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QUADRATIC QUATERNARY RINGS WITH VALUATION AND AFFINE BUILDINGS OF TYPE $\tilde{C}_2$

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We introduce the notion of a quadratic quaternary ring with valuation and show that it defines a unique affine building of type $\tilde{C}_2$ by an explicit construction. This way, we obtain examples of explicitly defined (non classical) affine buildings of type $\tilde{C}_2$.

INTRODUCTION

This paper is the third in a sequence of four which have as purpose to characterize the class of affine buildings of type $\tilde{C}_2$, the first two papers of this sequence being [7] and [8].

Let us briefly describe the situation. J.Tits classified in 1984 all affine buildings of rank ≥4 by showing they all arise from algebraic groups over a local field (see [2],[16]). This is certainly not the case for the affine buildings of rank 3, because there are counter examples (see e.g. [11],[14],[17]), the so called *non classical* buildings. There are three classes of affine buildings of rank 3. They correspond to the following diagrams.

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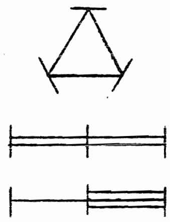

type $\tilde{A}_2$

type $\tilde{C}_2$

type $\tilde{G}_2$

An algebraic characterization of affine buildings of type $\tilde{A}_2$ is given in [17] and [18], by means of the notion of a *planar ternary ring with valuation*. This implied a lot of explicitly defined non classical examples of affine buildings of type $\tilde{A}_2$. It is our ambition to prove a similar characterization for the class of affine buildings of type $\tilde{C}_2$. The present paper is the crucial step in this characterization. Namely, we put a valuation on *quadratic quaternary rings*, the algebraic coordinatizing structures of a generalized quadrangle, and show that they give rise to an affine building of type $\tilde{C}_2$. Therefore, we have to use the results of [7]. So in a forthcoming paper, there remains to show that, given a quadratic quaternary ring with valuation $\mathcal{R}$, the spherical building at infinity (which is a generalized quadrangle ; see [16]) of the corresponding affine building of type $\tilde{C}_2$ can be coordinatized by $\mathcal{R}$.

The paper is organised as follows. In a first section, we recall everything that we need from [7] (e.g. the notion of a *Hjelmslev quadrangle of level n*), we define the notion of a quadratic quaternary ring with valuation and state our main theorem. Section 2 is devoted to the proof of the main result. We investigate some properties of quadratic quaternary rings with valuation, recoordinatize the corresponding generalized quadrangle $\mathcal{L}$, introduce a partial *-valuation, homogeneous coordinates and a partial valuation on $\mathcal{L}$, define certain quotient geometries and show that the latter are Hjelmslev quadrangles of level *n*. We quote [7] to finish off our proof. Section 3 finally deals with examples. Although the notion of a quadratic quaternary ring with valuation seems awful akward, there are a lot of non classical examples.

At this point, we can make the same remark as in the introduction of [17]. Namely, Ronan's construction of buildings in [14] contains a universal construction of affine buildings of type $\tilde{C}_2$. How do we then motivate our study ? Well, in our construction, we know a lot of additional information, e.g. the building at infinity. In the case of type $\tilde{A}_2$, we already proved that one of our examples in [18] admits a vertex transitive automorphism group (see [21]). Results in that style can certainly be proved for the examples of section 3.

We would like to remark that the proof of the main theorem in section 2 is not complete. Indeed, a complete detailed proof would require over 1000 pages. But what we left out can be reconstructed by any interested reader, since we only ommited proofs *similar* to other proofs we give. But we warn the reader that "*a similar proof*" here is not equivalent to "*a copy of*".

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1. DEFINITION, NOTATION AND MAIN RESULT

1.1. **Definition of a Hjelmslev quadrangle of level** n.

1.1.1. Notation.

Suppose $X = (\mathcal{P}(X), \mathcal{L}(X), I)$ is a point-line incidence geometry with point set $\mathcal{P}(X)$ and line set $\mathcal{L}(X)$. We denote the set of points incident with a given line $\mathcal{L}$ by $\sigma(\mathcal{L})$ and call it the *shadow* (*of* $\mathcal{L}$) (see Buekenhout[3]). Suppose $\mathcal{L}_1$ and $\mathcal{L}_2$ are two distinct lines of X. If there is a point incident with both $\mathcal{L}_1$ and $\mathcal{L}_2$, then we call $\mathcal{L}_1$ and $\mathcal{L}_2$ *concurrent* and we denote "$\mathcal{L}_1 \perp \mathcal{L}_2$" . Suppose $\mathcal{P}_1$ and $\mathcal{P}_2$ are two distinct points of X. If there is a line incident with both $\mathcal{P}_1$ and $\mathcal{P}_2$, then we call $\mathcal{P}_1$ and $\mathcal{P}_2$ *collinear* and we denote "$\mathcal{P}_1 \perp \mathcal{P}_2$". The negation of $\perp$ is denoted by $\not\perp$. A *flag* in X is an incident point-line pair of X. The set of flags of X is denoted by $\mathcal{F}(X)$. A *morphism* from X to some other point-line incidence geometry $X' = (\mathcal{P}(X'), \mathcal{L}(X'), I)$ maps $\mathcal{P}(X)$ to $\mathcal{P}(X')$, $\mathcal{L}(X)$ to $\mathcal{L}(X')$ and the map induced on $\mathcal{F}(X)$ maps $\mathcal{F}(X)$ to $\mathcal{F}(X')$. An *epimorphism* is a morphism which is surjective on the set of flags. The geometry X is called *thick* if every line is incident with at least three points and every point is incident with at least three lines.

Suppose $\mathcal{A}$ is an arbitrary set and $\mathbf{P}_1(\mathcal{A})$ and $\mathbf{P}_2(\mathcal{A})$ are two arbitrary partitions of $\mathcal{A}$. then we say that $\mathbf{P}_1(\mathcal{A})$ is *properly finer than* $\mathbf{P}_2(\mathcal{A})$ if *every* class of $\mathbf{P}_2(\mathcal{A})$ is the union of at least *two* classes of $\mathbf{P}_1(\mathcal{A})$. In that case, we denote

$$\dot{\mathbf{P}}_2(\mathcal{A})/\mathbf{P}_1(\mathcal{A}) = \{\{\mathcal{C} \in \mathbf{P}_1(\mathcal{A}) \mid \mathcal{C} \subseteq \mathcal{D}\} \mid \mathcal{D} \in \mathbf{P}_2(\mathcal{A})\},$$

which is a partition of $\mathbf{P}_1(\mathcal{A})$. If $\mathcal{D} \in \mathbf{P}_2(\mathcal{A})$, then we call $\{\mathcal{C} \in \mathbf{P}_1(\mathcal{A}) \mid \mathcal{C} \subseteq \mathcal{D}\}$ the *canonical image of* $\mathcal{D}$ *in* $\mathbf{P}_2(\mathcal{A})/\mathbf{P}_1(\mathcal{A})$.

1.1.2. Generalized quadrangles.

Let $\mathcal{S} = (\mathcal{P}(\mathcal{S}), \mathcal{L}(\mathcal{S}), I)$ a be point-line incidence geometry. Then we call $\mathcal{S}$ a *generalized quadrangle* if there exist positive integers $s \geq 1$ and $t \geq 1$ (s and/or t may also be infinite) such that the following axioms hold.

(Q1) *Every point of $\mathcal{S}$ is incident with $1+t$ lines and two distinct points are incident with at most one line.*

(Q2) *Every line of $\mathcal{S}$ is incident with $1+s$ points and two distinct lines are incident with at most one point.*

(Q3) *If $\mathcal{P} \in \mathcal{P}(\mathcal{S})$ and $\mathcal{L} \in \mathcal{L}(\mathcal{S})$ are not incident, then there exists a unique flag $(\mathcal{Q}, \mathcal{M}) \in \mathcal{F}(\mathcal{S})$ such that $\mathcal{P}\ I\ \mathcal{M}\ I\ \mathcal{Q}\ I\ \mathcal{L}$.*

Generalized quadrangles were introduced by J.Tits in his celebrated paper [15]. More information about generalized quadrangles can be found in e.g. Payne-Thas [13] or in the survey paper of W.M.Kantor [11]. In this paper, we will always assume that every generalized quadrangle is thick. One can check that an axiom system for thick generalized quadrangles can be given as follows.

(QQ1) *Every point is incident with at least two lines and there exists a point incident with at least three lines.*

(QQ2) *Every line is incident with at least two points and there exists a line incident with at least three points.*

(QQ3) *There exists a non-incident point-line pair.*

(QQ4) *If* $\mathcal{P} \in \mathcal{P}(\mathcal{S})$ *and* $\mathcal{L} \in \mathcal{L}(\mathcal{S})$ *are not incident, then there exists a unique flag* $(\mathcal{Q},\mathcal{M}) \in \mathcal{F}(\mathcal{S})$ *such that* $\mathcal{P}\ I\ \mathcal{M}\ I\ \mathcal{Q}\ I\ \mathcal{L}$.

The reason why we introduce this axiom system is because of the fact that (QQ1) up to (QQ4) is easier to check than (Q1) up to (Q3).

1.1.3. *Definition of a Hjelmslev quadrangle of level* n.

Throughout, n, i, j and k denote positive integers.

We define a Hjelmslev quadrangle of level n by induction on n. The induction will start with $n=1$. We give a separate definition for level 0. We abbreviate "Hjelmslev quadrangle of level n" by "level n HQ".

A *level* 0 *HQ* is any trivial geometry $\mathcal{V}_0 = (\{*\},\{*\},=)$, where * is any arbitrary (but twice the same) symbol.

A *level* 1 *HQ* is any 6-tuple

$$\mathcal{V}_1 = (\mathcal{P}(\mathcal{V}_1),\mathcal{L}(\mathcal{V}_1), I, (\mathbf{P}_i(\mathcal{V}_1))_{i\leq 1}, (\mathbf{L}_i(\mathcal{V}_1))_{i\leq 1}, (\mathcal{W}_0(\mathcal{V}_1,\{\mathcal{P}\}),\{\mathcal{P}\})_{\mathcal{P}\in\mathcal{P}(\mathcal{V}_1)}),$$

where $(\mathcal{P}(\mathcal{V}_1), \mathcal{L}(\mathcal{V}_1), I)$ is an arbitrary generalized quadrangle ; $\mathbf{P}_0(\mathcal{V}_1)$ is the partition of $\mathcal{V}_1$ determined by: every class is a singleton ; $\mathbf{P}_1(\mathcal{V}_1)$ is the partition of $\mathcal{P}(\mathcal{V}_1)$ consisting of one class ; similar for $(\mathbf{L}_i(\mathcal{V}_n))_{i\leq 1}$, and for every $\mathcal{P}\in\mathcal{P}(\mathcal{V}_1)$, $\mathcal{W}_0(\mathcal{V}_1,\{\mathcal{P}\}) = (\{\mathcal{P}\},\{\mathcal{P}\},=)$. The last three elements of $\mathcal{V}_1$ do not add more structure to the generalized quadrangle, but they are necessary to start the induction. So in fact, a level 1 HQ "is" a generalized quadrangle.

Now suppose $n \geq 2$. Let $(\mathcal{P}(\mathcal{V}_n), \mathcal{L}(\mathcal{V}_n), I)$ be a thick incidence geometry. Suppose $(\mathbf{P}_i(\mathcal{V}_n))_{i\leq n}$, resp. $(\mathbf{L}_i(\mathcal{V}_n))_{i\leq n}$ is a family of partitions of $\mathcal{P}(\mathcal{V}_n)$, resp. $\mathcal{L}(\mathcal{V}_n)$ satisfying :

(PS1) $\mathbf{P}_0(\mathcal{V}_n) = \{\{\mathcal{P}\} \mid \mathcal{P}\in\mathcal{P}(\mathcal{V}_n)\}$; $\mathbf{P}_n(\mathcal{V}_n) = \{\mathcal{P}(\mathcal{V}_n)\}$,

(PS2) $\mathbf{L}_0(\mathcal{V}_n) = \{\{\mathcal{L}\} \mid \mathcal{L}\in\mathcal{L}(\mathcal{V}_n)\}$; $\mathbf{L}_n(\mathcal{V}_n) = \{\mathcal{L}(\mathcal{V}_n)\}$,

(PS3) $\mathbf{P}_i(\mathcal{V}_n)$ is properly finer than $\mathbf{P}_{i+1}(\mathcal{V}_n)$, for all $i < n$,

(PS4) $\mathbf{L}_i(\mathcal{V}_n)$ is properly finer than $\mathbf{L}_{i+1}(\mathcal{V}_n)$, for all $i < n$,

The elements of $\mathbf{P}_i(\mathcal{V}_n)$, resp $\mathbf{L}_i(\mathcal{V}_n)$ are called *i-point-neighbourhoods*, resp. *i-line-neighbourhoods (of their elements)*. An i-point-neighbourhood is also called a *point-neighbourhood*, an *i-neighbourhood* or briefly a *neighbourhood*. Similar definitions for i-line-neighbourhoods. If $\mathcal{P}\in\mathcal{P}(\mathcal{V}_n)$ and $\mathcal{L}\in\mathcal{L}(\mathcal{V}_n)$, then we denote by $\mathcal{O}^i(\mathcal{P})$, resp. $\mathcal{O}^i(\mathcal{L})$ the unique i-point-neighbourhood of $\mathcal{P}$, resp. i-line- neighbourhood of $\mathcal{L}$.

Suppose for every $\mathcal{C} \in \mathbf{P}_{n-1}(\mathcal{V}_n)$, we have a level $(n-1)$ HQ, denoted by $\mathcal{W}_{n-1}(\mathcal{V}_n, \mathcal{C})$ (this is an element of a well-defined class of objects by induction) and select in every $\mathcal{W}_{n-1}(\mathcal{V}_n, \mathcal{C})$ an $(n-2)$-point-neighbourhood $\mathcal{N}_{\mathcal{C}}$. Then we call the 6-tuple

$$\mathcal{V}_n = (\mathcal{P}(\mathcal{V}_n), \mathcal{L}(\mathcal{V}_n), I, (\mathbf{P}_i(\mathcal{V}_n))_{i \leq n}, (\mathbf{L}_i(\mathcal{V}_n))_{i \leq n}, (\mathcal{W}_{n-1}(\mathcal{V}_n, \mathcal{C}), \mathcal{N}_{\mathcal{C}})_{\mathcal{C} \in \mathbf{P}_{n-1}(\mathcal{V}_n)})$$

a *level n HQ* if $\mathcal{V}_n$ satisfies the axioms (IS),(GQ) and (NP) below. The geometry $(\mathcal{P}(\mathcal{V}_n), \mathcal{L}(\mathcal{V}_n), I)$ is called the *base geometry* of $\mathcal{V}_n$. Before stating the actual axioms, we need some preliminaries.

We first define the canonical $(n-1)$-image of $\mathcal{V}_n$ by induction on n. The *canonical 0-image* of a level 1 HQ $\mathcal{V}_1$ is by definition the trivial geometry $(\{\mathcal{P}(\mathcal{V}_1)\}, \{\mathcal{P}(\mathcal{V}_1)\}, =)$. Now let $n \geq 2$. Define the geometry $(\mathbf{P}_1(\mathcal{V}_n), (\mathbf{L}_1(\mathcal{V}_n), I)$ as follows. If $\mathcal{C} \in \mathbf{P}_1(\mathcal{V}_n)$ and $\mathcal{D} \in (\mathbf{L}_1(\mathcal{V}_n)$, then $\mathcal{C}\ I\ \mathcal{D}$ if and only if there exist $\mathcal{P} \in \mathcal{C}$ and $\mathcal{L} \in \mathcal{D}$ which are incident in $(\mathcal{P}(\mathcal{V}_n), \mathcal{L}(\mathcal{V}_n), I)$. Furthermore, denote by $\mathcal{W}_{n-2}(\mathcal{V}_n, \mathcal{C})$ the canonical $(n-2)$-image of $\mathcal{W}_{n-1}(\mathcal{V}_n, \mathcal{C})$ (well-defined by the induction hypothesis). Denote by $\mathcal{N}'_{\mathcal{C}}$ the canonical image of $\mathcal{N}_{\mathcal{C}}$ in $\mathbf{P}_{n-2}(\mathcal{W}_{n-1}(\mathcal{V}_n, \mathcal{C}))/\mathbf{P}_1(\mathcal{W}_{n-1}(\mathcal{V}_n, \mathcal{C}))$ if $n > 2$ and $\mathcal{N}'_{\mathcal{C}} = \{\mathcal{P}(\mathcal{W}_1(\mathcal{V}_2, \mathcal{C}))\}$ if $n = 2$. Obviously, there is a bijective correspondence between $\mathbf{P}_{n-1}(\mathcal{V}_n)$ and $\mathbf{P}_{n-1}(\mathcal{V}_n)/\mathbf{P}_1(\mathcal{V}_n)$ and the unique element of $\mathbf{P}_{n-1}(\mathcal{V}_n)/\mathbf{P}_1(\mathcal{V}_n)$ corresponding to the element $\mathcal{C}$ of $\mathbf{P}_{n-1}(\mathcal{V}_n)$ is denoted by $\mathcal{C}^*$. In particular, all elements of $\mathbf{P}_{n-1}(\mathcal{V}_n)/\mathbf{P}_1(\mathcal{V}_n)$ are denoted with a *. We define the *canonical* $(n-1)$-*image* of $\mathcal{V}_n$ as the 6-tuple

$$\mathcal{V}_{n-1} = (\mathbf{P}_1(\mathcal{V}_n), (\mathbf{L}_1(\mathcal{V}_n), I, (\mathbf{P}_{i+1}(\mathcal{V}_n)/\mathbf{P}_1(\mathcal{V}_n))_{i \leq n-1}, (\mathbf{L}_{i+1}(\mathcal{V}_n)/\mathbf{L}_1(\mathcal{V}_n))_{i \leq n-1},$$
$$(\mathcal{W}_{n-2}(\mathcal{V}_n, \mathcal{C}), \mathcal{N}'_{\mathcal{C}})_{\mathcal{C}^* \in \mathbf{P}_{n-1}(\mathcal{V}_n)/\mathbf{P}_1(\mathcal{V}_n)}).$$

We can now state the very natural axiom (IS).

(IS) *The canonical* $(n-1)$*-image* $\mathcal{V}_{n-1}$ *of* $\mathcal{V}_n$ *is a level* $n-1$ *HQ.*

Using a similar notation for $\mathcal{V}_{n-1}$ as for $\mathcal{V}_n$, (IS) implies e.g. $\mathbf{P}_i(\mathcal{V}_{n-1}) = \mathbf{P}_{i+1}(\mathcal{V}_n)/\mathbf{P}_1(\mathcal{V}_n)$ and similarly for the line-partitions.

Define inductively the *canonical* $(n-j)$*-image* of $\mathcal{V}_n$ $(0 < j \le n)$ as the canonical $(n-j)$-image $\mathcal{V}_{n-j}$ of the canonical $(n-j+1)$-image $\mathcal{V}_{n-j+1}$ of $\mathcal{V}_n$, or as $\mathcal{V}_n$ (for $j = 0$). Note that $\mathcal{O}^1$ defines a mapping from $\mathcal{P}(\mathcal{V}_n)$ to $\mathcal{P}(\mathcal{V}_{n-1})$ and from $\mathcal{L}(\mathcal{V}_n)$ to $\mathcal{L}(\mathcal{V}_{n-1})$. By the definition of the incidence relation in $\mathcal{V}_{n-1}$, we can see that this mapping is an epimorphism from $(\mathcal{P}(\mathcal{V}_n), \mathcal{L}(\mathcal{V}_n), I)$ onto $(\mathbf{P}_1(\mathcal{V}_n), (\mathbf{L}_1(\mathcal{V}_n), I)$. We denote this epimorphism by Π^n_{n-1}. By the induction hypothesis, a similar epimorphism exists from the base geometry of $\mathcal{V}_{n-j+1}$ onto the base geometry of $\mathcal{V}_{n-j}$ and we denote it by Π^{n-j+1}_{n-j} By induction, we can put

$$\Pi^n_{n-j} = \Pi^{n-j+1}_{n-j} \circ \Pi^n_{n-j+1}.$$

From now on, we denote the canonical j-image $\mathcal{V}_j$ of $\mathcal{V}_n$ by

$$(\mathcal{P}_i(\mathcal{V}_j), \mathcal{L}_i(\mathcal{V}_j), I, (\mathbf{P}_i(\mathcal{V}_j))_{i \le j}, (\mathbf{L}_i(\mathcal{V}_j))_{i \le j}, (\mathcal{W}_{j-1}(\mathcal{V}_j, \mathcal{C}), \mathcal{N}_{\mathcal{C}})_{\mathcal{C} \in \mathbf{P}_{j-1}(\mathcal{V}_j)}),$$

for all j, $0 < j \le n$. The epimorphism Π^n_j is called the *projection*. We define the *valuation map* $u : (\mathcal{P}(\mathcal{V}_n) \cup \mathcal{L}(\mathcal{V}_n)) \times (\mathcal{P}(\mathcal{V}_n) \cup \mathcal{L}(\mathcal{V}_n)) \to \mathbf{N}$ as follows. Let x, y be either both points or both lines of $\mathcal{V}_n$, then

$$u(x,y) = sup\{j\leq n \mid \Pi'_j(x) = \Pi'_j(y)\}$$

If $\mathcal{P}\in\mathcal{P}(\mathcal{V}_n)$ and $\mathcal{L}\in\mathcal{L}(\mathcal{V}_n)$, then

$$u(\mathcal{P},\mathcal{L}) = u(\mathcal{L},\mathcal{P}) = (u_1(\mathcal{P},\mathcal{L}), u_2(\mathcal{P},\mathcal{L}))$$

with

$$u_1(\mathcal{P},\mathcal{L}) = u_1(\mathcal{L},\mathcal{P}) = sup\{j\leq n \mid \exists Q\ I\ \mathcal{L} \text{ such that } \Pi'_j(Q)=\Pi'_j(\mathcal{P}),\ Q\in\mathcal{P}(\mathcal{V}_n)\}$$

and

$$u_2(\mathcal{P},\mathcal{L}) = u_2(\mathcal{L},\mathcal{P}) = sup\{j\leq n \mid \exists \mathcal{M}\ I\ \mathcal{P} \text{ such that } \Pi'_j(\mathcal{M})=\Pi'_j(\mathcal{L}),\ \mathcal{M}\in\mathcal{L}(\mathcal{V}_n)\}$$

We now write down the axiom (GQ), consisting of two statements (GQ1) and (GQ2).

(GQ1) *If* $\mathcal{P},Q\in\mathcal{P}(\mathcal{V}_n)$, $\mathcal{L},\mathcal{M}\in\mathcal{L}(\mathcal{V}_n)$, $Q\ I\ \mathcal{L}\ I\ \mathcal{P}\ I\ \mathcal{M}$, $u(\mathcal{P},Q)=0$ *and* $\mathcal{L}\neq\mathcal{M}$, *then*

$$\sigma^{n-j}(Q)\cap\sigma(\mathcal{M})\neq\emptyset \iff 2.j\leq u(\mathcal{L},\mathcal{M})$$

(GQ2) *If* $\mathcal{P}\in\mathcal{P}(\mathcal{V}_n)$, $\mathcal{L}\in\mathcal{L}(\mathcal{V}_n)$ *and* $u(\mathcal{P},\mathcal{L}) = (k,2k)$ *for some* $k\leq\frac{n}{2}$, *then there exists a unique* $\mathcal{M}\in\mathcal{L}(\mathcal{V}_n)$ *such that* $\mathcal{P}\ I\ \mathcal{M}\perp\mathcal{L}$. *Moreover,* $u(\mathcal{L},\mathcal{M}) = 2k$ *and* $u(\mathcal{P},Q) = 0$, *for all* $Q\in\sigma(\mathcal{L})\cap\sigma(\mathcal{M})$. *If* $k=0$, *then* $u(Q_1,Q_2)\geq\frac{n}{2}$, *for all* $Q_1,Q_2\in\sigma(\mathcal{L})\cap\sigma(\mathcal{M})$.

We now define an affine structure on level j HQs. Suppose X_j is a level j HQ, $0<j<n$, with

$$X_j = (\mathcal{P}_i(X_j), \mathcal{L}_i(X_j), I, (\mathbf{P}_i(X_j))_{i\leq j}, (\mathbf{L}_i(X_j))_{i\leq j}, (\mathcal{W}_{j-1}(X_j,\mathcal{C}), \mathcal{N}_{\mathcal{C}})_{\mathcal{C}\in\mathbf{P}_{j-1}(X_j)}).$$

Let $X_1 = (\mathcal{P}(X_1),\ldots)$ be its canonical 1-image. Let $\mathcal{D}\in\mathbf{P}_{j-1}(X_j)$ be arbitrary. We denote:

* $\mathcal{L}_{\mathfrak{D}}^{\infty} = \{\mathcal{L} \in \mathcal{L}(X_j) \mid \sigma(\mathcal{L}) \cap \mathfrak{D} \neq \emptyset\}$,
* $\mathcal{P}_{\mathfrak{D}}^{\infty} = \{\mathcal{P} \in \mathcal{P}(X_j) \mid \exists \mathcal{L} \in \mathcal{L}_{\mathfrak{D}}^{\infty}$ such that $\mathcal{P}\ I\ \mathcal{L}\}$,
* $\mathcal{AP}(X_j, \mathfrak{D}) = \mathcal{P}(X_j) - \mathcal{P}_{\mathfrak{D}}^{\infty}$,
* $\mathcal{AL}(X_j, \mathfrak{D}) = \mathcal{L}(X_j) - \mathcal{L}_{\mathfrak{D}}^{\infty}$.

We call the elements of $\mathcal{AP}(X_j, \mathfrak{D})$ the *affine points* (*of* $(X_j, \mathfrak{D})$, if there is confusion possible) and the elements of $\mathcal{AL}(X_j, \mathfrak{D})$ the *affine lines* (*of* $(X_j, \mathfrak{D})$). The elements of $\mathcal{P}_{\mathfrak{D}}^{\infty} - \mathfrak{D}$, resp. of $\mathcal{L}_{\mathfrak{D}}^{\infty}$ are called the *points*, resp. the *lines at infinity* (*of* $(X_j, \mathfrak{D})$). The elements of $\mathfrak{D}$ are the *hyperpoints* (*of* $(X_j, \mathfrak{D})$). The pair $(X_j, \mathfrak{D})$ is called an *affine HQ* (*of level* n). The following lemma is proved in [7].

LEMMA(1.3). *Let* $(X_j, \mathfrak{D})$ *be as above. Every element of the* $(j-1)$*-point-neighbourhood of any affine point is again an affine point. Hence every element of the* $(j-1)$*-point-neighbourhood of any point at infinity, resp. hyperpoint, is again a point at infinity, resp. hyperpoint.*

This lemma will give sense to axiom (NP) below.

We now introduce the notion of a "strip of width i" in an affine HQ $(X_j, \mathfrak{D})$. Suppose $\mathcal{P} \in \mathcal{P}(X_j)$ is a point at infinity of $(X_j, \mathfrak{D})$ and $\mathcal{L} \in \mathcal{L}(X_j)$ is an affine line incident with $\mathcal{P}$. If $i<j$, then we call the set

$$\{Q \in \mathcal{AP}(X_j, \mathfrak{D}) \mid Q\ I\ \mathcal{M}\ I\ \mathcal{P} \text{ for some } \mathcal{M} \in \mathcal{O}^i(\mathcal{L})\}$$

a *strip of width* i (*in* $(X_j, \mathfrak{D})$). If $i \geq j$, then the set

$$\{Q \in \mathcal{AP}(X_j, \mathcal{D}) \mid Q \perp \mathcal{P}\}$$

is called a *strip of width* i *(in* $(X_j, \mathcal{D})$*)*. In every case, we call $\mathcal{P}$ a *base point (of the strip)*. It is not necessarily unique, even if the strip has width > 0 (cp.[7],property(2.26)).

We can now state the first part of (NP).

(NP1) *If* $\mathcal{C} \in \mathbf{P}_{n-1}(\mathcal{V}_n)$, *then* $\mathcal{AP}(\mathcal{W}_{n-1}(\mathcal{V}_n, \mathcal{C}), \mathcal{N}_{\mathcal{C}}) = \mathcal{C}$. *Moreover, the* i*-point-neighbourhood of any point* $\mathcal{P} \in \mathcal{C}$ *in* $\mathcal{W}_{n-1}(\mathcal{V}_n, \mathcal{C})$ *coincides with the* i*-point-neighbourhood of* $\mathcal{P}$ *in* $\mathcal{V}_n$, *for all* $i \leq n-2$.

Suppose $\mathcal{C}_{n-j} \in \mathbf{P}_{n-j}(\mathcal{V}_n)$ and let $\mathcal{C}_{n-k}$ be the unique element of $\mathbf{P}_{n-k}(\mathcal{V}_n)$ containing $\mathcal{C}_{n-j}$ as a subset, $0 \leq k \leq j < n$. By (NP1),

$$\mathcal{C}_{n-2} \in \mathbf{P}_{n-2}(\mathcal{W}_{n-1}(\mathcal{V}_n, \mathcal{C}_{n-1})),$$
$$\mathcal{C}_{n-3} \in \mathbf{P}_{n-3}(\mathcal{W}_{n-2}(\mathcal{W}_{n-1}(\mathcal{V}_n, \mathcal{C}_{n-1}), \mathcal{C}_{n-2})), \text{ etc....}$$

This way, we justify the following notation.

$$\mathcal{W}_{n-j}(\mathcal{V}_n, \mathcal{C}_{n-j}) = \mathcal{W}_{n-j}(\mathcal{W}_{n-j+1}(\ldots(\mathcal{W}_{n-1}(\mathcal{V}_n, \mathcal{C}_{n-1}), \ldots), \mathcal{C}_{n-j+1}), \mathcal{C}_{n-j}).$$

Moreover, $\mathcal{C}_{n-j} = \mathcal{AP}(\mathcal{W}_{n-j}(\mathcal{V}_n, \mathcal{C}_{n-j}), \mathcal{N}_{\mathcal{C}_{n-j}})$.

The axiom (NP1) was about points of the point-neighbourhoods. The last axiom, (NP2), which we call the *strip axiom*, says something about the lines in the affine HQs corresponding to these neighbourhoods.

(NP2) *If* $\mathcal{P}\in\mathcal{P}(\mathcal{V}_n)$, $\mathcal{L}\in\mathcal{L}(\mathcal{V}_n)$, $0<j<n$ *and* $\sigma(\mathcal{L})\cap\mathcal{O}^{n-j}(\mathcal{P})\neq\emptyset$, *then the set*

$$S_j^n(\mathcal{P},\mathcal{L})=\sigma(\mathcal{L})\cap\mathcal{O}^{n-j}(\mathcal{P})$$

is a strip of width j *in* $(\mathcal{W}_{n-j}(\mathcal{V}_n,\mathcal{O}^{n-j}(\mathcal{P})),\mathcal{N}_{\mathcal{O}^{n-j}(\mathcal{P})})$. *Every strip of width* 1 *in* $(\mathcal{W}_{n-1}(\mathcal{V}_n,\mathcal{O}^{n-1}(\mathcal{P})),\mathcal{N}_{\mathcal{O}^{n-1}(\mathcal{P})})$ *can be obtained in this way (putting* $j=1$*).*

This completes our list of axioms for a level n HQ.

We keep the same notation as above. Suppose $\mathcal{M}$ is an affine line of $(\mathcal{W}_{n-j}(\mathcal{V}_n,\mathcal{O}^{n-j}(\mathcal{P})),\mathcal{N}_{\mathcal{O}^{n-j}(\mathcal{P})})$ such that the set of affine points of $\mathcal{M}$ is a subset of $\sigma(\mathcal{L})\cap\mathcal{O}^{n-j}(\mathcal{P})$ (with the notation of (NP2) above), then we call $\mathcal{M}$ a *component of* $\mathcal{L}$, or a *component of the strip* $S_j^n(\mathcal{P},\mathcal{L})$ and we denote $\mathcal{M}<\mathcal{L}$. The set of all components of $S_j^n(\mathcal{P},\mathcal{L})$ is denoted by $C_j^n(\mathcal{P},\mathcal{L})$. The set of affine points of $\mathcal{M}$ is called the *affine shadow of* $\mathcal{M}$. As an extension, we call every point of $\mathcal{V}_n$ incident with $\mathcal{L}$ a *component of* $\mathcal{L}$.

Now let $\mathcal{V}_n'=(\mathcal{P}(\mathcal{V}_n'),\mathcal{L}(\mathcal{V}_n'),\ldots)$ be a second level n HQ and suppose

$$\Psi:(\mathcal{P}(\mathcal{V}_n),\mathcal{L}(\mathcal{V}_n),I)\to(\mathcal{P}(\mathcal{V}_n'),\mathcal{L}(\mathcal{V}_n'),I)$$

is an isomorphism of incidence geometries mapping the affine shadow of every component of any line $\mathcal{L}$ onto the affine shadow of a component of $\Psi(\mathcal{L})$ and mapping i-neighbourhoods onto i-neighbourhoods, for all i, $0<i\leq n$, then we call $\mathcal{V}_n$ and $\mathcal{V}_n'$ *equivalent*. This way, we can extend Ψ

to the set of all components of all lines of $\mathcal{V}_n$ and this extended map, which we also denote by Ψ, preserves "being component of". We call Ψ an *equivalence*.

