by ξ_x. By assumption (W), $\xi_x \neq \mathbf{PG}(d, q)$ for all $x \in P$.

Lemma 2.2 *For any $x \in P$ the space ξ_x has dimension $d - 1$ and does not contain any point opposite x.*

Proof. See proof of Lemma 3 in THAS & VAN MALDEGHEM [1996b]. $\square$

Corollary 2.3 *For $x, y \in P$, $x \neq y$, we have $\xi_x \neq \xi_y$.*

Proof. See proof of Corollary 4 in THAS & VAN MALDEGHEM [1996b]. $\square$

Corollary 2.4 *If L is a line of $\mathcal{H}$ and if L' is the corresponding line of $\mathbf{PG}(6, q)$, then the points of $\mathcal{H}$ on L' are exactly the $s + 1$ points of L.*

Proof. Assume, by way of contradiction, that x is a point of $\mathcal{H}$ on L', but not on L. If $y \in L$, then $x \in \xi_y$, so $d(x, y) \leq 4$. It immediately follows that $x \in L$, a contradiction. $\square$

For any $x \in P$, we denote by π_x the unique plane in $\mathbf{PG}(d, q)$ spanned by all points polycollinear with x.

Lemma 2.5 *For every $x \in P$, the plane π_x does not contain points of $\mathcal{H}$ not polycollinear with x.*

Proof. Let $u \in P \cap \pi_x$ be not collinear with x. If u is opposite x, then $u \in \pi_x \subseteq \xi_x$, contradicting Lemma 2.2. So u is not opposite x. Then the unique line L of $\mathcal{H}$ through u nearest to x lies in π_x. Let y be polycollinear with x and at distance 5 from L. Then u and y are opposite. As ξ_y contains all points polycollinear with x, it also contains π_x, hence also u, a contradiction. $\square$

Lemma 2.6 *$\mathcal{H}$ is a classical generalized hexagon. Hence also every thick ideal subhexagon of $\mathcal{H}$ is classical.*

Proof. See proof of Lemma 6 in Thas & Van Maldeghem [1996b]. $\square$

If x and y are opposite points of $\mathcal{H}$, then the set $\pi_x \cap \xi_y \cap P$ is called an *ideal line* in Ronan [1980], or a *distance-2-trace* (or briefly *trace*) in Van Maldeghem [1995, 19**].

Theorem 2.7 *We have $5 \leq d \leq 7$. Also, if $\mathcal{H} \cong H(s)$, then $d \neq 7$. If $\mathcal{H} \cong T(s, \sqrt[3]{s})$, then no subhexagon of $\mathcal{H}$ isomorphic with $H(\sqrt[3]{s})$ is contained in a $\mathbf{PG}(d-2, q)$.*

Proof. It is clear that $d \geq 3$. If $d = 3$, then for every point $x \in P$ we have $\pi_x = \xi_x$, contradicting Lemma 2.5. Now suppose that $d = 4$. If x and y are distinct collinear points of $\mathcal{H}$, then $\xi_x \cap \xi_y$ is a plane and so $\xi_x \cap \xi_y = \pi_x = \pi_y$, a contradiction. Hence $d \geq 5$.

Consider an apartment Σ in $\mathcal{H}$ and a line L in $\mathcal{H}$ concurrent with exactly one line of Σ. Let L and Σ generate a $\mathbf{PG}(m, q)$. Then $m \leq 6$. Let $\mathcal{H}'$ be the ideal subhexagon induced by $\mathbf{PG}(m, q)$. Then the order of $\mathcal{H}'$ is (s', t), with $2 \leq s' \leq s$. If $s = s'$, then $m = d \leq 6$ and we are done. So suppose $s' < s$. Then there is a line M of $\mathcal{H}$ which does not lie in $\mathbf{PG}(m, q)$, but which contains a point on a line of Σ. Let M and $\mathbf{PG}(m, q)$ generate $\mathbf{PG}(m+1, q)$ and let $\mathbf{PG}(m+1, q)$ induce an ideal subhexagon $\mathcal{H}''$ of order (s'', t),

$s' < s'' \le s$. Note that $\mathcal{H}'$ is an ideal subhexagon of $\mathcal{H}''$. If $s'' = s$, then $d = m+1 \le 7$ and we are done again. If $s'' < s$, then it follows from THAS [1976] that $s \ge s''^2 t$, $s'' \ge s'^2 t$, and hence $s \ge s'^4 t^3$. Now by HAEMERS & ROOS [1981] we have $s \le t^3$. This implies $s = t^3$ and $s' = 1$, a contradiction. We conclude that $d \le 7$.

Assume that $\mathcal{H} \cong H(s)$. Consider again the ideal subhexagon $\mathcal{H}'$ of $\mathcal{H}$. By THAS [1976], either $s = s'$ or $s \ge s'^2 s$. As $s' \ge 2$, necessarily $s = s'$ and so $m = d \le 6$.

Now let $\mathcal{H}^*$ be a proper thick ideal subhexagon of $\mathcal{H}$, contained in a $\mathbf{PG}(d-2, q)$. The subhexagon induced by the subspace $\mathbf{PG}(d-1, q)$ generated by this $\mathbf{PG}(d-2, q)$ and any line of $\mathcal{H}$ not in $\mathcal{H}^*$ but concurrent with a line of $\mathcal{H}^*$ must coincide with $\mathcal{H}$ (as above), a contradiction. In particular, if $\mathcal{H} \cong T(s, \sqrt[3]{s})$, then no subhexagon of $\mathcal{H}$ isomorphic with $H(\sqrt[3]{s})$ is contained in a $\mathbf{PG}(d-2, q)$. $\square$

Theorem 2.8 *If the thick generalized hexagon $\mathcal{H}$ of order (s, t) is regularly lax embedded in $\mathbf{PG}(5, q)$, then $s = t$, s is even, and $\mathcal{H}$ is a natural embedding of $H(s)$ in a subspace $\mathbf{PG}(5, s)$ of $\mathbf{PG}(5, q)$ for some subfield $\mathbf{GF}(s)$ of $\mathbf{GF}(q)$.*

Proof. Let L, M be two opposite lines of $\mathcal{H}$. Let x_1, x_2 be two different points of L, and let y_i, $i = 1, 2$, be on M and not opposite x_i. Let $x_i \mathrm{~I~} L_i \mathrm{~I~} z_i \mathrm{~I~} M_i \mathrm{~I~} y_i$ in $\mathcal{H}$. We claim that the subspace U of $\mathbf{PG}(5, q)$ generated by $x_1, x_2, y_1, y_2, z_1, z_2$ has dimension 5. Indeed, suppose that U has dimension $k \le 4$. Then without loss of generality, we may assume that U is generated by x_1, x_2, y_1, y_2, z_1. Since all the latter points lie in ξ_{z_1}, we have $U \subseteq \xi_{z_1}$, which implies $z_2 \in \xi_{z_1}$, contradicting Lemma 2.2. This proves our claim. Now let S be the set of points of $\mathcal{H}$ polycollinear with some point on L and with some point on M. Note that $|S| = s + 1$. Since the elements of $L_1 \setminus \{x_1, z_1\}$ belong to $\xi_{x_1} \cap \xi_{y_1} \cap \xi_{x_2}$ but not to ξ_{y_2}, the hyperplane ξ_{y_2} is linearly independent of the hyperplanes $\xi_{x_1}, \xi_{y_1}, \xi_{x_2}$. Similarly, every element of $\{\xi_{x_1}, \xi_{x_2}, \xi_{y_1}, \xi_{y_2}\}$ is linearly independent of the other three. Hence $\zeta = \xi_{x_1} \cap \xi_{y_1} \cap \xi_{x_2} \cap \xi_{y_2}$ is a line of $\mathbf{PG}(5, q)$ containing all elements of S. It follows that for any two opposite points v, w of $\mathcal{H}$ we have $|\{v, w\}^{\perp\!\!\perp\perp\!\!\perp}| = s+1$, where $\{v, w\}^{\perp\!\!\perp\perp\!\!\perp}$ is the set of points not opposite every point of $\{v, w\}^{\perp\!\!\perp} = v^{\perp\!\!\perp} \cap w^{\perp\!\!\perp}$. By VAN MALDEGHEM [19**](6.5.6), $s = t$ and $\mathcal{H} \cong H(s)$ with s even. Now the points of $\mathcal{H}$ and the lines of $\mathcal{H}$ together with the distance-2-traces form the symplectic polar space $W_5(s)$. Since the lines of $W_5(s)$ through a fixed point x are contained in ξ_x, we see that $W_5(s)$ is (sub-)weakly embedded in $\mathbf{PG}(5, q)$, and hence it is fully embedded in a subspace $\mathbf{PG}(5, s)$ over the subfield $\mathbf{GF}(s)$ of $\mathbf{GF}(q)$, by Theorem 1 of THAS & VAN MALDEGHEM [1996a]. Hence also $\mathcal{H}$ is fully embedded in $\mathbf{PG}(5, s)$ and the theorem follows from THAS & VAN MALDEGHEM [1996b]. $\square$

Theorem 2.9 *If the thick generalized hexagon $\mathcal{H}$ of order (s, t), $s \ne t^3$, is regularly lax embedded in $\mathbf{PG}(6, q)$, then $s = t$ and $\mathcal{H}$ is a natural embedding of $H(s)$ in a subspace $\mathbf{PG}(6, s)$ of $\mathbf{PG}(6, q)$ for some subfield $\mathbf{GF}(s)$ of $\mathbf{GF}(q)$.*

Proof. Let the thick generalized hexagon $\mathcal{H}$ of order (s,t), $s \neq t^3$, be regularly lax embedded in $\mathbf{PG}(6,q)$. As $\mathcal{H}$ is classical and $s \neq t^3$ we have $s = t$. The points of $\mathcal{H}$ together with the lines and distance-2-traces of $\mathcal{H}$ form a polar space $Q(6,s)$ which is weakly embedded in $\mathbf{PG}(6,q)$. Hence by THAS & VAN MALDEGHEM [1996a] $Q(6,s)$ is fully embedded in a subspace $\mathbf{PG}(6,s)$ of $\mathbf{PG}(6,q)$ for some subfield $\mathbf{GF}(s)$ of $\mathbf{GF}(q)$. It follows that $\mathcal{H}$ is fully embedded in $\mathbf{PG}(6,s)$. As it is clear that $\mathcal{H}$ is regularly embedded in $\mathbf{PG}(6,s)$, we conclude that $\mathcal{H}$ is a natural embedding of $H(s)$ in $\mathbf{PG}(6,s)$. $\square$

We now show a fairly general lemma, also valid in the infinite case. For a given Moufang hexagon, we call the group generated by all root elations the *little projective group*, see VAN MALDEGHEM [19**].

Lemma 2.10 *If the thick generalized hexagon $\mathcal{H}$ is regularly lax embedded in $\mathbf{PG}(d,\mathbb{K})$, for some field $\mathbb{K}$, then $\mathcal{H}$ is a Moufang hexagon and the collineation group inherited from $\mathbf{PG}(d,\mathbb{K})$ contains the little projective group of $\mathcal{H}$.*

Proof. First let $\mathcal{H} \cong H(2)$. The same argument as in the proof of Theorem 2.7 shows that $5 \leq d \leq 6$. For $d = 5$, the proof of Theorem 2.8 remains valid for infinite projective spaces. For $d = 6$, the same holds for Theorem 2.9. Hence, in this case, the embedding is the natural one in a subspace over a subfield. Consequently, the full collineation group of $\mathcal{H}$ is inherited from the projective space. So we may assume that $\mathcal{H} \ncong H(2)$.