We now define by induction the notion of an isomorphism between $\mathcal{V}_n$ and $\mathcal{V}_n' = (\mathcal{P}(\mathcal{V}_n'), \mathcal{L}(\mathcal{V}_n'), \ldots)$. If $n=1$, then $\mathcal{V}_1$ and $\mathcal{V}_1'$ are called *isomorphic* if their base geometries are isomorphic generalized quadrangles. Now let $n\geq 2$, then we call $\mathcal{V}_n$ and $\mathcal{V}_n'$ *isomorphic* if they are equivalent (denote in that case the corresponding equivalence by Ψ) and if for all $\mathcal{C} \in \mathbf{P}_{n-1}(\mathcal{V}_n)$, $\mathcal{W}_{n-1}(\mathcal{V}_n, \mathcal{C})$ is isomorphic to $\mathcal{W}_{n-1}(\mathcal{V}_n', \Psi(\mathcal{C}))$ and this isomorphism $\Psi_{\mathcal{C}}$ coincides with $\Psi/\mathcal{C}$ over $\mathcal{C}$. We can now extend Ψ with every $\Psi_{\mathcal{C}}$ and if we denote this extension still by Ψ, then we call Ψ an *isomorphism*. Obviously, isomorphic level n HQs are also equivalent.

Recall that Π_{n-1}^{n} is the projection mapping the base geometry of $\mathcal{V}_n$ onto the base geometry of the canonical $(n-1)$-image $\mathcal{V}_{n-1} = (\mathcal{P}(\mathcal{V}_{n-1}), \ldots)$. We can extend Π_{n-1}^{n} to all $\mathcal{W}_{n-j}(\mathcal{V}_n, \mathcal{C})$, $\mathcal{C} \in \mathbf{P}_{n-j}(\mathcal{V}_n)$ and $0<j<n$, with the projection of $\mathcal{W}_{n-j}(\mathcal{V}_n, \mathcal{C})$ onto $\mathcal{W}_{n-j-1}(\mathcal{V}_{n-1}, \Pi_{n-1}^{n}(\mathcal{C}))$. We denote that extension still by Π_{n-1}^{n}. Suppose now that $\mathcal{V}_{n-1}$ is isomorphic to some level n-1 HQ X_{n-1} and call the corresponding isomorphism Ψ. Then we call $\Psi \circ \Pi_{n-1}^{n}$ a *HQ-epimorphism*. Suppose now that $(X_n, \nabla_n^{n+1})_{n\in\mathbf{N}}$ is an infinite sequence with X_n a level n HQ and ∇_n^{n+1} an HQ-epimorphism from X_{n+1} onto X_n, then we call $(X_n, \nabla_n^{n+1})_{n\in\mathbf{N}}$ an *HQ-Artmann-sequence*. This name is inspired by the work of Artmann [1], who studied similar sequences of level n Hjelmslev planes, giving rise to affine buildings of type $\tilde{A}_2$ (by [9],[17]).

If Z_n is the base geometry of X_n, for all $n \in \mathbf{N}$, then we call the sequence $(Z_n, \nabla_n^{n+1}/Z_{n+1})_{n \in \mathbf{N}}$ the *base sequence of* $(X_n, \nabla_n^{n+1})_{n \in \mathbf{N}}$.

From [7], we recall :

THEOREM(1.1.3). *Every HQ-Artmann-sequence gives rise to an explicitly defined affine building of type* $\tilde{C}_2$.

1.2. **Quadratic quaternary rings with valuation.**

Quadratic quaternary rings are in fact the coordinatizing algebraic structures of generalized quadrangles (see [5]). But for the purpose of this paper, we do not need the background of this coordinatization theory. We will give the algebraic definition of a quadratic quaternary ring and show how one relates to every such structure a generalized quadrangle. The fact that every generalized quadrangle arises in this way is irrelevant for our purposes.

1.2.1. Quadratic quaternary rings.

Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two sets intersecting in the set $\{0,1\}$ of distinct elements 0,1 and both not containing the symbol ∞. Let Q_1 and Q_2 be two quaternary operations with

$$Q_1 : \mathcal{R}_2 \times \mathcal{R}_1 \times \mathcal{R}_2 \times \mathcal{R}_1 \to \mathcal{R}_1,$$
$$Q_2 : \mathcal{R}_1 \times \mathcal{R}_2 \times \mathcal{R}_1 \times \mathcal{R}_2 \to \mathcal{R}_2.$$

The quadruple $(\mathcal{R}_1, \mathcal{R}_2, Q_1, Q_2)$ is called a *quadratic quaternary ring* if it satisfies (0), $(\overline{0})$, (1), $(\overline{1})$, (A), $(\overline{A})$, (B), $(\overline{B})$ and (C) below.

(0) $Q_1(k,0,0,a') = a' = Q_1(0,a,k,a')$.

$(\overline{0})$ $Q_2(a,0,0,k') = k' = Q_2(0,k,a,k')$.

(1) $Q_1(1,a,0,0) = a$.

$(\overline{1})$ $Q_2(1,k,0,0) = k$.

(A) *There exists exactly one* $x \in \mathcal{R}_1$ *such that* $Q_1(k,a,\ell,x) = b$.

$(\overline{A})$ *There exists exactly one* $p \in \mathcal{R}_2$ *such that* $Q_2(a,k,b,p) = \ell$.

(B) *If* $k \neq \ell$, *there exists exactly one pair* $(x,y) \in \mathcal{R}_1 \times \mathcal{R}_1$ *such that*

$$Q_1(k,x,Q_2(x,k,a,k'),y) = a,$$
$$Q_1(\ell,x,Q_2(x,k,a,k'),y) = b.$$

$(\overline{B})$ *If* $a \neq b$, *there exists exactly one pair* $(p,q) \in \mathcal{R}_2 \times \mathcal{R}_2$ *such that*

$$Q_2(a,p,Q_1(p,a,k,a'),q) = k,$$
$$Q_2(b,p,Q_1(p,a,k,a'),q) = \ell.$$

(C) *If*

$$Q_1(k,a,\ell,a') \neq b \quad \text{(C1)}$$
$$Q_2(a,k,b,k') \neq \ell \quad \text{(C2)}$$

then there exists a unique quadruple $(x,x',p,p') \in \mathcal{R}_1 \times \mathcal{R}_1 \times \mathcal{R}_2 \times \mathcal{R}_2$ *such that*

$$Q_1(k,x,Q_2(x,k,b,k'),x') = b,$$
$$Q_1(p,x,Q_2(x,k,b,k'),x') = Q_1(p,a,\ell,a'),$$

$$Q_2(a,p,Q_1(p,a,\ell,a'),p') = \ell,$$

$$Q_2(x,p,Q_1(p,a,\ell,a'),p') = Q_2(x,k,b,k').$$

If exactly one of the conditions (C1) or (C2) holds, then there exists no quadruple (x,x',p,p') having the above properties.

We abbreviate the term *quadratic quaternary ring* to *QQR*.

THEOREM(1.2.1.1) (Hanssens-Van Maldeghem [5]). *Let $(\mathcal{R}_1,\mathcal{R}_2,Q_1,Q_2)$ be a QQR. The following point-line geometry $\mathcal{S}$ is a generalized quadrangle. The points of $\mathcal{S}$ are the elements of $\mathcal{R}_1 \cup \mathcal{R}_2 \times \mathcal{R}_1 \cup \mathcal{R}_1 \times \mathcal{R}_2 \times \mathcal{R}_1$, denoted with round brackets, together with the symbol (∞). The lines of $\mathcal{S}$ are the elements of $\mathcal{R}_2 \cup \mathcal{R}_1 \times \mathcal{R}_2 \cup \mathcal{R}_2 \times \mathcal{R}_1 \times \mathcal{R}_2$, denoted with square brackets, together with the symbol $[\infty]$. Incidence is defined as follows.*

$(a,\ell,a')\ I\ [k,b,k']$ *if and only if* $Q_1(k,a,\ell,a') = b$
and $Q_2(a,k,b,k') = \ell$,

$(a,\ell,a')\ I\ [a,\ell]$,

$(k,a)\ I\ [k,a,k']$,

$(k,a)\ I\ [k]$,

$(a)\ I\ [a,k]$,

$(a)\ I\ [\infty]$,

$(\infty)\ I\ [k]$,

$(\infty)\ I\ [\infty]$, *for all* $a,a',b \in \mathcal{R}_1$; $k,k',\ell \in \mathcal{R}_2$.

There are no other incidences.

Suppose $(\mathcal{R}_1, \mathcal{R}_2, Q_1, Q_2)$ is a QQR. We define two additions and four multiplications as follows. For all $a, b \in \mathcal{R}_1$ and all $k, \ell \in \mathcal{R}_2$, we define

$$a + b = Q_1(1, a, 0, b) \in \mathcal{R}_1,$$

$$k + \ell = Q_2(1, k, 0, \ell) \in \mathcal{R}_2, \quad \text{and we read } \textit{plus},$$

$$k \,.\, a = Q_1(k, a, 0, 0) \in \mathcal{R}_1,$$

$$a \,.\, k = Q_2(a, k, 0, 0) \in \mathcal{R}_2, \quad \text{and we read } \textit{times},$$

$$k \times \ell = Q_1(k, 0, \ell, 0) \in \mathcal{R}_1,$$

$$a \times b = Q_2(a, 0, b, 0) \in \mathcal{R}_2, \quad \text{and we read } \textit{cross}.$$

The next theorem is a direct consequence of the definition.

THEOREM(1.2.1.2). *Let $(\mathcal{R}_1, \mathcal{R}_2, Q_1, Q_2)$ be a QQR, then the following properties hold for all $a, b \in \mathcal{R}_1$ and all $k, \ell \in \mathcal{R}_2$.*

(i) $a + 0 = 0 + a = a$,

(ii) $k + 0 = 0 + k = k$,

(iii) $a + x = b$ *has a unique solution for x,*

(iv) $k + p = \ell$ *has a unique solution for x,*

(v) $1 \,.\, a = a$,

(vi) $1 \,.\, k = k$,

(vii) $k \,.\, x = a$ *has a unique solution for x if* $k \neq 0$,

(viii) $a \,.\, p = k$ *has a unique solution for x if* $a \neq 0$,

(ix) $0 \,.\, a = k \,.\, 0 = 0$,

(x) $0 \,.\, k = a \,.\, 0 = 0$,

(xi) $0 \times a = a \times 0 = 0$,

(xii) $0 \times k = k \times 0 = 0$.

A QQR $(\mathcal{R}_1, \mathcal{R}_2, Q_1, Q_2)$ is called *quasi-classical* if it satisfies (QC1) up to (QC7).

(QC1) $(\mathcal{R}_1,+)$ and $(\mathcal{R}_2,+)$ are groups.

(QC2) Both *cross*-multiplications are bi-additive.

(QC3) Both *times*-multiplications are additive in their second argument.

(QC4) $(a + b).k = b.k + a.k - b \times (k.a)$ and
$(k + \ell).a = \ell.a + k.a - \ell \times (a.k)$, for all $a, b \in \mathcal{R}_1$
and all $k, \ell \in \mathcal{R}_2$,

(QC5) $k + (a \times b) = (a \times b) + k$ and
$a + (k \times \ell) = (k \times \ell) + a$, for all $a, b \in \mathcal{R}_1$ and all $k, \ell \in \mathcal{R}_2$.

(QC6) $(k \times (a \times b)) = 0$, for all $a, b \in \mathcal{R}_1$ and all $k, \ell \in \mathcal{R}_2$.

(QC7) $Q_1(k, a, \ell, a') = k.a + (k \times \ell) + a'$ and
$Q_2(a, k, b, k') = a.k + (a \times b) + k'$, for all $a, a', b \in \mathcal{R}_1$
and all $k, k', \ell \in \mathcal{R}_2$.

All Moufang generalized quadrangles are coordinatized by a quasi-classical QQR. Quasi-classical QQRs play a similar rôle in the theory of the coordinatization of generalized quadrangles as *division rings* (in the sense of [10], see also [6]) in the theory of the coordinatization of projective planes. One can prove that in every quasi-classical QQR, $(\mathcal{R}_1,+)$ is commutative (unpublished, but we will not use this result).

Now let $\mathcal{S} = (\mathcal{P}(\mathcal{S}),\mathcal{L}(\mathcal{S}),I)$ be a generalized quadrangle and let $p \in \mathcal{P}(\mathcal{S})$. We call p a *regular* point if for every point q not collinear with p and every triplet $\{p_1,p_2,p_3\}$ of points, all collinear with both p and q, we have : every point collinear with both p_1 and p_2 is also collinear with p_3. Dually, one defines a *regular* line.

THEOREM(1.2.1.3) (Hanssens-Van Maldeghem [5]). *Let $\mathcal{S}$ be a generalized quadrangle coordinatized by the QQR $(\mathcal{R}_1,\mathcal{R}_2,Q_1,Q_2)$ as in theorem*(1.2.1.1). *The point (∞) is regular if and only if Q_1 is independent of its third argument. Dually, the line $[\infty]$ is regular if and only if Q_2 is independent of its third argument.*

In particular, $k \times \ell = 0$ for all $k,\ell \in \mathcal{R}_2$ if (∞) is regular. Combining this with the definition of quasi-classical QQR, we obtain :

THEOREM(1.2.1.4). *Let $\mathcal{S}$ be a generalized quadrangle coordinatized by a quasi-classical QQR $(\mathcal{R}_1,\mathcal{R}_2,Q_1,Q_2)$, then the point (∞) is regular if and only if the cross-product of two arbitrary elements of $\mathcal{R}_2$ is zero. Dually, the line $[\infty]$ is regular if and only if the cross-product of two arbitrary elements of $\mathcal{R}_1$ is zero. In both cases, $(\mathcal{R}_1,+)$ and $(\mathcal{R}_2,+)$ are commutative.*

A quasi-classical QQR in which the cross-product of every two elements of $\mathcal{R}_1$ is zero, is called a *regular* QQR. The dual of a regular QQR is a *dually regular* QQR. A QQR which is both regular and dually regular is called *doubly regular*.

We will now give an example of a doubly regular QQR. Let $\mathcal{F}$ be any field of characteristic 2 and $\mathcal{F}'$ a subfield of $\mathcal{F}$. Suppose $\mathcal{F}^2 \subseteq \mathcal{F}'$. We define $(\mathcal{F}', \mathcal{F}, Q_1, Q_2)$ as follows.

$$Q_1(k,a,\ell,a') = k^2.a + a',$$

$$Q_2(a,k,b,k') = a.k + k',$$

where the addition and the multiplication of the field coincide with the respective addition and multiplication in the QQR as defined above. In the finite case, we always have $\mathcal{F} = \mathcal{F}' = \mathcal{F}^2 = \mathrm{GF}(q)$, q even, and the corresponding generalized quadrangle is the *symplectic generalized quadrangle* W(q) (see [4],[13]).

1.2.2. Quadratic quaternary rings with valuation.

Let $(\mathcal{R}_1, \mathcal{R}_2, Q_1, Q_2)$ be a QQR and v a map from $\mathcal{R}_1 \times \mathcal{R}_1 \cup \mathcal{R}_2 \times \mathcal{R}_2$ to $\mathbf{Z} \cup \{+\infty\}$. Then we call $(\mathcal{R}_1, \mathcal{R}_2, Q_1, Q_2, v)$ a *QQR with valuation* if $(\mathcal{R}_1, \mathcal{R}_2, Q_1, Q_2)$ and v satisfy

(v1) *$v(x,y) = +\infty$ if and only if $x = y$, for all suitable* (x,y).

(v2) *For all suitable x, y and z,* $\mathrm{v}(x,z) \geq \inf\{v(x,y), v(z,y)\}$ *and if $v(x,y) \neq v(z,y)$, then equality holds.*

(v3) $v/\mathcal{R}_1 \times \mathcal{R}_1$ *and* $v/\mathcal{R}_2 \times \mathcal{R}_2$ *are both surjective.*

(v4) *If*

$$Q_1(k_1, a_1, \ell_1, a_1') = Q_1(k_1, a_2, \ell_2, a_2') = b_1,$$

$$Q_2(a_1, k_1, b_1, k_1') = Q_2(a_1, k_2, b_2, k_2') = \ell_1,$$

$$Q_1(k_2, a_1, \ell_1, a_1') = Q_1(k_2, a_3, \ell_3, a_3') = b_2,$$

$$Q_2(a_3, k_2, b_2, k_2') = Q_2(a_3, k_3, b_3, k_3') = \ell_3,$$

$$Q_1(k_3, a_3, \ell_3, a_3') = Q_1(k_3, a_2, \ell_2, a_2') = b_3,$$

$$Q_2(a_2, k_3, b_3, k_3') = Q_2(a_2, k_1, b_1, k_4') = \ell_2,$$

then

$$v(k_1, k_2) + v(k_1', k_4') = v(k_1, k_3) + v(k_2, k_3) + v(a_2, a_3).$$

(v5) *If*

$$Q_1(k_1, a_1, \ell_1, a_1') = Q_1(k_1, a_2, \ell_2, a_2') = b_1,$$

$$Q_2(a_1, k_1, b_1, k_1') = Q_2(a_1, k_2, b_2, k_2') = \ell_1,$$

$$Q_1(k_2, a_1, \ell_1, a_3') = Q_1(k_2, a_2, \ell_2, a_2') = b_2,$$

$$Q_2(a_2, k_2, b_2, k_2') = Q_2(a_2, k_1, b_1, k_3') = \ell_2,$$

then

$$v(k_1', k_3') = v(k_1, k_2) + v(a_1, a_2),$$

$$v(a_1', a_3') = v(a_1, a_2) + 2.v(k_1, k_2).$$

(v6) *If*

$$Q_1(k, a, \ell, a_1') = b_1 \quad \text{and} \quad Q_1(k, a, \ell, a_2') = b_2,$$

then

$$v(a_1', a_2') = v(b_1, b_2).$$

(v7) *If*

$$Q_2(a, k, b, k_1') = \ell_1 \quad \text{and} \quad Q_2(a, k, b, k_2') = \ell_2,$$

then

$$v(k_1', k_2') = v(\ell_1, \ell_2).$$

We abbreviate *QQR with valuation* by V-QQR. Let $(\mathcal{R}_1, \mathcal{R}_2, Q_1, Q_2, v)$ be a V-QQR, the we can define the following metric in $\mathcal{R}_1$.

$$\delta_1 : \mathcal{R}_1 \times \mathcal{R}_1 \to \mathbf{R} : (x, y) \to \mathbf{e}^{-v(x,y)},$$

where $\mathbf{e} \in \mathbf{R}$ denotes the exponential base number. Similarly, one can define

$$\delta_2 : \mathcal{R}_2 \times \mathcal{R}_2 \to \mathbf{R} : (x, y) \to \mathbf{e}^{-v(x,y)}$$

We call $(\mathcal{R}_1, \mathcal{R}_2, Q_1, Q_2, v)$ *complete* if $(\mathcal{R}_1, \delta_1)$ en $(\mathcal{R}_2, \delta_2)$ are *complete metric spaces*, i.e. every *Cauchy-sequence* converges. We abbreviate a *complete V-QQR* to *CV-QQR*.

We usually write $v(x,0)$ as $v(x)$. We will show in the next section that v is symmetric and hence $v(x) = v(0,x) = v(x,0)$, for all $x \in \mathcal{R}_1 \cup \mathcal{R}_2$.

1.3. The main theorem.

MAIN THEOREM. *Every generalized quadrangle that can be coordinatized by some V-QQR, resp. CV-QQR, is isomorphic to the generalized quadrangle at infinity of some symmetric, resp. complete, affine building of type* $\tilde{C}_2$.

This will be proved by constructing an HQ-Artmann-sequence $\mathcal{H}$ related to any V-QQR $(\mathcal{R}_1, \mathcal{R}_2, Q_1, Q_2, v)$. Then we show that the corresponding generalized quadrangle $\mathcal{V}$ is a subquadrangle of the inverse limit $\mathcal{S}$ of $\mathcal{H}$. By [8], $\mathcal{S}$ is isomorphic to the generalized quadrangle at infinity of some complete affine building Δ of type $\tilde{C}_2$. Leaving out the right apartments of Δ, we obtain a symmetric affine building of type $\tilde{C}_2$ whose generalized quadrangle at infinity is isomorphic to $\mathcal{V}$.

2. PROOF OF THE MAIN THEOREM

2.1. Properties of generalized quadrangles coordinatized by a V-QQR.

2.1.1. Properties of quadratic quaternary rings with valuation. Throughout this section, we denote by $(\mathcal{R}_1, \mathcal{R}_2, Q_1, Q_2, v)$ a fixed quadratic quaternary ring with valuation.

THEOREM(2.1.1.1). *The V-QQR $(\mathcal{R}_1, \mathcal{R}_2, Q_1, Q_2, v)$ has the following properties.*

(va1) *v is symmetric, i.e. $v(x,y) = v(y,x)$, for all $(x,y) \in \mathcal{R}_1^2 \cup \mathcal{R}_2^2$.*

(va2) $v(1) = 0$ and $v(0) = +\infty$.

(va3) If

$$Q_1(k_1, a_1, \ell_1, a_1') = Q_1(k_1, a_2, \ell_2, a_2') = b_1,$$
$$Q_2(a_1, k_1, b_1, k_1') = Q_2(a_1, k_2, \ell_2, k_2') = \ell_1,$$
$$Q_1(k_2, a_1, \ell_1, a_1') = Q_1(k_2, a_3, \ell_3, a_3') = b_2,$$
$$Q_2(a_3, k_2, b_2, k_2') = Q_2(a_3, k_3, \ell_3, k_3') = \ell_3,$$

$$Q_1(k_3, a_3, \ell_3, a_3') = Q_1(k_3, a_2, \ell_2, a_2') = b_3,$$

$$Q_2(a_2, k_3, b_3, k_3') = Q_2(a_2, k_1, b_1, k_4') = \ell_2,$$

then

$$v(k_1, k_2) + v(k_1', k_4') = v(k_1, k_3) + v(k_2, k_3) + v(a_2, a_3),$$

$$v(k_1, k_3) + v(k_1', k_4') = v(k_1, k_2) + v(k_2, k_3) + v(a_1, a_3),$$

$$2.v(k_1', k_4') = 2.v(k_2, k_3) + v(a_1, a_3) + v(a_2, a_3),$$

$$2.v(k_1, k_2) + v(a_1, a_3) = 2.v(k_1, k_3) + v(a_2, a_3).$$

(va4) *If*

$$Q_1(k_1, a_1, \ell_1, a_1') = Q_1(k_1, a_2, \ell_2, a_2') = b_1,$$

$$Q_2(a_1, k_1, b_1, k_1') = Q_2(a_1, k_2, b_2, k_2') = \ell_1,$$

$$Q_1(k_2, a_1, \ell_1, a_4') = Q_1(k_2, a_3, \ell_3, a_3') = b_2,$$

$$Q_2(a_3, k_2, b_2, k_2') = Q_2(a_3, k_3, b_3, k_3') = \ell_3,$$

$$Q_1(k_3, a_3, \ell_3, a_3') = Q_1(k_3, a_2, \ell_2, a_2') = b_3,$$

$$Q_2(a_2, k_3, b_3, k_3') = Q_2(a_2, k_1, b_1, k_1') = \ell_2,$$

then

$$v(a_1, a_2) + v(a_1', a_4') = v(a_1, a_3) + v(a_2, a_3) + 2.v(k_2, k_3),$$

$$v(a_1, a_3) + v(a_1', a_4') = v(a_1, a_2) + v(a_2, a_3) + 2.v(k_1, k_3),$$

$$v(a_1', a_4') = v(a_2, a_3) + v(k_1, k_3) + v(k_2, k_3),$$

$$v(a_1, a_2) + v(k_1, k_3) = v(a_1, a_3) + v(k_2, k_3).$$

(va5) *If*

$$Q_1(k_1, a_1, \ell_1, a_1') = Q_1(k_1, a_2, \ell_2, a_2') = b_1,$$

$$Q_2(a_1, k_1, b_1, k_1') = Q_2(a_1, k_2, b_2, k_2') = \ell_1,$$

$$Q_1(k_2, a_1, \ell_1, a_3') = Q_1(k_2, a_2, \ell_2, a_2') = b_2,$$

$$Q_2(a_2, k_2, b_2, k_2') = Q_2(a_2, k_1, b_1, k_3') = \ell_2,$$

then

$$v(k_1', k_3') = v(k_1, k_2) + v(a_1, a_2),$$

$$v(a_1', a_3') = v(a_1, a_2) + 2.v(k_1, k_2),$$

$$2.v(k_1', k_3') = v(a_1', a_3') + v(a_1, a_2),$$

$$v(a_1', a_3') = v(k_1', k_3') + v(k_1, k_2).$$

(va6) *If*

$$Q_1(k_1, a_1, \ell_1, a_1') = b_1,$$

$$Q_1(k_1, a_2, \ell_2, a_2') = b_3,$$

$$Q_2(a_1, k_2, b_2, k_2') = \ell_1,$$

$$Q_2(a_2, k_2, b_2, k_2') = \ell_2,$$

$$Q_1(k_2, a_1, \ell_1, a_1') = Q_1(k_2, a_2, \ell_2, a_2') = b_2,$$

then

$$v(b_1, b_3) = v(a_1, a_2) + 2.v(k_1, k_2).$$

(va7) *If*

$$Q_2(a_1, k_1, b_1, k_1') = \ell_1,$$

$$Q_2(a_1, k_2, b_2, k_2') = \ell_3,$$

$$Q_1(k_1, a_2, \ell_2, a_2') = b_1,$$

$$Q_1(k_2, a_2, \ell_2, a_2') = b_2,$$

$$Q_2(a_2, k_1, b_1, k_1') = Q_2(a_2, k_2, b_2, k_2') = \ell_2,$$

then

$$v(\ell_1, \ell_3) = v(k_1, k_2) + v(a_1, a_2).$$

(va8) *If*

$$Q_1(k, a, \ell, a_1') = b_1,$$

$$Q_1(k, a, \ell, a_2') = b_2,$$

then

$$v(a_1', a_2') = v(b_1, b_2).$$

(va9) *If*

$$Q_2(a,k,b,k_1') = \ell_1,$$

$$Q_2(a,k,b,k_2') = \ell_2,$$

then

$$v(k_1',k_2') = v(\ell_1,\ell_2).$$

(va10) *If*

$$Q_1(k_1,a_1,\ell_1,a_1') = Q_1(k_1,a_2,\ell_2,a_2') = b_1,$$

$$Q_2(a_1,k_1,b_1,k_1') = Q_2(a_1,k_2,b_2,k_2') = \ell_1,$$

$$Q_1(k_2,a_1,\ell_1,a_3') = Q_1(k_2,a_2,\ell_2,a_4') = b_2,$$

$$Q_2(a_2,k_1,b_1,k_1') = Q_2(a_2,k_2,b_2,k_2') = \ell_2,$$

then

$$v(a_1',a_3') = v(a_2',a_4').$$

(va11) *If*

$$Q_1(k_1,a_1,\ell_1,a_1') = Q_1(k_1,a_2,\ell_2,a_2') = b_1,$$

$$Q_2(a_1,k_1,b_1,k_1') = Q_2(a_1,k_2,b_2,k_2') = \ell_1,$$

$$Q_1(k_2,a_1,\ell_1,a_1') = Q_1(k_2,a_2,\ell_2,a_2') = b_2,$$

$$Q_2(a_2,k_1,b_1,k_3') = Q_2(a_2,k_2,b_2,k_4') = \ell_2,$$

then

$$v(k_1',k_3') = v(k_2',k_4').$$

(va12) *If*

$$Q_1(k_1,a_1,\ell_1,a_1') = Q_1(k_1,a_2,\ell_2,a_2') = b_1,$$

$$Q_2(a_1,k_1,b_1,k_1') = Q_2(a_1,k_4,b_4,k_4') = \ell_1,$$

$$Q_1(k_2,a_2,\ell_2,a_2') = Q_1(k_2,a_3,\ell_3,a_3') = b_2,$$

$$Q_2(a_2,k_2,b_2,k_2') = Q_2(a_2,k_1,b_1,k_1') = \ell_2,$$

$$Q_1(k_3,a_3,\ell_3,a_3') = Q_1(k_3,a_4,\ell_4,a_4') = b_3,$$

$$Q_2(a_3,k_2,b_2,k_2') = Q_2(a_3,k_3,b_3,k_3') = \ell_3,$$

$$Q_1(k_4,a_4,\ell_4,a_4') = Q_1(k_4,a_1,\ell_1,a_1') = b_4,$$

$$Q_2(a_4,k_3,b_3,k_3') = Q_2(a_4,k_4,b_4,k_4') = \ell_4,$$

then

$$v(a_1,a_2)+v(k_1,k_4)+v(k_1,k_2) = v(a_3,a_4)+v(k_2,k_3)+v(k_3,k_4),$$

$$v(a_1,a_4)+v(k_3,k_4)+v(k_1,k_4) = v(a_2,a_3)+v(k_1,k_2)+v(k_2,k_3),$$

$$2.v(k_1,k_2)+v(a_1,a_2)+v(a_2,a_3) = 2.v(k_3,k_4)+v(a_3,a_4)+v(a_1,a_4),$$

$$2.v(k_1,k_4)+v(a_1,a_4)+v(a_1,a_2) = 2.v(k_2,k_3)+v(a_2,a_3)+v(a_3,a_4).$$

PROOF.

(va1) Put $x=y$ in (v2) and we obtain $v(x,z) = \inf\{v(x,x),v(z,x)\} = v(z,x)$ by (v1), since if $z\neq x$, then $v(z,x) < v(x,x) = +\infty$.

(va2) By (v1), $v(0) = v(0,0) = +\infty$. Consider now the identities

$$Q_1(0,0,0,0) = Q_1(0,a,0,0) = 0,$$

$$Q_2(0,0,0,0) = Q_2(0,1,a,0) = 0,$$

$$Q_1(1,0,0,a) = Q_1(1,a,0,0) = a.$$

By (v5), $v(a) = v(a)+2.v(1)$ and since we can choose $a\in\mathcal{R}_1$ distinct from 0, v(1) = 0. Note that this argument together with its dual makes v well defined, since it implies $v(1)=0$, regardless whether we consider 1 as an element of either $\mathcal{R}_1$ or $\mathcal{R}_2$.

(va3) In (v4), we switch a_1 with a_2, ℓ_1 with ℓ_2, a_1' with a_2', k_2 with k_3, b_2 with b_3, k_2' with k_3' and k_1' with k_4' and obtain this way the same set of equalities. Hence we have

$$v(k_1,k_3') + v(k_1',k_4') = v(k_1,k_2) + v(k_2,k_3) + v(a_1,a_3).$$

The other two equalities follow by taking linear combinations of the first two.

Before proving (va4), we first show (va5) and (va6).

(va5) The first two equalities follow from (v5) and the last two are a linear combination of the first two.

(va6) Given

$$Q_1(k_1,a_1,\ell_1,a_1') = b_1 \quad (1)$$

$$Q_1(k_1,a_2,\ell_2,a_2') = b_3 \quad (2)$$

$$Q_2(a_1,k_2,b_2,k_2') = \ell_1 \quad (3)$$

$$Q_2(a_2,k_2,b_2,k_2') = \ell_2 \quad (4)$$

$$Q_1(k_2,a_1,\ell_1,a_1') = Q_1(k_2,a_2,\ell_2,a_2') = b_2 \quad (5)$$

Consider the quadruple (k_1,b_3,a_1,ℓ_1). Then by (A) and $(\bar{A})$, there exists a unique pair (x,p) such that

$$Q_1(k_1,a_1,\ell_1,x) = b_3 \quad (6)$$

$$Q_2(a_1,k_1,b_3,p) = \ell_1 \quad (7)$$

By (1),(6) and (v6), $v(x,a_1') = v(b_1,b_3)$ and by (2),(3),(4),(5),(6),(7) and (v5), $v(x,a_1') = v(a_1,a_2) + 2.v(k_1,k_2)$. Hence $v(b_1,b_3) = v(a_1,a_2) + 2.v(k_1,k_2)$.

(va4) Given

$$Q_1(k_1,a_1,\ell_1,a_1') = Q_1(k_1,a_2,\ell_2,a_2') = b_1 \tag{1}$$

$$Q_2(a_1,k_1,b_1,k_1') = Q_2(a_1,k_2,b_2,k_2') = \ell_1 \tag{2}$$

$$Q_1(k_2,a_1,\ell_1,a_4') = Q_1(k_2,a_3,\ell_3,a_3') = b_2 \tag{3}$$

$$Q_2(a_3,k_2,b_2,k_2') = Q_2(a_3,k_3,b_3,k_3') = \ell_3 \tag{4}$$

$$Q_1(k_3,a_3,\ell_3,a_3') = Q_1(k_3,a_2,\ell_2,a_2') = b_3 \tag{5}$$

$$Q_2(a_2,k_3,b_3,k_3') = Q_2(a_2,k_1,b_1,k_1') = \ell_2 \tag{6}$$

Consider the quadruple (k_3,b_3,a_1,ℓ_1). By (A) and $(\bar{A})$, there exists a unique pair (x,p) such that

$$Q_1(k_3,a_1,\ell_1,x) = b_3 \tag{7}$$

$$Q_2(a_1,k_3,b_3,p) = \ell_1 \tag{8}$$

By (1),(2),(5),(6),(7),(8) and (v5), $v(p,k_3') = v(k_1,k_3) + v(a_1,a_2)$.
By (2),(3),(4),(5),(7),(8) and (v5), $v(p,k_3') = v(k_2,k_3) + v(a_1,a_3)$.
Hence $v(k_1,k_3) + v(a_1,a_2) = v(k_2,k_3) + v(a_1,a_3)$.

To show the other equalities, we consider two possibilities :

First possibility : $k_1 = k_2$.

By (1),(3) and (v6), $v(a_1',a_4') = v(b_1,b_2)$.
By (1),(3),(4),(5),(6) and (va6), $v(b_1,b_2) = v(a_2,a_3) + 2.v(k_1,k_3)$.
Hence $v(a_1',a_4') = v(a_2,a_3) + 2.v(k_1,k_3) = v(a_2,a_3) + v(k_1,k_3) + v(k_2,k_3)$.

Second possibility : $k_1 \neq k_2$.

By (B), there exist unique $x,y \in \mathcal{R}_1$, $p \in \mathcal{R}_2$ such that

$$Q_1(k_2, x, p, y) = b_2 \quad (9)$$

$$Q_1(k_1, x, p, y) = b_1 \quad (10)$$

$$Q_2(x, k_2, b_2, k_2') = p \quad (11)$$

Furthermore, we define $q \in \mathcal{R}_2$ such that (cp. (A)) :

$$Q_2(x, k_1, b_1, q) = p \quad (12)$$

By (1),(2),(3),(9),(10),(11),(12) and (va5), $v(a_1', a_4') = v(k_1, k_2) + v(q, k_1')$.

By (1),(3),(4),(5),(6),(9),(10),(11),(12) and (va3), $v(a_2, a_3) + v(k_1, k_3) + v(k_2, k_3) = v(k_1, k_2) + v(q, k_1')$.

Hence, $v(a_1', a_4') = v(a_2, a_3) + v(k_1, k_3) + v(k_2, k_3)$.

This showes the third formula of (va4) completely. The first two formulas follow by taking linear combinations of the last two.

(va7) Similarly to (va6), now using (v7) and (v5). It is a kind of "dual" of (va6). Later on, we shall define this "duallity" in a more formal way.

(va8) = (v6).

(va9) = (v7).

(va10) Given (we renumber) :

$$Q_1(k_1, a_1, \ell_1, a_1') = Q_1(k_1, a_2, \ell_2, a_2') = b_1 \quad (1)$$

$$Q_2(a_1, k_1, b_1, k_1') = Q_2(a_1, k_2, b_2, k_2') = \ell_1 \quad (2)$$

$$Q_1(k_2, a_1, \ell_1, a_3') = Q_1(k_2, a_2, \ell_2, a_4') = b_2 \tag{3}$$

$$Q_2(a_2, k_1, b_1, k_1') = Q_2(a_2, k_2, b_2, k_2') = \ell_2 \tag{4}$$

Again, there are two cases to consider here.

First possibility : $k_1 = k_2$.

By (1),(3) and (v6), $v(a_1', a_3') = v(b_1, b_2) = v(a_2', a_4')$.

Second possibility : $k_1 \neq k_2$.

By (B) there exist unique $x, y \in \mathcal{R}_1$, $p \in \mathcal{R}_2$ such that

$$Q_1(k_2, x, p, y) = b_2 \tag{5}$$

$$Q_1(k_1, x, p, y) = b_1 \tag{6}$$

$$Q_2(x, k_2, b_2, k_2') = p \tag{7}$$

Further more we define $q \in \mathcal{R}_2$ such that (cp. (A)) :

$$Q_2(x, k_1, b_1, q) = p. \tag{8}$$

By (1),(2),(3),(5),(6),(7),(8) and (va5), $v(a_1', a_3') = v(q, k_1') + v(k_1, k_2)$.

By (1),(3),(4),(5),(6),(7),(8) and (va5), $v(a_2', a_4') = v(q, k_1') + v(k_1, k_2)$.

Hence, $v(a_1', a_3') = v(a_2', a_4')$.

(va11) Similarly to the previous proof, now using properties $(\overline{B})$ and $(\overline{A})$ in stead of (B) resp. (A). Again, (va11) is a kind of "dual" of (va10).

(va12) Given (we renumber again) :

$$Q_1(k_1, a_1, \ell_1, a_1') = Q_1(k_1, a_2, \ell_2, a_2') = b_1 \quad (1)$$

$$Q_2(a_1, k_1, b_1, k_1') = Q_2(a_1, k_4, b_4, k_4') = \ell_1 \quad (2)$$

$$Q_1(k_2, a_2, \ell_2, a_2') = Q_1(k_2, a_3, \ell_3, a_3') = b_2 \quad (3)$$

$$Q_2(a_2, k_1, b_1, k_1') = Q_2(a_2, k_2, b_2, k_2') = \ell_2 \quad (4)$$

$$Q_1(k_3, a_3, \ell_3, a_3') = Q_1(k_3, a_4, \ell_4, a_4') = b_3 \quad (5)$$

$$Q_2(a_3, k_2, b_2, k_2') = Q_2(a_3, k_3, b_3, k_3') = \ell_3 \quad (6)$$

$$Q_1(k_4, a_4, \ell_4, a_4') = Q_1(k_4, a_1, \ell_1, a_1') = b_4 \quad (7)$$

$$Q_2(a_4, k_3, b_3, k_3') = Q_2(a_4, k_4, b_4, k_4') = \ell_4 \quad (8)$$

We consider again two possibilities.

First possibility : $k_1 = k_3$.

By (1),(2),(5),(7),(8) and (va6), $v(b_1, b_3) = v(a_1, a_4) + 2.v(k_1, k_4)$.

By (1),(3),(4),(5),(6) and (va6), $v(b_1, b_3) = v(a_2, a_3) + 2.v(k_1, k_2)$.

Hence, $v(a_1, a_4) + v(k_1, k_4) + v(k_3, k_4) = v(a_1, a_4) + 2.v(k_1, k_4) =$

$$v(a_2, a_3) + 2.v(k_1, k_2) = v(a_2, a_3) + v(k_1, k_2) + v(k_2, k_3).$$

This is the second equality.

Second possibility : $k_1 \neq k_3$.

By (B), there exist unique $x, y \in \mathcal{R}_1$, $p \in \mathcal{R}_2$ such that

$$Q_1(k_3, x, p, y) = b_3 \quad (9)$$

$$Q_1(k_1, x, p, y) = b_1 \quad (10)$$

$$Q_2(x, k_3, b_3, k_3') = p \quad (11)$$

Furthermore, we define $q \in \mathcal{R}_2$ such that (cp. (A)) :

$$Q_2(x,k_1,b_1,q) = p \tag{12}$$

By (1),(2),(5),(7),(8),(9),(10),(11),(12) and (v4), $v(q,k_1') + v(k_1,k_3) = v(a_1,a_4) + v(k_1,k_4) + v(k_3,k_4)$.

By (1),(3),(4),(5),(6),(9),(10),(11),(12) and (v4), $v(q,k_1') + v(k_1,k_3) = v(a_2,a_3) + v(k_1,k_2) + v(k_2,k_3)$.

Hence, $v(a_1,a_4) + v(k_3,k_4) + v(k_1,k_4) = v(a_2,a_3) + v(k_1,k_2) + v(k_2,k_3)$, the second formula.

The first formula now follows from the second one by the cyclic permutation $1\to2\to3\to4$ of the indices, taking (va1) into account. The last two equalities then follow from the first two by linear combination.

Q.E.D.

Let $\mathcal{V}$ be the generalized quadrangle coordinatized by the V-QQR $(\mathcal{R}_1,\mathcal{R}_2,Q_1,Q_2,v)$. We fix the notation $\mathcal{V}$ for the rest of this section. We can consider $v(x,y)$ as $v((x),(y))$ if $x,y\in\mathcal{R}_1$, and as $v([x],[y])$ if x, $y\in\mathcal{R}_2$. In both cases, it is considered as the valuation of a pair of varieties, incident with a common variety. In this point of view, we are going to extend v. First to pairs of points $(\mathcal{P}_1,\mathcal{P}_2)$ incident with a common line $\mathcal{L}$. If $\mathcal{P}_1$ and $\mathcal{P}_2$ have a distinct number of coordinates, then by definition $v(\mathcal{P}_1,\mathcal{P}_2) = 0$. If $\mathcal{P}_1 = \mathcal{P}_2$, then by definition $v(\mathcal{P}_1,\mathcal{P}_2) = +\infty$. If $\mathcal{P}_i = (a_i)$, i=1,2, then $v(\mathcal{P}_1,\mathcal{P}_2) = v(a_1,a_2)$ as above. If $\mathcal{P}_i = (k,a_i)$, i=1,2, then we put $v(\mathcal{P}_1,\mathcal{P}_2) = v(a_1,a_2)$. If $\mathcal{P}_i = (a_i,\ell_i,a_i')$, i=1,2, then we put $v(\mathcal{P}_1,\mathcal{P}_2) = v(a_1,a_2)$ if $\mathcal{L}$ has three coordinates and $v(\mathcal{P}_1,\mathcal{P}_2) = v(a_1',a_2')$ if $\mathcal{L}$ has two coordinates. The definition of the valuation of a pair of lines incident with a commom line is completely similar. We now state a theorem which we will call ***the main property*** of a generalized quadrangle coordinatized by a V-QQR.

THEOREM(2.1.1.2). *Let* $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ *be four points of* $\mathcal{V}$ *and* $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4$ *four lines of* $\mathcal{V}$ *with* $\mathcal{P}_1\ I\ \mathcal{L}_1\ I\ \mathcal{P}_2\ I\ \mathcal{L}_2\ I\ \mathcal{P}_3\ I\ \mathcal{L}_3\ I\ \mathcal{P}_4\ I\ \mathcal{L}_4\ I\ \mathcal{P}_1$, *then we have :*

(1) $\quad v(\mathcal{P}_1,\mathcal{P}_2) + v(\mathcal{L}_1,\mathcal{L}_4) + v(\mathcal{L}_1,\mathcal{L}_2) = v(\mathcal{P}_3,\mathcal{P}_4) + v(\mathcal{L}_2,\mathcal{L}_3) + v(\mathcal{L}_3,\mathcal{L}_4),$

(2) $\quad v(\mathcal{P}_1,\mathcal{P}_4) + v(\mathcal{L}_3,\mathcal{L}_4) + v(\mathcal{L}_1,\mathcal{L}_4) = v(\mathcal{P}_2,\mathcal{P}_3) + v(\mathcal{L}_1,\mathcal{L}_2) + v(\mathcal{L}_2,\mathcal{L}_3),$

(3) $\quad 2.v(\mathcal{L}_1,\mathcal{L}_2) + v(\mathcal{P}_1,\mathcal{P}_2) + v(\mathcal{P}_2,\mathcal{P}_3) = 2.v(\mathcal{L}_3,\mathcal{L}_4) + v(\mathcal{P}_3,\mathcal{P}_4) + v(\mathcal{P}_1,\mathcal{P}_4),$

(4) $\quad 2.v(\mathcal{L}_1,\mathcal{L}_4) + v(\mathcal{P}_1,\mathcal{P}_4) + v(\mathcal{P}_1,\mathcal{P}_2) = 2.v(\mathcal{L}_2,\mathcal{L}_3) + v(\mathcal{P}_2,\mathcal{P}_3) + v(\mathcal{P}_3,\mathcal{P}_4).$

PROOF. Except for the trivial cases, all possibilities are given by interpreting the suitable one-,two- and three-tuples in (va3) up to (va12) as the coordinates of the points $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ and the lines $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4$. One can check that (va3) up to (va12) imply exactly (1),(2),(3) and (4). Q.E.D.

Let $\mathcal{P}_1\ I\ \mathcal{L}_1\ I\ \mathcal{P}_2\ I\ \mathcal{L}_2\ I\ \mathcal{P}_3\ I\ \mathcal{L}_3\ I\ \mathcal{P}_4\ I\ \mathcal{L}_4\ I\ \mathcal{P}_1$ be as in the previous theorem, then we express that we use formula (1) of theorem(2.1.1.2) by saying that we ***apply the main property in the quadrangle*** $\mathcal{P}_1\ I\ \mathcal{L}_1\ I\ \mathcal{P}_2\ I\ \mathcal{L}_2\ I\ \mathcal{P}_3\ I\ \mathcal{L}_3\ I\ \mathcal{P}_4\ I\ \mathcal{L}_4\ I\ \mathcal{P}_1$ ***on the side*** $\mathcal{L}_1$ (or $\mathcal{L}_3$). Indeed, the right hand side of (1) is the sum of the valuations of pairs of points lying on $\mathcal{L}_1$ and the valuations of the pairs of lines of the (ordinary) quadrangle containing $\mathcal{L}_1$. Similarly for the right hand side. In the same way, we express that we use (3) by saying that we ***apply the main property in the quadrangle*** $\mathcal{P}_1\ I\ \mathcal{L}_1\ I\ \mathcal{P}_2\ I\ \mathcal{L}_2\ I\ \mathcal{P}_3\ I\ \mathcal{L}_3\ I\ \mathcal{P}_4\ I\ \mathcal{L}_4\ I\ \mathcal{P}_1$ ***on the vertex*** $\mathcal{P}_2$ (or $\mathcal{P}_4$). In this way, (3) is the dual of (1) (or of (2)), provided we double the valuations of pairs of lines when applying the main property on a vertex. Now note that the main property contains the only non trivial identities in valuations of pairs of points and pairs of lines. Hence every true equality has to follow from the main property. So we can define the dual of a formula as the formula obtained by interchanging in the proof the words side and vertex, point and line, etc... Note also that the formulas (3) and (4) (resp. (1) and (2)) follow from (1) and (2) (resp. (3) and (4)) by linear combination.

We now define $w(x,y) = v(x,y) - v(x) - v(y)$ for all pairs (x,y) in the domain of v and such that $v(x) < 0$ and $v(y) < 0$ (cp. the analogue for planar ternary rings with valuation [17], definition (2.2)).

LEMMA (2.1.1.3). *Let x_1, x_2, x_3, x_4 be four elements of either $\mathcal{R}_1$ or $\mathcal{R}_2$.*

If

$$v(x_1,x_2) + v(x_3,x_4) < v(x_1,x_3) + v(x_2,x_4)$$

then

$$v(x_1,x_2) + v(x_3,x_4) = v(x_1,x_4) + v(x_2,x_3).$$

PROOF. Suppose that $v(x_1,x_2) + v(x_3,x_4) \neq v(x_1,x_4) + v(x_2,x_3)$. Without loss of generality, we can assume that $v(x_1,x_2) + v(x_3,x_4) < v(x_1,x_4) + v(x_2,x_3)$ and $v(x_1,x_2) < v(x_1,x_3)$. By (v2), $v(x_1,x_2) = v(x_2,x_3)$ and so $v(x_3,x_4) < v(x_1,x_4)$, hence $v(x_3,x_4) = v(x_1,x_3)$. Thus, by assumption, $v(x_1,x_2) < v(x_2,x_4)$ implying $v(x_1,x_2) = v(x_1,x_4)$. But then $v(x_3,x_4) < v(x_2,x_3)$, so $v(x_3,x_4) = v(x_2,x_4)$. Now, we can rewrite $v(x_1,x_2) + v(x_3,x_4) < v(x_1,x_4) + v(x_2,x_3)$ as $v(x_1,x_2) + v(x_1,x_3) < v(x_1,x_2) + v(x_1,x_2)$ or $v(x_1,x_3) < v(x_1,x_2)$. But similarly we can rewrite $v(x_1,x_2) + v(x_3,x_4) < v(x_1,x_3) + v(x_2,x_4)$ as $v(x_1,x_2) < v(x_1,x_3)$, a contradiction. The result follows. **Q.E.D.**

PROPOSITION(2.1.1.4). *With the same notation as above, one has :*

(w1) $w(x,y) > 0$, *for all* (x,y) *in the domain of* w.

(w2) $w(x,y) = +\infty$ *if and only if* $x=y$.

(w3) w *is symmetric, i.e.* $w(x,y) = w(y,x)$ *for all* (x,y) *in the domain of* w.

(w4) $w(x,z) \geq \inf\{w(x,y),w(z,y)\}$ *and if* $v(x,y) \neq v(z,y)$, *then the equality holds (for all suitable* x,y,z*).*

PROOF. (w2) and (w3) follow directly from the analogous properties for v. We now show (w1). So let $v(x) < 0$ and $v(y) < 0$. If $v(x) \neq v(y)$, then $v(x,y) = \inf\{v(x),v(y)\}$ by (v2). Hence $w(x,y) = -\sup\{v(x),v(y)\} > 0$. So suppose $v(x) = v(y)$. Then $w(x,y) = v(x,y)-2.v(x) \geq v(x)-2.v(x) = -v(x) > 0$. We now show ($w$4). Note that by ($w$3), it suffices to show that, if $w(x,y) < w(z,y)$, then $w(x,z) = w(x,y)$. So let $w(x,y) < w(z,y)$. Then $v(x,y) - v(x) < v(z,y) - v(z)$ and hence $v(x,y) + v(z) < v(z,y) + v(x)$. But by the preceding lemma, applies on $x,y,z,0$, we get $v(x,y) + v(z) = v(x,z) + v(y)$ or $v(x,y) - v(y) = v(x,z) - v(z)$, hence $w(x,y) = v(x,y) - v(x) - v(y) = v(x,z) - v(x) - v(z) = w(x,z)$. **Q.E.D.**

Note that this proof does not use the algebraic operations we are dealing with, in contradistiction to the the proof of the analogous result for planar ternary rings in [17],remark(2.4). In this way, the above proof is more universal and will hold for every system with valuation satisfying (v1),(v2),(v3).

2.1.2. Recoordinatization of the generalized quadrangle V.

As the title of this paragraph indecates, we will recoordinatize $\mathcal{V}$. However, this recoordinatization heavily uses the properties of the valuation. In fact it is the complete analogue of the recoordinatization of projective planes coordinatized by a planar ternary ring with valuation described in [18],§5, but the way to get these new coordinates is in the present case much more complicated. Essentially, this is because a generalized quadrangle is not a linear space and hence, on the level of valuation, we can only "compare" varieties incident with a common variety (cp. the definition of valuation on pairs of lines and pairs of points in the previous paragraph) and there is no way to get around this.

Let (b,k,c) be the coordinates of a point $\mathcal{P}$ of $\mathcal{V}$ and suppose first that $b\neq 0$, $k\neq 0$ and $c\neq 0$. So $\mathcal{P}$ is not incident with [0,0]. Hence there exist unique $n \in \mathcal{R}_2$, $e \in \mathcal{R}_1$, such that $\mathcal{P}$ *I* $[n,e,0]$ *I* $(0,0,e)$ *I* [0,0]. Since $c\neq 0$, $\mathcal{P}$ is not incident with the line [0,0,0]. Since $k\neq 0$, the line $[b,k]$ does not meet [0,0,0]. Hence there exist unique $\ell, p \in \mathcal{R}_2$, $a,d \in \mathcal{R}_1$ such that $\mathcal{P}$ *I* $[p,d,\ell]$ *I* $(a,0,0)$ *I* [0,0,0]. Note that $\mathcal{P}$ is also not incident with [0] and not concurrent with (∞), hence there exists a unique $q \in \mathcal{R}_2$ such that $\mathcal{P}$ *I* $[0,c,q]$ *I* $(0,c)$ *I* [0].

PROPOSITION(2.1.2.1). *With the above notation, exactly one of the following statements holds :*

(A) $v(k), v(b), v(c) \geq 0$.

(B) $v(n), v(e) \geq 0$; $v(b) < 0$.

(C) $v(a) \geq 0$; $v(c), v(p) < 0$.

(D) $v(a), v(e), v(\ell) < 0$.

PROOF. We start by proving some properties concerning the valuation of the above defined elements of $\mathcal{R}_1$ and $\mathcal{R}_2$.

We apply the main property in the quadrangle $(0,c)\ I\ [0,c,0]\ I\ (0,0,c)\ I\ [0,0]\ I\ (0,0,e)\ I\ [n,e,0]\ I\ \mathcal{P}\ I\ [0,c,q]\ I\ (0,c)$ on the side $[0,0]$ and the vertex $(0,0,c)$:

$$v(c,e) = v(q) + v(n) = v(b) + 2.v(n) \tag{1}$$

We apply the main property in the quadrangle $\mathcal{P}\ I\ [n,e,0]\ I\ (0,0,e)\ I\ [0,0]\ I\ (0)\ I\ [0,\ell] \perp [p,d,\ell]\ I\ \mathcal{P}$ on the side $[0,0]$:

$$v(\ell) = v(b) + v(p,n) \tag{2}$$

We apply the main property in the quadrangle $\mathcal{P}\ I\ [p,d,\ell]\ I\ (a,0,0)\ I\ [0,0,0]\ I\ (0,0)\ I\ [0,0,k]\ I\ (b,k,0)\ I\ [b,k]\ I\ \mathcal{P}$ on the side $[b,k]$ and on the side $[p,d,\ell]$:

$$v(c) = v(k) + v(p) \tag{3}$$

$$v(a,b) = v(k) - v(p) = 2.v(k) - v(c) \tag{4}$$

We apply the main property in the quadrangle $\mathcal{P}\ I\ [n,e,0]\ I\ (0,0,e)\ I\ [0,0]\ I\ (0,0,0)\ I\ [0,0,0]\ I\ (a,0,0)\ I\ [p,d,\ell]\ \ I\ \mathcal{P}$ on the side $[0,0]$:

$$v(e) = v(p) + v(a,b) + v(p,n) \tag{5}$$

We apply the main property in the quadrangle $(a,0,0)$ I $[p,d,\ell]$ I (p,d) I $[p,d,0]$ I $(0,0,d)$ I $[0,0]$ I $(0,0,0)$ I $[0,0,0]$ I $(a,0,0)$ on the side $[p,d,0]$:

$$v(\ell) = v(a) + v(p) \tag{6}$$

We apply the main property in the quadrangle $\mathcal{P}$ I $[p,d,\ell]$ $\perp$ $[0,\ell]$ I (0) I $[0,q]$ I $(0,q,c)$ I $[0,c,q]$ I $\mathcal{P}$ on the side $[0,\ell]$:

$$v(q,\ell) = v(p) + v(b) \tag{7}$$

We apply the main property in the quadrangle $\mathcal{P}$ I $[n,e,0]$ I $(0,0,e)$ I $[0,0]$ I (0) I $[0,q]$ I $(0,q,c)$ I $(0,c,q)$ I $\mathcal{P}$ on the side $[0,0]$:

$$v(q) = v(n) + v(b) \tag{8}$$

We apply the main property in the quadrangle $\mathcal{P}$ I $[n,e,0]$ I $(0,0,e)$ I $[0,0]$ I $(0,0,d)$ I $[p,d,0]$ I (p,d) I $[p,d,q]$ I $\mathcal{P}$ on the side $[0,0]$:

$$v(d,e) = v(p,n) + v(\ell) \tag{9}$$

We apply the main property in the quadrangle $(a,0,0)$ I $[p,d,\ell]$ I (p,d) I $[p,d,0]$ I $(0,0,d)$ I $[0,0]$ I $(0,0,0)$ I $[0,0,0]$ I $(a,0,0)$ on the side $[0,0]$:

$$v(d) = v(p) + v(\ell) \tag{10}$$

We now show that at most one of the statements (A),(B),(C),(D) holds. Therefore, note (A) and (B), resp. (A) and (C) ; (B) and (D) ; (C) and (D), are mutually exclusive. So there remains to show that (A) and (D) resp. (B) and (C) can not occur simultanously.

Suppose first that both (A) and (D) hold. Then $v(k), v(b), v(c) \geq 0$ and $v(a), v(e), v(\ell) < 0$. So $v(a,b) = v(a)$ and by (4), $v(p) + v(a) = v(k) \geq 0$ and by (6), $v(p) + v(a) = v(\ell) < 0$, a contradiction.

Now suppose that both (B) and (C) hold. So $v(n), v(a), v(e) \geq 0$ and $v(b)$, $v(c), v(p) < 0$. By (5), $v(e) = v(a,b) + v(p) + v(p,n) = v(b) + 2.v(p) < 0$, contradicting our assumptions.

We now show that at least one of the following statements occurs :

(A') $v(b), v(c) \geq 0$.

(B') $v(e) \geq 0$ and $v(b) < 0$.

(C') $v(a) \geq 0$ and $v(c) < 0$.

(D') $v(a), v(e) < 0$.

Suppose that none of the statements (A'),(B'),(C'),(D') holds. In particular, (A') does not hold. There are two possibilities :

First possibility : $v(b) < 0$.

Since (B') does not hold, $v(e) < 0$, but since (D') does not hold, $v(a) \geq 0$ and since (C') does not hold, $v(c) \geq 0$.

By (3) and (4), $2.v(p) = v(c) - v(a,b) = v(c) - v(b) > 0$.

By (6), $v(\ell) = v(a) + v(p) > 0$.

By (10), $v(d) = v(p) + v(\ell) > 0$ and so $v(d,e) = v(e)$.

By (9), $v(p,n) = v(d,e) - v(\ell) = v(e) - v(\ell) < 0$.

By (2), we have $v(p,n) = v(\ell) - v(b) > 0$, a contradiction.

Second possibility : $v(b) \geq 0$.

Similarly as above, $v(c) < 0$, $v(a) < 0$ and $v(e) \geq 0$. If $v(p) < 0$, then by (6), $v(\ell) < 0$. By (2), $v(p,n) = v(\ell) - v(b) < 0$. By (5), $v(e) = v(a,b) + v(p) + v(p,n) < 0$, a contradiction. Hence $v(p) \geq 0$.

By (1), $2.v(n) = v(c,e) - v(b) = v(c) - v(b) < 0$, so $v(p,n) = v(n) < 0$.

By (4) and (5), $v(k) = v(e) - v(p,n) > 0$. By (4), $2.v(k) = v(a,b) + v(c) < 0$, a contradiction.

Hence at least one of the statements (A'),(B'),(C'),(D') holds.

Now we suppose that none of the statements (A),(B),(C),(D) holds. There are four distinct possibilities :

First possibility : (A') holds.

Then $v(b), v(c) \geq 0$ and since (A) does not hold, $v(k) < 0$.

By (4), $v(a,b) = 2.v(k) - v(c) < 0$, so $v(a,b) = v(a) < 0$.

By (3), $v(p) = v(c) - v(k) > 0$.

By (4) and (6), $v(\ell) = v(a) + v(p) = v(a,b) + v(p) = v(k) < 0$.

By (7), $v(q,\ell) = v(p) + v(b) > 0$, so $v(q) = v(\ell) < 0$.

By (8), $v(n) = v(q) - v(b) < 0$.

By (1), $v(c,e) = v(q) + v(n) < 0$, so $v(c,e) = v(e) < 0$.

But all this implies that (D) holds, a contradiction.

Second possibility : (B') holds.

Then $v(e) \geq 0$, $v(b) < 0$ and since (B) does not hold, $v(n) < 0$.

By (1), $v(c,e) = v(b) + 2.v(n) < 0$, so $v(c,e) = v(c) < 0$.

Suppose first that $v(k) < 0$.

By (4) and (5), $v(p,n) = v(e) - v(k) > 0$, so $v(p) = v(n) < 0$.

Since (C) does not hold, $v(a) < 0$.

By (6), $v(\ell) = v(a) + v(p) < 0$.

By (10), $v(d) = v(p) + v(\ell) < 0$ and so $v(d,e) = v(d) < 0$.

By (9), $v(p,n) = v(d) - v(\ell) = v(p) < 0$, contradicting the earlier value $v(p,n) > 0$. Hence $v(k) \geq 0$.

By (3), $v(p) = v(c) - v(k) < 0$. Since (C) does not hold, $v(a) < 0$ and similarly as above, $v(p,n) = v(p)$.

By (4) and (5), $v(k) + v(p,n) = v(k) + v(p) = v(e) \geq 0$.

By (3), $v(k) + v(p) = v(c) < 0$, contradicting the preceding line.

Third possibility : (C') holds.

Then $v(a) \geq 0$, $v(c) < 0$ and since (C) does not hold, $v(p) \geq 0$.

By (3), $v(k) = v(c) - v(p) < 0$.

By (4), $v(a,b) = v(k) - v(p) < 0$, so $v(b) = v(a,b) < 0$.

By (6), $v(\ell) = v(a) + v(p) \geq 0$.

By (2), $v(p,n) = v(\ell) - v(b) > 0$, so $v(n) > 0$.

Since (B) does not hold, $v(e) < 0$.

By (10), $v(d) = v(p) + v(\ell) \geq 0$, so $v(d,e) = v(e) < 0$.

By (5) and (9), $v(\ell) - v(p) = v(b) < 0$.

By (6), $v(\ell) - v(p) = v(a) \geq 0$, a contradiction.

Fourth possibility : (D') holds.

Then $v(a), v(e) < 0$ and since (D) does not hold, $v(\ell) \geq 0$.

By (6), $v(p) = v(\ell) - v(a) > 0$.

By (10), $v(d) = v(p) + v(\ell) > 0$, so $v(d,e) = v(e) < 0$.

By (9), $v(p,n) = v(e) - v(\ell) < 0$, so $v(n) = v(p,n) < 0$.

By (2), $v(b) = v(\ell) - v(n) > 0$, so $v(a,b) = v(a) < 0$.

By (6), $v(a) + v(p) = v(\ell) \geq 0$.

By (4), $v(k) = v(a) + v(p) \geq 0$.

By (3), $v(c) = v(k) + v(p) > 0$.

All this implies that (A) holds, a contradiction.

This completes the proof of the proposition. **Q.E.D.**

We now introduce the new coordinates for points of $\mathcal{V}$ having coordinates (b,k,c) with $b \neq 0$, $k \neq 0$ and $c \neq 0$. Denoting the new coordinates with double parentheses, we define (with the above notation) :

$$
\begin{aligned}
(b,k,c) &= ((b,k,c)) \text{ if } v(b), v(k), v(c) \geq 0, \\
&= ((b,n,e)) \text{ if } v(n), v(e) \geq 0 \text{ and } v(b) < 0, \\
&= ((a,p,c)) \text{ if } v(a) \geq 0 \text{ and } v(p), v(c) < 0, \\
&= ((a,\ell,e)) \text{ if } v(a), v(\ell), v(e) < 0.
\end{aligned}
$$

This is well defined by the previous proposition.

Our next goal is to recoordinatize the points having old coordinates $(0,k,c)$ with $k \neq 0$ and $c \neq 0$. So let $\mathcal{P}$ be a point of $\mathcal{V}$ with coordinates $(0,k,c)$ with $k \neq 0$ and $c \neq 0$. Then $\mathcal{P}$ is neither incident with the line $[0,0,0]$ nor collinear with the point $(0,0)$. The line $[0,k]$ does not meet the line $[0,0,0]$. Consequently, there exist unique $a,d \in \mathcal{R}_1$, $p \in \mathcal{R}_2$ such that $\mathcal{P}\ I\ [p,d,k]\ I\ (a,0,0)\ I\ [0,0,0]$.

PROPOSITION (2.1.2.2). *With the above notation,* ***exactly one*** *of the following statements holds :*

(E) $v(k), v(c) \geq 0$.

(F) $v(a) \geq 0$; $v(p), v(c) < 0$.

(G) $v(a), v(k) < 0$.

PROOF. Apparently, these statements are pairwise mutually exclusive. We now show that at least one of them must hold.

We apply the main property in the quadrangle $\mathcal{P}$ I $[0,k]$ I (0) I $[0,0]$ I $(0,0,0)$ I $[0,0,0]$ I $(a,0,0)$ I $[p,d,k]$ I $\mathcal{P}$ on the side $[0,0]$:

$$v(k) = v(a) + v(p) \tag{1}$$

We apply the main property in the quadrangle $\mathcal{P}$ I $[0,c,k]$ I $(0,c)$ I $[0]$ I $(0,0)$ I $[0,0,0]$ I $(a,0,0)$ I $[p,d,k]$ I $\mathcal{P}$ on the side $[0]$:

$$v(c) = v(a) + 2.v(p) \tag{2}$$

Eliminating $v(a)$ resp. $v(p)$ in (1) and (2), we obtain

$$v(c) = v(k) + v(p) \tag{3}$$

$$v(c) = 2.v(k) - v(a) \tag{4}$$

Now suppose that none of the statements (E),(F),(G) holds. In particular, (E) does not hold. There are three possibilities to consider.

First possibility : $v(k), v(c) < 0$.

Since (G) does not hold, $v(a) \geq 0$ and since (F) does not hold, $v(p) \geq 0$. But by (1), $v(k) = v(a) + v(p) \geq 0$, a contradiction.

Second possibility : $v(k) < 0$ and $v(c) \geq 0$.

Since (G) does not hold, $v(a) \geq 0$. But by (4), $v(c) = 2.v(k) - v(a) < 0$, a contradiction.

Third possibility : $v(k) \geq 0$ and $v(c) < 0$.

By (3), $v(p) = v(c) - v(k) < 0$. Since (F) does not hold, $v(a) < 0$. But by (1), $v(a) = v(k) - v(p) > 0$, a contradiction. Q.E.D.

We recoordinatize $\mathcal{P} = (0,k,c)$ with $k \neq 0$ and $c \neq 0$ as follows (using the above notation).

$$(0,k,c) = \begin{cases} ((0,k,c)) & \text{if } v(k),v(c) \geq 0, \\ ((a,p,c)) & \text{if } v(a) \geq 0 \text{ and } v(p),v(c) < 0, \\ ((a,k,\infty)) & \text{if } v(a),v(k) < 0. \end{cases}$$

By the preceding proposition, this is well defined.

We will now recoordinatize points $\mathcal{P}$ having three coordinates $(b,0,c)$ with $b \neq 0$ and $c \neq 0$. Since $b \neq 0$, $\mathcal{P}$ is not collinear with (0) and hence there exist unique $e \in \mathcal{R}_1$, $n \in \mathcal{R}_2$ such that $\mathcal{P}\ I\ [n,e,0]\ I\ (0,0,e)\ I\ [0,0]$. Since $\mathcal{P}$ is not collinear with (∞), there exists a unique $q \in \mathcal{R}_2$ such that $\mathcal{P}\ I\ [0,c,q]\ I\ (0,c)\ I\ [0]$.