Let x be any point of $\mathcal{H}$. Suppose that a subspace U contains $\langle x^{\perp\!\!\perp}\rangle$ and a point y of $\mathcal{H}$ opposite x. Clearly, U contains all apartments of $\mathcal{H}$ through x and y, hence U induces a subhexagon $\mathcal{H}'$ which contains all lines of $\mathcal{H}$ through x and which contains all points of $\mathcal{H}$ polycollinear with x. By VAN MALDEGHEM [19**](2.8.2), $\mathcal{H}' = \mathcal{H}$, hence $U = \mathbf{PG}(d,\mathbb{K})$, $\xi_x := \langle x^{\perp\!\!\perp}\rangle$ is a hyperplane and ξ_x does not contain any point of $\mathcal{H}$ opposite x. Also, it is now clear that Lemma 2.5 is valid in the infinite case. Hence the proof of Lemma 2.6 implies that $\mathcal{H}$ is a Moufang hexagon.

Now we consider a line L of $\mathcal{H}$ and we show that there is a collineation of $\mathbf{PG}(d,\mathbb{K})$ which preserves $\mathcal{H}$ and induces in $\mathcal{H}$ an axial collineation with axis L in the sense of e.g. RONAN [1980]. Choose $x, y \in L$, $x \neq y$. Then $\xi_x \neq \xi_y$ since there are clearly points of $\mathcal{H}$ (at distance 4 from x) contained in ξ_x, but not in ξ_y. Hence $\zeta_L := \xi_x \cap \xi_y$ is a $(d-2)$-dimensional subspace of $\mathbf{PG}(d,\mathbb{K})$. It clearly contains all points of $\mathcal{H}$ at distance ≤ 3 from L. We claim that these points generate ζ_L. Indeed, if they generate a strictly smaller space U, then by considering a line M of $\mathcal{H}$ opposite L, we see that the subspace $\langle U, M\rangle$ of dimension at most $d-1$ would induce an ideal subhexagon which contains all points of $\mathcal{H}$ on L; by VAN MALDEGHEM [19**](2.8.2), this subhexagon coincides with $\mathcal{H}$, a contradiction. The claim follows. Hence $\xi_v \cap \xi_w = \zeta_L$ for every two distinct points $v, w \in L$.

Now let M_1 be any line of $\mathcal{H}$ opposite L. Let a, b be two distinct points both at distance 3 from both L and M_1. Let M_2, $M_2 \neq L$ be any line at distance 3 from both a and

b. Clearly neither M_1 nor M_2 meets ζ_L (otherwise $\mathcal{H}$ is induced in a hyperplane of $\mathbf{PG}(d,\mathbb{K})$, a contradiction). Also, the lines L, M_1, M_2 are contained in the 3-dimensional space generated by a^b and b^a. Hence there is a unique projective linear collineation α of $\mathbf{PG}(d,\mathbb{K})$ fixing all points of ζ_L, stabilizing all hyperplanes containing L, and mapping M_1' onto M_2'. Let c_1 respectively c_2, a' be the point on M_1 respectively M_2, L polycollinear with a. Then $c_1, c_2, a' \in a^b$ and hence they are collinear in $\mathbf{PG}(d,\mathbb{K})$. Moreover, they lie in the plane $\langle L, c_1\rangle$, which implies that $c_1^\alpha = (M_1' \cap \langle L, c_1\rangle)^\alpha = M_2' \cap \langle L, c_1\rangle = c_2$. So the line ac_1 of $\mathbf{PG}(d,\mathbb{K})$ is mapped onto the line ac_2. Varying a, we conclude that every point of M_1 is mapped by α onto a point of M_2. It is also clear that α induces in π_a an elation with axis aa' and center a'. Hence all points of $\mathcal{H}$ on ac_1 are mapped onto points of $\mathcal{H}$ on ac_2.

Next we note that, if u is any point of $\mathcal{H}$ at distance 3 from L, but not polycollinear with a' and not opposite c_1, then the spaces $\langle \pi_u, L\rangle$ and π_{c_1} meet in the unique point d of $\mathcal{H}$ polycollinear with both c_1 and u. Indeed, it is clear that d belongs to the intersection; conversely, if this intersection would contain more, then $V := \langle \pi_u, L, \pi_{c_1}\rangle$ is contained in a d'-dimensional space, with $d' \leq 4$. Hence 5 points of the apartment determined by L, u, c_1 generate V, and so V is contained in ξ_x for some point x, a contradiction. Since clearly $(\langle \pi_u, L\rangle)^\alpha = \langle \pi_u, L\rangle$, and since $\pi_{c_1}^\alpha = \pi_{c_2}$, we see that d^α is the unique point of $\mathcal{H}$ polycollinear with both c_2 and u. Hence, similarly as above (interchanging roles of M_1 respectively M_2 and c_1d respectively c_2d^α), all points of $\mathcal{H}$ on the line c_1d are mapped by α onto points of $\mathcal{H}$ on the line c_2d^α. Since c_1d can be considered as a general line of $\mathcal{H}$ opposite L and meeting M, we may conclude with Proposition 7 of ABRAMENKO [1996] (see also ABRAMENKO & VAN MALDEGHEM [1997]) that α preserves $\mathcal{H}$ and induces an axial collineation with axis L. Since the group generated by all axial collineations is a normal subgroup of the little projective group, and since the latter is simple if $\mathcal{H} \not\cong H(2)$, the result follows. $\square$

The next corollary is also valid for infinite Moufang hexagons, but for reasons of simplicity, we only state it for finite classical hexagons.

Corollary 2.11 *If the thick generalized hexagon $\mathcal{H}$ of order (s,t) is regularly lax embedded in $\mathbf{PG}(d,q)$, for some prime power q, then the points on any line of $\mathcal{H}$ form a projective subline over $\mathbf{GF}(s)$ of $\mathbf{PG}(1,q)$. In particular, $\mathbf{GF}(s)$ is a subfield of $\mathbf{GF}(q)$.*

Proof. Consider two opposite lines L, M of $\mathcal{H}$. By the proof of Lemma 2.10, the projectivity $[L;M] : L \to M : x \mapsto y$, where $x \in L$, $y \in M$ and x, y not opposite, is induced by an element of the automorphism group $\mathbf{PGL}_{d+1}(q)$ of $\mathbf{PG}(d,q)$. Hence the group G of projectivities of L in $\mathcal{H}$ is a subgroup of $\mathbf{PGL}_2(q)$ in its natural action on $L' = \mathbf{PG}(1,q)$, having an orbit of length $s+1$. By KNARR [1988], $G \cong \mathbf{PGL}_2(s)$. If $s \neq 2$, then the result follows from Lemma 3 of THAS & VAN MALDEGHEM [19**b]. If $s = 2$, then, since by the proof of Lemma 2.10 $\mathcal{H}$ is classical, $t = 2$, and hence $\mathcal{H} \cong H(2)$. The result follows from Theorem 2.8 and Theorem 2.9. $\square$

Theorem 2.12 *A thick generalized hexagon of order* (t^3, t) *cannot be regularly lax embedded in* $\mathbf{PG}(6, q)$.

Proof. Assume, by way of contradiction, that the thick generalized hexagon $\mathcal{H}$ of order (t^3, t) is regularly lax embedded in $\mathbf{PG}(6, q)$. Then $\mathcal{H}$ is classical. Let $\mathcal{H}'$ be a subhexagon of order (t, t). By Theorem 2.7 $\mathcal{H}'$ is not contained in a $\mathbf{PG}(4, q)$, so $\mathcal{H}'$ is regularly lax embedded in $\mathbf{PG}(6, q)$ or in a hyperplane $\mathbf{PG}(5, q)$. From Theorems 2.8 and 2.9 it then follows that any apartment of $\mathcal{H}$ generates a hyperplane of $\mathbf{PG}(6, q)$.

First, suppose that q is even (and note that this is equivalent with t even). Let Σ be an apartment of $\mathcal{H}$ and let $\mathcal{H}'$ be a subhexagon of order (t, t) containing Σ. If $\mathcal{H}'$ is laxly embedded in a hyperplane $\mathbf{PG}(5, q)$, then for any two distinct polycollinear points x, y of Σ the space $\mathbf{PG}(5, q)$ contains exactly $t^2 + t + 1$ lines of $\mathcal{H}$ concurrent with xy (including xy itself) as otherwise $\mathcal{H}$ would be contained in $\mathbf{PG}(5, q)$. It follows that at most one subhexagon of order (t, t) containing Σ, generates a hyperplane of $\mathbf{PG}(6, q)$. Let $\mathcal{H}''$ be any subhexagon of order (t, t) containing Σ, which generates $\mathbf{PG}(6, q)$. The point set of $\mathcal{H}''$ is the point set of a non-singular quadric $Q(6, t)$ in some subspace $\mathbf{PG}(6, t)$ of $\mathbf{PG}(6, q)$. If x is a point of Σ and if θ_x is the tangent space (in $\mathbf{PG}(6, t)$) of $Q(6, t)$ at x, then the hyperplane ξ_x of $\mathbf{PG}(6, q)$ generated by θ_x is independent of the choice of $\mathcal{H}''$. Hence the nucleus n of the quadric $Q(6, t)$ is independent of the choice of $\mathcal{H}''$ (n is the intersection of the 6 hyperplanes ξ_z, with z in Σ). Now let $y \neq x$ be a point of Σ, with x and y collinear in $\mathcal{H}$. Then $\xi_x \cap \xi_y =: \xi_{xy}$ is 4-dimensional, contains all points of $\mathcal{H}$ at distance at most 3 from the line xy of $\mathcal{H}$, and is generated by these points (as ξ_{xy} is generated by all points of $\mathcal{H}''$ at distance at most 3 from the line xy). Also, $n \in \xi_{xy}$. If z is any point of $\mathcal{H}$ on xy, then ξ_z contains ξ_{xy}, so contains n. Now let u be any point of $\mathcal{H}$ not on a line $\bar{N}$ of $\mathcal{H}$, with $\bar{N}$ containing a line N of $\mathcal{H}''$. If L, M are distinct lines of $\mathcal{H}$ containing u, then by 6.5 of Thas [1995] $\mathcal{H}''$ contains two lines L'', M'' whose extensions $\bar{L}'', \bar{M}''$ to $\mathcal{H}$ are concurrent respectively with L, M. If $L \cap \bar{L}'' = \{l\}$ and $M \cap \bar{M}'' = \{m\}$, then we choose a window Ω containing u, l, m, two distinct points l_1, l_2 on L'' ($l_i \neq l$, $i = 1, 2$), and two distinct points m_1, m_2 on M'' ($m_i \neq m$, $i = 1, 2$), where l_i is at distance 4 from m_i, $i = 1, 2$. Also, l_1, l_2 can be chosen in such a way that $\langle \pi_l, \pi_{l_1}, \pi_{l_2} \rangle$ is 4-dimensional. Then the subhexagon $\mathcal{H}'''$ of order (t, t) defined by the window Ω generates $\mathbf{PG}(6, q)$. As $\mathcal{H}''$ and $\mathcal{H}'''$ share an apartment, the corresponding quadrics $Q(6, t)$ and $Q'(6, t)$ have the common nucleus n. Hence ξ_u contains n. Now let u be any point of $\mathcal{H}$ not in $\mathcal{H}''$, but on a line $\bar{N}$ of $\mathcal{H}$ where $\bar{N}$ contains a line N of $\mathcal{H}''$. Let z_1, z_2 be distinct points of N and let Σ' be an apartment in $\mathcal{H}''$ containing z_1, z_2. Then, by the foregoing, the hyperplane ξ_u contains the nucleus n of $Q(6, t)$. So it follows that for any point u of $\mathcal{H}$ the hyperplane ξ_u contains n. Now assume, by way of contradiction, that there is a line W of $\mathbf{PG}(6, q)$

through n, which contains distinct points v and w of $\mathcal{H}$. Let r be a point of $\mathcal{H}$ at distance 4 from v and 6 from w. Then ξ_r contains n and v, so contains w, a contradiction. Next, assume that there is a plane $\pi_b = \langle b^\perp \rangle$, with b in $\mathcal{H}$, which contains n. If c is a point of $\mathcal{H}$ at distance 4 from b, then, as ξ_c contains n, it also contains π_b, clearly a contradiction. Next, let $\mathbf{PG}(5,q)$ be a hyperplane of $\mathbf{PG}(6,q)$ not containing n. Now we project $\mathcal{H}$ from n onto $\mathbf{PG}(5,q)$. Then the projection $\mathcal{H}^*$ of $\mathcal{H}$ is regularly lax embedded in $\mathbf{PG}(5,q)$, contradicting Theorem 2.8 as $s \neq t$. This proves the theorem for q (or equivalently t) even.

Next, note that by Corollary 2.11, any line of $\mathcal{H}$ is a subline $\mathbf{PG}(1,s)$ of some line of $\mathbf{PG}(6,q)$.

Now let q be odd. Consider an apartment Σ of $\mathcal{H}$, and let $\mathbf{PG}(5,q)$ be the hyperplane generated by the 6 points of Σ. Further, let $\mathcal{H}_1, \mathcal{H}_2$ be distinct subhexagons of order (t,t) containing Σ; by Theorem 2.8 the point set of $\mathcal{H}_i$ generates $\mathbf{PG}(6,q)$, $i = 1,2$. Let $\mathbf{PG}^{(i)}(6,t)$ be the subspace over $\mathbf{GF}(t)$ of $\mathbf{PG}(6,q)$ in which $\mathcal{H}_i$ is fully embedded by Theorem 2.9, and let $Q^{(i)}(6,t)$ be the quadric of $\mathbf{PG}^{(i)}(6,t)$ whose point set coincides with the point set of $\mathcal{H}_i$, $i = 1,2$. Further, let $\mathbf{PG}(5,q) \cap \mathbf{PG}^{(i)}(6,t) = \mathbf{PG}^{(i)}(5,t)$, $i = 1,2$. If x is a point of Σ and if $\theta_x^{(i)}$ is the tangent space in $\mathbf{PG}^{(i)}(6,t)$ of $Q^{(i)}(6,t)$ at x, then ξ_x is the hyperplane of $\mathbf{PG}(6,q)$ generated by $\theta_x^{(i)}$. So the pole of $\mathbf{PG}^{(1)}(5,t)$ with respect to $Q^{(1)}(6,t)$ coincides with the pole of $\mathbf{PG}^{(2)}(5,t)$ with respect to $Q^{(2)}(6,t)$; let p be this common point. Hence $p \in \mathbf{PG}^{(1)}(6,t) \cap \mathbf{PG}^{(2)}(6,t)$. Clearly p does not belong to $\mathcal{H}$. Let $x_1, x_2, \ldots, x_6$ be the 6 points of Σ, where $x_j x_{j+1}$ is a line of $\mathcal{H}$ (subscripts being taken modulo 6). The lines $x_j x_{j+1}$ of $\mathcal{H}$ belong to the extension $\mathbf{PG}^{(i)}(5,s)$ of $\mathbf{PG}^{(i)}(5,t)$ to $\mathbf{GF}(s)$. It immediately follows that $\mathbf{PG}^{(1)}(5,s) = \mathbf{PG}^{(2)}(5,s)$; this common space will be denoted by $\mathbf{PG}(5,s)$. Now assume, by way of contradiction, that $\mathbf{PG}^{(1)}(6,s) = \mathbf{PG}^{(2)}(6,s)$, with $\mathbf{PG}^{(i)}(6,s)$ the extension of $\mathbf{PG}^{(i)}(6,t)$ to $\mathbf{GF}(s)$, $i = 1,2$. Let u be a point of $\mathcal{H}$ not in $\mathbf{PG}^{(1)}(6,s)$, and let V be a line of $\mathcal{H}$ containing u. By 6.5 of Thas [1995], $\mathcal{H}_i$ contains a line V_i, whose extension $\bar{V}_i$ to $\mathcal{H}$ contains a point of V. Put $\bar{V}_i \cap V = \{a_i\}$, $i = 1,2$. First, assume that $a_1 = a_2$. If $\bar{V}_1 \neq \bar{V}_2$, then the plane over $\mathbf{GF}(s)$ containing $\bar{V}_1 \cup \bar{V}_2$ belongs to $\mathbf{PG}^{(1)}(6,s)$, so u belongs to $\mathbf{PG}^{(1)}(6,s)$, a contradiction. Hence necessarily $\bar{V}_1 = \bar{V}_2$. Assume, by way of contradiction, that $a_1 \notin \mathbf{PG}(5,s)$. Then $\bar{V}_1$ has exactly one point b in common with $\mathbf{PG}(5,s)$. So $\{b\} = V_i \cap \mathbf{PG}^{(i)}(5,t)$, $i = 1,2$. Hence b belongs to $\mathcal{H}_1 \cap \mathcal{H}_2$, and consequently V_1 and V_2 belong to $\mathbf{PG}(5,s)$. It follows that $a_1 \in \mathbf{PG}(5,s)$, a contradiction. Consequently $a_1 \in \mathbf{PG}(5,s)$. Also, a_1 is on exactly one line of $\mathcal{H}_i$ containing two distinct points of $\mathcal{H}_1 \cap \mathcal{H}_2$, $i = 1,2$. It easily follows that $a_1 = a_2$ for either zero, or one or all lines V of $\mathcal{H}$ through u. If $a_1 \neq a_2$, then V contains at least two points of $\mathbf{PG}^{(1)}(6,s)$. Hence if $a_1 = a_2$ for either zero or one line V, then u belongs to $\mathbf{PG}^{(1)}(6,s)$. Now suppose that $a_1 = a_2$ for all lines V of $\mathcal{H}$ containing u. Then there is a subhexagon $\mathcal{H}^*$ of order (t,t) containing Σ and u. Let L be a line of $\mathcal{H}$ through u and let v be a point of $\mathcal{H}$ on L which does not belong to $\mathcal{H}^*$. Then there is no subhexagon of order (t,t) containing Σ and v. It follows that v is a point of $\mathbf{PG}^{(1)}(6,s)$. Now clearly every point of L belongs to $\mathbf{PG}^{(1)}(6,s)$, and so u is a point of $\mathbf{PG}^{(1)}(6,s)$. Consequently $\mathcal{H}$ is contained in $\mathbf{PG}^{(1)}(6,s)$, that is, $\mathcal{H}$ is regularly full embedded in $\mathbf{PG}^{(1)}(6,s)$, contradicting the Main Result of Thas & Van Maldeghem [1996b]. We conclude that $\mathbf{PG}^{(1)}(6,s) \neq \mathbf{PG}^{(2)}(6,s)$. Hence $\mathbf{PG}^{(1)}(6,s) \cap \mathbf{PG}^{(2)}(6,s) = \mathbf{PG}(5,s) \cup \{p\}$. Now we consider a projective space $\mathbf{PG}(7,q)$ containing $\mathbf{PG}(6,q)$, and in $\mathbf{PG}(7,q)$ a subspace $\mathbf{PG}(7,s)$ such that the projection Φ of $\mathbf{PG}(7,s)$ from some point $y \in \mathbf{PG}(7,q) \setminus \mathbf{PG}(6,q)$ contains $\mathbf{PG}^{(1)}(6,s) \cup \mathbf{PG}^{(2)}(6,s)$. Then there is a line M in $\mathbf{PG}(7,s)$ whose extension M' to $\mathbf{GF}(q)$ contains y and p. Let u be a point of $\mathcal{H}$ not in $\mathbf{PG}^{(1)}(6,s) \cup \mathbf{PG}^{(2)}(6,s)$, and let V be a line of $\mathcal{H}$ containing u. By 6.5 of Thas [1995] $\mathcal{H}_i$ contains a line V_i whose extension $\bar{V}_i$ to $\mathcal{H}$ contains a point of V, $i = 1,2$. Put $\bar{V}_i \cap V = \{a_i\}, i = 1,2$. First, assume that $a_1 = a_2$. Then $a_1 \in \mathbf{PG}^{(1)}(6,s) \cap \mathbf{PG}^{(2)}(6,s)$, and so $a_1 \in \mathbf{PG}(5,s)$. Hence a_1 is on exactly one line of $\mathcal{H}_i$ containing two distinct points of $\mathcal{H}_1 \cap \mathcal{H}_2$, $i = 1,2$. It easily follows that $a_1 = a_2$ for either zero, or one or all lines V of $\mathcal{H}$ through u. Assume, by way of contradiction, that $\pi_u = \langle u^\perp \rangle$ contains p. Let V be any line of $\mathcal{H}$ through u whose extension tot $\mathbf{GF}(q)$ does not contain p, and let V_1 be the line of $\mathcal{H}_1$ whose extension $\bar{V}_1$ to $\mathbf{GF}(s)$ contains a point a_1 of V. Let v_1 be a common point of V_1 and $\mathbf{PG}^{(1)}(5,t)$. Then ξ_{v_1} contains V and p, so contains π_u. It follows that $v_1 \in V$, so $u \in \mathbf{PG}^{(1)}(6,s)$, a contradiction. Consequently π_u does not contain p. If $a_1 \neq a_2$, then $\langle a_1, a_2 \rangle \cap \Phi$ is a line over $\mathbf{GF}(s)$ which is the projection from y of a uniquely defined line $a_1^* a_2^*$ of $\mathbf{PG}(7,s)$, with $\{a_i^*\} = \langle a_i, y \rangle \cap \mathbf{PG}(7,s)$. Now assume that $a_1 = a_2$ for either zero or one line V of $\mathcal{H}$ through u. Then we obtain at least two lines over $\mathbf{GF}(s)$ in Φ which are the projections from y of uniquely defined lines $a_1^* a_2^*$ and $b_1^* b_2^*$ of $\mathbf{PG}(7,s)$. As π_u does not contain p, one easily shows that $a_1^* a_2^*$ and $b_1^* b_2^*$ have exactly one point u^* in common and that u is the projection of u^* from y onto $\mathbf{PG}(6,q)$. So u is the projection from y of some point of $\mathbf{PG}(7,s)$. Now assume that $a_1 = a_2$ for all lines of $\mathcal{H}$ containing u. Then there is a subhexagon $\widetilde{\mathcal{H}}$ of order (t,t) containing Σ and u. Let L be a line of $\mathcal{H}$ through u and let v be a point of $\mathcal{H}$ on L which does not belong to $\widetilde{\mathcal{H}}$. Then there is no subhexagon of order (t,t) containing Σ and v. Hence v is the projection from y onto $\mathbf{PG}(6,q)$ of some point v^* of $\mathbf{PG}(7,s)$. As the extension L' of L to $\mathbf{GF}(q)$ does not contain p, the line L is the projection from y of a uniquely defined line L^* of $\mathbf{PG}(7,s)$. It now follows that u is the projection from y onto $\mathbf{PG}(6,q)$ of a uniquely defined point u^* of $\mathbf{PG}(7,s)$. Now we show that, putting $\{u^*\} = \langle u, y \rangle \cap \mathbf{PG}(7,s)$ for any point u of $\mathcal{H}$, one obtains a fully embedded generalized hexagon $\bar{\mathcal{H}}$ in $\mathbf{PG}(7,s)$. Let M be any line of $\mathcal{H}$. If M contains a point $u \notin \mathbf{PG}^{(1)}(6,s) \cup \mathbf{PG}^{(2)}(6,s)$, then, as π_u does not contain p, it easily follows that $M^* = \{u^* \parallel u \in M\}$ is a line of $\mathbf{PG}(7,s)$. Now assume that M is contained in $\mathbf{PG}^{(1)}(6,s) \cup \mathbf{PG}^{(2)}(6,s)$, say M is contained in $\mathbf{PG}^{(1)}(6,s)$. As the extension M' of M to $\mathbf{GF}(q)$ intersects $\mathbf{PG}^{(1)}(6,s)$ in M and $p \notin M$, we also have $p \notin M'$, and so $M^* = \{u^* \parallel u \in M\}$ is a line of $\mathbf{PG}(7,s)$. It follows that there arises a fully embedded generalized hexagon $\bar{\mathcal{H}}$ in $\mathbf{PG}(7,s)$. Let u^* be any point of $\bar{\mathcal{H}}$. Assume, by way of contradiction, that $u^{*\perp}$ is not contained in a plane of $\mathbf{PG}(7,s)$. Then π_u, with $\{u\} = \mathbf{PG}(6,q) \cap < y, u^* >$, contains p. From the foregoing, we necessarily have $u \in \mathbf{PG}^{(1)}(6,s) \cup \mathbf{PG}^{(2)}(6,s)$, with u not in $\mathcal{H}_1$ or $\mathcal{H}_2$. Let V be any line of $\mathcal{H}$ through u which does not contain a line of $\mathcal{H}_1$, nor p, and let V_1 be the line of $\mathcal{H}_1$ whose extension $\bar{V_1}$ to $\mathbf{GF}(s)$ contains a point a_1 of V. Let v_1 be a common point of V_1 and $\mathbf{PG}^{(1)}(5,t)$. Then ξ_{v_1} contains V and p, so contains π_u. It follows that $v_1 \in V$, so V contains a line of $\mathcal{H}_1$, a contradiction. We conclude that $u^{*\perp}$ is contained in a plane of $\mathbf{PG}(7,s)$. So $\bar{\mathcal{H}}$ is flatly full embedded in $\mathbf{PG}(7,s)$. As $\mathcal{H}$ is weakly embedded in $\mathbf{PG}(6,q)$, it is immediate that $\bar{\mathcal{H}}$ is weakly embedded in $\mathbf{PG}(7,s)$. Hence $\bar{\mathcal{H}}$ is regularly full embedded in $\mathbf{PG}(7,s)$. Then by Thas & Van Maldeghem [1996b] $\bar{\mathcal{H}}$ is a natural embedding of $T(t^3,t)$ in $\mathbf{PG}(7,s)$. Let $Q^*(7,s)$ be the quadric which contains the points of $\bar{\mathcal{H}}$. As the projection $\mathcal{H}$ of $\bar{\mathcal{H}}$ from y onto $\mathbf{PG}(6,q)$ is weakly embedded in $\mathbf{PG}(6,q)$, it follows that for any point u^* of $\bar{\mathcal{H}}$ the hyperplane ξ_{u^*} of $\mathbf{PG}(7,q)$ contains y. As ξ_{u^*} is the extension to $\mathbf{GF}(q)$ of the tangent hyperplane of $Q^+(7,s)$ at u^*, we have that $y \in \mathbf{PG}(7,s)$ and that $\bar{\mathcal{H}}$ is contained in a hyperplane, clearly a contradiction.