PROPOSITION(2.1.2.3). *With the above notation,* ***exactly one*** *of the following statements holds :*

(H) $v(b),v(c) \geq 0$,

(I) $v(e),v(n) \geq 0\ ;\ v(b) < 0$,

(J) $v(b) \geq 0\ ;\ v(c) < 0$,

(K) $v(b), v(e) < 0$.

PROOF. Apparently, these statements are pairwise mutually exclusive. We show that at least one of them must hold. Therefore, we apply the main property in the quadrangle $\mathcal{P}\ I\ [n,e,0]\ I\ (0,0,e)\ I\ [0,0]\ I\ (0,0,0)\ I\ [0,0,0]\ I\ (b,0,0)\ I\ [b,0]\ I\ \mathcal{P}$ on the side $[0,0]$ and get $v(e) = v(c)$. Putting

(I') $v(e) \geq 0\ ;\ v(b) < 0$,

then since $v(e) = v(c)$, exactly one of the statements (H), (I'), (J), (K) holds. So if we show that (I) is equivalent to (I'), then we are done. Well, (I) implies (I'). Suppose now (I') holds, i.e. $v(e) = v(c) \geq 0$ and $v(b) < 0$. Then also $v(c,e) \geq 0$. We apply the main property in the quadrangle $\mathcal{P}\ I\ [n,e,0]\ I\ (0,0,e)\ I\ [0,0]\ I\ (0,0,c)\ I\ [0,c,0]\ I\ (0,c)\ I\ [0,c,q]\ I\ \mathcal{P}$ on the side $[0,0]$ and we get $v(c,e) = v(b) + 2.v(n)$, hence $2.v(n) = v(c,e) - v(b) \geq 0$ and (I) follows.

This completes the proof of the proposition. **Q.E.D.**

We recoordinatize the points $(b,0,c)$ with $b \neq 0$ and $c \neq 0$ as follows (using the notation above) :

$$\begin{aligned}(b,0,c) &= ((b,0,c)) \text{ if } v(b), v(c) \geq 0,\\ &= ((b,n,e)) \text{ if } v(e), v(n) \geq 0 \text{ and } v(b) < 0,\\ &= ((b,\infty,c)) \text{ if } v(b) \geq 0 \text{ and } v(c) < 0,\\ &= ((b,\infty,e)) \text{ if } v(b), v(e) < 0.\end{aligned}$$

This is well defined by the previous proposition.

Note that the above proof tells us that in the second case, the valuation of n is always positive (see below).

We now consider points having coordinates $(b,k,0)$ with $b\neq 0$ and $k\neq 0$. So let $\mathcal{P}$ be such a point. Then $\mathcal{P}$ is not concurrent with (0) and hence there exist unique $e\in\mathcal{R}_1$, $n\in\mathcal{R}_2$, such that $\mathcal{P}\ I\ [n,e,0]\ I\ (0,0,e)\ I\ [0,0]$.

PROPOSITION(2.1.2.4). *With the above notation,* ***exactly one*** *of the following statements holds :*

(L) $v(k), v(b) \geq 0$,

(M) $v(n), v(e) \geq 0$; $v(b) < 0$,

(N) $v(k), v(e) < 0$.

PROOF. Clearly, these statements are pairwise mutually exclusive. We show that at least one of them must hold. We apply the main property in the quadrangle $\mathcal{P}\ I\ [n,e,0]\ I\ (0,0,e)\ I\ [0,0]\ I\ (0,0,0)\ I\ [0,0,0]\ I\ (0,0)\ I\ (0,0,k)\ I\ \mathcal{P}$ on the side $[0,0]$ resp. the side $[0,0,0]$ and get :

$$v(e) = v(k) + v(n) \tag{1}$$

$$v(k) = v(b) + v(n) \tag{2}$$

Suppose that none of the statements (L),(M),(N) holds. Since in particular (L) does not hold, there are three distinct possibilities to consider (concerning the valuation of k and b) :

First possibility : $v(k), v(b) < 0$.

Since (N) does not hold, $v(e) \geq 0$ and since (M) does not hold, $v(n) < 0$.

But by (1), $v(n) = v(e) - v(k) > 0$, a contradiction.

Second possibility : $v(k) < 0$ and $v(b) \geq 0$.

Since (N) does not hold, $v(e) \geq 0$. By (1), $v(n) = v(e) - v(k) > 0$, but by (2), $v(n) = v(k) - v(b) < 0$, a contradiction.

Third possibility : $v(k) \geq 0$ and $v(b) < 0$.

By (2), $v(n) = v(k) - v(b) > 0$. Since (M) does not hold, $v(e) < 0$. But by (1), $v(e) = v(k) + v(n) > 0$, a contradiction. **Q.E.D.**

We recoordinatize the points having coordinates $(b,k,0)$ with $b\neq 0$ and $k\neq 0$ as follows (with the notation above).

$$\begin{aligned}(b,k,0) &= ((b,k,0)) \quad \text{if} \quad v(k),v(b) \geq 0,\\ &= ((b,n,e)) \quad \text{if} \quad v(n),v(e) \geq 0 \text{ and } v(b) < 0,\\ &= ((\infty,k,e)) \quad \text{if} \quad v(k),v(e) < 0.\end{aligned}$$

All other points having three coordinates can be recoordinatized without preceding proposition as follows.

$$(b,0,0) = ((b,0,0)),$$

$$\begin{aligned}(0,k,0) &= ((0,k,0)) \quad \text{if} \quad v(k) \geq 0,\\ &= ((\infty,k,\infty)) \quad \text{if} \quad v(\ell) < 0,\end{aligned}$$

$$\begin{aligned}(0,0,c) &= ((0,0,c)) \quad \text{if} \quad v(c) \geq 0,\\ &= ((0,\infty,c)) \quad \text{if} \quad v(c) < 0.\end{aligned}$$

We next consider points with two coordinates. Let first $\mathcal{P}$ be a point having coordinates (k,b) with $k\neq 0$ and $b\neq 0$. Since $k\neq 0$, there exist unique $a \in \mathcal{R}_1$, $\ell \in \mathcal{R}_2$ such that $\mathcal{P}\ I\ [k,b,\ell]\ I\ (a,0,0)\ I\ [0,0,0]$.

PROPOSITION(2.1.2.5). *With the above notation,* ***exactly one*** *of the following statements holds :*

(P) $v(k), v(b) \geq 0$,

(Q) $v(a) \geq 0$; $v(k) < 0$,

(R) $v(a), v(b), v(\ell) < 0$.

PROOF. Clearly, these statements are pairwise mutually exclusive. We now show that at least one of them must hold. We apply the main property in the quadrangle $\mathcal{P}\ I\ [k,b,0]\ I\ (0,0,b)\ I\ [0.0]\ I\ (0,0,0)\ I\ [0,0,0]\ I\ (a,0,0)\ I\ [k,b,\ell]\ I\ \mathcal{P}$ on the side $[0,0,0]$ resp. the side $[0,0]$ and we get :

$$v(\ell) = v(a) + v(k) \quad (1)$$

$$v(b) = v(\ell) + v(k) \quad (2)$$

Suppose that none of the statements (P),(Q),(R) holds. Since (P) does not hold, there are three distinct possibilities concerning the valuation of k and b :

First possibility *:* $v(k), v(b) < 0$.

Since (Q) does not hold, $v(a) < 0$ and since (R) does not hold, $v(\ell) \geq 0$. But by (1), $v(\ell) = v(a) + v(k) < 0$, a contradiction.

Second possibility : $v(k) < 0$ and $v(b) \geq 0$.

Since (Q) does not hold, again $v(a) < 0$. By (1), $v(\ell) = v(a) + v(k) < 0$. But by (2), $v(\ell) = v(b) - v(k) > 0$, a contradiction.

Third possibility : $v(k) \geq 0$ and $v(b) < 0$.

By (2), $v(\ell) = v(b) - v(k) < 0$, hence $v(a) \geq 0$ since (R) does not hold. But by (1), $v(a) = v(\ell) - v(k) < 0$, a contadiction. This completes the proof of the proposition. **Q.E.D.**

With the notation above, we put :

$$
\begin{aligned}
(k,b) &= ((\infty,k,b)) \quad \text{if} \quad v(k),v(b) \geq 0,\\
&= ((a,k,\infty)) \quad \text{if} \quad v(a) \geq 0 \text{ and } v(k) < 0,\\
&= ((a,\ell,b)) \quad \text{if} \quad v(a),v(\ell),v(b) < 0.
\end{aligned}
$$

Furthermore, we define

$$
\begin{aligned}
(0,b) &= ((\infty,0,b)) \quad \text{if} \quad v(b) \geq 0,\\
&= ((\infty,\infty,b)) \quad \text{if} \quad v(b) < 0,\\
(k,0) &= ((\infty,k,0)) \quad \text{if} \quad v(k) \geq 0,\\
&= ((0,k,\infty)) \quad \text{if} \quad v(k) < 0,
\end{aligned}
$$

For points with exactly one (old) coordinate, we put :

$$(x) = ((x,\infty,\infty)), \qquad x \in \mathcal{R}_1 \cup \{\infty\}.$$

Hence we have now recoordinatized all points by means of three-tuples. Let us take a closer look at this recoordinatization. Therefore, we put

$v(\infty) = -\infty < 0$. We introduce the following notation. If $x \in \mathcal{R}_1 \cup \mathcal{R}_2 \cup \{\infty\}$ and $v(x) < 0$, then we write sometimes x^-. If $v(x) \geq 0$, then we denote sometimes x^+. If we write simply x, then we leave the two possibilities open. We now identify the following objects.

$$(0) \longleftrightarrow (0,0,\infty)$$
$$(0,0) \longleftrightarrow (\infty,0,0)$$
$$(\infty) \longleftrightarrow (0,\infty)$$

$$[0] \longleftrightarrow [0,0,\infty]$$
$$[0,0] \longleftrightarrow [\infty,0,0]$$
$$[\infty] \longleftrightarrow [0,\infty]$$

Note that these identifications are very natural. For instance, for all $a \in \mathcal{R}_1$, $(a,0,0)$ is collinear with (a). Extending this to $\mathcal{R}_1 \cup \{\infty\}$, this becomes : $(\infty,0,0) = (0,0)$ is collinear with (∞).

In view of the previous propositions, there are exactly ***four*** distinct kinds of coordinate three-tuples for points :

(1) $((a^+, \ell^+, a'^+))$

(2) $((a^-, \ell^+, a'^+))$

(3) $((a^+, \ell^-, a'^-))$

(4) $((a^-, \ell^-, a'^-))$

We leave it to the reader to show that conversely, every three-tuple as above defines a unique point. The following short discussion may be helpful. In the meantime, we also define the ***shape*** of a point.

(1) Let $\mathcal{P}$ be a point with coordinates $((a^+, \ell^+, a'^+))$. Then $\mathcal{P}$ has shape (+,+,+) and we denote $\nu(\mathcal{P})$ = (+,+,+). In this case, old and new coordinates coincide. Hence $\mathcal{P}$ is collinear with both (a) and $(0,a')$ and the line through (0,0) meeting the joining line of $\mathcal{P}$ and (a) is the line $[0,0,\ell]$ (see figure 1).

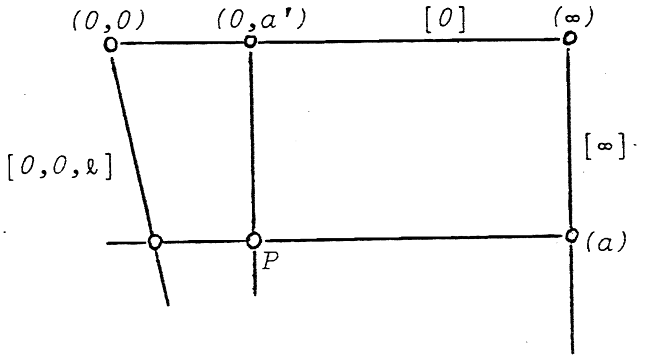

FIGURE 1

(2) Let $\mathcal{P}$ be a point with coordinates $((a^-, \ell^+, a'^+))$. Then $\mathcal{P}$ has shape (-,+,+) and we denote $\nu(\mathcal{P})$ = (-,+,+). Here, $\mathcal{P}$ is collinear with both (a) (a might be ∞ !) and $(0,0,a')$ and the line through (∞) meeting the joining line of $\mathcal{P}$ and $(0,0,a')$ is the line $[\ell]$ (see figure 2).

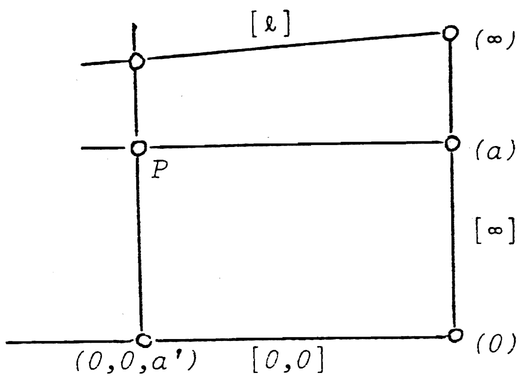

FIGURE 2

(3) Let $\mathcal{P}$ be a point with coordinates $((a^+, \ell^-, a'^-))$. Then $\mathcal{P}$ has shape (+,−,−) and we denote $\nu(\mathcal{P})$ = (+,−,−). Here, $\mathcal{P}$ is collinear with both $(a,0,0)$ and $(0,a')$ and the line passing through (∞) meeting the joining line of $\mathcal{P}$ and $(a,0,0)$ is the line $[\ell]$ (possibly $[\infty]$!) (see figure 3).

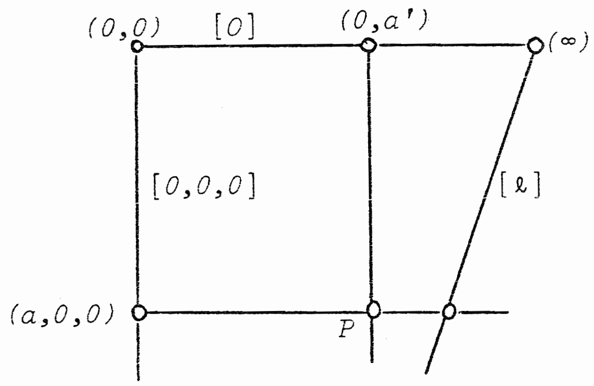

FIGURE 3

(4) Let $\mathcal{P}$ be a point with coordinates $((a^-, \ell^-, a'^-))$. Then $\mathcal{P}$ has shape (−,−,−) and we denote $\nu(\mathcal{P})$ = (−,−,−). Here, $\mathcal{P}$ is collinear with both $(a,0,0)$ and $(0,0,a')$ and the line passing through (0) meeting the joining line of $\mathcal{P}$ and $(a,0,0)$ is the line $[0,\ell]$ (see figure 4).

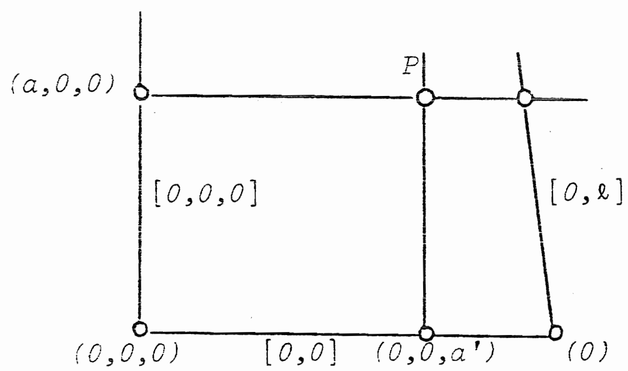

FIGURE 4

Similarly, one can recoordinatize the lines of $\mathcal{V}$. In a completely dual way, one shows results analoguously to propositions(2.1.2.1),(2.1.2.2), (2.1.2.3),(2.1.2.4) and (2.1.2.5). We obtain exactly ***four*** distinct types of coordinate three-tuples for lines :

(1) $[[k^+, b^+, k'^+]]$

(2) $[[k^-, b^+, k'^+]]$

(3) $[[k^+, b^-, k'^-]]$

(4) $[[k^-, b^-, k'^-]]$

Again, every three-tuple of one of the types above defines a unique line. We discuss briefly these types and define the ***shape*** of a line.

(1) Let $\mathcal{L}$ be a line with coordinates $[[k^+, b^+, k'^+]]$. Then $\mathcal{L}$ has shape [+,+,+] and we denote $\nu(\mathcal{L})$ = [+,+,+]. Old and new coordinates coincide. Hence $\mathcal{L}$ is concurrent with both $[k]$ and $[0,k']$ and the meeting point of $\mathcal{L}$ and $[k]$ is collinear with $(0,0,b)$ (see figure 5).

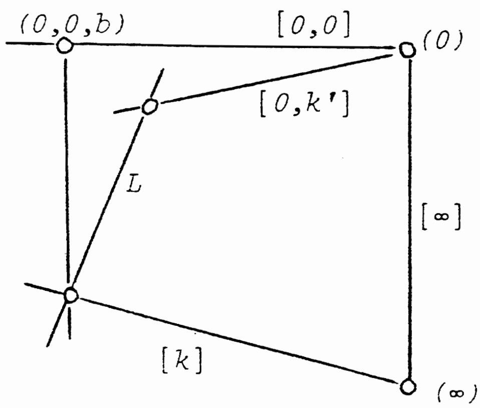

FIGURE 5

(2) Let $\mathcal{L}$ be a line with coordinates $[[k^-, b^+, k'^+]]$. Then $\mathcal{L}$ has shape [-,+,+] and we denote $\nu(\mathcal{L})$ = [-,+,+]. Here, $\mathcal{L}$ is concurrent with both $[k]$ and $[0,0,k']$ and the meeting point of $\mathcal{L}$ and $[0,0,k']$ is collinear with (b) (see figure 6).

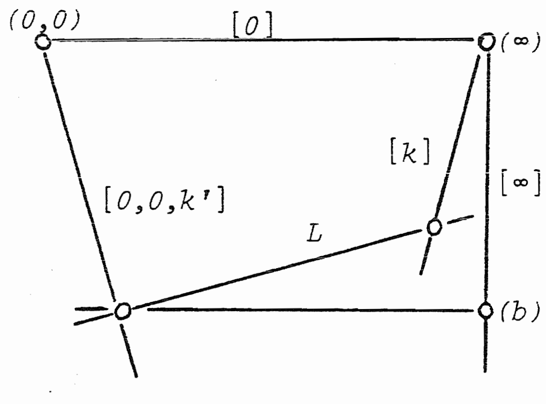

FIGURE 6

(3) Let $\mathcal{L}$ be a line with coordinates $[[k^+, b^-, k'^-]]$. Then $\mathcal{L}$ has shape [+,-,-] and we denote $\nu(\mathcal{L})$ = [+,-,-]. Here, $\mathcal{L}$ is concurrent with both $[k,0,0]$ and $[0,k']$ and the meeting point of $\mathcal{L}$ and $[k,0,0]$ is collinear with (b) (see figure 7).

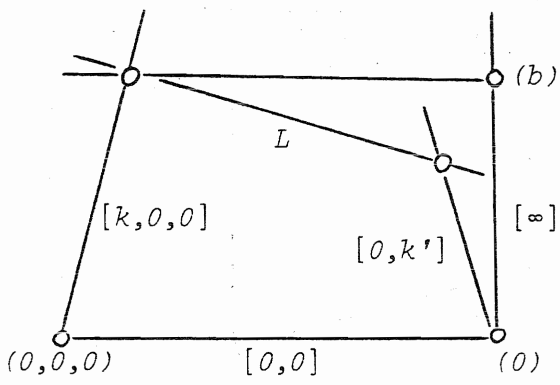

FIGURE 7

(4) Let $\mathcal{L}$ be a line with coordinates $[[k^-, b^-, k'^-]]$. Then $\mathcal{L}$ has shape [-,-,-] and we denote $\nu(\mathcal{L})$ = [-,-,-]. Here, $\mathcal{L}$ is concurrent with both $[k,0,0]$ and $[0,0,k']$ and the meeting point of $\mathcal{L}$ and $[k,0,0]$ is collinear with $(0,b)$ (see figure 8).

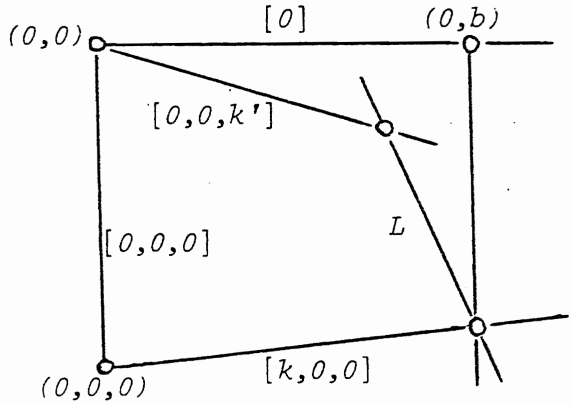

FIGURE 8

This recoordinatization will unable us to define a *partial valuation* on the set of pairs of points and pairs of lines of $\mathcal{V}$, similarly to the case of a projective plane (see [17],[18]). But in the latter case, the n^{th} floor was obtained by identifying all points resp. all lines whose mutual partial valuation was bigger than or equal to n. But here, in the case of generalized quadrangles, we will need a different partial valuation (which, if generalized to projective planes, coincides in that case with the original partial valuation). To make the distinction between the two partial valuations, we will denote the first one by u^* and call it the *partial *-valuation*. The reason for this strange behaviour (compared to the case of projective planes) is essentially due to the fact that a generalized quadrangle is not a linear space (see below for more explanation).

*2.1.3. The partial *-valuation.*

In this paragraph, we define the partial *-valuation we mentioned above. We will prove some properties about it and later, we will use these to investigate the *proper* partial valuation. Our goal is namely to extend the main property to the partial *-valuation and later on, we will prove a kind of dual main property for the partial valuation.

Throughout, $\mathcal{P}$ and $\mathcal{Q}$ denote points and $\mathcal{L}$ and $\mathcal{M}$ denote lines of $\mathcal{V}$. In the first paragraph of this section, we introduced the notation $w(x,y)$ for elements with negative valuation. Since we introduced in the previous paragraph $v(\infty) = -\infty$, we extend w to pairs containing ∞ as follows.

$$w(x,\infty) = w(\infty,x) = -v(x)$$

and hence in particular $w(\infty,\infty) = +\infty$. We now define the ***partial *-valuation*** u^* as follows. Let $a,a',b,c,c',d \in \mathcal{R}_1 \cup\{\infty\}$, $k,k',\ell,m,n,n' \in \mathcal{R}_2 \cup\{\infty\}$, then we put

$$u^*(((a^+,\ell^+,a'^+)),((c^+,m^+,c'^+))) = \inf\{v(a,c),2.v(\ell,m),v(a',c')\},$$

$$u^*(((a^-,\ell^+,a'^+)),((c^-,m^+,c'^+))) = \inf\{w(a,c),2.v(\ell,m),v(a',c')\},$$

$$u^*(((a^+,\ell^-,a'^-)),((c^+,m^-,c'^-))) = \inf\{v(a,c),2.w(\ell,m),w(a',c')\},$$

If $\mathcal{P}$ and $\mathcal{Q}$ have distinct shape, then $u(\mathcal{P},\mathcal{Q}) = 0$,

$$u^*([[k^+,b^+,k'^+]],[[n^+,d^+,n'^+]]) = \inf\{v(k,n),v(b,d),v(k',n')\},$$

$$u^*([[k^-,b^+,k'^+]],[[n^-,d^+,n'^+]]) = \inf\{w(k,n),v(b,d),v(k',n')\},$$

$$u^*([[k^+,b^-,k'^-]],[[n^+,d^-,n'^-]]) = \inf\{v(k,n),w(b,d),w(k',n')\},$$

If $\mathcal{L}$ and $\mathcal{M}$ have distinct shape, then $u(\mathcal{L},\mathcal{M}) = 0$.

PROPOSITION(2.1.3.1). *Let* $\mathcal{L} = [[k^+,b^+,k'^+]]$, $\mathcal{M} = [[n^-,d^+,n'^+]]$ *and suppose* $\mathcal{L}\ I\ \mathcal{P}\ I\ \mathcal{M}$ *with* $\nu(\mathcal{P}) = (+,+,+)$. *Then* $u^*(\mathcal{L},\mathcal{M}) = v(k,n)$.

PROOF. Let $\mathcal{P} = ((a,\ell,a'))$. We apply the main property in the quadrangle $\mathcal{P}\ I\ \mathcal{L} \perp [0,k']\ I\ (0)\ I\ [0,n'] \perp \mathcal{M}\ I\ \mathcal{P}$ on the side $[0,n']$ and get

$$v(k',n') = v(k,n) + v(a) \geq v(k,n).$$

Consider the line $[n,b,0]$. If $\mathcal{P}\ I\ [n,b,0]$, then $\mathcal{P} \perp (n,b) \perp (n,d) \perp \mathcal{P}$ is a triangle unless $d=b$, but then $v(b,d) = \infty > v(k,n)$. So we may assume that $\mathcal{P}$ is not incident with $[n,b,0]$. Let $\mathcal{K}$ and Q be such that $\mathcal{P}\ I\ \mathcal{K}\ I\ Q\ I\ [n,b,0]$. Note that Q has three (old) coordinates, so put $Q = (A,B,C)$. But for $\mathcal{K}$, there are two distinct possibilities.

First possibility : $\mathcal{K} = [X,Y,Z]$.

We apply the main property in the quadrangle $\mathcal{P}\ I\ \mathcal{K}\ I\ Q\ I\ [n,b,0]\ I\ (n,b)\ I\ [n]\ I\ (n,d)\ I\ \mathcal{M}\ I\ \mathcal{P}$ on the side $[n]$:

$$v(b,d) = v(a,A) + 2.v(n,X) \tag{1}$$

We apply the main property in the quadrangle $\mathcal{P}\ I\ \mathcal{K}\ I\ Q\ I\ [n,b,0]\ I\ (0,0,b)\ I\ [k,b,0]\ I\ (k,b)\ I\ \mathcal{L}\ I\ \mathcal{P}$ on the vertex Q resp. the vertex $\mathcal{P}$:

$$2.v(k') = 2.v(n,X) + v(a,A) + v(A) \tag{2}$$

$$2.v(n,k) + v(A) = 2.v(k,X) + v(a,A) \tag{3}$$

By (2) and (3), $v(a,A) = v(n,k) + v(k') - v(n,X) - v(k,X)$. Substituting this in (1), we get

$$v(d,b) = v(n,k) + v(k') + v(n,X) - v(k,X) \tag{4}$$

If $v(n,X) \geq v(k,X)$, then since $v(k') \geq 0$ and by (4), $v(d,b) \geq v(n,k)$.

Hence in this case the result follows. So let $v(n,X) < v(k,X)$. Then $v(n,X) = v(n,k)$ and (3) implies $v(A) > v(a,A)$ and thus $v(A) > v(a,A) = v(a) \geq 0$. But by (1), $v(b,d) = 2.v(n,k) + v(a) \geq v(n,k)$.

Second possibility : $\mathcal{K} = [X,Y]$.

We apply the main property in the quadrangle $\mathcal{P}\ I\ \mathcal{K}\ I\ Q\ I\ [n,b,0]\ I\ (n,b)\ I\ [n]\ I\ (n,d)\ I\ \mathcal{M}\ I\ \mathcal{P}$ on the side $[n]$:

$$v(b,d) = v(a',C) \qquad (5)$$

We apply the main property in the quadrangle $\mathcal{P}\ I\ \mathcal{K}\ I\ Q\ I\ [n,b,0]\ I\ (0,0,b)\ I\ [k,b,0]\ I\ (k,b)\ I\ \mathcal{L}\ I\ \mathcal{P}$ on the side $\mathcal{K}$:

$$v(a',C) = v(k,n) + v(k') \qquad (6)$$

Eliminating $v(a',C)$ in (5) and (6), one gets $v(b,d) = v(k,n) + v(k') \geq v(k,n)$. This completes the proof of the theorem. **Q.E.D.**

PROPOSITION(2.1.3.2). *Let* $\mathcal{L} = [[k^-, b^+, k'^+]]$, $\mathcal{M} = [[n^-, d^+, n'^+]]$ *and suppose* $\mathcal{L}\ I\ \mathcal{P}\ I\ \mathcal{M}$ *met* $\nu(\mathcal{P}) = (+,+,+)$. *Then* $u^*(\mathcal{L},\mathcal{M}) = w(k,n)$.

PROOF. Put $\mathcal{P} = ((a,\ell,a'))$ and suppose first that $k \neq \infty \neq n$. We apply the main property in the quadrangle $\mathcal{P}\ I\ \mathcal{M}\ I\ (d,n',0)\ I\ [0,0,n']\ I\ (0,0)\ I\ [0,0,k']\ I\ (b,0,0)\ I\ \mathcal{L}\ I\ \mathcal{P}$ on the side $[0,0,k']$:

$$v(k',n') + v(k) = v(k,n) + v(a,d) + v(n) \qquad (1)$$

We apply the main property in the quadrangle $\mathcal{P}\ I\ \mathcal{M}\ I\ (d,n',0)\ I\ [0,0,n']\ I\ (0,0)\ I\ [0]\ I\ (0,a')\ \perp\ \mathcal{P}$ on the side $[0]$:

$$v(a') = v(a,d) + 2.v(n) \tag{2}$$

Eliminating $v(a,d)$ in (1) and (2), we get

$$\begin{aligned} v(k',n') &= v(k,n) - v(k) - v(n) + v(a') \\ &= w(k,n) + v(a') \geq w(k,n) \end{aligned} \tag{3}$$

Consider the line $[d,k']$. Suppose first that $\mathcal{P}$ is not incident with $[d,k']$. Let $\mathcal{K} = [X,Y,Z]$ be such that $\mathcal{P}\ I\ \mathcal{K} \perp [d,k']$. We apply the main property in the quadrangle $\mathcal{P}\ I\ \mathcal{M}\ I\ (d,n',0)\ I\ [d,n']\ I\ (d)\ I\ [d,k'] \perp \mathcal{K}\ I\ \mathcal{P}$ on the side $\mathcal{M}$:

$$v(a,d) + v(X,n) = v(k',n') \tag{4}$$

Next, we apply the main property in the quadrangle $\mathcal{P}\ I\ \mathcal{L}\ I\ (b,k',0)\ I\ [0,0,k']\ I\ (d,k',0)\ I\ [d,k'] \perp \mathcal{K}\ I\ \mathcal{P}$ on the side $[0,0,k']$:

$$v(b,d) + v(k) = v(a,d) + v(X,k) \tag{5}$$

We eliminate $v(a,d)$ in (4) and (5) and obtain

$$v(b,d) = v(k',n') - v(X,n) + v(X,k) - v(k) \tag{6}$$

If $v(X,k) \geq v(X,n)$, then (6) implies $v(b,d) \geq v(k',n') - v(k) > v(k',n')$ and the result follows. So suppose $v(X,k) < v(X,n)$. Then $v(k,n) = v(X,k)$. We plug this in in (6) :

$$\begin{aligned} v(b,d) &= v(k',n') - v(X,n) + v(k,n) - v(k) - v(n) + v(n) \\ &= v(k',n') - v(X,n) + w(k,n) + v(n) \end{aligned} \tag{7}$$

By (4) resp.(2) we can rewrite (7) as

$$v(b,d) = v(a,d) + w(k,n) + v(n)$$

$$= w(k,n) + v(a') - v(n) > w(k,n).$$

This shows the proposition for $k \neq \infty \neq n$ completely.

Now suppose $k = \infty \neq n$. Then $\mathcal{L} = [b,k']$. We apply the main property in the quadrangle $\mathcal{P}\ I\ \mathcal{L}\ I\ (b,k',0)\ I\ [0,0,k']\ I\ (0,0)\ I\ [0,0,n']\ I\ (d,n',0)\ I\ \mathcal{M}\ I\ \mathcal{P}$ on the side $[0,0,k']$:

$$v(k',n') = v(b,d) + v(n) = v(b,d) - w(k,n) \qquad (1)$$

(because $v(n) = -w(\infty,n) = -w(k,n)$). Hence

$$v(b,d) = w(k,n) + v(k',n') \geq w(k,n).$$

We apply the main property in the quadrangle $\mathcal{P} \perp (0,a')\ I\ [0]\ I\ (0,0)\ I\ [0,0,n']\ I\ (d,n',0)\ I\ \mathcal{M}\ I\ \mathcal{P}$ on the side $[0]$:

$$v(a') = v(b,d) + 2.v(n) \qquad (2)$$

Eliminating $v(b,d)$ in (1) and (2), we obtain :

$$v(k',n') = w(k,n) + v(a') \geq w(k,n)$$

and the result follows again.

If $k \neq \infty = n$, then we interchange the rôles of k and n in the previous argument.