Now the theorem is completely proved. $\square$

Theorem 2.13 *If the thick generalized hexagon $\mathcal{H}$ of order (s,t) is regularly lax embedded in $\mathbf{PG}(7,q)$, then $s = t^3$ and $\mathcal{H}$ is a natural embedding of $T(s, \sqrt[3]{s})$ in a subspace $\mathbf{PG}(7,s)$ of $\mathbf{PG}(7,q)$ for some subfield $\mathbf{GF}(s)$ of $\mathbf{GF}(q)$.*

Proof. Let the thick generalized hexagon $\mathcal{H}$ of order (s,t) be regularly lax embedded in $\mathbf{PG}(7,q)$. As $\mathcal{H}$ is classical and $s \neq t$ by Theorem 2.7, we have $s = t^3$. Consider a subhexagon $\mathcal{H}'$ of order (t,t) of $\mathcal{H}$. Remark that for any point x of $\mathcal{H}'$ the space ξ_x does not contain a point y of $\mathcal{H}'$ opposite x. Then by Theorem 2.7 $\mathcal{H}'$ is a regular lax embedding of $H(t)$ in a hyperplane $\mathbf{PG}(6,q)$ of $\mathbf{PG}(7,q)$. Now by Theorem 2.9 $\mathcal{H}'$ is a natural embedding of $H(t)$ in a subspace $\mathbf{PG}(6,t)$ of $\mathbf{PG}(6,q)$ for some subfield $\mathbf{GF}(t)$ of $\mathbf{GF}(q)$.

By Corollary 2.11, any line L of $\mathcal{H}$ is a subline $\mathbf{PG}(1,s)$ of the line $L' = \mathbf{PG}(1,q)$ of $\mathbf{PG}(6,q)$ which contains L.

Consider a subhexagon $\mathcal{H}'$ of order (t,t) of $\mathcal{H}$ and the subspace $\mathbf{PG}(6,t)$ containing it. Let $\mathbf{PG}(6,s)$ be the 6-dimensional space over $\mathbf{GF}(s)$ containing $\mathbf{PG}(6,t)$. Then $\mathbf{PG}(6,s)$ contains all lines of $\mathcal{H}$ which intersect $\mathbf{PG}(6,t)$ in a line of $\mathcal{H}'$. Let Σ be an apartment of $\mathcal{H}'$. Then Σ is contained in a unique hyperplane $\mathbf{PG}(5,s)$ of $\mathbf{PG}(6,s)$. Now we consider a subhexagon $\mathcal{H}'' \neq \mathcal{H}'$ of order (t,t) of $\mathcal{H}$ which also contains the points of Σ. Then $\mathcal{H}''$ is a natural embedding of $H(t)$ in a subspace $\mathbf{PG}'(6,t)$; also $\mathbf{PG}'(6,t)$ extends uniquely to a $\mathbf{PG}'(6,s)$. The apartment Σ is contained in a unique hyperplane $\mathbf{PG}'(5,s)$ of $\mathbf{PG}'(6,s)$. As $\mathbf{PG}(5,s) \cap \mathbf{PG}'(5,s)$ contains Σ and also the six lines of $\mathcal{H}$ defined by Σ, we have that $\mathbf{PG}(5,s) = \mathbf{PG}'(5,s)$. If $\mathbf{PG}(6,s) \cap \mathbf{PG}'(6,s) \neq \mathbf{PG}(5,s)$, then $\mathbf{PG}(6,s)$ and $\mathbf{PG}'(6,s)$ belong to a common $\mathbf{PG}(6,q)$. As $\mathcal{H}'$ and $\mathcal{H}''$ belong to $\mathbf{PG}(6,q)$, it easily follows that

$\mathcal{H}$ belongs to $\mathbf{PG}(6,q)$, a contradiction. Consequently $\mathbf{PG}(6,s) \cap \mathbf{PG}'(6,s) = \mathbf{PG}(5,s)$. Also $\mathbf{PG}(6,s) \cup \mathbf{PG}'(6,s)$ generates $\mathbf{PG}(7,q)$, and so $\mathbf{PG}(6,s) \cup \mathbf{PG}'(6,s)$ is contained in a unique subspace $\mathbf{PG}(7,s)$ of $\mathbf{PG}(7,q)$.

Now let y be a point of $\mathcal{H}$ not in $\mathbf{PG}(6,s) \cup \mathbf{PG}'(6,s)$. If the line L of $\mathcal{H}$ contains y, then by Section 6.5 of Thas [1995] the subhexagon $\mathcal{H}'$ respectively $\mathcal{H}''$ contains exactly one line L' respectively L'' whose extension to $\mathbf{GF}(s)$ has a point z' respectively z'' in common with L. If for at least two distinct lines L of $\mathcal{H}$ through y the corresponding points z', z'' are distinct, then y clearly belongs to $\mathbf{PG}(7,s)$.

Now suppose that for at least t lines L of $\mathcal{H}$ through y we have $z' = z''$. For these lines $z' = z''$ belongs to $\mathbf{PG}(5,s)$. Let L be a line of $\mathcal{H}$ through y for which $z' = z''$. The hyperplane $\mathbf{PG}''(6,q) = \langle \mathbf{PG}(5,q), y\rangle$, with $\mathbf{PG}(5,s) \subseteq \mathbf{PG}(5,q)$, induces a subhexagon $\mathcal{H}'''$ of order (t,t) (if y is contained in $\mathbf{PG}(5,q)$, then y belongs to the subhexagon of order $(1,t)$ containing Σ, so y belongs to $\mathcal{H}'$ and hence to $\mathbf{PG}(5,s)$, a contradiction). Through each point of L not in $\mathcal{H}'''$ there is a line of $\mathcal{H}$ which is not contained in $\mathbf{PG}''(6,q)$, so which does not contain a point of $\mathbf{PG}(5,s)$. From the foregoing it follows that each of these $t^3 - t$ points of L are contained in $\mathbf{PG}(7,s)$. Hence L, and consequently y, is contained in $\mathbf{PG}(7,s)$.

We conclude that $\mathcal{H}$ is fully embedded in $\mathbf{PG}(7,s)$. As it is clear that $\mathcal{H}$ is regularly embedded in $\mathbf{PG}(7,s)$, we conclude that $\mathcal{H}$ is a natural embedding of $T(t^3,t)$ in $\mathbf{PG}(7,s)$. $\square$

3 Flat full embeddings

In this section, we assume that the thick generalized hexagon $\mathcal{H}$ of order (q,t) is flatly and fully embedded in a projective space $\mathbf{PG}(d,q)$.

For any point x of $\mathcal{H}$, we denote by π_x the plane in $\mathbf{PG}(d,q)$ generated by the points of $\mathcal{H}$ polycollinear with x.

Theorem 3.1 *We have $4 \leq d \leq 7$. Also, we have $t \leq q$, and if $d = 7$, then $\mathcal{H} \cong T(q, \sqrt[3]{q})$.*

Proof. Clearly $d \geq 4$ because the number of points of $\mathcal{H}$, which is equal to $(1+q)(1+tq+t^2q^2)$, is always larger than the number q^3+q^2+q+1 of points in $\mathbf{PG}(3,q)$. Since the number of lines through a point in a plane of $\mathbf{PG}(d,q)$ is equal to the number $q+1$ of points on a line of $\mathbf{PG}(d,q)$, it immediately follows that $t \leq q$.

The proof of the fact that $d \leq 7$ in Theorem 2.7 can be copied here. Also, that proof reveals that in case $d = 7$ every window is contained in a proper ideal subhexagon. By De Smet & Van Maldeghem [1993], $\mathcal{H}$ is isomorphic to $T(q, \sqrt[3]{q})$. $\square$

We now take a closer look at the dimensions $d = 5, 6, 7$.

Theorem 3.2 *If $d = 5$ and $q = t$, then the flat embedding is a regular embedding and hence a natural embedding of $H(q)$.*

Proof. If $d = 5$ and $q = t$, then the point set of the flatly embedded hexagon $\mathcal{H}$ concides with the point set of $\mathbf{PG}(5, q)$. Consider any hyperplane U of $\mathbf{PG}(5, q)$. We show that there are exactly $q^3 + q^2 + q + 1$ lines of $\mathcal{H}$ in U. For any point x of U, either all lines of $\mathcal{H}$ incident with x lie in U, or exactly one such line lies in U. Let a be the number of points x of U for which $x^\perp$ is contained in U and let b be the number of points of U for which this does not hold. Then

$$a + b = q^4 + q^3 + q^2 + q + 1.$$

Also, the number of lines of $\mathcal{H}$ in U, respectively not in U, is equal to

$$\frac{a(q+1)+b}{q+1}, \text{ respectively } bq.$$

Hence we have

$$\frac{a(q+1)+b}{q+1} + bq = q^5 + q^4 + q^3 + q^2 + q + 1.$$

Solving the system of equations thus obtained, we obtain $a = q^2 + q + 1$ and $b = q^4 + q^3$. Hence the number of lines of $\mathcal{H}$ in U is

$$\frac{a(q+1)+b}{q+1} = q^3 + q^2 + q + 1.$$

We claim that every apartment spans $\mathbf{PG}(5, q)$. Indeed, if not, then by 6.5 of Thas [1995] there is a full subhexagon of order $(1, q)$ in a hyperplane. But such a subhexagon contains $(q + 1)(q^2 + q + 1)$ lines, contradicting the previous paragraph. The claim is proved.

Now consider two opposite lines L_1 and L_2 of $\mathcal{H}$, a point x at distance 3 from both L_1 and L_2, and two points y_1 and y_2 at distance 4 from x, at distance 4 from each other and such that y_i is on L_i, $i = 1, 2$. Since every apartment spans $\mathbf{PG}(5, q)$, the elements y_1, L_1, x, L_2, y_2 span a hyperplane U. Suppose that U contains a point v of $\mathcal{H}$ opposite x. Then U contains a line $M = \pi_v \cap U$ of $\mathcal{H}$ at distance 5 from x. Let z be polycollinear with both x and y_1. By possibly interchanging the roles of y_1 and y_2, we may suppose that M is at distance 5 from z. We now consider the apartment Σ defined by x, M, z. Since $\pi_x \subseteq U$, $\pi_z \subseteq U$ and $M \subseteq U$, we conclude that Σ is in U, a contradiction. Hence no point opposite x is in U. As there are exactly q^5 points of $\mathcal{H}$ opposite x, the set of these points coincides with $\mathbf{PG}(5, q) \setminus U$. So U is the set of the $q^4 + q^3 + q^2 + q + 1$ points of $\mathcal{H}$ which are not opposite x. Hence the embedding is regular and the result follows. □

Theorem 3.3 *If $d = 6$ and if every hyperplane of* $\mathbf{PG}(6, q)$ *containing an apartment of* $\mathcal{H}$ *induces a non-thick subhexagon (this happens for instance automatically if $q^3 < t^5$, in particular if $q = t$), then the flat embedding is a regular embedding and hence a natural embedding of $H(q)$.*

Proof. Consider any point x of $\mathcal{H}$ and let L and M be two opposite lines at distance 3 from x. For every point y on L, y not collinear with x, there exists a unique apartment Σ_y containing x, L, M, y. Let U_y be the projective subspace generated by Σ_y. If U_y had dimension ≤ 4, then, by Lemma 2.1, U_y would be contained in a hyperplane inducing a subhexagon of order (q, q), a contradiction. So U_y is a hyperplane of $\mathbf{PG}(6, q)$. It is clear that $U = \langle x, L, M\rangle$ is a projective 4-space. Hence U_y is a hyperplane in $\mathbf{PG}(6, q)$ containing U. Remark that distinct points y define distinct hyperplanes U_y as otherwise U_y would not induce a subhexagon of order $(1, t)$. Since there are q choices for y, there is a unique hyperplane U_∞ containing x, L, M and not containing any point opposite x at distance 3 from both L and M. Now consider any line N of $\mathcal{H}$ at distance 3 from x and distance 4 from L or M. Then N is not contained in any U_y since otherwise U_y does not induce a subhexagon of order $(1, t)$. Hence N is contained in U_∞. It easily follows that U_y does not contain any apartment of $\mathcal{H}$. Note that, if z is the point on L polycollinear with x, then also all lines of $\mathcal{H}$ through z are contained in U_∞. Now we note that U_∞ is in fact uniquely defined by x, L^*, M^*, where L^* is the line of $\mathcal{H}$ incident with x and z, and M^* is the line of $\mathcal{H}$ through x meeting M. Hence we may rewrite U_∞ as U_{x,L^*,M^*}. Put $L^* = L_0$ and let $\{L_i : i \in \{0, 1, 2, \ldots, t\}\}$ be the set of lines of $\mathcal{H}$ through x, with $L_t = M^*$. Also, let $\{x_j : j \in \{1, 2, \ldots, q\}\}$ be the set of points of $\mathcal{H}$ on M^*, different from x. For each point x_j, we choose a line N_j through x_j different from M^*. We put $x = x_0$ and $N_0 = L_k$, for some arbitrary $k \in \{0, 1, \ldots, t-1\}$. If $U_{x_j,N_j,M^*} = U_{x_\ell,N_\ell,M^*}$, $j \neq \ell$, then U_{x_j,N_j,M^*} contains an apartment through N_J and N_ℓ, a contradiction. In particular, U_{x_0,N_0,M^*} is distinct from the q different hyperplanes U_{x_j,N_j,M^*}, $j = 1, 2, \ldots, q$, which all contain the 4-dimensional space generated by the points of $\mathcal{H}$ at distance 3 from M^* (this is indeed a 4-dimensional space since it is contained in at least two different hyperplanes, and since by adding a line opposite M^*, one generates a subspace inducing $\mathcal{H}$ itself). In particular U_{x_0,N_0,M^*} does not depend on k and so it follows that U_{x,L^*,M^*} only depends on x and contains all lines at distance ≤ 3 from x. Consequently Axiom (W) holds and we have a full regular embedding. The theorem is proved. $\square$

We now prove a lemma for flatly lax embedded hexagons. We will need this weaker form in the next section.

Lemma 3.4 *If $\mathcal{H}$ is a thick generalized hexagon of order (s, t) which is flatly lax embedded in* $\mathbf{PG}(7, q)$*, then $s = t^3$, $\mathcal{H} \cong T(s, t)$ and every distance-2-trace of $\mathcal{H}$ is contained in a line of* $\mathbf{PG}(7, q)$*.*

Proof. Consider an apartment Σ of $\mathcal{H}$. The six points of Σ span a $\mathbf{PG}(m,q)$, with $m \le 5$. Now consider a line L of $\mathcal{H}$ meeting a line of Σ but not containing a point of Σ. Then L and Σ span a $\mathbf{PG}(m',q)$, with $m' \le m+1 \le 6$. The space $\mathbf{PG}(m',q)$ induces a subhexagon $\mathcal{H}'$ of order (s',t), $s' \ge 2$. If $s' = s$, then $\mathcal{H}' = \mathcal{H}$ and $m' = 7$, a contradiction. Hence $s' < s$. Next we consider a line M of $\mathcal{H}$, with $L \ne M$, and containing a point of L not in $\mathcal{H}'$. Then M and $\mathbf{PG}(m',q)$ generate a $\mathbf{PG}(m'',q)$, with $m'' \le m'+1 \le 7$. The space $\mathbf{PG}(m'',q)$ induces a subhexagon $\mathcal{H}''$ of order (s'',t), $s' < s''$. If $s'' < s$, then it follows from Thas [1976] that $s \ge {s''}^2 t, s'' \ge {s'}^2 t$, and hence $s \ge {s'}^4 t^3$. Now by Haemers & Roos [1981], we have $s \le t^3$. This implies $s = t^3$ and $s' = 1$, a contradiction. Hence $s'' = s, \mathcal{H}'' = \mathcal{H}$, and $m'' = 7$. Also, $m' = 6$ and $m = 5$. So every apartment of $\mathcal{H}$ generates a 5-dimensional space. The subspace $\mathbf{PG}(m,q) = \mathbf{PG}(5,q)$ induces a subhexagon $\mathcal{H}^*$ of order (s^*,t). As $\mathbf{PG}(5,q) = \mathbf{PG}(m,q) \ne \mathbf{PG}(m',q) = \mathbf{PG}(6,q)$ we have $\mathcal{H}^* \ne \mathcal{H}'$, and so $s^* < s'$. Now it follows from Thas [1976] that $s \ge {s'}^2 t, s' \ge {s^*}^2 t$, and hence $s \ge {s^*}^4 t^3$. Now by Haemers & Roos [1981], we have $s \le t^3$. This implies $s = t^3, s^* = 1$ and $s' = t$.

The previous paragraph shows that every window of $\mathcal{H}$ is contained in a proper ideal subhexagon. Hence, by De Smet & Van Maldeghem [1993], $\mathcal{H}$ is isomorphic to the classical hexagon $T(t^3,t)$.

We now prove that each distance-2-trace of $\mathcal{H}$ is subset of a line of $\mathbf{PG}(7,q)$. Suppose that x and y are opposite points in $\mathcal{H}$ and that x^y is not contained in a line of $\mathbf{PG}(7,q)$. Then x^y spans the plane π_x. Let K_1, K_2, K_3 be 3 lines of $\mathcal{H}$ at distance 3 from both x and y, and suppose that the points y_1, y_2, y_3 nearest y on K_1, K_2, K_3 respectively, are on a line N of $\mathbf{PG}(7,q)$. As $\mathcal{H}$ satisfies the regulus condition (see Ronan [1980]), there exists a point z of $\mathcal{H}$ at distance 3 from each of K_1, K_2, K_3, where $z \notin \{x,y\}$. If the points z_1, z_2, z_3 nearest z on K_1, K_2, K_3 respectively are contained in a line N' of $\mathbf{PG}(7,q)$, then $\langle N, N' \rangle = \langle K_1, K_2, K_3 \rangle$ is at most 3-dimensional, so $\langle K_1, K_2, K_3, x, y \rangle$ is at most 4-dimensional, a contradiction as this last space contains an apartment. It follows that z_1, z_2, z_3 are not on a common line of $\mathbf{PG}(7,q)$. Hence z_1, z_2, z_3 span the plane π_z. As $\langle K_1, K_2, K_3 \rangle$ is at most 4-dimensional, we then have that $\langle K_1, K_2, K_3, x, z \rangle = \langle K_1, K_2, K_3 \rangle$ is at most 4-dimensional. As $\langle K_1, K_2, K_3, x, z \rangle$ contains an apartment we have again a contradiction. It follows that y_1, y_2, y_3 are not collinear. If $\mathbf{PG}(r,q) = \langle K_1, K_2, K_3 \rangle$, then it is now clear that this space contains all points at distance 3 from each of K_1, K_2, K_3. Clearly $r \le 5$. Hence $\mathbf{PG}(r,q)$ induces an ideal subhexagon $\mathcal{H}'$ containing all points of K_1. Consequently $\mathcal{H}' = \mathcal{H}$, contradicting $r \le 5$. We conclude that each trace of $\mathcal{H}$ is a subset of a line of $\mathbf{PG}(7,q)$. $\square$