If $k = n = \infty$, then $\mathcal{L} = \mathcal{M}$ because $\mathcal{P}$ is not incident with $[\infty]$. This completes the proof of the proposition. **Q.E.D.**

PROPOSITION(2.1.3.3). *Suppose* $\nu(\mathcal{P}) = (+,+,+)$ and $\nu(\mathcal{L}) = [+,-,-]$. *Then* $\mathcal{P}$ *and* $\mathcal{L}$ *are not incident with each other.*

PROOF. Let $\mathcal{P} = ((a^{+},\ell^{+},a'^{+}))$ and $\mathcal{L} = [[k^{+},b^{-},k'^{-}]]$ and suppose $\mathcal{P}\ I\ \mathcal{L}$. Note that (∞) is not incident with $\mathcal{L}$, otherwise $a = \infty$. If $[\infty] \perp \mathcal{L}$, then by definition of b, $b = a$, contradicting $v(b) < 0$ and $v(a) \geq 0$. Hence we may assume that $\mathcal{L}$ has three (old) coordinates, say $\mathcal{L} = [s,\ldots]$. Furthermore, let $[a,p]$ be the unique line through (a) meeting $[k,0,0]$. We apply the main property in the quadrangle $(0)\ I\ [0,0]\ I\ (0,0,0)\ I\ [k,0,0] \perp \mathcal{L} \perp [0,k']\ I\ (0)$ on the side $[0,0]$:

$$v(k') = v(b) + v(k,s) \tag{1}$$

We apply the main property in the quadrangle $(0,0,0)\ I\ [k,0,0] \perp [a,p]\ I\ (a,p,0)\ I\ [0,0,p]\ I\ (0,0)\ I\ [0,0,0]\ I\ (0,0,0)$ on the side $[0,0,p]$:

$$v(p) = v(a) + v(k) \geq 0 \tag{2}$$

Finally, we apply the main property in the quadrangle $\mathcal{P}\ I\ [a,\ell]\ I\ (a)\ I\ [a,p] \perp [k,0,0] \perp \mathcal{L}\ I\ \mathcal{P}$ on the side $[a,p]$:

$$\begin{aligned} v(\ell,p) = v(a,b) + v(k,s) &= v(b) + v(k,s) \\ &= v(k') < 0 \end{aligned}$$

by (1) and the fact that $v(a,b) = v(b)$. But $v(\ell,p) < 0$ contradicts $v(\ell) \geq 0$ and (2). Hence $\mathcal{P}$ is not incident with $\mathcal{L}$. **Q.E.D.**

PROPOSITION(2.1.3.4). *Suppose* $\nu(\mathcal{P}) = (+,+,+)$ and $\nu(\mathcal{L}) = [-,-,-]$. *Then* $\mathcal{P}$ *and* $\mathcal{L}$ *are not incident with each other.*

PROOF. Let $\mathcal{P} = ((a^+, \ell^+, a'^+))$ and $\mathcal{L} = [[k^-, b^-, k'^-]]$ and suppose $\mathcal{P}\ I\ \mathcal{L}$. Note that, if $k' = \infty$, then $a' = b$, a contradiction. Hence $k' \neq \infty$. If $b = \infty$, then we apply the main property in the quadrangle $\mathcal{P}\ I\ \mathcal{L} \perp [0,0,k']\ I\ (0,0)\ I\ [0,0,\ell]\ I\ (a,\ell,0)\ I\ [a,\ell]\ I\ \mathcal{P}$ on the side $[a,\ell]$ and take into account the fact that, since $b = \infty$, $\mathcal{L} = [k, \ldots]$:

$$v(a') = v(k',\ell) + v(k) = v(k') + v(k) < 0,$$

a contradiction. So we can assume that neither b nor k' equals ∞. Suppose now $k = \infty$. Since $b \neq \infty$, $[\infty]$ does not meet $\mathcal{L}$ and we can put $\mathcal{L} = [s, \ldots]$. So we can substitute s for k in the equality above :

$$v(a') = v(k') + v(s) \tag{1}$$

We apply the main property in the quadrangle $\mathcal{L}\ I\ (0,0,b)\ I\ [0,0]\ I\ (0,0,0)\ I\ [0,0,0]\ I\ (0,0)\ I\ [0,0,k'] \perp \mathcal{L}$ on the side $[0,0]$:

$$v(b) = v(k') + v(s) \tag{2}$$

By (1) and (2), $v(a') = v(b)$, contradicting our assumptions. So we can consider the general case, where no (new) coordinate of $\mathcal{L}$ is ∞. Note first that from $\mathcal{L} \perp [\infty]$ follows that $\ell = k'$, a contradiction. Hence $\mathcal{L}$ does not meet $[\infty]$ and we can put $\mathcal{L} = [s, \ldots]$. Let Q be the meeting point of $\mathcal{L}$ and $[k,0,0]$. Since $b \neq \infty$, Q is not collinear with (∞), and so we can put $Q = (c, \ldots, b)$. Note that formula (1) remains valid, so

$$v(s) = v(a') - v(k') > 0.$$

We apply the main property in the quadrangle $\mathcal{P}\ I\ \mathcal{L}\ I\ Q \perp (0,b)\ I\ [0]\ I$

$(0,a) \perp \mathcal{P}$ on the side $[0]$:

$$v(a',b) = v(b) = v(a,c) + 2.v(s) \tag{3}$$

Hence $v(a,c) = v(b) - 2.v(s) < 0$ and so $v(a,c) = v(c)$ and by (3),

$$v(b) = v(c) + 2.v(s) \tag{4}$$

But applying the main property in the quadrangle $Q \; I \; [k,0,0] \; I \; (0,0,0) \; I \; [0,0,0] \; I \; (0,0) \; I \; [0] \; I \; (0,b) \perp Q$ on the side $[0]$, we obtain :

$$v(b) = v(c) + 2.v(k).$$

So by (4), $v(k) = v(s) \geq 0$, a contradiction. **Q.E.D.**

Similarly, one shows the dual of the propositions(2.1.3.1),(2.1.3.2), (2.1.3.3) and (2.1.3.4). We summarize all results in the next theorem.

THEOREM(2.1.3.5). **(I)** *Let* $\mathcal{P} = ((a^+, \ell^+, a'^+))$, $\mathcal{L} = [[k,b,k']]$, $\mathcal{M} = [[n,d,n']]$ *and suppose* $\mathcal{L} \; I \; \mathcal{P} \; I \; \mathcal{M}$ *in* $\mathcal{V}$. *Then* ***exactly one*** *of the following occurs :*

(1) $\nu(\mathcal{L}) = \nu(\mathcal{M}) = [+,+,+]$ *and* $u^*(\mathcal{L},\mathcal{M}) = v(k,n)$,

(2) $\nu(\mathcal{L}) = \nu(\mathcal{M}) = [-,+,+]$ *and* $u^*(\mathcal{L},\mathcal{M}) = w(k,n)$,

(3) $\{\nu(\mathcal{L}),\nu(\mathcal{M})\} = \{[+,+,+],[-,+,+]\}$ *and* $u^*(\mathcal{L},\mathcal{M}) = 0$.

(II) Let $\mathcal{L} = [[k^+, b^+, k'^+]]$, $\mathcal{P} = ((a,\ell,a'))$, $Q = ((c,p,c'))$ and suppose $\mathcal{P} \; I \; \mathcal{L} \; I \; Q$ *in* $\mathcal{V}$. Then ***exactly one of the*** *following occurs :*

(1) $\nu(\mathcal{P}) = \nu(Q) = (+,+,+)$ *and* $u^*(\mathcal{P},Q) = v(a,c)$,

(2) $\nu(\mathcal{P}) = \nu(Q) = (-,+,+)$ *and* $u^*(\mathcal{P},Q) = w(a,c)$,

(3) $\{\nu(\mathcal{P}),\nu(Q)\} = \{(+,+,+),(-,+,+)\}$ *and* $u^*(\mathcal{P},Q) = 0$.

PROPOSITION(2.1.3.6). *Let* $\mathcal{L} = [[k^+,b^+,k'^+]]$, $\mathcal{M} = [[n^+,d^+,n'^+]]$ *and suppose* $\mathcal{L}\ I\ \mathcal{P}\ I\ \mathcal{M}$ *with* $\nu(\mathcal{P}) = (-,+,+)$. *Then* $u^*(\mathcal{L},\mathcal{M}) = v(k',n')$.

PROOF. Put $\mathcal{P} = ((a,\ell,a'))$. If $a=\infty$, then $k=n$ and $b=d$ and the result follows. So we can assume $a\neq\infty$. We apply the main property in the quadrangle $\mathcal{P}\ I\ \mathcal{L} \perp [0,k']\ I\ (0)\ I\ [0,n'] \perp \mathcal{M}\ I\ \mathcal{P}$ on the side $[0,k']$:

$$v(k',n') = v(a) + v(k,n) \tag{1}$$

hence $v(k,n) = v(k',n') - v(a) > v(k',n')$. We now show that $v(b,d) \geq v(k',n')$, from which the result follows.
Consider the line $[k,d,0]$. If $\mathcal{P}$ is incident with this line, then $b=d=a'$, hence the result. So we may assume that $\mathcal{P}$ is not incident with $[k,d,0]$. Define the line $\mathcal{K}$ and the point $\mathcal{R}$ in $\mathcal{V}$ as $\mathcal{P}\ I\ \mathcal{K}\ I\ \mathcal{R}\ I\ [k,d,0]$. There are two possibilities.

First possibility : $\mathcal{K}$ is not concurrent with $[\infty]$.
Put $\mathcal{K} = [X,\ldots]$. If $\mathcal{R} = (k,d)$, then again $b=d$. So we may assume that $\mathcal{R}$ has three (old) coordinates and we put $\mathcal{R} = (A,\ldots)$. We apply the main property in the quadrangle $\mathcal{P}\ I\ \mathcal{K}\ I\ \mathcal{R}\ I\ [k,d,0]\ I\ (0,0,d)\ I\ [n,d,0]\ I\ (n,d)\ I\ \mathcal{M}\ I\ \mathcal{P}$ on the side $[n,d,0]$ resp. the vertex $(0,0,d)$:

$$v(n') + v(k,n) = v(A,a) + v(X,k) + v(X,n) \tag{2}$$

$$v(A) + 2.v(k,n) = v(A,a) + 2.v(X,n) \tag{3}$$

We apply the main property in the quadrangle $\mathcal{P}$ I $\mathcal{K}$ I $\mathcal{R}$ I $[k,d,0]$ I (k,d) I $[k]$ I (k,b) I $\mathcal{L}$ I $\mathcal{P}$ on the side $[k]$:

$$v(b,d) = v(A,a) + 2.v(X,k) \qquad (4)$$

By (2) and (4), one has

$$v(b,d) = v(k,n) + v(n') + v(X,k) - v(X,n) \qquad (5)$$

If $v(X,k) \geq v(X,n)$, then by (5), $v(b,d) \geq v(k,n) + v(n') \geq v(k,n) > v(k',n')$ (by (1)), proving the result. So suppose $v(X,k) < v(X,n)$. Then $v(X,k) = v(k,n) < v(X,n)$. Then (3) implies $v(A) > v(A,a)$ and thus $v(A,a) = v(a)$. But (4) implies that $v(b,d) = 2.v(k,n) + v(a)$ and substiting (1) in this, we obtain $v(b,d) = v(k',n') + v(k,n) > v(k',n')$.

Second possibility *:* $\mathcal{K}$ is concurrent with $[\infty]$.
We apply the main property in the same quadrangles as above on the same sides and vertices, then the same equalities remain valid, provided that we omit the terms where X appears. Thus, (5) leads to $v(b,d) = v(k,n) + v(n') > v(k',n')$ by (1). This completes the proof of the proposition.

Q.E.D.

PROPOSITION(2.1.3.7). *Let* $\mathcal{L} = [[k^+, b^-, k'^-]]$, $\mathcal{M} = [[n^+, d^-, n'^-]]$ *and suppose* $\mathcal{L}$ I $\mathcal{P}$ I $\mathcal{M}$ *with* $\mathrm{v}(\mathcal{P}) = (-,+,+)$. *then* $u^*(\mathcal{L},\mathcal{M}) = w(k',n')$.

PROOF. Put $\mathcal{P} = ((a,\ell,a'))$. This time, we will only prove the general case, where no coordinate of $\mathcal{P}$,$\mathcal{L}$ or $\mathcal{M}$ is the symbol ∞. The other cases always have a simpler, if not a trivial, proof. In fact, the proof of the general case is universal in the sense that the proofs of all other cases are copies (that means, we apply the main property in the same quadrangles on the same vertices and sides) of the general case, but the resulting equalities are simpler and sometimes trivial (since the quadrangles itselves are sometimes degenerate).

So we show the general case. Since $k' \neq \infty \neq n'$, neither $\mathcal{L}$ nor $\mathcal{M}$ meets $[\infty]$. Hence $\mathcal{L}$ and $\mathcal{M}$ have three (old) coordinates. Put $\mathcal{L} = [k^*,\ldots]$ and $\mathcal{M} = [n^*,\ldots]$. We apply the main property in the quadrangle $\mathcal{P}\ I\ \mathcal{L} \perp [0,k']\ I\ (0)\ I\ [0,n'] \perp \mathcal{M}\ I\ \mathcal{P}$ on the side $[0,k']$:

$$v(k',n') = v(a) + v(k^*,n^*) \tag{1}$$

We apply the main property in the quadrangle $\mathcal{P}\ I\ \mathcal{L} \perp [k,0,0]\ I\ (0,0,0)\ I\ [n,0,0] \perp \mathcal{M}\ I\ \mathcal{P}$ on the vertex $(0,0,0)$:

$$2.v(k,n) + v(b) + v(d) = 2.v(k^*,n^*) + v(a,b) + v(a,d) \tag{2}$$

We apply the main property in the quadrangles $(0)\ I\ [0,k'] \perp \mathcal{L} \perp [k,0,0]\ I\ (0,0,0)\ I\ [0,0]\ I\ (0)$ and $(0)\ I\ [0,n'] \perp \mathcal{M} \perp [n,0,0]\ I\ (0,0,0)\ I\ [0,0]\ I\ (0)$ on the side $[0,0]$:

$$v(b) = v(k') - v(k,k^*) \tag{3}$$

$$v(d) = v(n') - v(n,n^*) \tag{4}$$

We apply the main property in the quadrangles $\mathcal{P}\ I\ \mathcal{L} \perp [k,0,0]\ I\ (0,0,0)\ I\ [0,0]\ I\ (0,0,a')\ I\ [\ell,a',0]\ I\ \mathcal{P}$ and $\mathcal{P}\ I\ \mathcal{M} \perp [n,0,0]\ I\ (0,0,0)\ I\ [0,0]\ I\ (0,0,a')\ I\ [\ell,a',0]\ I\ \mathcal{P}$ on the side $[0,0]$:

$$v(a,b) = v(a') - v(k,k^*) - v(k^*,\ell) \tag{5}$$

$$v(a,d) = v(a') - v(n,n^*) - v(n^*,\ell) \qquad (6)$$

We apply the main property in the quadrangles $\mathcal{P}\ I\ \mathcal{L} \perp [0,k']\ I\ (0)\ I\ [0,0]\ I\ (0,0,a')\ I\ [\ell,a',0]\ I\ \mathcal{P}$ and $\mathcal{P}\ I\ \mathcal{L} \perp [0,k']\ I\ (0)\ I\ [0,0]\ I\ (0,0,a')\ I\ [\ell,a',0]\ I\ \mathcal{P}$ on the side $[0,0]$:

$$v(k^*,\ell) = v(k') - v(a) \qquad (7)$$

$$v(n^*,\ell) = v(n') - v(a) \qquad (8)$$

Substituting (3) and (4) in (2) ; (7) in (5) ; (8) in (6), afterwards (5) and (6) in (2) and eliminating $v(k^*,n^*)$ in (1) and (2), we obtain :

$$v(k,n) = w(k',n') + v(a') \geq w(k',n') \qquad (9)$$

Next we show $w(b,d) \geq w(k',n')$. Consider the line $[[k,d,\infty]]$ and call it $\mathcal{K}$. If $\mathcal{P}$ is incident with $\mathcal{K}$, then either $k' = \infty$, or $\mathcal{P}\ I\ [k,0,0]$ and then $k = n$ and also $b = d$ and the result follows. So we can assume that $\mathcal{P}$ is not incident with $\mathcal{K}$. Let $\mathcal{N}$ be the unique line incident with $\mathcal{P}$ and meeting $\mathcal{K}$. If $\mathcal{N}$ meets $[\infty]$, then $a = d$ and we interchange the rôles of $\mathcal{L}$ and $\mathcal{M}$ in the definition of $\mathcal{K}$ and $\mathcal{N}$. If $\mathcal{N}$ still meets $[\infty]$, then $a = b$ and hence $w(b,d) = \infty$. So we may assume that $\mathcal{N}$ (with the first definition) does not meet $[\infty]$ and we can write $\mathcal{N} = [X,\ldots]$. We apply the main property in the quadrangle $\mathcal{P}\ I\ \mathcal{N} \perp \mathcal{K} \perp [k,0,0] \perp \mathcal{L}\ I\ \mathcal{P}$ on the side $\mathcal{N}$:

$$v(b,d) = v(a,d) + v(X,k^*) - v(k,k^*) \qquad (10)$$

We apply the main property in the quadrangle $\mathcal{P}\ I\ \mathcal{M}\ I\ [b,Q_2(d,n,0,0)] \perp (d)\ I\ \mathcal{K} \perp \mathcal{N}\ I\ \mathcal{P}$ on the side $\mathcal{N}$:

$$v(a,d) + v(X,n^*) = v(Q_2(d,n,0,0), Q_2(d,k,0,0))$$

$$= v(k,n) + v(d) \qquad (11)$$

The last equality holds by the main property applied in the quadrangle $(d)\ I\ [d, Q_2(d,k,0,0)] \perp [k,0,0]\ I\ (0,0,0)\ I\ [n,0,0] \perp [d, Q_2(d,n,0,0)]\ I\ (d)$ on any side. We eliminate $v(a,d)$ and $v(k^*,k)$ in (10), (11) and (3) and obtain :

$$w(b,d) = v(k,n) + v(X,k^*) - v(X,n^*) - v(k') \qquad (12)$$

If $v(X,k^*) \geq v(X,n^*)$, then by (12), $w(b,d) \geq v(k,n) - v(k') > v(k,n) \geq w(k',n')$. Suppose now that $v(X,k^*) < v(X,n^*)$, then $v(X,k^*) = v(k^*,n^*)$. We compute $w(b,d)$:

$$w(b,d) = v(b,d) - v(b) - v(d) \qquad \text{(by definition)}$$

$$= v(a,d) + v(k^*,n^*) - v(k,k^*) - v(b) - v(d) \qquad \text{(by (10))}$$

$$= v(a,d) + v(k^*,n^*) - v(k') - v(d) \qquad \text{(by (3))}$$

$$= v(a,d) + v(k',n') - v(a) - v(k') - v(d) \qquad \text{(by (1))}$$

$$= v(a') - v(n,n^*) - v(n^*,\ell) + v(k',n') - v(a) - v(k') - v(d) \qquad \text{(by (6))}$$

$$= v(a') - v(n') - v(n^*,\ell) + v(k',n') - v(a) - v(k') \qquad \text{(by (4))}$$

$$= v(a') - v(n^*,\ell) + w(k',n') - v(a) \qquad \text{(by definition)}$$

$$= w(k',n') + v(a') - v(n') \qquad \text{(by (8))}$$

$$> w(k',n')$$

This completes the proof of the proposition. Q.E.D.

PROPOSITION(2.1.3.8). *Suppose* $\nu(\mathcal{P}) = (-,+,+)$ and $\nu(\mathcal{L}) = [-,+,+]$. *Then* $\mathcal{P}$ *and* $\mathcal{L}$ *are not incident with each other.*

PROOF. Let $\mathcal{P} = ((a^-, \ell^+, a'^+))$ and $\mathcal{L} = [[k^-, b^+, k'^+]]$ and suppose $\mathcal{P}\ I\ \mathcal{L}$. If $k = \infty$, then $a^- = b^+$, a contradiction. So we may assume that $k \neq \infty$. Now suppose that $a = \infty$, then $k^- = \ell^+$ by the definition of the new coordinates, again a contradiction. Hence we may assume that $a \neq \infty$. So $\mathcal{P}$ has three (old) coordinates, say $\mathcal{P} = (a, p, a")$. Note that $v(a,b) = v(a)$. We apply the main property in the quadrangle $\mathcal{P}\ I\ \mathcal{L}\ I\ (b,k',0)\ I\ [0,0,k']\ I\ (0,0)\ I\ [0,0,p]\ I\ (a,p,0)\ I\ [a,p]\ I\ \mathcal{P}$ on the side $\mathcal{L}$:

$$v(k',p) = v(a,b) + v(k) = v(a) + v(k) < 0.$$

Hence $v(k',p) = v(p)$ and

$$v(p) = v(a) + v(k) \qquad (1)$$

We apply the main property in the same quadrangle on the side $[a,p]$:

$$v(a") = v(k',p) + v(k) = v(p) + v(k) < 0 \qquad (2)$$

Hence $v(a',a") = v(a")$. Now by (1) and (2),

$$v(a") = v(a) + 2.v(k) \qquad (3)$$

Finally, we apply the main property in the quadrangle $\mathcal{P}\ I\ [\ell,a',0]\ I\ (0,0,a')\ I\ [0,a',0]\ I\ (0,a')\ I\ [0]\ I\ (0,a")\ \perp\ \mathcal{P}$ on the side $[0]$:

$$v(a") = v(a',a") = v(a) + 2.v(\ell) \qquad (4)$$

By (3) and (4), $v(k^-) = v(\ell^+)$, a contradiction.

This completes the proof of the proposition. **Q.E.D.**

$$v(b) = v(c) + 2.v(k) \tag{5}$$

$$v(a'',b) = v(a,c) + 2.v(p) \tag{6}$$

Finally, we apply the main property in the quadrangle $\mathcal{P} \perp (0,a'')\ I\ [0,a'',0]\ I\ (0,0,a'')\ I\ [0,0]\ I\ (0,0,a')\ I\ [\ell,a',0]\ I\ \mathcal{P}$ on the vertex $\mathcal{P}$:

$$v(a',a'') = v(a) + 2.v(\ell) \tag{7}$$

Suppose for a moment that $v(c) \geq 0$. Then $v(a,c) = v(a)$ and by (3), $2.v(\ell,p) = v(c^+) + v(a'^+) - 2.v(a^-) > 0$. So $v(p) \geq 0$, hence $v(k,p) = v(k)$. But (4) implies $v(a') = v(c) + 2.v(k)$ and by (5), $v(a'^+) = v(b^-)$, a contradiction. Hence $v(c) < 0$. There are three distinct possibilities.

First possibility *:* $v(c) < v(a)$.

In this case $v(a,c) = v(c)$. By (3), $2.v(\ell,p) = v(a'^+) - v(a^-) > 0$ such that $v(p) \geq 0$ and hence $v(k) = v(k,p)$. But by (1) and (2), $v(k,p) > v(\ell,p)$, contradicting the previous equalities.

Second possibility *:* $v(c) > v(a)$.

In this case $v(a,c) = v(a)$. By (4), $2.v(k,p) = v(a'^+) - v(c^-) > 0$ such that $v(k) = v(p)$ and hence $v(\ell,p) = v(p) < 0 < v(k,p)$. Consequently $v(c) + v(k,p) > v(a) + v(\ell,p)$, contradicting (1) and (2).

Third possibility *:* $v(c) = v(a)$.

By (1) and (2), $v(k,p) = v(\ell,p)$. One checks that this implies that $v(\ell,p) = v(p) = v(k,p) \leq v(k)$. Eliminating $v(a,c)$ in (3) and (6), we obtain $v(a') = v(a'',b)$ and since this is bigger then or equal to zero, $v(a'') = v(b)$ and $v(a',a'') = v(a'') = v(b)$. Hence by (5) and (7), we find

$2.v(k^-) = v(b) - v(a) = 2.v(\ell^+)$, a contradiction.

Suppose now that $\mathcal{L}$ meets $[\infty]$. If $(\infty)\ I\ \mathcal{L}$, then by the definition of the (new) coordinates, $k^- = \ell^+$, a contradiction. Hence $\mathcal{L}$ meets $[\infty]$ in the point (a). The elements a'' and c are still well defined as above and we have $a=c$. The equality (3) becomes $v(a'',b) = v(a') \geq 0$, hence $v(a'') = v(b)$ and $v(a',a'') = v(a'')$. Note that the equalities (5) and (7) remain valid and hence since $v(a',a'') = v(b)$ and $a = c$, this implies $v(k^-) = v(\ell^+)$, a contradiction. So $\mathcal{L}$ does not meet $[\infty]$.

Suppose now that $k = \infty$. By the definition of the new coordinates, clearly $a'^+ = b^-$, a contradiction. Hence $k \neq \infty$.

Next suppose $k' = \infty$. Note that $b \neq \infty$ since otherwise $\mathcal{L}$ meets (∞). So p and p' are still well defined as above and moreover $p = 0$. Consequently the equalities (1) and (5) remain valid :

$$v(p') = v(c) + v(k) \tag{8}$$

$$v(b) = v(c) + 2.v(k) \tag{9}$$

We apply the main propertie in the quadrangle $\mathcal{P}\ I\ \mathcal{L}\ I\ (0,b)\ I\ [0,b,0]\ I\ (0,0,b)\ I\ [0,0]\ I\ (0,0,a')\ I\ [\ell,a',0]\ I\ \mathcal{P}$ on the side $[0,0]$:

$$v(a',b) = v(p') + v(\ell) \tag{10}$$

Eliminating $v(p')$ in (8) and (10) and taking into account the fact that $v(a',b) = v(b)$, one has

$$v(b) = v(c) + v(k) + v(\ell) \tag{11}$$

Substracting (9) and (11) side by side, we have $v(k^-) = v(\ell^+)$, a contradiction. Hence $k' \neq \infty$.

Suppose now that $b = \infty$. In this case, apparently $p = k$ and $c = \infty$. The equality (3) becomes $v(a') = v(a) + 2.v(\ell,k) = v(a) + 2.v(k) < 0$, contradicting $v(a') \geq 0$.

Finally, suppose $a = \infty$. Note that c and p are still well defined and we have $p = \ell$. Hence the formula (4) becomes $v(a') = v(c) + 2.v(\ell,k) = v(c) + 2.v(k) = v(b)$ by (5) (which holds here!). This contradicts $v(b) < 0 \leq v(a')$. The result follows. **Q.E.D.**

The four foregoing propositions can be summarized and dualized in the following theorem.

THEOREM(2.1.3.10). **(I)** *Let* $\mathcal{P} = ((a^-,\ell^+,a'^+))$, $\mathcal{L} = [[k,b,k']]$, $\mathcal{M} = [[n,d,n']]$ *and suppose* $\mathcal{L}\ I\ \mathcal{P}\ I\ \mathcal{M}$ *in* $\mathcal{V}$. *Then* ***exactly one*** *of the following occurs :*

(1) $\nu(\mathcal{L}) = \nu(\mathcal{M}) = [+,+,+]$ *and* $u^*(\mathcal{L},\mathcal{M}) = v(k',n')$,

(2) $\nu(\mathcal{L}) = \nu(\mathcal{M}) = [+,-,-]$ *and* $u^*(\mathcal{L},\mathcal{M}) = w(k',n')$,

(3) $\{\nu(\mathcal{L}),\nu(\mathcal{M})\} = \{[+,+,+],[+,-,-]\}$ *and* $u^*(\mathcal{L},\mathcal{M}) = 0$.

(II) *Let* $\mathcal{L} = [[k^-,b^+,k'^+]]$, $\mathcal{P} = ((a,\ell,a'))$, $Q = ((c,p,c'))$ *and suppose* $\mathcal{P}\ I\ \mathcal{L}\ I\ Q$. *Then* ***exactly one*** *of the following occurs :*

(1) $\nu(\mathcal{P}) = \nu(Q) = (+,+,+)$ *and* $u^*(\mathcal{P},Q) = v(a',c')$,

(2) $\nu(\mathcal{P}) = \nu(Q) = (+,-,-)$ *and* $u^*(\mathcal{P},Q) = w(a',c')$,

(3) $\{\nu(\mathcal{P}),\nu(Q)\} = \{(+,+,+),(+,-,-)\}$ *and* $u^*(\mathcal{P},Q) = 0$.

There remain to prove two theorems similar to theorem(2.1.3.10). We will not prove them completely because of two reasons. Firstly, the proofs are very long and they do not give more insight in the theory. Secondly, the proofs are similar to the proof of the above mentioned theorem. Nevertheless, the reconstruction of these proofs is not trivial. Hence we will show the most tricky case.

Here are the theorems in question.

THEOREM(2.1.3.11). **(I)** *Let* $\mathcal{P} = ((a^+,\ell^-,a'^-))$, $\mathcal{L} = [[k,b,k']]$, $\mathcal{M} = [[n,d,n']]$ *and suppose* $\mathcal{L}\ I\ \mathcal{P}\ I\ \mathcal{M}$ *in* $\mathcal{V}$. *Then* ***exactly one*** *of the following occurs :*

(1) $\nu(\mathcal{L}) = \nu(\mathcal{M}) = [-,+,+]$ *and* $u^*(\mathcal{L},\mathcal{M}) = v(k',n')$,

(2) $\nu(\mathcal{L}) = \nu(\mathcal{M}) = [-,-,-]$ *and* $u^*(\mathcal{L},\mathcal{M}) = w(k',n')$,

(3) $\{\nu(\mathcal{L}),\nu(\mathcal{M})\} = \{[-,+,+],[-,-,-]\}$ *and* $u^*(\mathcal{L},\mathcal{M}) = 0$.

(II) *Let* $\mathcal{L} = [[k^+,b^-,k'^-]]$, $\mathcal{P} = ((a,\ell,a'))$, $Q = ((c,p,c'))$ *and suppose* $\mathcal{P}\ I\ \mathcal{L}\ I\ Q$ *in* $\mathcal{V}$. *Then* ***exactly one*** *of the following occurs :*

(1) $\nu(\mathcal{P}) = \nu(Q) = (-,+,+)$ *and* $u^*(\mathcal{P},Q) = v(a',c')$,

(2) $\nu(\mathcal{P}) = \nu(Q) = (-,-,-)$ *and* $u^*(\mathcal{P},Q) = w(a',c')$,

(3) $\{\nu(\mathcal{P}),\nu(Q)\} = \{(-,+,+),(-,-,-)\}$ *and* $u^*(\mathcal{P},Q) = 0$.

THEOREM(2.1.3.12). **(I)** *Let* $\mathcal{P} = ((a^-, \ell^-, a'^-))$, $\mathcal{L} = [[k, b, k']]$, $\mathcal{M} = [[n, d, n']]$ *and suppose* $\mathcal{L}$ *I* $\mathcal{P}$ *I* $\mathcal{M}$ *in* $\mathcal{V}$. *Then* ***exactly one*** *of the following occurs :*

(1) $\nu(\mathcal{L}) = \nu(\mathcal{M}) = [+,-,-]$ *and* $u^*(\mathcal{L},\mathcal{M}) = v(k,n)$,

(2) $\nu(\mathcal{L}) = \nu(\mathcal{M}) = [-,-,-]$ *and* $u^*(\mathcal{L},\mathcal{M}) = w(k,n)$,

(3) $\{\nu(\mathcal{L}), \nu(\mathcal{M})\} = \{[+,-,-],[-,-,-]\}$ *and* $u^*(\mathcal{L},\mathcal{M}) = 0$.

(II) *Let* $\mathcal{L} = [[k^-, b^-, k'^-]]$, $\mathcal{P} = ((a, \ell, a'))$, $Q = ((c, p, c'))$ *and suppose* $\mathcal{P}$ *I* $\mathcal{L}$ *I* Q *in* $\mathcal{V}$. *Then* ***exactly one*** *of the following occurs :*

(1) $\nu(\mathcal{P}) = \nu(Q) = (+,-,-)$ *and* $u^*(\mathcal{P},Q) = v(a,c)$,

(2) $\nu(\mathcal{P}) = \nu(Q) = (-,-,-)$ *and* $u^*(\mathcal{P},Q) = w(a,c)$,

(3) $\{\nu(\mathcal{P}), \nu(Q)\} = \{(+,-,-),(-,-,-)\}$ *and* $u^*(\mathcal{P},Q) = 0$.

PROOF. As mentioned above, we show the most difficult case, namely the general case of theorem(2.1.3.12)(I)(2).

Let $\mathcal{P} = ((a^-, \ell^-, a'^-))$, $\mathcal{L} = [[k^-, b^-, k'^-]]$ and $\mathcal{M} = [[n^-, d^-, n'^-]]$. We suppose that none of these coordinates equals ∞, that $\mathcal{P}$ is not collinear with (∞) and that neither $\mathcal{L}$ nor $\mathcal{M}$ is concurrent with $[\infty]$. This is the general case. Put $\mathcal{P} = (a'', \ldots)$; $\mathcal{L} = [k^*, \ldots]$ and $\mathcal{M} = [n^*, \ldots]$. Let $\mathcal{R}_{\mathcal{L}}$ be the meeting point of $\mathcal{L}$ and $[k,0,0]$. Since $b \neq \infty$, $\mathcal{R}_{\mathcal{L}}$ is not collinear with (∞). So put $\mathcal{R}_{\mathcal{L}} = (b', \ldots, b)$. Similarly we can put $\mathcal{R}_{\mathcal{M}} = (d', \ldots, d)$ with $\mathcal{R}_{\mathcal{M}}$ the meeting point of $\mathcal{M}$ and $[n,0,0]$. Let $\mathcal{R}'_{\mathcal{L}}$ be the meeting point of $\mathcal{L}$ and $[0,0,k']$. Since $k' \neq \infty$, $\mathcal{R}'_{\mathcal{L}}$ is not collinear with (∞). So put $\mathcal{R}'_{\mathcal{L}} = (b'', k', 0)$. Similarly, we can put $\mathcal{R}'_{\mathcal{M}} = (d'', n', 0)$, where $\mathcal{R}'_{\mathcal{M}}$ is the meeting point of $\mathcal{M}$ and $[0,0,n']$. Now let $\mathcal{K}_{\mathcal{P}}$ be the line joining $\mathcal{P}$ and $(a,0,0)$. Since $\ell \neq \infty$, $\mathcal{K}_{\mathcal{P}}$ is not concurrent with $[\infty]$ and hence we can put $\mathcal{K}_{\mathcal{P}} = [\ell',\ldots,\ell]$.