Theorem 3.5 *If $\mathcal{H}$ is a thick generalized hexagon of order (q,t) which is flatly full embedded in $\mathbf{PG}(7,q)$, then $q = t^3$ and $\mathcal{H}$ is a natural embedding of $T(q, \sqrt[3]{q})$ in $\mathbf{PG}(7,q)$.*

Proof. By Lemma 3.4, $\mathcal{H}$ is a classical hexagon isomorphic to $T(q, \sqrt[3]{q})$, and each distance-2-trace is contained in some line of $\mathbf{PG}(7,q)$. Let x be any point of $\mathcal{H}$. If $\mathcal{H}'$ is any subhexagon of order (t,t), then $\mathcal{H}' \cong H(t)$. The lines and distance-2-traces of $\mathcal{H}'$ are the lines of a polar space $Q(6,t)$ which, by the proof of Lemma 3.4, is laxly embedded in some $\mathbf{PG}(6,q)$. Hence by THAS & VAN MALDEGHEM [19**a], the polar space is fully embedded in some subspace $\mathbf{PG}(6,t)$ of $\mathbf{PG}(6,q)$, and so the set of points $x^{\perp\!\!\perp}$ of $\mathcal{H}'$ at distance ≤ 4 from x is contained in a 5-dimensional space U. Let y be any point of $\mathcal{H} \setminus \mathcal{H}'$ polycollinear with x and let L be the line of $\mathcal{H}$ incident with x and y. Then π_y is not contained in U, otherwise $\mathbf{PG}(6,q)$ induces $\mathcal{H}$, a contradicton. Hence $V := \langle U, \pi_y \rangle$ is a hyperplane of $\mathcal{H}$. Now let z_1 and z_2 be two distinct polycollinear points of $\mathcal{H}'$ polycollinear with x, but not with y (in $\mathcal{H}$). Let $\mathcal{H}''$ be a subhexagon of order (t,t) containing z_1, z_2, y. The points of $\mathcal{H}''$ at distance ≤ 4 from x are contained in a 5-dimensional space U' which is generated by $\pi_{z_1}, \pi_{z_2}, \pi_y$ (as $\mathcal{H}''$ is naturally embedded in some subspace $\mathbf{PG}'(6,t)$ of $\mathbf{PG}(6,q)$). This implies that $U' \subseteq V$. Let R be the set of points of $x^\perp$ with the property: $z \in R$ if $\pi_z \subseteq V$. Now let z_1', z_2' be distinct polycollinear points of R which are polycollinear with x. Further, let y' be a point of R polycollinear with x, but not with z_1' nor with z_2'. If $\mathcal{H}'''$ is a subhexagon of order (t,t) containing $z_1', z_2', , y'$, then again all points of $\mathcal{H}'''$ at distance ≤ 4 from x are contained in V.

Now the geometry of distance-2-traces contained in the set $x^\perp$ of points polycollinear with x is a dual net $\mathcal{N}$ which clearly satisfies Veblen's axiom (indeed, any two distinct intersecting traces generate in $\mathcal{N}$ a dual affine plane of order t, see RONAN [1980]), hence, by THAS & DE CLERCK [1977], the dual net $\mathcal{N}$ is isomorphic to the dual net H_t^q. So the points of $x^\perp$ are the points of a 4-dimensional projective space $\mathbf{PG}(4,t)$ (not related to $\mathbf{PG}(6,q)$) off a plane $\mathbf{PG}(2,t)$ of $\mathbf{PG}(4,t)$, the traces in $x^\perp$ are the lines in $\mathbf{PG}(4,t)$ skew to $\mathbf{PG}(2,t)$, and incidence is the natural one. The points in that model corresponding to the points of $\mathcal{H}'$ polycollinear with x are the points of a dual affine plane π in $\mathbf{PG}(4,t)$ (the projective plane defined by π contains exactly one point of $\mathbf{PG}(2,t)$). The point y is a point off π, and so π and y generate in $\mathcal{N}$ a dual net whose point set is of the form $\mathbf{PG}(3,t) \setminus \mathbf{PG}(2,t)$, with $\mathbf{PG}(3,t)$ some hyperplane of $\mathbf{PG}(4,t)$ (remark that $\mathbf{PG}(3,t) \cap \mathbf{PG}(2,t)$ is a line). From the foregoing paragraph it now follows that all points of $x^\perp$ which are points of $\mathbf{PG}(3,t)$, are contained in R. Let R' be the set of points of $x^\perp$ contained in $\mathbf{PG}(3,t)$. So we have $R' \subseteq R$. Every line of $\mathcal{H}$ through x contains exactly t^2 elements of R' and R' is closed with respect to collinearity in $\mathcal{N}$.

Now let w be a point polycollinear with x but not contained in R'. There are exactly t^3 traces through w in $x^\perp$. Every such trace contains at most one point of R'. On the other hand, there are t^3 points of R' not polycollinear with w, and so there are at least t^3 traces which do contain a point of R'. It follows that every trace in $x^\perp$ through w contains a unique element of R'. Now let N be a line at distance 3 from w, but not concurrent with the line wx. Let u be the unique point of N polycollinear with w. Let u' be any other point of N and suppose that V contains u'. The trace $x^{u'}$ contains w and hence it contains some unique point z of R'. Since π_z is contained in V, the point z' polycollinear with both z and u' is contained in V, and hence so is the line $u'z'$. Consideration of any point u'' on

$u'z'$ at distance 4 from any point z'' of R' not polycollinear with z leads to an apartment in V, a contradiction as otherwise V would induce $\mathcal{H}$. Hence u' is not contained in V. So the hyperplane V meets the line N necessarily in the point u. Consequently the line uw belongs to V and hence $\pi_w \subseteq V$.

We have shown that the embedding is regular and the theorem now follows from THAS & VAN MALDEGHEM [1996b]. $\square$

4 Flat lax embeddings

In this section, we assume that $\mathcal{H}$ is a thick generalized hexagon flatly lax embedded in $\mathbf{PG}(d,q)$.

For any $x \in P$, with P the point set of $\mathcal{H}$, we denote by π_x the unique plane in $\mathbf{PG}(d,q)$ spanned by all points polycollinear with x.

Theorem 4.1 *If $\mathcal{H}$ is a thick generalized hexagon which is flatly lax embedded in the projective space $\mathbf{PG}(d,q)$, then $d \leq 7$.*

Proof. See proof of Theorem 2.7. $\square$

We first deal with $d = 7$ and with the smallest possible case $(s,t) = (8,2)$.

Theorem 4.2 *If $\mathcal{H}$ is generalized hexagon of order $(8,2)$ which is flatly lax embedded in $\mathbf{PG}(7,q)$, then $\mathcal{H}$ is a natural embedding of $T(8,2)$ in a subspace $\mathbf{PG}(7,8)$ of $\mathbf{PG}(7,q)$ (in particular $\mathbf{GF}(8)$ is a subfield of $\mathbf{GF}(q)$).*

Proof. By Lemma 3.4, $\mathcal{H}$ is a classical hexagon isomorphic to $T(8,2)$, and all distance-2-traces are subsets of lines of $\mathbf{PG}(7,q)$. Note that for any point x of $\mathcal{H}$ and any subhexagon $\mathcal{H}'$ of order $(2,2)$ containing x, the geometry of distance-2-traces of $\mathcal{H}'$ in $x^\perp$ together with the lines of $\mathcal{H}'$ through x, is a projective plane of order 2 which is embedded in some $\mathbf{PG}(2,q)$. Hence q is even.

Now let L be any line of $\mathcal{H}$ containing the points x_i, $i = 0,1,2,\ldots,8$, and choose coordinates in $\mathbf{PG}(7,q)$ in such a way that $x_0 = (0,1,0^6)$, $x_1 = (1,0^7)$, $x_2 = (1,1,0^6)$ and $x_3 = (1,a,0^6)$, for some $a \in \mathbf{GF}(q)$, where 0^i is an abbreviation for $0,0,\ldots,0$ (i zeros). If L_1 and L_2 are the other two lines of $\mathcal{H}$ through x_0, then we can choose coordinates in such a way that the points $y_1 = (1,0,1,0^5)$ and $y_2 = (1,1,1,0^5)$ belong to L_1, and $z_1 = (0,0,1,0^5)$ and $z_2 = (0,1,1,0^5)$ belong to L_2. Expressing that the traces in $x_0^\perp$ are subsets of lines of the plane $\langle L, L_1\rangle$, we see that we may assume that $x_4 = (1, a+1, 0^6)$ ($\{x_4\} = y_2r \cap L$ with $\{r\} = L_2 \cap y_1x_3$). This means that, whenever u,v,w are three points of $\mathcal{H}$ on L, and $\mathbf{PG}(1,q)$ is the line of $\mathbf{PG}(7,q)$ containing L, then the translation σ of $\mathbf{PG}(1,q)$ fixing u and mapping v to w preserves the set of points of $\mathcal{H}$ on $\mathbf{PG}(1,q)$. It is easily seen, however, that, considering the natural embedding $\widetilde{\mathcal{H}}$ of $\mathcal{H}$ in some $\mathbf{PG}'(7,8)$, with such a translation σ there corresponds a translation $\widetilde{\sigma}$ of the line $\widetilde{L}$ of $\widetilde{\mathcal{H}}$ which corresponds to L. Hence, varying x_0 over L, these translations generate $\mathbf{PGL_2}(8)$, which consequently is a subgroup of $\mathbf{PGL_2}(q)$ and which has an orbit of length 9 (namely, the points of L) in $\mathbf{PG}(1,q)$. Hence, by Lemma 3 of THAS & VAN MALDEGHEM [19**b], L is a projective subline of $\mathbf{PG}(1,q)$ over the field $\mathbf{GF}(8)$. Now the proof of Theorem 2.13, from the third paragraph on, can be copied to show that $\mathcal{H}$ is flatly full embedded in a subspace $\mathbf{PG}(7,8)$ of $\mathbf{PG}(7,q)$. By Theorem 3.5, $\mathcal{H}$ is a natural embedding in $\mathbf{PG}(7,8)$. $\square$

Lemma 4.3 *For $t \neq 2$, the $3-(t^3+1,t+1,1)$ design $\mathcal{C}$ formed by the points of $\mathbf{PG}(1,t^3)$ together with the sublines of $\mathbf{PG}(1,t^3)$ over $\mathbf{GF}(t)$, is generated by a block $\mathbf{PG}(1,t)$ and a point y of $\mathbf{PG}(1,t^3)$ not in $\mathbf{PG}(1,t)$.*