We apply the main property in the quadrangle $\mathcal{P}\ I\ \mathcal{L}\ I\ \mathcal{R}_{\mathcal{L}}\ I\ [k,0,0]\ I\ (0,0,0)\ I\ [n,0,0]\ I\ \mathcal{R}_{\mathcal{M}}\ I\ \mathcal{M}\ I\ \mathcal{P}$ on the vertex $(0,0,0)$:

$$2.v(k,n) + v(d') + v(b') = 2.v(k^*.n^*) + v(a",b') + v(a",d') \tag{1}$$

We apply the main property in the quadrangle $\mathcal{P}\ I\ \mathcal{L}\ I\ \mathcal{R}'_{\mathcal{L}}\ I\ [0,0,k']\ I\ (0,0)\ I\ [0,0,n']\ I\ \mathcal{R}'_{\mathcal{M}}\ I\ \mathcal{M}\ I\ \mathcal{P}$ on the vertex $(0,0)$:

$$2.v(k',n') = 2.v(k^*,n^*) + v(a",b") + v(a",d") \tag{2}$$

We apply the main property in the quadrangle $\mathcal{P}\ I\ \mathcal{K}_{\mathcal{P}}\ I\ (a,0,0)\ I\ [0,0,0]\ I\ (0,0,0)\ I\ [k,0,0]\ I\ \mathcal{R}_{\mathcal{L}}\ I\ \mathcal{L}\ I\ \mathcal{P}$, resp. the quadrangle $\mathcal{P}\ I\ \mathcal{K}_{\mathcal{P}}\ I\ (a,0,0)\ I\ [0,0,0]\ I\ (0,0,0)\ I\ [n,0,0]\ I\ \mathcal{R}_{\mathcal{M}}\ I\ \mathcal{M}\ I\ \mathcal{P}$ on the side $[0,0,0]$:

$$v(a) + v(k) + v(\ell') = v(a",b') + v(k,k^*) + v(\ell',k^*) \tag{3}$$

$$v(a) + v(n) + v(\ell') = v(a",d') + v(n,n^*) + v(\ell',n^*) \tag{4}$$

We apply the main property in the quadrangle $\mathcal{P}\ I\ \mathcal{K}_{\mathcal{P}}\ I\ (a,0,0)\ I\ [0,0,0]\ I\ (0,0)\ I\ [0,0,k']\ I\ \mathcal{R}'_{\mathcal{L}}\ I\ \mathcal{L}\ I\ \mathcal{P}$, resp. the quadrangle $\mathcal{P}\ I\ \mathcal{K}_{\mathcal{P}}\ I\ (a,0,0)\ I\ [0,0,0]\ I\ (0,0)\ I\ [0,0,n']\ I\ \mathcal{R}'_{\mathcal{M}}\ I\ \mathcal{M}\ I\ \mathcal{P}$ on the side $[0,0,0]$:

$$v(k') + v(\ell') = v(a",b") + v(k^*) + v(\ell',k^*) \tag{5}$$

$$v(n') + v(\ell') = v(a",d") + v(n^*) + v(\ell',n^*) \tag{6}$$

Finally, we apply the main property in the quadrangle $[0,0,0]\ I\ (0,0,0)\ I\ [k,0,0]\ I\ \mathcal{R}_{\mathcal{L}}\ I\ \mathcal{L}\ I\ \mathcal{R}'_{\mathcal{L}}\ I\ [0,0,k']\ I\ (0,0)\ I\ [0,0,0]$, resp. in the quadrangle $[0,0,0]\ I\ (0,0,0)\ I\ [n,0,0]\ I\ \mathcal{R}_{\mathcal{M}}\ I\ \mathcal{M}\ I\ \mathcal{R}'_{\mathcal{M}}\ I\ [0,0,n']\ I\ (0,0)$

I [0,0,0] on the side $[k,0,0]$, resp. $[n,0,0]$:

$$v(b') + v(k) + v(k,k^*) = v(k') + v(k^*) \qquad (7)$$

$$v(d') + v(n) + v(n,n^*) = v(n') + v(n^*) \qquad (8)$$

In (1) and (2), we eliminate $v(k^*,n^*)$ and obtain :

$$2.v(k',n') = 2.v(k,n) + v(b') + v(d') + v(a'',b'') + v(a'',d'') - v(a'',b') - v(a'',d') \qquad (9)$$

We eliminate the last six terms in the right hand side of (9) by using for $v(b')$, resp. $v(d'),v(a'',b''),v(a'',d''),v(a'',b'),v(a'',d')$ its value in the equality (7), resp. (8),(5),(6),(3),(4) and we obtain :

$$\begin{aligned} 2.v(k',n') = 2.v(k,n) & \\ & + v(k') + v(k^*) - v(k) - v(k,k^*) && (= v(b')) \\ & + v(n') + v(n^*) - v(n) - v(n,n^*) && (= v(d')) \\ & + v(k') + v(\ell') - v(k^*) - v(\ell',k^*) && (= v(a'',b'')) \\ & + v(n') + v(\ell') - v(n^*) - v(\ell',n^*) && (= v(a'',d'')) \\ & + v(k,k^*) + v(\ell',k^*) - v(a) - v(k) - v(\ell') && (=-v(a'',b')) \\ & + v(n,n^*) + v(\ell',n^*) - v(a) - v(n) - v(\ell') && (=-v(a'',d')) \end{aligned}$$

$$= 2.v(k,n) + 2.v(k') - 2.v(k) + 2.v(n') - 2.v(n) - 2.v(a)$$

Hence $w(k',n') = w(k,n) - v(a) > w(k,n)$.

We now show $w(b,d) > w(k,n)$.

Denote the line joining $\mathcal{R}_{\mathcal{L}}$ and $(0,b)$ by $\mathcal{L}_1 = [0,b,z_1]$ and consider the line $\mathcal{L}_2 = [0,b,z_2]$ passing through $(0,b)$ and meeting $[n,0,0]$. The meeting point $\mathcal{P}^*$ of $[0,b,z_2]$ and $[n,0,0]$ can be given coordinates $\mathcal{P}^* =$

$(b^*,\ldots,b)$. Since $n' \neq \infty$, $\mathcal{P}$ is not incident with $\mathcal{L}_2$, and so there exist a unique line $\mathcal{K}$ and a unique point $\mathcal{R}$ such that $\mathcal{P}\ I\ \mathcal{K}\ I\ \mathcal{R}\ I\ \mathcal{L}_2$. We assume first that $\mathcal{K}$ does not meet $[\infty]$. We can put $\mathcal{K} = [X,\ldots,Z]$ and $\mathcal{R} = (A,\ldots)$. Note that $\mathcal{R} \neq (0,b)$ because otherwise $\mathcal{R} \perp (0,b) \perp \mathcal{R}_{\mathcal{L}} \perp \mathcal{R}$ forms a triangle. Hence $X \neq 0$. We apply the main property in the quadrangle $[0]\ I\ (0,b)\ I\ \mathcal{L}_1\ I\ \mathcal{P}^*\ I\ [n,0,0]\ I\ \mathcal{R}_{\mathcal{M}} \perp (0,d)\ I\ [0]$ on the side $[0]$:

$$v(b,d) = v(b^*,d') + 2.v(n) \tag{10}$$

We apply the main property in the quadrangle $\mathcal{P}\ I\ \mathcal{K}\ I\ \mathcal{R}\ I\ \mathcal{L}_2\ I\ \mathcal{P}^*\ I\ [n,0,0]\ I\ \mathcal{R}_{\mathcal{M}}\ I\ \mathcal{M}\ I\ \mathcal{P}$ on the side $[n,0,0]$:

$$v(b^*,d') + v(n,n^*) + v(n) = v(A,a'') + v(X) + v(X,n^*) \tag{11}$$

We apply the main property in the quadrangle $\mathcal{P}\ I\ \mathcal{K}\ I\ \mathcal{R}\ I\ \mathcal{L}_2\ I\ (0,b)\ I\ \mathcal{L}_1\ I\ \mathcal{R}_{\mathcal{L}}\ I\ \mathcal{L}\ I\ \mathcal{P}$ on the side $\mathcal{K}$:

$$v(A,a') + v(X,k^*) + v(X) = v(Z_1,Z_2) + v(k^*) \tag{12}$$

We apply the main property in the quadrangle $(0,b)\ I\ \mathcal{L}_1\ I\ \mathcal{R}_{\mathcal{L}}\ I\ [k,0,0]\ I\ (0,0,0)\ I\ [n,0,0]\ I\ \mathcal{P}^*\ I\ \mathcal{L}_2\ I\ (0,b)$ on the vertex $(0,b)$:

$$2.v(Z_1,Z_2) = 2.v(k,n) + v(b^*) + v(b') \tag{13}$$

We apply the main property in the quadrangle $[0]\ I\ (0,0)\ I\ [0,0,0]\ I\ (0,0,0)\ I\ [k,0,0]\ I\ \mathcal{R}_{\mathcal{L}}\ I$ e $\mathcal{L}_1\ I\ (0,b)\ I\ [0]$, resp. $[0]\ I\ (0,0)\ I\ [0,0,0]\ I\ (0,0,0)\ I\ [n,0,0]\ I\ \mathcal{R}_{\mathcal{M}} \perp (0,d)\ I\ [0]$; $[0]\ I\ (0,0)\ I\ [0,0,0]\ I\ (0,0,0)\ I\ [n,0,0]\ I\ \mathcal{P}^*\ I\ \mathcal{L}_2\ I\ (0,b)\ I\ [0]$ on the side $[0]$:

$$v(b) = v(b') + 2.v(k) \tag{14}$$

$$v(d) = v(d') + 2.v(n) \tag{15}$$

$$v(b) = v(b^*) + 2.v(n) \tag{16}$$

Eliminating $v(b^*,d'), v(A,a''), v(n,n^*), v(Z_1,Z_2), v(b'), v(b^*)$ and $v(d')$ in the equalities (8),(10),(11),(12),(13),(14),(15) and (16), we obtain

$$w(b,d) = w(k,n) - v(n') + [v(X,n^*)+v(k^*)] - [v(X,k^*)+v(n^*)] \tag{17}$$

If $v(X,n^*)+v(k^*) \geq v(X,k^*)+v(n^*)$, then $w(b,d) \geq w(k,n) - v(n') > w(k,n)$ and this shows the result. So we may assume that $v(X,n^*)+v(k^*) < v(X,k^*)+v(n^*)$. By lemma(2.1.1.3), $v(X,n^*)+v(k^*) = v(k^*,n^*)+v(X)$. Keeping this in mind eliminating $v(b^*,d')$ in (10) and (11), we obtain

$$v(b,d) = v(A,a'') + 2.v(X) + v(k^*,n^*) - v(k^*) - v(n,n^*) + v(n) \tag{18}$$

We apply the main property in the quadrangle $\mathcal{P}$ I $\mathcal{L}$ I $\mathcal{R}_{\mathcal{L}}$ I $[k,0,0]$ I $(0,0,0)$ I $[n,0,0]$ I $\mathcal{R}_{\mathcal{M}}$ I $\mathcal{M}$ I $\mathcal{P}$ on the side $\mathcal{L}$:

$$v(a'',b') + v(k,k^*) + v(k^*,n^*) = v(d') + v(k,n) + v(n,n^*) \tag{19}$$

We apply the main property in the quadrangle $\mathcal{P}$ I $\mathcal{K}$ I $\mathcal{R}$ I $\mathcal{L}_2$ I $(0,b)$ I $\mathcal{L}_1$ I $\mathcal{R}_{\mathcal{L}}$ I $\mathcal{L}$ I $\mathcal{P}$ on the vertex $\mathcal{R}$:

$$v(A,a'') + 2.v(X) = v(a'',b') + 2.v(k^*) \tag{20}$$

We eliminate $v(A,a'')$ and $v(a'',b')$ in (18),(19) and (20) :

$$v(b,d) = v(k,n) + v(d') - v(k,k^*) + v(k^*) + v(n) \tag{21}$$

We eliminate $v(k,k^*), v(b')$ and $v(d')$ in (7),(14),(15) and (21) :

$$w(b,d) = w(k,n) - v(k') > w(k,n)$$

Assume $\mathcal{K} = [a'',Y]$. Then we interchange the rôles of $\mathcal{L}$ and $\mathcal{M}$ and we call $\mathcal{K}'$ the line through $\mathcal{P}$ meeting $[[k,d,\infty]]$ (note that $[[k,d,\infty]]$ is the unique line through $(0,d)$ meeting $[k,0,0]$). If $\mathcal{K}'$ is not concurrent with $[\infty]$, then the result follows from similar argument as above (with $\mathcal{L}$ and $\mathcal{M}$ interchanged). Suppose $\mathcal{K}' = [a'',Y]$. One can check that the equality (17) remains valid provided that we omit the terms containing X. One can show this by applying the main property in the same quadrangles on the same vertices and sides as in the case that $\mathcal{K}$ does not meet $[\infty]$. Hence formula (17) becomes

$$w(b,d) = w(k,n) - v(n') + v(k^*) - v(n^*) \qquad (22)$$

Similarly

$$w(b,d) = w(k,n) - v(k') + v(n^*) - v(k^*) \qquad (23)$$

Adding (22) and (23) side by side, the result follows.
All other cases are handled similarly. **Q.E.D.**

We summarize this as follows :

THEOREM(2.1.3.13). **(I).** *Suppose* $\mathcal{L}$ *I* $\mathcal{P}$ *I* $\mathcal{M}$ *and* $\nu(\mathcal{L}) = \nu(\mathcal{M})$. *Let* $\mathcal{L} = [[k,b,k']]$ *and* $\mathcal{M} = [[n,d,n']]$. *Then* $u^*(\mathcal{L},\mathcal{M})$ *is summed up in table 1.*

$\nu(\mathcal{P})\backslash\nu(\mathcal{L})$	[+,+,+]	[−,+,+]	[+,−,−]	[−,−,−]
(+,+,+)	$v(k,n)$	$w(k,n)$	–	–
(−,+,+)	$v(k',n')$	–	$w(k',n')$	–
(+,−,−)	–	$v(k',n')$	–	$w(k',n')$
(−,−,−)	–	–	$v(k,n)$	$w(k,n)$

TABLE 1

(II). *Suppose $\mathcal{P}$ I $\mathcal{L}$ I Q and $\nu(\mathcal{P}) = \nu(Q)$. Let $\mathcal{P} =$ $((a,\ell,a'))$ and $Q = ((c,p,c'))$. Then $u^*(\mathcal{P},Q)$ is summed up in table 2.*

$\nu(\mathcal{L})\backslash\nu(\mathcal{P})$	(+,+,+)	(−,+,+)	(+,−,−)	(−,−,−)
[+,+,+]	$v(a,c)$	$w(a,c)$	–	–
[−,+,+]	$v(a',c')$	–	$w(a',c')$	–
[+,−,−]	–	$v(a',c')$	–	$w(a',c')$
[−,−,−]	–	–	$v(a,c)$	$w(a,c)$

TABLE 2

REMARK(2.1.3.14). Let us omit for a moment the claim of surjectivity of the valuation map v, then for every QQR, we can define a trivial valuation by sending all non-identical pairs to 0 and all identical pairs to $+\infty$. We can also recoordinatize the corresponding generalized quadrangle as above. There exists a bijective correspondence between the set of points with three old coordinates and the set of points of shape (+,+,+) : the old coordinates are the new ones. Similarly, the points of shape (-,+,+) are exactly the points with two old coordinates and $((a^-,\ell^+,a'^+)) = (\ell,a')$ with $a=\infty$ automatically. Also, the points of shape (+,-,-) are exactly the points with one old coordinate and $((a^+,\ell^-,a'^-)) = (a)$ with ℓ and a' automatically equal to ∞. Finally, there is a unique point of shape (-,-,-), namely $((\infty,\infty,\infty)) = (\infty)$. Similarly for the lines. Hence, the number of plus signs in the shape of a point (or line) is equal to the number of old coordinates of that point (or line). We can interpret tables 1 and 2 as follows. Table 1 tells us when two lines having a certain number of coordinates can meet, e.g. two different lines having one coordinate can never meet, etc... Dually for table 2. But this interpretation is also valid for surjective valuations. We will explain this in remark(2.2.3.12).

THEOREM(2.1.3.15). *Let $\mathcal{P}_1,\mathcal{P}_2,\mathcal{P}_3,\mathcal{P}_4$ be four points of $\mathcal{V}$ and $\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\mathcal{L}_4$ four lines of $\mathcal{V}$ with $\mathcal{P}_1\ I\ \mathcal{L}_1\ I\ \mathcal{P}_2\ I\ \mathcal{L}_2\ I\ \mathcal{P}_3\ I\ \mathcal{L}_3\ I\ \mathcal{P}_4\ I\ \mathcal{L}_4\ I\ \mathcal{P}_1$ and suppose that only $\mathcal{L}_1 \perp [\infty]$ and that $v(\mathcal{L}_1) = [-,+,+]$. Then we have :*

[1] $v(\mathcal{P}_1,\mathcal{P}_2) + v(\mathcal{L}_1,\mathcal{L}_4) + v(\mathcal{L}_1,\mathcal{L}_2) = v(\mathcal{P}_3,\mathcal{P}_4) + v(\mathcal{L}_2,\mathcal{L}_3) + v(\mathcal{L}_3,\mathcal{L}_4),$

[2] $v(\mathcal{P}_1,\mathcal{P}_4) + v(\mathcal{L}_3,\mathcal{L}_4) + v(\mathcal{L}_1,\mathcal{L}_4) = v(\mathcal{P}_2,\mathcal{P}_3) + v(\mathcal{L}_1,\mathcal{L}_2) + v(\mathcal{L}_2,\mathcal{L}_3),$

[3] $2.v(\mathcal{L}_1,\mathcal{L}_2) + v(\mathcal{P}_1,\mathcal{P}_2) + v(\mathcal{P}_2,\mathcal{P}_3) = 2.v(\mathcal{L}_3,\mathcal{L}_4) + v(\mathcal{P}_3,\mathcal{P}_4) + v(\mathcal{P}_1,\mathcal{P}_4),$

[4] $2.v(\mathcal{L}_1,\mathcal{L}_4) + v(\mathcal{P}_1,\mathcal{P}_4) + v(\mathcal{P}_1,\mathcal{P}_2) = 2.v(\mathcal{L}_2,\mathcal{L}_3) + v(\mathcal{P}_2,\mathcal{P}_3) + v(\mathcal{P}_3,\mathcal{P}_4).$

PROOF. We have to check all possible cases with respect to the shapes of $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4, \mathcal{L}_2, \mathcal{L}_3$, and $\mathcal{L}_4$. In view of the previous theorem, there are 72 cases. We enumerate them all below, using the following conventions. We abbreviate the shape of a point or a line by the number of minus signs it contains. We take the following order : $\mathcal{P}_2, \mathcal{L}_2, \mathcal{P}_3, \mathcal{L}_3, \mathcal{P}_4, \mathcal{L}_4, \mathcal{P}_1$. So for instance $0^2 123^2 2$ means : $\nu(\mathcal{P}_2) = 0 = (0,0,0)$; $\nu(\mathcal{L}_2) = 0 = [0,0,0]$; $\nu(\mathcal{P}_3) = 1 = (-,+,+)$; $\nu(\mathcal{L}_3) = 2 = [+,-,-]$; $\nu(\mathcal{P}_4) = 3 = (-,-,-)$; $\nu(\mathcal{L}_4) = 3 = [-,-,-]$ en $\nu(\mathcal{P}_1) = (+,-,-)$. Note that in the example, we used the self-explaining convention $0012332 = 0^2 123^2 2$. The 72 possiblities are :

0^7	$010^3 10$	21210^3	0121210	2121212	2323210
$0^4 10^2$	$010^3 12$	$0^3 1232$	0121012	2121232	$0123^3 2$
$0^2 10^4$	$210^3 10$	23210^3	2101210	2321212	$23^3 210$
$0^5 10$	$210^3 12$	0101010	0121212	2321232	2123212
010^5	$0^3 10^3$	0101210	2121210	$0^2 1210^2$	2123232
$0^5 12$	$0^3 1010$	0121010	0121232	$0^2 123^2 2$	2323212
210^5	01010^3	0101012	2321210	$23^2 210^2$	$2123^3 2$
$0^2 1010^2$	$0^3 1210$	2101010	2101012	$23^2 23^2 2$	$23^3 212$
$0^2 10^2 10$	01210^3	0101212	2101212	0123210	2323232
$010^2 10^2$	$0^3 1012$	2121010	2121012	0123212	$2323^3 2$
$0^2 10^2 12$	21010^3	0101232	2101232	2123210	$23^3 232$
$210^2 10^2$	$0^3 1212$	2321010	2321012	0123232	$23^5 2$

If we define ***the symmetric 7-tuple*** of a given 7-tuple as the one obtained by reversing the order, then in the enumeration above, the symmetric 7-tuple follows its original. Now note that $[^3]$ and $[^4]$

follow from [1] and [2] by a simple linear combination. Also, [2] for a certain 7-tuple is equivalent to [1] for the symmetric 7-tuple. Hence it suffices to show [1] for all the 72 cases. We are not going to write down all 72 proofs, because this would cost us over 100 pages telling us nothing essential. So instead, we will show four examples which are representative for all others. We prove successively $0^3 1010$, 210^3, 23210^3 and $2323^3 2$.

Put $\mathcal{P}_i = (a_i, \ell_i, a_i')$, $i=1,2,3,4$ and $\mathcal{L}_j = [k_j, b_j k_j']$, $j=2,3,4$. Note that $\mathcal{L}_1$ is not incident with (∞) by theorem(2.1.3.13).

First example : $0^3 1010$.

We apply the main property in the given quadrangle on the vertex $\mathcal{P}_2$:

$$v(a_1', a_2') + v(a_2, a_3) = 2.v(k_3, k_4) + v(a_3, a_4) + v(a_1, a_4) \tag{1}$$

Note that $u^*(\mathcal{L}_1, \mathcal{L}_2) = 0$, $u^*(\mathcal{P}_1, \mathcal{P}_2) = v(a_1', a_2')$, $u^*(\mathcal{P}_2, \mathcal{P}_3) = v(a_2, a_3)$, $u^*(\mathcal{L}_3, \mathcal{L}_4) = w(k_3, k_4)$, $u^*(\mathcal{P}_3, \mathcal{P}_4) = v(a_3', a_4')$ en $u^*(\mathcal{P}_1, \mathcal{P}_4) = v(a_1', a_4')$. So we must show :

$$v(a_1', a_2') + v(a_2, a_3) = 2.w(k_3, k_4) + v(a_3', a_4') + v(a_1', a_4') \tag{2}$$

We apply the main property in the quadrangle $[0]\ I\ (0, a_3')\ \perp\ \mathcal{P}_3\ I\ \mathcal{L}_3\ I\ \mathcal{P}_4$ $\perp\ (0, a_4')\ I\ [0]$, resp. the quadrangle $[0]\ I\ (0, a_1')\ \perp\ \mathcal{P}_1\ I\ \mathcal{L}_4\ I\ \mathcal{P}_4\ \perp$ $(0, a_4')\ I\ [0]$ on the side $[0]$:

$$v(a_3, a_4) = v(a_3', a_4') - 2.v(k_3) \tag{3}$$

$$v(a_1, a_4) = v(a_1', a_4') - 2.v(k_4) \tag{4}$$

Substituting (3) and (4) in (1), (2) follows directly.

Second example : 210^5.

Put $\mathcal{L}_j = [[\ldots, n_j']]$, $j=1,2$. We must show :

$$2.v(n_1', n_2') = 2.v(k_3, k_4) + v(a_3, a_4) + v(a_1, a_4) \qquad (5)$$

First, we remark that $v(a_2') < 0$ and so $v(a_1', a_2') = v(a_2')$ since $v(a_1') \geq 0$. Remember that formula (1) is still valid :

$$v(a_1', a_2') + v(a_2, a_3) = 2.v(k_3, k_4) + v(a_3, a_4) + v(a_1, a_4) \qquad (1)$$

Let $\mathcal{T}$ be the meeting point of $\mathcal{L}_2$ and $[0,0,n_2']$. Since $v(n_2') \geq 0$, $\nu([0,0,n_2']) = [+,+,+]$. Since $\nu(\mathcal{L}_2) = [-,+,+]$, the only possibility (by theorem(2.1.3.13)) is $\nu(\mathcal{T}) = (+,+,+)$. Put $\mathcal{T} = (a_{\mathcal{T}}, \ell_{\mathcal{T}}, a_{\mathcal{T}}')$. We apply the main property in the quadrangle $[0]\ I\ (0,a_2') \perp \mathcal{P}_2\ I\ \mathcal{L}_2\ I\ \mathcal{P}_3 \perp (0,a_3')\ I\ [0]$ resp. the quadrangle $[0]\ I\ (0,a_2') \perp \mathcal{P}_2\ I\ \mathcal{L}_2\ I\ \mathcal{T} \perp (0,a_{\mathcal{T}}')\ I\ [0]$ on the side $[0]$:

$$v(a_2', a_3') = v(a_2, a_3) + 2.v(k_2) \qquad (6)$$

$$v(a_2', a_{\mathcal{T}}') = v(a_2, a_{\mathcal{T}}) + 2.v(k_2) \qquad (7)$$

We apply the main property in the quadrangle $(0,0)\ I\ [0,0,n_1']\ I\ (a_2, n_1', 0)\ I\ \mathcal{L}_1\ I\ \mathcal{P}_2\ I\ \mathcal{L}_2\ I\ \mathcal{T}\ I\ [0,0,n_2']\ I\ (0,0)$ on the vertex $(0,0)$:

$$2.v(n_1', n_2') = v(a_2') + v(a_2, a_{\mathcal{T}}) \qquad (8)$$

Since $v(a_2', a_3') = v(a_2', a_{\mathcal{T}}') = v(a_2')$ (indeed, $v(a_2') < 0$ and $v(a_3'), v(a_{\mathcal{T}}') \geq 0$), (6) and (7) imply $v(a_2, a_3) = v(a_2, a_{\mathcal{T}})$. Plugging this in into (8) and eliminating $v(a_2') + v(a_2, a_3)$ in (8) and (1), (5) follows.

Third example : 23210^3.

Put $\mathcal{P}_i = ((c_i^+, p_i^-, a_i^{!-}))$, $i=2,3$. Then $u^*(\mathcal{L}_1,\mathcal{L}_2) = u^*(\mathcal{P}_1,\mathcal{P}_2) = u^*(\mathcal{L}_3,\mathcal{L}_4) = u^*(\mathcal{P}_3,\mathcal{P}_4) = 0$, $u^*(\mathcal{P}_1,\mathcal{P}_4) = v(a_1,a_4)$ and $u^*(\mathcal{P}_2,\mathcal{P}_3) = v(c_2,c_3)$. We must prove

$$v(a_1,a_4) = v(c_2,c_3) \tag{9}$$

Let $\mathcal{T}_i = (d_i, n_i, 0)$ be the unique point of $\mathcal{V}$ incident with $\mathcal{L}_i$ and collinear with $(0,0)$, $i=1,2,3$. Then $[0,0,n_i]$ is the line joining $(0,0)$ and $\mathcal{T}_i$, $i=1,2,3$. Since $\nu(\mathcal{L}_2) = [-,-,-]$, $v(n_2) < 0$. Since $\nu(\mathcal{L}_3) = \nu(\mathcal{L}_1) = [-,+,+]$, $v(n_3), v(n_1) \geq 0$ and so $v(n_2,n_3) = v(n_2) = v(n_2,n_1)$. Let $\mathcal{K}_j$ be the line joining $\mathcal{P}_j$ and $(c_j,0,0)$, $j=2,3$. Then $\mathcal{K}_j = [p_j, \ldots]$. We apply the main property in the given quadrangle on the side $\mathcal{L}_2$:

$$v(a_2,a_3) + v(k_2,k_3) = v(a_1,a_4) + v(k_3,k_4) \tag{10}$$

We apply the main property in the quadrangle $[0,0,0]\ I\ (c_2,0,0)\ I\ \mathcal{K}_2\ I\ \mathcal{P}_2\ I\ \mathcal{L}_2\ I\ \mathcal{P}_3\ I\ \mathcal{K}_3\ I\ (c_3,0,0)\ I\ [0,0,0]$ on the side $\mathcal{L}_2$:

$$v(a_2,a_3) = v(c_2,c_3) + v(p_2) + v(p_3) - v(p_2,k_2) - v(p_3,k_2) \tag{11}$$

We apply the main property in the quadrangle $(0,0)\ I\ [0,0,n_2]\ I\ \mathcal{T}_2\ I\ \mathcal{L}_2\ I\ \mathcal{P}_3\ I\ \mathcal{K}_3\ I\ (c_3,0,0)\ I\ [0,0,0]\ I\ (0,0)$ resp. $(0,0)\ I\ [0,0,n_2]\ I\ \mathcal{T}_2\ I\ \mathcal{L}_2\ I\ \mathcal{P}_2\ I\ \mathcal{K}_2\ I\ (c_2,0,0)\ I\ [0,0,0]\ I\ (0,0)$; $(0,0)\ I\ [0,0,n_2]\ I\ \mathcal{T}_2\ I\ \mathcal{L}_2\ I\ \mathcal{P}_3\ I\ \mathcal{L}_3\ I\ \mathcal{T}_3\ I\ [0,0,n_3]\ I\ (0,0)$ on the side $\mathcal{L}_2$:

$$v(n_2) + v(p_3) = v(p_3,k_2) + v(a_3,d_2) + v(k_2) \tag{12}$$

$$v(n_2) + v(p_2) = v(p_2,k_2) + v(a_2,d_2) + v(k_2) \tag{13}$$

$$v(n_2) + v(k_3) = v(k_2,k_3) + v(a_3,d_2) + v(k_2) \tag{14}$$

We apply the main property in the quadrangle $(0,0)\ I\ [0,0,n_2]\ I\ \mathcal{T}_2\ I\ \mathcal{L}_2$ $I\ \mathcal{P}_2\ I\ \mathcal{L}_1\ I\ \mathcal{T}_1\ I\ [0,0,n_1]\ I\ (0,0)$ on the side $\mathcal{L}_2$:

$$v(n_2) = v(a_2,d_2) + v(k_2) \qquad (15)$$

We remark that $v(k_3) < 0 \le v(k_4)$ and hence $v(k_3,k_4) = v(k_3)$. Taking this into account and making the following operation side-by-side : (10) - (11) - (12) - (13) + (14) + (15), then we obtain (9). Except for some special cases (containing ∞'s), this shows the result.

Fourth example : $2323^3 2$

Here, $u^*(\mathcal{L}_1,\mathcal{L}_2) = u^*(\mathcal{P}_3,\mathcal{P}_4) = u^*(\mathcal{P}_1,\mathcal{P}_4) = 0$. Let $Q_i = (c_i,0,0)$ be collinear with $\mathcal{P}_i$ (this defines Q_i unambiguously) and let $\mathcal{M}_i = [n_i,\ldots]$ be the line joining $\mathcal{P}_i$ and Q_i, i=1,2,3,4. We assume that $\mathcal{M}_i$ does not meet $[\infty]$ (if it does, then the proof is similar ; see also the remark at the end of the proof of theorem(2.1.3.12)). Suppose $\mathcal{K}_j$ is incident with (0,0,0) and meets $\mathcal{L}_j$ and let $\mathcal{K}_j = [\ell_j,\ldots]$. The meeting point of $\mathcal{K}_j$ and $\mathcal{L}_j$ will be denoted by $\mathcal{T}_j = (d_j,\ldots)$, j=3,4. Finally, we put $\mathcal{L}_1 = [a_1,k_1]$, $\mathcal{L}_2 = [[\ldots,\ell_2']]$ and $\mathcal{P}^* = (a^*,\ldots)$ is the meeting point of $[0,0,\ell_2']$ and $\mathcal{L}_2$.

By theorem(2.1.3.13), $u^*(\mathcal{P}_1,\mathcal{P}_2) = w(a_1',a_2')$, $u^*(\mathcal{P}_2,\mathcal{P}_3) = v(c_2,c_3)$ and $u^*(\mathcal{L}_3,\mathcal{L}_4) = w(\ell_3,\ell_4)$. So we have to show :

$$w(a_1',a_2') + v(c_2,c_3) = 2.w(\ell_3,\ell_4) \qquad (16)$$

We remark that $v(\ell_2') < 0 \le v(k_1)$ and so $v(k_1,\ell_2') = v(\ell_2')$. Also, $v(c_4) < 0 \le v(c_1), v(c_2), v(c_3)$, thus $v(c_1,c_4) = v(c_2,c_4) = v(c_3,c_4) = v(c_4)$. Note that (1) is still valid :

$$v(a_1', a_2') + v(a_2, a_3) = 2.v(k_3, k_4) + v(a_3, a_4) + v(a_1, a_4) \quad (1)$$

We apply the main property in the quadrangle $\mathcal{P}_4$ I $\mathcal{L}_4$ I $\mathcal{T}_4$ I $\mathcal{K}_4$ I (0,0,0) I $\mathcal{K}_3$ I $\mathcal{T}_3$ I $\mathcal{L}_3$ I $\mathcal{P}_4$ on the vertex $\mathcal{P}_4$:

$$2.v(k_3, k_4) + v(a_4, d_3) + v(a_4, d_4) = 2.v(\ell_3, \ell_4) + v(d_3) + v(d_4) \quad (17)$$

We apply the main property in the quadrangle $\mathcal{P}_2$ I $\mathcal{L}_2$ I $\mathcal{P}_3$ I $\mathcal{M}_3$ I Q_3 I [0,0,0] I Q_2 I $\mathcal{M}_2$ I $\mathcal{P}_2$ on the vertex $\mathcal{P}_2$:

$$v(c_2, c_3) + v(a_3, c_3) + 2.v(n_3) = v(a_2, a_3) + v(a_2, c_2) + 2.v(k_2, n_2) \quad (18)$$

We apply the main property in the quadrangle $\mathcal{P}_4$ I $\mathcal{L}_3$ I $\mathcal{P}_3$ I $\mathcal{M}_3$ I Q_3 I [0,0,0] I Q_4 I $\mathcal{M}_4$ I $\mathcal{P}_4$ resp. the quadrangle $\mathcal{P}_4$ I $\mathcal{L}_4$ I $\mathcal{P}_1$ I $\mathcal{M}_1$ I Q_1 I [0,0,0] I Q_4 I $\mathcal{M}_4$ I $\mathcal{P}_4$ on the vertex $\mathcal{P}_4$:

$$v(a_3, a_4) + v(a_4, c_4) + 2.v(k_3, n_4) = 2.v(n_3) + v(c_4) + v(a_3, c_3) \quad (19)$$

$$v(a_1, a_4) + v(a_4, c_4) + 2.v(k_4, n_4) = 2.v(n_1) + v(c_4) + v(a_1, c_1) \quad (20)$$

We apply the main property in the quadrangle $\mathcal{P}_4$ I $\mathcal{L}_3$ I $\mathcal{T}_3$ I $\mathcal{K}_3$ I (0,0,0) I [0,0,0] I Q_4 I $\mathcal{M}_4$ I $\mathcal{P}_4$ resp. $\mathcal{P}_4$ I $\mathcal{L}_4$ I $\mathcal{T}_4$ I $\mathcal{K}_4$ I (0,0,0) I [0,0,0] I Q_4 I $\mathcal{M}_4$ I $\mathcal{P}_4$ on the vertex $\mathcal{P}_4$:

$$2.v(\ell_3) + v(d_3) + v(c_4) = 2.v(k_3, n_4) + v(a_4, c_4) + v(a_4, d_3) \quad (21)$$

$$2.v(\ell_4) + v(d_4) + v(c_4) = 2.v(k_4, n_4) + v(a_4, c_4) + v(a_4, d_4) \quad (22)$$

We apply the main property in the quadrangle [0] I (0,0) I [0,0,0] I Q_1 I $\mathcal{M}_1$ I $\mathcal{P}_1$ $\perp$ $(0, a_1')$ I [0], resp. [0] I (0,0) I [0,0,0] I Q_2 I $\mathcal{M}_2$ I $\mathcal{P}_2$ $\perp$ $(0, a_2')$ I [0] on the side [0] :

$$v(a_1, c_1) + 2.v(n_1) = v(a_1') \tag{23}$$

$$v(a_2, c_2) + 2.v(n_2) = v(a_2') \tag{24}$$

We apply the main property in the quadrangle $[0,0,0]\ I\ Q_2\ I\ M_2\ I\ \mathcal{P}_2\ I\ \mathcal{L}_2\ I\ \mathcal{P}^*\ I\ [0,0,\ell_2']\ I\ (0,0)\ I\ [0,0,0]$ on the side $[0,0,0]$:

$$v(a_2, a^*) + v(k_2, n_2) + v(k_2) = v(\ell_2') + v(n_2) \tag{25}$$

Finally, we apply the main property in the quadrangle $\mathcal{L}_2\ I\ \mathcal{P}_2\ I\ \mathcal{L}_1 \perp [0,0,k_1]\ I\ (0,0)\ I\ [0,0,\ell_2']\ I\ \mathcal{P}^*\ I\ \mathcal{L}_2$ on the side $\mathcal{L}_2$:

$$v(\ell_2') = v(a_2, a^*) + v(k_2) \tag{26}$$

Carrying out the operation (1) + (17) + (18) + ... + (26), we obtain (16), and the result follows. **Q.E.D.**

Similarly, one can prove the same property for $v(\mathcal{L}_1) = [+,-,-]$ and for $v(\mathcal{L}_1) = [-,-,-]$. Since the proofs do not give us deeper insight and since they are anyway very long and tiresome, we omit them. We summarize all results in the next proposition :

PROPOSITION(2.1.3.16). *Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ be four points of $\mathcal{V}$ and $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4$ be four lines of $\mathcal{V}$ with $\mathcal{P}_1\ I\ \mathcal{L}_1\ I\ \mathcal{P}_2\ I\ \mathcal{L}_2\ I\ \mathcal{P}_3\ I\ \mathcal{L}_3\ I\ \mathcal{P}_4\ I\ \mathcal{L}_4\ I\ \mathcal{P}_1$ and suppose that only $\mathcal{L}_1 \perp [\infty]$ (but not incident with (∞)). Then we have*

[1] $2.u^*(\mathcal{L}_1,\mathcal{L}_2) + u^*(\mathcal{P}_1,\mathcal{P}_2) + u^*(\mathcal{P}_2,\mathcal{P}_3) = 2.u^*(\mathcal{L}_3,\mathcal{L}_4) + u^*(\mathcal{P}_3,\mathcal{P}_4) + u^*(\mathcal{P}_1,\mathcal{P}_4)$,

[2] $2.u^*(\mathcal{L}_1,\mathcal{L}_4) + u^*(\mathcal{P}_1,\mathcal{P}_4) + u^*(\mathcal{P}_1,\mathcal{P}_2) = 2.u^*(\mathcal{L}_2,\mathcal{L}_3) + u^*(\mathcal{P}_2,\mathcal{P}_3) + u^*(\mathcal{P}_3,\mathcal{P}_4)$,

[3] $u^*(\mathcal{P}_1,\mathcal{P}_2) + u^*(\mathcal{L}_1,\mathcal{L}_4) + u^*(\mathcal{L}_1,\mathcal{L}_2) = u^*(\mathcal{P}_3,\mathcal{P}_4) + u^*(\mathcal{L}_2,\mathcal{L}_3) + u^*(\mathcal{L}_3,\mathcal{L}_4)$,

[4] $u^*(\mathcal{P}_1,\mathcal{P}_4) + u^*(\mathcal{L}_3,\mathcal{L}_4) + u^*(\mathcal{L}_1,\mathcal{L}_4) = u^*(\mathcal{P}_2,\mathcal{P}_3) + u^*(\mathcal{L}_1,\mathcal{L}_2) + u^*(\mathcal{L}_2,\mathcal{L}_3)$.

Dually, we have :

PROPOSITION(2.1.3.17). *Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ be four points of $\mathcal{V}$ and $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4$ be four lines of $\mathcal{V}$ with $\mathcal{P}_1 \ I\ \mathcal{L}_1 \ I\ \mathcal{P}_2 \ I\ \mathcal{L}_2 \ I\ \mathcal{P}_3 \ I\ \mathcal{L}_3 \ I\ \mathcal{P}_4 \ I\ \mathcal{L}_4 \ I\ \mathcal{P}_1$ and suppose that only $\mathcal{P}_1 \perp (\infty)$ (but not incident with $[\infty]$). Then we have*

[1] $2.u^*(\mathcal{L}_1,\mathcal{L}_2) + u^*(\mathcal{P}_1,\mathcal{P}_2) + u^*(\mathcal{P}_2,\mathcal{P}_3) = 2.u^*(\mathcal{L}_3,\mathcal{L}_4) + u^*(\mathcal{P}_3,\mathcal{P}_4) + u^*(\mathcal{P}_1,\mathcal{P}_4)$,

[2] $2.u^*(\mathcal{L}_1,\mathcal{L}_4) + u^*(\mathcal{P}_1,\mathcal{P}_4) + u^*(\mathcal{P}_1,\mathcal{P}_2) = 2.u^*(\mathcal{L}_2,\mathcal{L}_3) + u^*(\mathcal{P}_2,\mathcal{P}_3) + u^*(\mathcal{P}_3,\mathcal{P}_4)$,

[3] $u^*(\mathcal{P}_1,\mathcal{P}_2) + u^*(\mathcal{L}_1,\mathcal{L}_4) + u^*(\mathcal{L}_1,\mathcal{L}_2) = u^*(\mathcal{P}_3,\mathcal{P}_4) + u^*(\mathcal{L}_2,\mathcal{L}_3) + u^*(\mathcal{L}_3,\mathcal{L}_4)$,

[4] $u^*(\mathcal{P}_1,\mathcal{P}_4) + u^*(\mathcal{L}_3,\mathcal{L}_4) + u^*(\mathcal{L}_1,\mathcal{L}_4) = u^*(\mathcal{P}_2,\mathcal{P}_3) + u^*(\mathcal{L}_1,\mathcal{L}_2) + u^*(\mathcal{L}_2,\mathcal{L}_3)$.

PROPOSITION(2.1.3.18). *Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ be four points of $\mathcal{V}$ and $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4$ be four lines of $\mathcal{V}$ with $\mathcal{P}_1 \ I\ \mathcal{L}_1 \ I\ \mathcal{P}_2 \ I\ \mathcal{L}_2 \ I\ \mathcal{P}_3 \ I\ \mathcal{L}_3 \ I\ \mathcal{P}_4 \ I\ \mathcal{L}_4 \ I\ \mathcal{P}_1$ and suppose that only $\mathcal{L}_1 \perp [\infty]$ and only $\mathcal{P}_3 \perp (\infty)$. Then we have*

[1] $2.u^*(\mathcal{L}_1,\mathcal{L}_2) + u^*(\mathcal{P}_1,\mathcal{P}_2) + u^*(\mathcal{P}_2,\mathcal{P}_3) = 2.u^*(\mathcal{L}_3,\mathcal{L}_4) + u^*(\mathcal{P}_3,\mathcal{P}_4) + u^*(\mathcal{P}_1,\mathcal{P}_4)$,

[2] $2.u^*(\mathcal{L}_1,\mathcal{L}_4) + u^*(\mathcal{P}_1,\mathcal{P}_4) + u^*(\mathcal{P}_1,\mathcal{P}_2) = 2.u^*(\mathcal{L}_2,\mathcal{L}_3) + u^*(\mathcal{P}_2,\mathcal{P}_3) + u^*(\mathcal{P}_3,\mathcal{P}_4)$,

[3] $u^*(\mathcal{P}_1,\mathcal{P}_2) + u^*(\mathcal{L}_1,\mathcal{L}_4) + u^*(\mathcal{L}_1,\mathcal{L}_2) = u^*(\mathcal{P}_3,\mathcal{P}_4) + u^*(\mathcal{L}_2,\mathcal{L}_3) + u^*(\mathcal{L}_3,\mathcal{L}_4)$,

[4] $u^*(\mathcal{P}_1,\mathcal{P}_4) + u^*(\mathcal{L}_3,\mathcal{L}_4) + u^*(\mathcal{L}_1,\mathcal{L}_4) = u^*(\mathcal{P}_2,\mathcal{P}_3) + u^*(\mathcal{L}_1,\mathcal{L}_2) + u^*(\mathcal{L}_2,\mathcal{L}_3)$.

PROOF. The proof is in fact the same as for proposition (2.1.3.16), only the ***result*** of applying the main property will give simpler expressions and hence one will obtain the result quicker (see also the remark at the end of the proof of theorem(2.1.3.13)). **Q.E.D.**

The same thing can be said about the case where $\mathcal{L}_1$ *I* (∞) and the rest is arbitrary or $\mathcal{L}_1 \perp [\infty]$ and not incident with (∞) and the rest is arbitrary. And of course, also the dual can be proved. We summarize :

PROPOSITION(2.1.3.19). *Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ be four points of $\mathcal{V}$ and $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4$ be four lines of $\mathcal{V}$ with $\mathcal{P}_1$ I $\mathcal{L}_1$ I $\mathcal{P}_2$ I $\mathcal{L}_2$ I $\mathcal{P}_3$ I $\mathcal{L}_3$ I $\mathcal{P}_4$ I $\mathcal{L}_4$ I $\mathcal{P}_1$ and suppose that $\mathcal{L}_1 \perp [\infty]$ or $\mathcal{P}_1 \perp (\infty)$. Then we have*

$$[1]\quad 2.u^*(\mathcal{L}_1,\mathcal{L}_2) + u^*(\mathcal{P}_1,\mathcal{P}_2) + u^*(\mathcal{P}_2,\mathcal{P}_3) = 2.u^*(\mathcal{L}_3,\mathcal{L}_4) + u^*(\mathcal{P}_3,\mathcal{P}_4) + u^*(\mathcal{P}_1,\mathcal{P}_4),$$

$$[2]\quad 2.u^*(\mathcal{L}_1,\mathcal{L}_4) + u^*(\mathcal{P}_1,\mathcal{P}_4) + u^*(\mathcal{P}_1,\mathcal{P}_2) = 2.u^*(\mathcal{L}_2,\mathcal{L}_3) + u^*(\mathcal{P}_2,\mathcal{P}_3) + u^*(\mathcal{P}_3,\mathcal{P}_4),$$

$$[3]\quad u^*(\mathcal{P}_1,\mathcal{P}_2) + u^*(\mathcal{L}_1,\mathcal{L}_4) + u^*(\mathcal{L}_1,\mathcal{L}_2) = u^*(\mathcal{P}_3,\mathcal{P}_4) + u^*(\mathcal{L}_2,\mathcal{L}_3) + u^*(\mathcal{L}_3,\mathcal{L}_4),$$

$$[4]\quad u^*(\mathcal{P}_1,\mathcal{P}_4) + u^*(\mathcal{L}_3,\mathcal{L}_4) + u^*(\mathcal{L}_1,\mathcal{L}_4) = u^*(\mathcal{P}_2,\mathcal{P}_3) + u^*(\mathcal{L}_1,\mathcal{L}_2) + u^*(\mathcal{L}_2,\mathcal{L}_3).$$

We now show the most general case :

THEOREM(2.1.3.20). *Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ be four points of $\mathcal{V}$ and $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4$ be four lines of $\mathcal{V}$ with $\mathcal{P}_1$ I $\mathcal{L}_1$ I $\mathcal{P}_2$ I $\mathcal{L}_2$ I $\mathcal{P}_3$ I $\mathcal{L}_3$ I $\mathcal{P}_4$ I $\mathcal{L}_4$ I $\mathcal{P}_1$. Then we have*

$$[1]\quad 2.u^*(\mathcal{L}_1,\mathcal{L}_2) + u^*(\mathcal{P}_1,\mathcal{P}_2) + u^*(\mathcal{P}_2,\mathcal{P}_3) = 2.u^*(\mathcal{L}_3,\mathcal{L}_4) + u^*(\mathcal{P}_3,\mathcal{P}_4) + u^*(\mathcal{P}_1,\mathcal{P}_4),$$

$$[2]\quad 2.u^*(\mathcal{L}_1,\mathcal{L}_4) + u^*(\mathcal{P}_1,\mathcal{P}_4) + u^*(\mathcal{P}_1,\mathcal{P}_2) = 2.u^*(\mathcal{L}_2,\mathcal{L}_3) + u^*(\mathcal{P}_2,\mathcal{P}_3) + u^*(\mathcal{P}_3,\mathcal{P}_4),$$

$$[3]\quad u^*(\mathcal{P}_1,\mathcal{P}_2) + u^*(\mathcal{L}_1,\mathcal{L}_4) + u^*(\mathcal{L}_1,\mathcal{L}_2) = u^*(\mathcal{P}_3,\mathcal{P}_4) + u^*(\mathcal{L}_2,\mathcal{L}_3) + u^*(\mathcal{L}_3,\mathcal{L}_4),$$

$$[4]\quad u^*(\mathcal{P}_1,\mathcal{P}_4) + u^*(\mathcal{L}_3,\mathcal{L}_4) + u^*(\mathcal{L}_1,\mathcal{L}_4) = u^*(\mathcal{P}_2,\mathcal{P}_3) + u^*(\mathcal{L}_1,\mathcal{L}_2) + u^*(\mathcal{L}_2,\mathcal{L}_3).$$

PROOF. If one of the lines $\mathcal{L}_j$ meets $[\infty]$ or one of the points $\mathcal{P}_i$ is collinear with (∞), then the result follows from the previous propositions. So suppose that no point $\mathcal{P}_i$ is collinear with (∞) and that no line $\mathcal{L}_j$ is concurrent with $[\infty]$. Consider the unique line incident with (∞) and meeting $\mathcal{L}_4$, say in the point Q_1. If this line also meets $\mathcal{L}_2$, then call it $\mathcal{M}_1$. In the other case, we call $\mathcal{M}_1$ the unique line incident with Q_1 and meeting $\mathcal{L}_2$. Let Q_2 be the meeting point of $\mathcal{M}_1$ and $\mathcal{L}_2$. Consider the quadrangle Q_1 I $\mathcal{M}_1$ I Q_2 I $\mathcal{L}_2$ I $\mathcal{P}_3$ I $\mathcal{L}_3$

$I\ \mathcal{P}_4\ I\ \mathcal{L}_4\ I\ Q_1$. By proposition(2.1.2.19) [3], we have :

$$u^*(Q_1,Q_2) + u^*(\mathcal{M}_1,\mathcal{L}_4) + u^*(\mathcal{M}_1,\mathcal{L}_2) = u^*(\mathcal{P}_3,\mathcal{P}_4) + u^*(\mathcal{L}_2,\mathcal{L}_3) + u^*(\mathcal{L}_3,\mathcal{L}_4)$$

Now consider the quadrangle $Q_1\ I\ \mathcal{M}_1\ I\ Q_2\ I\ \mathcal{L}_2\ I\ \mathcal{P}_2\ I\ \mathcal{L}_1\ I\ \mathcal{P}_1\ I\ \mathcal{L}_4\ I\ Q_1$. By proposition(2.1.3.19) [3], we again have :

$$u^*(\mathcal{P}_1,\mathcal{P}_2) + u^*(\mathcal{L}_1,\mathcal{L}_4) + u^*(\mathcal{L}_1,\mathcal{L}_2) = u^*(Q_1,Q_2) + u^*(\mathcal{M}_1,\mathcal{L}_4) + u^*(\mathcal{M}_1,\mathcal{L}_2)$$

Combining these equations, we obtain the relation [3]. By cyclic permutation of the indices, [4] holds. Dually (or by linear combination), also [1] and [2] follow. Q.E.D.

We will call this property from now on ***the second main property***. The earlier main property will be called now ***the first main property***. We will also use self explaining expressions as "we apply the second main property in the quadrangle ... on the side/vertex ... ".

We will now define the partial valuation and the level *n* H-Q's derived from it.

2.2. The level *n* H-Q's derived from a V-QQR.

2.2.1 Homogeneous coordinates and the partial valuation.

In the definition of the new (and also the old) coordinates of, say, a point, it is conspicuous that the element of $\mathcal{R}_2$ plays a special rôle. Indeed, let $\mathcal{P}$ be a point with coordinates $((a,\ell,a'))$ and let $\mathcal{P}_a$ resp.

$\mathcal{P}_{a'}$ be the projection of $\mathcal{P}$ on the line responsible for the coordinate a, resp. a'. For instance, if $\nu(\mathcal{P}) = (+,+,+)$ or $(-,+,+)$, then $\mathcal{P}_a = (a)$, if $\nu(\mathcal{P}) = (+,-,-)$ or $(-,-,-)$, then $\mathcal{P}_a = (a,0,0)$. We call ***axis*** every line of the set $\{[\infty],[0],[0,0],[0,0,0]\}$ and dually, we call ***axispoint*** every point of the set $\{(\infty),(0).(0,0).(0,0,0)\}$. Now $\mathcal{P}$ is connected to two special axes, namely those axes on which we have to project $\mathcal{P}$ to find $\mathcal{P}_a$ and $\mathcal{P}_{a'}$. We call those axes respectivily ***the first axis of*** $\mathcal{P}$ and the ***second axis of*** $\mathcal{P}$. Note that these axes always meet in an axispoint, called from now on the ***central axispoint*** of $\mathcal{P}$. The other axispoint incident with the first, resp. the second axis of $\mathcal{P}$ will be called ***the first*** resp. ***the second axispoint of*** $\mathcal{P}$. Finally, the line joining $\mathcal{P}$ and $\mathcal{P}_a$ resp. $\mathcal{P}$ and $\mathcal{P}_{a'}$ will be called ***the first*** resp. ***the second coordinateline of*** $\mathcal{P}$. So the coordinate ℓ of $\mathcal{P}$ is determined by the line incident with the first (or second) axispoint of $\mathcal{P}$ and meeting the second (or first) coordinateline of $\mathcal{P}$. So clearly, we made a choice here and this choice was not intrinsicly determined by $\mathcal{P}$, but it was made as follows : choose the axispoint (first or second) containing the least number of zeros (in its coordinate-tuple). So if we want to make the coordinates homogenuous, we should also consider the other axispoint of $\mathcal{P}$ and add the corresponding coordinate to the coordinate-tuple of $\mathcal{P}$, obtaining a coordinate-*quadruple* for points (and dually for lines). Hence the following recoordinatization.

(1). $\nu(\mathcal{P}) = (+,+,+)$.

We define $\mathcal{P} = ((a,\ell;a',\ell'))$ where $[0,\ell']$ meets the second coordinateline of $\mathcal{P}$.

(2). $\nu(\mathcal{P}) = (-,+,+)$.

We define $\mathcal{P} = ((a,\ell';a',\ell))$ where $[\ell',0,0]$ meets the first coordinateline of $\mathcal{P}$.

(3). $\nu(\mathcal{P}) = (+,-,-)$.

We define $\mathcal{P} = ((a,\ell;a',\ell'))$ where $[\ell',0,0]$ again meets the second coordinateline of $\mathcal{P}$.

(4). $\nu(\mathcal{P}) = (-,-,-)$.

We define $\mathcal{P} = ((a,\ell;a',\ell'))$ where $[0,0,\ell']$ again meets the second coordinateline of $\mathcal{P}$.

So if $\mathcal{P} = ((a,\ell;a',\ell'))$, then ℓ is determined by the second axispoint of $\mathcal{P}$ and the first coordinateline of $\mathcal{P}$ and ℓ' by the first axispoint of $\mathcal{P}$ and the second coordinateline. Of course, not every quadruple $((a,\ell;a',\ell'))$ will define a point because ℓ' is completely determined by a,ℓ and a' and ℓ is completely determined by a,ℓ' and a'. What we get is a kind of symmetry in the coordinates : in all properties or definitions, we may switch (a,ℓ) and (a',ℓ') and we have a new property or definition. As an example, we have the following property.

PROPOSITION(2.2.1.1). *Let $\mathcal{P} = ((a,\ell;a',\ell'))$, then $v(\ell)$ and $v(\ell')$ are simultanously strictly negative or non-negative.*

PROOF. **(1).** $\nu(\mathcal{P}) = (+,+,+)$. Suppose $v(\ell') < 0$ and let $\mathcal{L}_1$ be the second coordinateline of $\mathcal{P}$. Since $\mathcal{L}_1$ meets $[0]$, $\mathcal{L}_1$ does not have the shape $[-,+,+]$ and since $[0,\ell']$ meets $\mathcal{L}_1$, $\mathcal{L}_1$ does not have the shape $[+,+,+]$. But $\mathcal{P}\ I\ \mathcal{L}_1$, contradicting theorem(2.1.3.5).

(2). $v(\mathcal{P}) = (-,+,+)$. Suppose $v(\ell) < 0$ and let $\mathcal{L}_1$ be the first coordinateline of $\mathcal{P}$. Since $\mathcal{L}_1$ meets $[\infty]$, $\mathcal{L}_1$ does not have the shape $[+,+,+]$ and since $[\ell',0,0]$ meets $\mathcal{L}_1$, $\mathcal{L}_1$ does not have the shape $[+,-,-]$. But $\mathcal{P}\ I\ \mathcal{L}_1$, contradicting theorem(2.1.3.10).

(3). $v(\mathcal{P}) = (+,-,-)$. Similarly as above.

(4). $v(\mathcal{P}) = (-,-,-)$. Similarly as above. **Q.E.D.**

Similarly, one can introduce homogenuous coordinates for lines. The usual notation will be $\mathcal{L} = [[k,b;k',b']]$. We also have dual definitions for the ***first*** and the ***second axispoint***, the ***central***, ***first*** and ***second axis*** and the ***first*** and ***second coordinatepoint of*** $\mathcal{L}$. Also the dual of the previous proposition holds :

PROPOSITION(2.2.1.2). *Let* $\mathcal{L} = [[k,b;k',b']]$, *then* $v(b)$ *and* $v(b')$ *are simultanously strictly negative or non-negative.*

PROPOSITION(2.2.1.3). *Let* $\mathcal{P}_i = ((a,\ell_i;a',\ell_i'))$ *be a point of* $\mathcal{V}$, $i=1,2$. *Then* $v(\ell_1,\ell_2) = v(\ell_1',\ell_2')$ *if* $v(\ell_1) \geq 0$,

$w(\ell_1,\ell_2) = w(\ell_1',\ell_2')$ *if* $v(\ell_1) < 0$.

PROOF. Let $\mathcal{L}_i$ be the first and $\mathcal{L}_i'$ be the second coordinateline of $\mathcal{P}_i$, $i=1,2$. Let Q_i be the first and Q_i' be the second axispoint of $\mathcal{P}_i$, $i=1,2$. Let $\mathcal{M}_i$ be the unique line through Q_i' meeting $\mathcal{L}_i$, and let $\mathcal{M}_i'$ be the unique line through Q_i meeting $\mathcal{L}_i'$, $i=1,2$. Let $\mathcal{T}_i$ be the meetingpoint of $\mathcal{M}_i$ and $\mathcal{L}_i$ and let $\mathcal{T}_i'$ be the meetingpoint of $\mathcal{M}_i'$ and $\mathcal{L}_i'$, $i=1,2$. One can check easily case by case that $u^*(Q_i,\mathcal{T}_i') = u^*(Q_i',\mathcal{T}_i) = u^*((\mathcal{P}_i)_a,\mathcal{P}_i) =$

$u^*((\mathcal{P}_i)_{a'},\mathcal{P}_i) = 0$, $i=1,2$, because the shapes of the points in each pair differ. Moreover $u^*(\mathcal{M}_1,\mathcal{M}_2)$ is precisely equal to $v(\ell_1,\ell_2)$ or $w(\ell_1,\ell_2)$ according to $v(\ell_1) \geq 0$ or $v(\ell_1) < 0$. Similarly, $u^*(\mathcal{M}_1',\mathcal{M}_2')$ equals $v(\ell_1',\ell_2')$ or $w(\ell_1',\ell_2')$ according to $v(\ell_1') \geq 0$ or $v(\ell_1') < 0$. We apply the second main property in the quadrangle $(\mathcal{P}_1)_a=(\mathcal{P}_2)_a \; I \; \mathcal{L}_2 \; I \; \mathcal{P}_2 \; I \; \mathcal{L}_2' \; I \; (\mathcal{P}_2)_{a'}=(\mathcal{P}_1)_{a'} \; I \; \mathcal{L}_1' \; I \; \mathcal{P}_1 \; I \; \mathcal{L}_1 \; I \; (\mathcal{P}_1)_a$ on the vertex $(\mathcal{P}_1)_a$:

$$u^*(\mathcal{L}_1,\mathcal{L}_2) = u^*(\mathcal{L}_1',\mathcal{L}_2') \qquad (1)$$

We apply the second main property in the quadrangle $(\mathcal{P}_1)_a \; I \; \mathcal{L}_1 \; I \; \mathcal{T}_1 \; I \; \mathcal{M}_1 \; I \; Q_1' \; I \; \mathcal{M}_2 \; I \; \mathcal{T}_2 \; I \; \mathcal{L}_2 \; I \; (\mathcal{P}_2)_a$ resp. the quadrangle $(\mathcal{P}_1)_{a'} \; I \; \mathcal{L}_1' \; I \; \mathcal{T}_1' \; I \; \mathcal{M}_1' \; I \; Q_1 \; I \; \mathcal{M}_2' \; I \; \mathcal{T}_2' \; I \; \mathcal{L}_2' \; I \; (\mathcal{P}_2')_a$ on the vertex Q_1' resp. Q_1 :

$$u^*(\mathcal{L}_1,\mathcal{L}_2) = u^*(\mathcal{M}_1,\mathcal{M}_2) \qquad (2)$$

$$u^*(\mathcal{L}_1',\mathcal{L}_2') = u^*(\mathcal{M}_1',\mathcal{M}_2') \qquad (3)$$

The result follows from the equalities (1),(2) and (3). **Q.E.D.**

The previous proof also implies immediatly :

COROLLARY(2.2.1.4). *Let* $\mathcal{P}_i = ((a_i,\ell_i;a_i',\ell_i'))$ *and let* $\mathcal{L}_i$ *be the first and* $\mathcal{L}_i'$ *the second coordinateline of* $\mathcal{P}_i$, $i=1,2$. *If* $a_1=a_2$, *then* $u^*(\mathcal{L}_1,\mathcal{L}_2) = z(\ell_1,\ell_2)$; *if* $a_1'=a_2'$, *then* $u^*(\mathcal{L}_1',\mathcal{L}_2') = z(\ell_1',\ell_2')$ *with* $z=v$ *or* $z=w$ *according to* $v(\ell_1) \geq 0$ *or* $v(\ell_1) < 0$.

Dually, we have :

PROPOSITION(2.2.1.5). *Let* $\mathcal{L}_i = [[k, b_i; k', b_i']]$ *be a line of* $\mathcal{V}$, $i=1,2$. *Then* $v(b_1, b_2) = v(b_1', b_2')$ *if* $v(b_1) \geq 0$,

$w(b_1, b_2) = w(b_1', b_2')$ *if* $v(b_1) < 0$.

COROLLARY(2.2.1.6). *Let* $\mathcal{L}_i = [[k_i, b_i; k_i', b_i']]$ *be a line of* $\mathcal{V}$ *and let* $\mathcal{P}_i$ *be the first and* $\mathcal{P}_i'$ *the second coordinatepoint of* $\mathcal{L}_i$, $i=1,2$. *If* $k_1=k_2$, *then* $u^*(\mathcal{P}_1, \mathcal{P}_2) = z(b_1, b_2)$; *if* $k_1'=k_2'$, *then* $u^*(\mathcal{P}_1', \mathcal{P}_2') = z(b_1', b_2')$ *with* $z=v$ *or* $z=w$ *according to* $v(b_1) \geq 0$ *or* $v(b_1) < 0$.

We define now for every natural number $n \neq 0$ three equivalence relations in the set of points of $\mathcal{V}$. Let $\mathcal{P}_i = ((a_i, \ell_i; a_i', \ell_i'))$, $i=1,2$.

$$(1) \quad \mathcal{P}_1 (\square 1)_n \mathcal{P}_2 \iff (a_1, \ell_1) = (a_2, \ell_2) \text{ and } \begin{cases} v(a_1', a_2') \geq n \text{ if } v(a_i') \geq 0 \\ w(a_1', a_2') \geq n \text{ if } v(a_i') < 0 \end{cases} (\forall i)$$

$$(2) \quad \mathcal{P}_1 (\square 2)_n \mathcal{P}_2 \iff (a_1', \ell_1') = (a_2', \ell_2') \text{ and } \begin{cases} v(a_1, a_2) \geq n \text{ if } v(a_i) \geq 0 \\ w(a_1, a_2) \geq n \text{ if } v(a_i) < 0 \end{cases} (\forall i)$$

$$(3) \quad \mathcal{P}_1 (\square 3)_n \mathcal{P}_2 \iff (a_1, a_1') = (a_2, a_2') \text{ and } \begin{cases} v(\ell_1, \ell_2) \geq n \text{ if } v(\ell_i) \geq 0 \\ w(\ell_1, \ell_2) \geq n \text{ if } v(\ell_i) < 0 \end{cases} (\forall i)$$

Similarly, one has a dual definition for lines. Because of a factor two appearing in that definition, we write it down explicitly. Let $\mathcal{L}_i = [[k_i, b_i; k_i', b_i']]$, $i=1,2$.

$$(1) \quad \mathcal{P}_1 [\square 1]_n \mathcal{P}_2 \iff (k_1, b_1) = (k_2, b_2) \text{ and } \begin{cases} 2v(k_1', k_2') \geq n \text{ if } v(k_i') \geq 0 \\ 2w(k_1', k_2') \geq n \text{ if } v(k_i') < 0 \end{cases} (\forall i)$$

$$(2) \quad \mathcal{P}_1 (\Box 2)_n \mathcal{P}_2 \iff (k_1', b_1') = (k_2', b_2') \text{ and } \begin{cases} 2v(k_1, k_2) \geq n \text{ if } v(k_i) \geq 0 \\ 2w(k_1, k_2) \geq n \text{ if } v(k_i) < 0 \end{cases} (\forall i)$$

$$(3) \quad \mathcal{P}_1 (\Box 3)_n \mathcal{P}_2 \iff (k_1, k_1') = (k_2, k_2') \text{ and } \begin{cases} v(b_1, b_2) \geq n \text{ if } v(b_i) \geq 0 \\ w(b_1, b_2) \geq n \text{ if } v(b_i) < 0 \end{cases} (\forall i)$$

We call each of the equivalence relations $(\Box 1)_n, (\Box 2)_n, (\Box 3)_n$ a ***partial n-equivalence***. We call two points $\mathcal{P}_1$ and $\mathcal{P}_2$ ***n-equivalent***, notation $\mathcal{P}_1 \; \Box_n \; \mathcal{P}_2$, if $\mathcal{P}_1$ and $\mathcal{P}_2$ can be joined by a sequence of partial n-equivalences. Similarly for lines. Note that this definition is symmetric in the homogenuous coordinates since by proposition(2.2.1.3), we can replace ℓ_i by ℓ_i' in $(\Box 3)_n$. For $n = 0$, we define $\mathcal{P}_1 \; \Box_0 \; \mathcal{P}_2$ for all points $\mathcal{P}_1, \mathcal{P}_2$ of $\mathcal{V}$. Similarly for lines.

If there is no confusion possible, we omit the index n in the notation $(\Box 1)_n, (\Box 2)_n, (\Box 3)_n$ and $\Box_n$.