Proof. Let $\mathbf{GF}(t^3) = \{a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 \parallel a_i \in \mathbf{GF}(t)\}$, with $\alpha_1, \alpha_2, \alpha_3$ suitable elements of $\mathbf{GF}(t^3)$. On $\mathbf{PG}(1,t^3)$ we choose affine coordinates in such a way that $y = (\infty)$. If the point $z \neq y$ has coordinate (a), $a \in \mathbf{GF}(t^3)$ and $a = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3$, with $a_1, a_2, a_3 \in \mathbf{GF}(t)$, then we put $z^\theta = (a_1, a_2, a_3) \in \mathbf{AG}(3,t)$, which unambiguously defines the map $\theta : \mathbf{PG}(1,t^3) \setminus \{y\} \to \mathbf{AG}(3,t)$. If D is a block of $\mathcal{C}$ containing y, then $(D \setminus \{y\})^\theta$ is an affine line of $\mathbf{AG}(3,t)$; so with the $t^2(t^2+t+1)$ blocks of $\mathcal{C}$ containing y there correspond the $t^2(t^2+t+1)$ affine lines of $\mathbf{AG}(3,t)$. With the $t^3(t^3-1)$ blocks D of $\mathcal{C}$ not containing y, there correspond the $t^3(t^3-1)$ twisted cubics of $\mathbf{PG}(3,t)$, with $\mathbf{PG}(3,t)$ the projective completion of $\mathbf{AG}(3,t)$, which contain 3 fixed non-collinear points c_1, c_2, c_3 of the extension $\mathbf{PG}(2,t^3)$ of the plane at infinity $\mathbf{PG}(2,t)$ of $\mathbf{AG}(3,t)$, where c_1, c_2, c_3 are conjugate with respect to the cubic extension $\mathbf{GF}(t^3)$ of $\mathbf{GF}(t)$. Let $\mathbf{PG}(1,t)^\theta$ be the twisted cubic C. If V is the point set of $\mathcal{C}$ generated by $\mathbf{PG}(1,t)$ and y, then for any two points z_1 and z_2 of V^θ the affine line z_1z_2 belongs to V^θ. Hence V^θ is an affine subspace of $\mathbf{AG}(3,t)$. As V^θ contains the twisted cubic C, we clearly have $V^\theta = \mathbf{AG}(3,t)$. We conclude that $\mathcal{C}$ is generated by $\mathbf{PG}(1,t)$ and y. $\square$

Theorem 4.4 *If $\mathcal{H}$ is a thick generalized hexagon of order (s,t) which is flatly lax embedded in $\mathbf{PG}(7,q)$, then $s = t^3$ and $\mathcal{H}$ is a natural embedding of $H(s, \sqrt[3]{s})$ in a subspace $\mathbf{PG}(7,s)$ of $\mathbf{PG}(7,q)$ for some subfield $\mathbf{GF}(s)$ of $\mathbf{GF}(q)$.*

Proof. By Lemma 3.4, $\mathcal{H}$ is a classical hexagon isomorphic to $T(s,t)$, with $s = t^3$. By Theorem 4.2, we may assume that $t > 2$. Let $\mathcal{H}'$ be a subhexagon of order (t,t) of $\mathcal{H}$. By the first paragraph of the proof of Lemma 3.4, the points of $\mathcal{H}'$ span a $\mathbf{PG}(6,q)$. By the second part of the proof of Lemma 3.4, the points of $\mathcal{H}'$ together with the lines and traces of $\mathcal{H}'$ form a polar space $Q(6,t)$ which is laxly embedded in $\mathbf{PG}(6,q)$. Hence, by Thas & Van Maldeghem [19**a], $Q(6,t)$ is fully embedded in some subspace $\mathbf{PG}(6,t)$ of $\mathbf{PG}(6,q)$. It follows that $\mathcal{H}'$ is regularly embedded in $\mathbf{PG}(6,t)$. Now by Cameron & Kantor [1979], see also Thas & Van Maldeghem [1996b], the subhexagon $\mathcal{H}'$ is a natural embedding of $H(t)$ in $\mathbf{PG}(6,t)$.

We now show that for any point x of $\mathcal{H}$ the set of all points of $\mathcal{H}$ not opposite x is contained in a hyperplane of $\mathbf{PG}(7,q)$. Let $L_1, L_2, \cdots, L_{t+1}$ be the lines of $\mathcal{H}$ containing x, let $\mathcal{H}'$ be a subhexagon of order (t,t) containing x, and let L_1^* be the line of $\mathcal{H}'$ contained in L_1. Further, let $y \in L_1 \setminus L_1^*$. As $\mathcal{H}'$ is a natural embedding of $H(t)$ in a subspace $\mathbf{PG}(6,t)$ of $\mathbf{PG}(6,q)$, the planes π_z, with z any point of $\mathcal{H}'$ polycollinear with x, are contained in a common $\mathbf{PG}(5,q)$. By the first paragraph of the proof of Lemma 3.4, no line $M \neq L_1$ of $\mathcal{H}$ containing y is contained in $\mathbf{PG}(6,q)$. Hence $\langle \mathbf{PG}(5,q), \pi_y \rangle$ is a 6-dimensional space which will be denoted by ξ_x. Consider points z_1, z_2, with $z_1 \neq z_2$, on L_1^*. Then y, z_1, z_2 are contained in a subhexagon $\mathcal{H}''$ of order (t,t). Let L_1^{**} be the line of $\mathcal{H}''$ contained in L_1. As $\mathcal{H}''$ is a natural embedding of $H(t)$ in some 6-dimensional space over $\mathbf{GF}(t)$, the planes $\pi_{z_1}, \pi_{z_2}, \pi_y$ span a $\mathbf{PG}(4,q)$ and moreover π_z, with $z \in L_1^{**}$, is a plane of $\mathbf{PG}(4,q)$. As $\mathbf{PG}(4,q) \subset \xi_x$, we have $\pi_z \subset \xi_x$. Let us consider the 3-$(t^3+1, t+1, 1)$ design $\mathcal{D}$ with point set L_1 and having as blocks the subsets of L_1 which are the lines of the subhexagons of order (t,t) containing a point of L_1. As $\mathcal{H}$ is classical , $\mathcal{D}$ is isomorphic to the design with point set $\mathbf{PG}(1,t^3)$ and having as blocks the lines over $\mathbf{GF}(t)$ contained in $\mathbf{PG}(1,t^3)$. By Lemma 4.3, for $t \neq 2$ the design $\mathcal{D}$ is generated by the block L_1^* and the point y. It immediately follows that π_z, with z any point of L_1, is a plane of ξ_x. Now we consider any point z of $\mathcal{H}$, $z \notin L_1$, polycollinear with x. Let z_1' and z_2' be distinct points of $\mathcal{H}'$, with z_1' and z_2' polycollinear with x, where $z_1' \notin L_1$ and $z_2' \notin L_1$, with z_1' and z_2' not polycollinear with z, and with z_1' polycollinear with z_2'. Then there is a subhexagon $\mathcal{H}''$ of order (t,t) containing z, x, z_1', z_2'. The trace defined by z and z_i' contains a point z_i'' of $L_1, i = 1,2$. The 5-dimensional space containing any plane π_u, with u any point of $\mathcal{H}''$ polycollinear with x, is spanned by $\pi_{z_1'}, \pi_{z_2'}, \pi_x, \pi_{z_1''}$. As $\pi_{z_1'}, \pi_{z_2'}, \pi_x, \pi_{z_1''}$ are contained in ξ_x, also π_z is contained in ξ_x. Consequently, for any point x of $\mathcal{H}$ the set of all points of $\mathcal{H}$ not opposite x is contained in a hyperplane ξ_x of $\mathbf{PG}(7,q)$.

We conclude that $\mathcal{H}$ is regularly lax embedded in $\mathbf{PG}(7,q)$, and so by Theorem 2.13, $\mathcal{H}$ is a natural embedding of $T(s, \sqrt[3]{s})$ in some subspace $\mathbf{PG}(7,t)$ of $\mathbf{PG}(7,q)$ for some subfield $\mathbf{GF}(t)$ of $\mathbf{GF}(q)$. $\square$

Theorem 4.5 *If $\mathcal{H}$ is a generalized hexagon of order (s,t), with $s \neq t^3$, isomorphic to a classical or dual classical generalized hexagon, which is flatly lax embedded in $\mathbf{PG}(6,q)$, then $\mathcal{H}$ is a natural embedding of $H(s)$ in a subspace $\mathbf{PG}(6,s)$ of $\mathbf{PG}(6,q)$ for some subfield $\mathbf{GF}(s)$ of $\mathbf{GF}(q)$.*

Proof. The generalized hexagon $\mathcal{H}$ clearly has a proper subhexagon of order (s',t). By Thas [1976] we have $s \geq s'^2 t$, and so $(s,t) \neq (s,s^3)$. Hence $s = t$.

Let Σ be an apartment of the hexagon $\mathcal{H}$. If Σ is contained in a $\mathbf{PG}(4,q)$, then $\mathbf{PG}(4,q)$ induces a subhexagon $\mathcal{H}'$ of order (s',s), $s'<s$. If L is a line of $\mathcal{H}$ concurrent in $\mathcal{H}$ with a line of Σ but not containing a point of $\mathcal{H}'$, then $\mathbf{PG}(4,q)$ and L generate a $\mathbf{PG}(5,q)$. The space $\mathbf{PG}(5,q)$ induces a subhexagon $\mathcal{H}''$ of order (s'',s), with $s'<s''<s$. By Thas [1976], we have $1 \geq s''^2$ and $s'' \geq s'^2 s$, clearly a contradiction. Consequently Σ generates a hyperplane $\mathbf{PG}(5,q)$ of $\mathbf{PG}(6,q)$.

Suppose that x and y are opposite points in $\mathcal{H}$. As in the proof of Lemma 3.4 we show that x^y is contained in a line of $\mathbf{PG}(6,q)$. It immediately follows that every trace of $\mathcal{H}$ is determined by two of its elements, and so $\mathcal{H} \cong H(s)$.

Then the points of $\mathcal{H}$ together with the lines and traces of $\mathcal{H}$ form a polar space $Q(6,s)$ which is laxly embedded in $\mathbf{PG}(6,q)$. Now by Thas & Van Maldeghem [19**a] $Q(6,s)$ is fully embedded in a subspace $\mathbf{PG}(6,s)$ of $\mathbf{PG}(6,q)$, for some subfield $\mathbf{GF}(s)$ of $\mathbf{GF}(q)$. It immediately follows that $\mathcal{H}$ is regularly embedded in $\mathbf{PG}(6,s)$, and so, by Thas & Van Maldeghem [1996b], $\mathcal{H}$ is a natural embedding of $H(s)$ in $\mathbf{PG}(6,s)$. $\square$

5 Weak embeddings

For weak embeddings of hexagons, we do not see a way at the moment to bound the dimension of the projective space. But we are able to classify all weak lax embeddings of thick generalized hexagons in dimension at most 5, and all weak full embeddings of thick generalized hexagons of order (s,t) with s odd in dimension 6.

For a weakly lax embedded hexagon $\mathcal{H}$ in $\mathbf{PG}(d,q)$, and for any point x of $\mathcal{H}$, we denote by ξ_x the subspace of $\mathbf{PG}(d,q)$ generated by $x^{\perp\!\!\perp}$. By assumption, this is a proper subspace, i.e., $\xi_x \neq \mathbf{PG}(d,q)$.

Theorem 5.1 *If $\mathcal{H}$ is a thick generalized hexagon which is weakly lax embedded in the projective space $\mathbf{PG}(d,q)$, $d \leq 5$, then $\mathcal{H}$ is a regular lax embedding of a classical hexagon $H(s)$, with s even, and hence a natural embedding in some subspace $\mathbf{PG}(5,s)$ over the subfield $\mathbf{GF}(s)$ of $\mathbf{GF}(q)$.*

Proof. First suppose that the order (s,t) of $\mathcal{H}$ is distinct from $(2,2)$. Let x be any point of $\mathcal{H}$ and let y be any point opposite x. Then it is immediately clear that all points z of $\mathcal{H} \setminus x^{\perp\!\!\perp}$ for which there exist points $y=u_0,u_1,\ldots,u_i=z$ opposite x, with $u_j \in u_{j-1}^{\perp}$, $1 \leq j \leq i$, belong to $\langle \xi_x, y\rangle$. By Brouwer [1993], all points z of $\mathcal{H}$ opposite x qualify, hence $\mathcal{H}$ is contained in $\langle \xi_x, y\rangle$. This shows that no point of $\mathcal{H}$ opposite x is contained in ξ_x, and that ξ_x is a hyperplane.

Now let x,y,z be three points of $\mathcal{H}$, with y and z polycollinear with x, and y at distance 4 from z. Since there are points of $\mathcal{H}$ opposite z which are not opposite x and not opposite

y, we easily see that $\xi_x \cap \xi_y \cap \xi_z$ is a subspace of dimension $d-3$. Since it contains the $t+1$ lines of $\mathcal{H}$ through x, it must be a plane, and so we have shown that $d=5$ and the embedding is flatly lax, hence regularly lax and we can apply Theorem 2.8 to finish the proof.

Now let $(s,t)=(2,2)$. Then $\mathcal{H}$ is either isomorphic to $H(2)$, or to its dual $H(2)^D$, see Tits [1959] or Cohen & Tits [1985]. If $\mathcal{H} \cong H(2)$, then we can again rely on Brouwer [1993] and copy the arguments of the previous paragraphs. Now let $\mathcal{H} \cong H(2)^D$. Let x,y,z be as above. Since the geometry $\mathcal{H}^{(y)}$ of points and lines at distance ≥ 5 from y has two components (see Brouwer [1993]), we see that ξ_y is either $(d-1)$-dimensional or $(d-2)$-dimensional. Suppose that $\xi_y \subseteq \xi_z$. There are at least 16 points of $\mathcal{H}$ in ξ_y opposite z, hence exactly 16 points as each connected component of $\mathcal{H}^{(z)}$ contains 16 points. Let y' be a point of $\mathcal{H}$ polycollinear with one of these 16 points, and also opposite both y and z. Then y' belongs to ξ_z. It follows that $\mathcal{H}^{(z)}$ has a connected component of at least 17 points, a contradiction. Hence $\xi_y \not\subseteq \xi_z$ and analogously $\xi_z \not\subseteq \xi_y$. If the dimension of one of ξ_y, ξ_z is $d-2$, then $\xi_z \cap \xi_y$ has dimension at most $d-3$, hence $d=5$ and $x^\perp$ is contained in a plane of $\mathbf{PG}(5,q)$. So suppose both spaces ξ_y and ξ_z are $(d-1)$-dimensional. If ξ_x does not contain $\xi_y \cap \xi_z$, then $x^\perp$ is contained in the $(d-3)$-space $\xi_x \cap \xi_y \cap \xi_z$ and so again $d=5$ and $x^\perp$ is contained in a plane. Hence suppose that ξ_x contains $\xi_y \cap \xi_z$. Then ξ_x contains the 8 points of $\mathcal{H}$ opposite x and at distance 4 from both y and z. But every such point is polycollinear with 3 points in the same connected component of $\mathcal{H}^{(x)}$. Moreover, one can check that these 8×3 points are distinct (using $s=t=2$). Hence ξ_x contains all 32 points opposite x, a contradiction.

The theorem is proved. □

Now we consider full weak embeddings of hexagons in $\mathbf{PG}(6,q)$. If q is odd, we have a complete classification. First we need a lemma.

Lemma 5.2 *Let $\mathcal{H}$ be a classical hexagon of order (s,t). Let L and M be opposite lines and let S be the set of points at distance 3 from both L and M. If s is odd, then there is a point x of $\mathcal{H}$ opposite every point of S. If s is even, then no point of $\mathcal{H}$ is opposite every point of S.*

Proof. First let $s=t$ and consider the natural embedding of $\mathcal{H}$ in $\mathbf{PG}(6,s)$. The set S is a conic and for every point x of $\mathcal{H}$, the subspace ξ_x meets S in 0, 1, 2 or $s+1$ points. Let $Q(6,s)$ be the quadric on which $\mathcal{H}$ is defined.

Let s be odd. Then we consider a point x of $\mathcal{H}$ for which the tangent hyperplane of $Q(6,s)$ at x contains exactly 2 points y,z of S. The point x is at distance at most 4 from y and z, and ξ_x contains no other points of S. It is easy to see that x is at distance exactly 4 from y and z. Let N be any line through x distinct from the two lines at distance 3 from y or z. Every element of S is at distance 4 from exactly one element of N, and x is at distance 4 from two elements of S. So there must be a point of N opposite every element of S.

Now let s be even. Let n be the nucleus of the quadric $Q(6,s)$. Then the plane π of S contains n. As ξ_x is the tangent hyperplane of $Q(6,s)$ at x, also ξ_x contains n. So either $S \subseteq \xi_x$ or $\pi \cap \xi_x$ is a line which contains exactly one point of S. Hence $\xi_x \cap S$ is not empty.

Now let $s = t^3$. We consider the natural embedding of $\mathcal{H}$ in $\mathbf{PG}(7,s)$.

Let s be odd. Consider a subhexagon $\mathcal{H}_1$ of order (t,t) intersecting L in a line L_1 of $\mathcal{H}_1$ and M in a line M_1 of $\mathcal{H}_1$. Then $\mathcal{H}_1$ contains $t+1$ points of S. By the foregoing, $\mathcal{H}_1$ contains a point x at distance 4 from exactly 2 elements a_0, a_1 of S in $\mathcal{H}_1$, and opposite every other element of S in $\mathcal{H}_1$. Then x is at distance at least 4 from any element of S. Suppose, by way of contradiction, that x is at distance 4 from a point a_2 of S not contained in $\mathcal{H}_1$. Let R be the line of $\mathcal{H}$ through a_2 at distance 3 from x. Then R contains a point r of a line W of $\mathcal{H}$ at distance 3 from every element of S. As W contains a line of $\mathcal{H}_1$, the point r belongs to $\mathcal{H}_1$. At least one of L, M is distinct from W, say $L \neq W$. As L contains a line of $\mathcal{H}_1$ and r is a point of $\mathcal{H}_1$, the unique point a_2 of $\mathcal{H}$ at distance 2 from r and at distance 3 from L, belongs to $\mathcal{H}_1$. Hence $a_2 \in \{a_0, a_1\}$, a contradiction. So x is at distance 4 from exactly two points of S. Now the same argument as above for $s = t$ completes the proof.

Now let s be even. Assume, by way of contradiction, that $\mathcal{H}$ contains a point x such that $\xi_x \cap S$ is empty. Let l be the point of L at distance 4 from x, and let m be the point of M at distance 4 from x. Further, let R be the line of $\mathcal{H}$ at distance 3 from x and at distance 2 from M, and let r be the point of R at distance 4 from l (possibly $r = m$). Then there is a subhexagon $\mathcal{H}_1$ of order (t,t) containing l, x, r and m. The hexagon $\mathcal{H}_1$ intersects L and M in lines of $\mathcal{H}_1$ and hence contains $t+1$ points of S. From the case $s = t$ it now follows that ξ_x contains at least one point of S in $\mathcal{H}_1$, a contradiction.

The lemma is proved. $\square$

Theorem 5.3 *If $\mathcal{H}$ is a generalized hexagon of order (q,t), with q odd, weakly embedded in $\mathbf{PG}(6,q)$, then $\mathcal{H}$ is a regular full (and hence natural) embedding of $H(q)$.*

Proof. As in the proof of Theorem 5.1, one shows easily that the subspace π_x of $\mathbf{PG}(6,q)$ generated by the points polycollinear with some point x of $\mathcal{H}$ is 2-dimensional or 3-dimensional. If π_x is 2-dimensional, then x is clearly a distance-2-regular point (since no space ξ_y generated by $y^{\perp\!\!\!\perp}$ for y opposite x can contain x). Note that each ξ_y is a hyperplane in $\mathbf{PG}(6,q)$. Also, $\xi_{y'} \neq \xi_{y''}$ for distinct points y', y''. Suppose now that π_x is 3-dimensional. Let L be a line of $\mathcal{H}$ containing x and let $y \in L \setminus \{x\}$. Then $\xi_x \cap \xi_y = U_L$ is 4-dimensional. Clearly U_L contains π_x and each line at distance 2 from L. Let M be a line at distance 3 from x. Then M is not contained in π_x, since $\pi_x \subseteq \xi_z$ for every point z polycollinear with x and $M \not\subseteq \xi_z$ for z at distance 5 from M. Hence the lines M at distance 2 from L generate U_L. Let w and z be two non-polycollinear points in $x^\perp$ and put $U = \xi_z \cap \xi_w$. Then U is a 4-dimensional space containing π_x. If L is any line containing x, then $U_L \neq U$ (U_L contains points opposite either w or z). So exactly one of the $q+1$ hyperplanes $\xi_u \supseteq U_L$, $u \in L$, contains U. If ξ_u, $u \in L$, contains U, then, as U contains points opposite x, we have $x \neq u$. Let v be any point of $\mathcal{H}$ at distance 4 from both z and w. Then $v \in U$, and hence $v \in \xi_u$, implying $u \in x^v$. Hence x is a distance-2-regular point. Consequently every point x of $\mathcal{H}$ is distance-2-regular and so it follows from Ronan [1980] that $\mathcal{H}$ is classical.

Still to prove is that the embedding is flat. Suppose by way of contradiction that it is not flat. Then there is a point x with π_x a 3-dimensional space (where again π_x is the space generated by the points polycollinear with x). Let L be a line of $\mathcal{H}$ at distance 3 from x, and let M be opposite L and also at distance 3 from x. We claim that $\langle \pi_x, L, M \rangle =: V$ is 5-dimensional. Indeed, if z is incident with L and polycollinear with x, then M is not contained in ξ_z, but the 4-dimensional space $U = \langle \pi_x, L \rangle$ belongs to ξ_z. Whence our claim. It follows that the space U' generated by L, M and x^a, where $a \neq x$ is a point of $\mathcal{H}$ at distance 3 from both L and M, is at least 4-dimensional. As U' is contained in both ξ_x and ξ_a, it must be 4-dimensional. Varying a over the set of points at distance 3 from both L and M, we obtain all hyperplanes ξ_a of $\mathbf{PG}(6, q)$ containing U'. Hence every point of $\mathcal{H}$ is at distance at most 4 of at least one such point, hence q is even by the previous lemma, a contradiction. The theorem is proved. $\square$

The previous theorem is not true in the even case. Without proof, we mention that there is a counterexample for $q = 2$.

Also remark that the property for s even stated in Lemma 5.2 characterizes the finite Moufang hexagons of order (s, t), $s \in \{t, t^3\}$ even. Indeed, in Govaert [1997], it is shown that, if in a finite thick generalized hexagon $\mathcal{H}$ every two opposite lines L and M have the property that any point x is not opposite at least one point at distance 3 from both L and M, then $\mathcal{H}$ is classical of order (s, t) with $s \in \{t, t^3\}$, and with s even.

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