We define finally the ***partial valuation*** on pairs of points and pairs of lines as follows. Let $\mathcal{P}_1, \mathcal{P}_2$ be two points of $\mathcal{V}$, then $u(\mathcal{P}_1, \mathcal{P}_2) = n$ if $\mathcal{P}_1 \; \Box_n \; \mathcal{P}_2$ and $\mathcal{P}_1$ and $\mathcal{P}_2$ are not $(n+1)$-equivalent. This implies implicitly $u(\mathcal{P}, \mathcal{P}) = +\infty$. Similarly for pairs of lines.

The partial *-valuation of the previous paragraphs is not good enough to use for the definition of the quotient-geometries just because one of the coordinates plays a special rôle. In the definition of partial valuation however, this is not the case anymore.
Note that the partial valuation is in fact more of a geometric nature, since the definition of partial n-equivalence breaks down to vary one

coordinate while fixing the others.

Note that u is a real valuation-map : the two smallest amongst $u(\mathcal{P}_1,\mathcal{P}_2)$, $u(\mathcal{P}_2,\mathcal{P}_3)$ and $u(\mathcal{P}_2,\mathcal{P}_3)$ have to be equal (we call this property from now on the ***triangle inequality***, it holds also for u^* and v); u is symmetric and $u(\mathcal{P}_1,\mathcal{P}_2) = +\infty \Longleftrightarrow \mathcal{P}_1 = \mathcal{P}_2$. Similarly for lines.

PROPOSITION(2.2.1.7). *Let* $\mathcal{P}_i = ((a,\ell_i;a_i',\ell_i'))$, $i=1,2$. *If* $\nu(\mathcal{P}_1) = \nu(\mathcal{P}_2)$, $v(a_1',a_2') \geq n$ *if* $v(a_1') \geq 0$ *and* $w(a_1',a_2') \geq n$ *if* $v(a_1') < 0$, *then*

$$u(\mathcal{P}_1,\mathcal{P}_2) \geq n \iff \begin{cases} v(\ell_1,\ell_2) \geq n \text{ if } v(\ell_1) \geq 0 \\ w(\ell_1,\ell_2) \geq n \text{ if } v(\ell_1) < 0 \end{cases}.$$

PROOF. ***First part :*** $\Longrightarrow$

Suppose $u(\mathcal{P}_1,\mathcal{P}_2) \geq n$. We show the result in two steps.

(1). Suppose $a_1' = a_2' = a'$. Then there is the symmetry $a \leftrightarrow a'$ in the assumptions. One can easily check that every sequence of partial n-equivalences joining $\mathcal{P}_1$ and $\mathcal{P}_2$ can be split up in partial sequences of the following kind :

$$\text{(A)}\quad Q_1\,(\square 1)\,Q_2\,(\square 2)\,Q_3\,(\square 3)\,Q_4\,(\square 2)\,Q_5\,(\square 1)\,Q_6 \text{ and } \begin{cases} Q_1 = ((a,\ldots;a',\ldots)) \\ Q_6 = ((a,\ldots;a',\ldots)) \end{cases},$$

$$\text{(B)}\quad Q_1\,(\square 2)\,Q_2\,(\square 1)\,Q_3\,(\square 3)\,Q_4\,(\square 1)\,Q_5\,(\square 2)\,Q_6 \text{ and } \begin{cases} Q_1 = ((a,\ldots;a',\ldots)) \\ Q_6 = ((a,\ldots;a',\ldots)) \end{cases},$$

$$\text{(C)}\quad Q_1\,(\square 1)\,Q_2\,(\square 2)\,Q_3\,(\square 1)\,Q_4\,(\square 2)\,Q_5 \text{ and } \begin{cases} Q_1 = ((a,\ldots;a',\ldots)) \\ Q_5 = ((a,\ldots;a',\ldots)) \end{cases},$$

(D) $Q_1\,(\Box 2)\,Q_2\,(\Box 1)\,Q_3\,(\Box 2)\,Q_4\,(\Box 1)\,Q_5$ and $\begin{cases} Q_1 = ((a,\ldots;a',\ldots)) \\ Q_5 = ((a,\ldots;a',\ldots)) \end{cases}$

By the symmetry $a \leftrightarrow a'$, we only need to investigate the cases (A) and (C).

(A). Let $Q_i = ((a,\ell_i;a',\ell_i'))$, i=1,6, then we must show that $v(\ell_1,\ell_6) \geq n$ or $w(\ell_1,\ell_6) \geq n$ according to $v(\ell_1) \geq 0$ or $\dot{v}(\ell_1) < 0$. Let $\mathcal{L}_j$ be the first and $\mathcal{L}_j'$ be the second coordinateline of Q_j, j=1,2,...,6. Then by corollary (2.2.1.4), it is sufficient and necessary to show $u^*(\mathcal{L}_1,\mathcal{L}_6) \geq n$. Our assumptions imply by corollary(2.2.1.4) consecutively $u^*(\mathcal{L}_1,\mathcal{L}_6) = u^*(\mathcal{L}_1',\mathcal{L}_6') = u^*(\mathcal{L}_2',\mathcal{L}_5') = u^*(\mathcal{L}_2,\mathcal{L}_5) = u^*(\mathcal{L}_3,\mathcal{L}_4) \geq n$ since $\mathcal{L}_1' = \mathcal{L}_2'$, $\mathcal{L}_6' = \mathcal{L}_5'$, $\mathcal{L}_2 = \mathcal{L}_3$ and $\mathcal{L}_5 = \mathcal{L}_4$.

(C). Let $\mathcal{L}_j$ again be the first and $\mathcal{L}_j'$ the second coordinateline of Q_j, j=1,2,3,4,5. Let $\mathcal{M}$ be the first and $\mathcal{M}'$ the second axis of Q_j, for all $j \in \{1,2,3,4,5\}$. Let $\mathcal{T}_j$ resp. $\mathcal{T}_j'$ be the meeting point of $\mathcal{L}_j$ resp. $\mathcal{L}_j'$ and $\mathcal{M}$ resp. $\mathcal{M}'$, j=1,...,5. Then by assumption, $\mathcal{T}_2 = \mathcal{T}_3$ and we put $\mathcal{T}_2 = \mathcal{T}_3 = \mathcal{T}_{23}$. Similarly, we can put $\mathcal{T}_1 = \mathcal{T}_4 = \mathcal{T}_5 = \mathcal{T}_{145}$, $\mathcal{T}_1' = \mathcal{T}_2' = \mathcal{T}_5' = \mathcal{T}_{125}'$, $\mathcal{T}_3' = \mathcal{T}_4' = \mathcal{T}_{34}'$, $\mathcal{L}_2 = \mathcal{L}_3 = \mathcal{L}_{23}$, $\mathcal{L}_4 = \mathcal{L}_5 = \mathcal{L}_{45}$, $\mathcal{L}_1' = \mathcal{L}_2' = \mathcal{L}_{12}'$, $\mathcal{L}_3' = \mathcal{L}_4' = \mathcal{L}_{34}'$. So we have to show that dat $u^*(\mathcal{L}_1,\mathcal{L}_5) \geq 0$. If $Q_4\ I\ \mathcal{L}_{12}'$, then $\mathcal{L}_1 = \mathcal{L}_5$. So suppose that Q_4 is not incident with $\mathcal{L}_{12}'$. Let $(\mathcal{P},\mathcal{L})$ be the unique point-line pair satisfying $Q_4\ I\ \mathcal{L}\ I\ \mathcal{P}\ I\ \mathcal{L}_{12}'$. We remark first that $u^*(\mathcal{L}_i,\mathcal{L}_j') = 0 = u^*(Q_k,\mathcal{T}_\ell) = u^*(Q_k,\mathcal{T}_\ell')$, for all $i,j,k,\ell \in \{1,2,3,4,5\}$. Furthermore, since $Q_2\,(\Box 2)\,Q_3$ and $Q_3\,(\Box 1)\,Q_4$,

$$u^*(Q_2,Q_3),u^*(Q_3,Q_4) \geq n \tag{1}$$

We apply the second main property in the quadrangle $\mathcal{T}_{145}\ I\ \mathcal{L}_{45}\ I\ Q_4\ I\ \mathcal{L}\ I\ \mathcal{P}\ I\ \mathcal{L}_{12}'\ I\ Q_1\ I\ \mathcal{L}_1\ I\ \mathcal{T}_{145}$ on the vertex $\mathcal{T}_{145}$:

$$2.u^*(\mathcal{L}_1,\mathcal{L}_{45}) = u^*(\mathcal{P},Q_1) + u^*(\mathcal{P},Q_4) + 2.u^*(\mathcal{L},\mathcal{L}'_{12}) \tag{2}$$

We apply the second main property in the quadrangle $\mathcal{P}\ I\ \mathcal{L}\ I\ Q_4\ I\ \mathcal{L}'_{34}\ I\ Q_3\ I\ \mathcal{L}_{23}\ I\ Q_2\ I\ \mathcal{L}'_{12}\ I\ \mathcal{P}$ on the sides $\mathcal{L}'_{34}$ and $\mathcal{L}$:

$$u^*(Q_3,Q_4) + u^*(\mathcal{L},\mathcal{L}'_{34}) = u^*(\mathcal{P},Q_2) + u^*(\mathcal{L},\mathcal{L}'_{12}) \tag{3}$$

$$u^*(Q_2,\mathcal{A}_3) = u^*(\mathcal{P},Q_4) + u^*(\mathcal{L},\mathcal{L}'_{12}) + u^*(\mathcal{L},\mathcal{L}'_{34}) \tag{4}$$

We combine the second main property applied in the quadrangles $\mathcal{M}\ I\ \mathcal{T}_{145}\ I\ \mathcal{L}_1\ I\ Q_1\ I\ \mathcal{L}'_{12}\ I\ Q_2\ I\ \mathcal{L}_{23}\ I\ \mathcal{T}_{23}\ I\ \mathcal{M}$ and $\mathcal{M}\ I\ \mathcal{T}_{145}\ I\ \mathcal{L}_{45}\ I\ Q_4\ I\ \mathcal{L}'_{34}\ I\ Q_3\ I\ \mathcal{L}_{23}\ I\ \mathcal{T}_{23}\ I\ \mathcal{M}$ on the side $\mathcal{M}$ to obtain :

$$u^*(Q_1,Q_2) = u^*(Q_3,Q_4) \tag{5}$$

Finally, we apply the second main property in the quadrangle $\mathcal{P}\ I\ \mathcal{L}\ I\ Q_4\ I\ \mathcal{L}'_{34}\ I\ \mathcal{T}'_{34}\ I\ \mathcal{M}'\ I\ \mathcal{T}'_{125}\ I\ \mathcal{L}'_{12}\ I\ \mathcal{P}$ on the side $\mathcal{L}'_{12}$:

$$u^*(\mathcal{L},\mathcal{L}'_{34}) = u^*(\mathcal{L},\mathcal{L}'_{12}) + u^*(\mathcal{P},\mathcal{T}'_{125}) \tag{6}$$

Now if $u^*(\mathcal{P},\mathcal{T}'_{125}) > 0$, then $u^*(\mathcal{P},Q_2) = 0$ since $u^*(Q_2,\mathcal{T}'_{125}) = 0$. We eliminate $u^*(\mathcal{L},\mathcal{L}'_{12})$ in (3) and (6) and obtain

$$u^*(\mathcal{L},\mathcal{L}'_{34}) = u^*(\mathcal{L},\mathcal{L}'_{34}) + u^*(Q_3,Q_4) + u^*(\mathcal{P},\mathcal{T}'_{125}).$$

Since the partial $*$-valuation is non-negative, $u^*(\mathcal{P},\mathcal{T}'_{125}) = 0$, a contradiction. Hence $u^*(\mathcal{P},\mathcal{T}'_{125}) = 0$ and by (6), $u^*(\mathcal{L},\mathcal{L}'_{34}) = u^*(\mathcal{L},\mathcal{L}'_{12})$. Plugging this in into (3), we have $u^*(\mathcal{P},Q_2) = u^*(Q_3,Q_4) = u^*(Q_1,Q_2)$ (by (5)). Hence by the triangle inequality,

$$u^*(\mathcal{P}, Q_1) \geq u^*(Q_1, Q_2) \geq n \tag{7}$$

The last inequality holds by (1) and (5). Eliminating $u^*(\mathcal{L}, \mathcal{L}'_{12})$ $(=u^*(\mathcal{L}, \mathcal{L}'_{34}))$ in (2) and (4), we obtain

$$2.u^*(\mathcal{L}_1, \mathcal{L}_{45}) = u^*(Q_2, Q_3) + u^*(\mathcal{P}, Q_1) \geq 2n$$

by (1). This shows **(1)**.

(2). Let $a'_1 \neq a'_2$. Note that $\mathfrak{z}(a'_1, a'_2) \geq n$ with $\mathfrak{z}=v$ or $\mathfrak{z}=w$ according to $v(a'_1) \geq 0$ or $v(a'_1) < 0$. Let $\mathcal{L}_2$ be the first coordinateline of $\mathcal{P}_2$ and let $\mathcal{T}'_1$ be the meeting point of the second axis of $\mathcal{P}_1$ and the second coordinateline of $\mathcal{P}_1$. Consider the unique point $\mathcal{P}\ I\ \mathcal{L}_2$ collinear with $\mathcal{T}'_1$. Well, $\mathcal{P}$ has the same shape as $\mathcal{P}_1$ and $\mathcal{P}_2$ and looking at the homogenuous coordinates of $\mathcal{P}$, we see that, by the remark above, $u(\mathcal{P}, \mathcal{P}_2)$ $= \mathfrak{z}(a'_1, a'_2) \geq n$. By the triangle inequality, $u(\mathcal{P}_1, \mathcal{P}) \geq n$. If $\mathcal{L}_1$ is the first coordinateline of $\mathcal{P}_1$, then by the first part **(1)** of the proof, $u^*(\mathcal{L}_1, \mathcal{L}_2) \geq n$. The result follows from corollary (2.2.1.4).

Second part : $\Leftarrow$

We adopt the same notation as above in **(2)**. Given now is $u^*(\mathcal{L}_1, \mathcal{L}_2) \geq n$. Hence $u(\mathcal{P}, \mathcal{P}_1) = u^*(\mathcal{L}_1, \mathcal{L}_2) \geq n$ (again using corollary (2.2.1.4)). But also $u(\mathcal{P}, \mathcal{P}_2) \geq n$ as above. By the triangle inequality, $u(\mathcal{P}_1, \mathcal{P}_2) \geq n$. This completes the proof of the proposition. **Q.E.D.**

Also the dual of the previous proposition holds.

PROPOSITION(2.2.1.8). *Let* $\mathcal{L}_i = ((k, b_i; k_i', b_i'))$ *,* $i=1,2$. *If* $\nu(\mathcal{L}_1) = \nu(\mathcal{L}_2)$, $2.v(k_1', k_2') \geq n$ *if* $v(k_1') \geq 0$ *and* $2.w(k_1', k_2') \geq n$ *if* $v(k_1') < 0$, *then*

$$u(\mathcal{L}_1, \mathcal{L}_2) \geq 0 \iff \begin{cases} v(b_1, b_2) \geq n \text{ if } v(b_1) \geq 0 \\ w(b_1, b_2) \geq n \text{ if } v(b_1) < 0 \end{cases}.$$

By symmetry $(a, \ell) \leftrightarrow (a', \ell')$, we also have :

PROPOSITION(2.2.1.9). *Let* $\mathcal{P}_i = ((a_i, \ell_i; a', \ell_i'))$ *,* $i=1,2$. *If* $\nu(\mathcal{P}_1) = \nu(\mathcal{P}_2)$, $v(a_1, a_2) \geq n$ *if* $v(a_1) \geq 0$ *and* $w(a_1, a_2) \geq n$ *if* $v(a_1) < 0$, *then*

$$u(\mathcal{P}_1, \mathcal{P}_2) \geq n \iff \begin{cases} v(\ell_1', \ell_2') \geq n \text{ if } v(\ell_1') \geq 0 \\ w(\ell_1', \ell_2') \geq n \text{ if } v(\ell_1') < 0 \end{cases}.$$

And dually

PROPOSITION(2.2.1.10). *Let* $\mathcal{L}_i = ((k_i, b_i; k', b_i'))$ *,* $i=1,2$. *If* $\nu(\mathcal{L}_1) = \nu(\mathcal{L}_2)$, $2.v(k_1, k_2) \geq n$ *if* $v(k_1) \geq 0$ *and* $2.w(k_1, k_2) \geq n$ *if* $v(k_1) < 0$, *then*

$$u(\mathcal{L}_1, \mathcal{L}_2) \geq n \iff \begin{cases} v(b_1', b_2') \geq n \text{ if } v(b_1') \geq 0 \\ w(b_1', b_2') \geq n \text{ if } v(b_1') < 0 \end{cases}.$$

We now show a theorem stating a connection between u^* and u in case pair of points are collinear or the pair of lines are concurrent. But first a definition to shorten the proof.

Let κ be the shape of a point and λ the shape of a line. Then we call κ ***incompatible with*** λ if no line of shape λ is incident with a point of shape κ.

PROPOSITION(2.2.1.11). **(I).** *Let* $\mathcal{L}_1$ *I* $\mathcal{P}$ *I* $\mathcal{L}_2$ *in* $\mathcal{V}$. *Suppose* $\nu(\mathcal{L}_1) = \nu(\mathcal{L}_2)$ *and put* $\mathcal{L}_i = [[k_i, b_i, k_i']]$, $i=1,2$. *Then* $u(\mathcal{L}_1, \mathcal{L}_2) = 2.u^*(\mathcal{L}_1, \mathcal{L}_2)$ *and* $u(\mathcal{L}_1, \mathcal{L}_2)$ *can be read off in table 3.*

$\nu(\mathcal{P}) \backslash \nu(\mathcal{L})$	[+,+,+]	[-,+,+]	[+,-,-]	[-,-,-]
(+,+,+)	$2v(k_1, k_2)$	$2w(k_1, k_2)$	-	-
(-,+,+)	$2v(k_1', k_2')$	-	$2w(k_1', k_2')$	-
(+,-,-)	-	$2v(k_1', k_2')$	-	$2w(k_1', k_2')$
(-,-,-)	-	-	$2v(k_1, k_2)$	$2w(k_1, k_2)$

TABLE 3

(II). *Let* $\mathcal{P}_1$ *I* $\mathcal{L}$ *I* $\mathcal{P}_2$ *in* $\mathcal{V}$. *Suppose* $\nu(\mathcal{P}_1) = \nu(\mathcal{P}_2)$ *and put* $\mathcal{P}_i = ((a_i, \ell_i, a_i'))$, $i=1,2$. *Then* $u(\mathcal{P}_1, \mathcal{P}_2) = u^*(\mathcal{P}_1, \mathcal{P}_2)$ *and* $u(\mathcal{P}_1, \mathcal{P}_2)$ *can be read off in table 4.*

$\nu(\mathcal{L}) \backslash \nu(\mathcal{P})$	(+,+,+)	(-,+,+)	(+,-,-)	(-,-,-)
[+,+,+]	$v(a_1, a_2)$	$w(a_1, a_2)$	-	-
[-,+,+]	$v(a_1', a_2')$	-	$w(a_1', a_2')$	-
[+,-,-]	-	$v(a_1', a_2')$	-	$w(a_1', a_2')$
[-,-,-]	-	-	$v(a_1, a_2)$	$w(a_1, a_2)$

TABLE 4

PROOF. We show for example **(II)**. The proof of **(I)** is dual. We again put $\mathfrak{z}=v$ or $\mathfrak{z}=w$ according to the fact that the pair of elements in the argument has non-negative or strictly negative valuation. Suppose without loss of generality $u^*(\mathcal{P}_1,\mathcal{P}_2) = \mathfrak{z}(a_1,a_2)$. This is allowed by theorem(2.1.3.13). Let $\mathcal{M}$ be the first and $\mathcal{M}'$ the second axis $\mathcal{P}_1$ and $\mathcal{P}_2$, $\mathcal{L}_i$ the first and $\mathcal{L}_i'$ the second coordinateline of $\mathcal{P}_i$, i=1,2 and let $\mathcal{T}_i$ resp. $\mathcal{T}_i'$ be the meeting point of $\mathcal{M}$ and $\mathcal{L}_i$ resp. $\mathcal{M}'$ and $\mathcal{L}_i'$, i=1,2. One can check that in every case, $\nu(\mathcal{L})$ is incompatible with $\nu(\mathcal{T}_i)$, i=1,2 (this is a general rule, also for projective planes and planar ternary rings with valuation). Hence, $\nu(\mathcal{L}) \neq \nu(\mathcal{L}_i)$, i=1,2 and consequently $u^*(\mathcal{L},\mathcal{L}_i) = 0$, i=1,2. Put $\mathfrak{z}(a_1,a_2) = u^*(\mathcal{P}_1,\mathcal{P}_2) = n$. Then $u(\mathcal{P}_1,\mathcal{P}_2) \leq n$ by definition. We show $u(\mathcal{P}_1,\mathcal{P}_2) \geq n$. Let $\mathcal{P}_3,\mathcal{L}_3'$ be such that $\mathcal{T}_2'\ I\ \mathcal{L}_3'\ I\ \mathcal{P}_3\ I\ \mathcal{L}_1$. Then $\mathcal{P}_3$ has the same shape as both $\mathcal{P}_2$ and $\mathcal{P}_1$ and moreover $\mathcal{L}_1$ is the first and $\mathcal{L}_3'$ the second coordinateline of $\mathcal{P}_3$. Now note that

$$u^*(\mathcal{P}_1,\mathcal{P}_3) = \mathfrak{z}(a_1',a_2') \geq n = u^*(\mathcal{P}_1,\mathcal{P}_2) \qquad (1)$$

(the first equality by applying the second main property in the quadrangle $\mathcal{P}_1\ I\ \mathcal{L}_1\ I\ \mathcal{P}_3\ I\ \mathcal{L}_3'\ I\ \mathcal{T}_2'\ I\ \mathcal{M}'\ I\ \mathcal{T}_1'\ I\ \mathcal{L}_1'\ I\ \mathcal{P}_1$ on the side $\mathcal{L}_1$, the inequality holds by assumption). So one has $\mathcal{P}_1(\square 2)_n\mathcal{P}_3$. We apply the second main property in the quadrangle $\mathcal{T}_2'\ I\ \mathcal{L}_2'\ I\ \mathcal{P}_2\ I\ \mathcal{L}\ I\ \mathcal{P}_1\ I\ \mathcal{L}_1\ I\ \mathcal{P}_3\ I\ \mathcal{L}_3'\ I\ \mathcal{T}_2'$ on the vertex $\mathcal{T}_2'$:

$$2.u^*(\mathcal{L}_2',\mathcal{L}_3') = u^*(\mathcal{P}_1,\mathcal{P}_2) + u^*(\mathcal{P}_1,\mathcal{P}_3) \geq 2.n \qquad (2)$$

by (1). Now, an indirect consequence of corollary(2.2.1.4) is the fact $u^*(\mathcal{L}_2',\mathcal{L}_3') = \mathfrak{z}(\ell_1',\ell_3')$, where $\mathcal{P}_3 = ((a_1,\ell_1;a_2',\ell_3'))$. Hence by (2), $\mathfrak{z}(\ell_1',\ell_3') \geq n$. By proposition(2.2.1.9), $u(\mathcal{P}_2,\mathcal{P}_3) \geq n$. Since we had also $\mathcal{P}_1(\square 2)_n\mathcal{P}_3$, we have $u(\mathcal{P}_1,\mathcal{P}_2) \geq n$. Q.E.D.

Another reason why the partial *-valuation is not the good one is the following. As in the case of a projective plane, the partial valuation will determine the valuation map in the quotient-geometries. But if the quotient-geometries are to be level n H-Qs, then the valuation of a pair of concurrent lines should have even valuation (see [7]). This would contradict theorem(2.1.3.13). Contrarily, proposition(2.2.1.11) is in conformity with that property.

We now state the ***third main property*** of a generalized quadrangle coordinatized by a V-QQR.

THEOREM(2.2.1.12). *Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ be four points of $\mathcal{V}$ and $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4$ four lines of $\mathcal{V}$ with $\mathcal{P}_1 \ I\ \mathcal{L}_1 \ I\ \mathcal{P}_2 \ I\ \mathcal{L}_2 \ I\ \mathcal{P}_3 \ I\ \mathcal{L}_3 \ I\ \mathcal{P}_4 \ I\ \mathcal{L}_4 \ I\ \mathcal{P}_1$. Then we have*

[1] $u(\mathcal{L}_1,\mathcal{L}_2) + u(\mathcal{P}_1,\mathcal{P}_2) + u(\mathcal{P}_2,\mathcal{P}_3) = u(\mathcal{L}_3,\mathcal{L}_4) + u(\mathcal{P}_3,\mathcal{P}_4) + u(\mathcal{P}_1,\mathcal{P}_4),$

[2] $u(\mathcal{L}_1,\mathcal{L}_4) + u(\mathcal{P}_1,\mathcal{P}_4) + u(\mathcal{P}_1,\mathcal{P}_2) = u(\mathcal{L}_2,\mathcal{L}_3) + u(\mathcal{P}_2,\mathcal{P}_3) + u(\mathcal{P}_3,\mathcal{P}_4),$

[3] $2.u(\mathcal{P}_1,\mathcal{P}_2) + u(\mathcal{L}_1,\mathcal{L}_4) + u(\mathcal{L}_1,\mathcal{L}_2) = 2.u(\mathcal{P}_3,\mathcal{P}_4) + u(\mathcal{L}_2,\mathcal{L}_3) + u(\mathcal{L}_3,\mathcal{L}_4),$

[4] $2.u(\mathcal{P}_1,\mathcal{P}_4) + u(\mathcal{L}_3,\mathcal{L}_4) + u(\mathcal{L}_1,\mathcal{L}_4) = 2.u(\mathcal{P}_2,\mathcal{P}_3) + u(\mathcal{L}_1,\mathcal{L}_2) + u(\mathcal{L}_2,\mathcal{L}_3).$

PROOF. This follows directly from theorem(2.1.3.20) and proposition (2.2.1.11). **Q.E.D.**

Let $\mathcal{P}$ be a point of $\mathcal{V}$, then we call the meeting point of the first coordinateline of $\mathcal{P}$ and the first axis of $\mathcal{P}$ the ***first coordinatepoint of*** $\mathcal{P}$. Similarly, we define the ***second coordinatepoint of*** $\mathcal{P}$ and dually, we define the ***first*** and ***second coordinateline*** of any line $\mathcal{L}$ of $\mathcal{V}$.

2.2.2. The quotient-geometries and their properties.

Throughout, we fix $n \in \mathbf{N}^*$. We will define the quotient-geometry $\mathcal{V}_n = (\mathcal{P}(\mathcal{V}_n), \mathcal{L}(\mathcal{V}_n), I)$ of $\mathcal{V}$ by identifying n-equivalent points and n-equivalent lines. So we have :

$\mathcal{P}(\mathcal{V}_n)$ = the set of n-equivalence-classes in the pointset of $\mathcal{V}$,

$\mathcal{L}(\mathcal{V}_n)$ = the set of n-equivalence-classes in the lineset of $\mathcal{V}$,

$(\forall \mathcal{P} \in \mathcal{P}(\mathcal{V}_n))(\forall \mathcal{L} \in \mathcal{L}(\mathcal{V}_n))(\mathcal{P}\ I\ \mathcal{L} \iff$ there exist respective representatives of $\mathcal{P}$ and $\mathcal{L}$ which are incident in $\mathcal{V}$)

We abbreviate the sentence "representative of X" by "$\hat{X}$". It will always be clear whether $\hat{X}$ is unique or arbitrary or belongs to a certain set, etc... We will also use notation like $\forall\hat{X}$, meaning : for all representatives of X.

We now prove a sequence of properties of $\mathcal{V}_n$ culminating in the axioms for a level n H-Q (for a suitable definition of neighbourhoods, still to give !). Throughout, we assume $\mathcal{P}_n, Q_n \in \mathcal{P}(\mathcal{V}_n)$ and $\mathcal{L}_n, \mathcal{M}_n \in \mathcal{L}(\mathcal{V}_n)$.

PROPOSITION(2.2.2.1). *If $\mathcal{P}_n\ I\ \mathcal{L}_n$, then for every $\hat{\mathcal{P}}_n$, there exists $\hat{\mathcal{L}}_n$ incident with $\hat{\mathcal{P}}_n$ in $\mathcal{V}$.*

PROOF. By assumption, there are representatives $\mathcal{P}, \mathcal{L}$ of respectivily $\mathcal{P}_n$ and $\mathcal{L}_n$ incident with each other. Let $\hat{\mathcal{P}}_n$ be arbitrary. Then $\mathcal{P}$ and $\hat{\mathcal{P}}_n$ are n-equivalent. By definition of n-equivalence, we can divide the proof in three pieces, corresponding to the three kinds of partial n-

equivalences. Without loss of generality, we can assume that the shape of $\mathcal{L}$ is incompatible with the shape of the first coordinatepoint of $\mathcal{P}$ and dually, that the shape of the first coordinateline of $\mathcal{L}$ is incompatible with the shape of $\mathcal{P}$.

(1). Suppose $\mathcal{P}(\square 1)_{\gamma}\hat{\mathcal{P}}_{\gamma}$.

Let $\mathcal{M}$ resp $\mathcal{M}'$ be the first resp. second coordinateline of $\mathcal{L}$. Let $\mathcal{L}^*$ be the unique line of $\mathcal{V}$ incident with $\hat{\mathcal{P}}_{\gamma}$ and concurrent with $\mathcal{M}$. Since $\nu(\mathcal{M})$ and $\nu(\mathcal{P})=\nu(\hat{\mathcal{P}}_{\gamma})$ are incompatible, it follows from proposition (2.2.1.11) that $\nu(\mathcal{L}^*) = \nu(\mathcal{L})$. So let $\mathcal{M}^*$ be the second coordinateline of $\mathcal{L}^*$ ($\mathcal{M}$ is its first). Also, let Q_i resp Q_i^* be the i-th coordinatepoint of $\mathcal{L}$ resp. $\mathcal{L}^*$, i=1,2. If $Q_2^*\ I\ \mathcal{L}$, then $\mathcal{L}=\mathcal{L}^*$, so suppose that Q_2^* is not incident with $\mathcal{L}$. Let $(Q,\mathcal{K})$ be the unique point-line pair such that $Q_2^*\ I\ \mathcal{K}\ I\ Q\ I\ \mathcal{L}$. We first show $u(Q_1,Q_1^*) \geq n$ and $u(\mathcal{M}',\mathcal{M}^*) \geq n$. We apply the third main property in the quadrangle $Q_1\ I\ \mathcal{M}\ I\ Q_1^*\ I\ \mathcal{L}^*\ I\ \hat{\mathcal{P}}_{\gamma}\ \perp\ \mathcal{P}\ I\ \mathcal{L}\ I\ Q_1$ on the side $\mathcal{M}$ and get :

$$u(Q_1,Q_1^*) = u(\mathcal{P},\hat{\mathcal{P}}_{\gamma}) \geq n \tag{1}$$

We apply the third main property in the quadrangle $\mathcal{K}\ I\ Q\ I\ \mathcal{L}\ I\ \mathcal{P}\ \perp\ \hat{\mathcal{P}}_{\gamma}\ I\ \mathcal{L}^*\ I\ Q_2^*\ I\ \mathcal{K}$ on the sides $\mathcal{M}$ and $\mathcal{L}$:

$$2.u(\mathcal{P},\hat{\mathcal{P}}_{\gamma}) = 2.u(Q,Q_2^*) + u(\mathcal{K},\mathcal{L}) + u(\mathcal{K},\mathcal{L}^*) \tag{2}$$

$$2.u(\mathcal{P},Q) + u(\mathcal{K},\mathcal{L}) = 2.u(\hat{\mathcal{P}}_{\gamma},Q_2^*) + u(\mathcal{L}^*,\mathcal{K}) \tag{3}$$

We apply the third main property in the quadrangle $\mathcal{M}\ I\ Q_1\ I\ \mathcal{L}\ I\ Q\ I\ \mathcal{K}\ I\ Q_2^*\ I\ \mathcal{L}^*\ I\ Q_1^*\ I\ \mathcal{M}$ on the side $\mathcal{L}$:

$$u(\mathcal{L}^*,\mathcal{K}) = 2.u(Q,Q_1) + u(\mathcal{K},\mathcal{L}) \tag{4}$$

Finally, we apply the third main property in the quadrangle $\mathcal{K}$ I Q I $\mathcal{L}$ I Q_2 I $\mathcal{M}'$ $\perp$ $\mathcal{M}^*$ I Q_2^* I $\mathcal{K}$ on the vertex Q :

$$u(\mathcal{M}',\mathcal{M}^*) = u(\mathcal{K},\mathcal{L}) + u(Q,Q_2) + u(Q,Q_2^*) \qquad (5)$$

The first inequality to prove follows from (1). To show the second, we assume that $u(Q,Q_1) > 0$. Since $u(\mathcal{P},Q_1) = 0$ ($\nu(\mathcal{P}) \neq \nu(Q_1)$ since Q_1 I $\mathcal{M}$ and $\nu(\mathcal{P})$ is incompatible with $\nu(\mathcal{M})$), $u(\mathcal{P},Q) = 0$. Eliminating $u(\mathcal{K},\mathcal{L})$ in (3) and (4) gives us $0 = 2.u(Q,Q_1) + 2.u(\hat{\mathcal{P}}_{\tau},Q_2^*)$, contradicting $u(Q,Q_1) > 0$. Hence $u(Q,Q_1) = 0$. By (4), $u(\mathcal{K},\mathcal{L}) = u(\mathcal{L}^*,\mathcal{K})$ and plugging this in into (2), we obtain

$$u(\mathcal{K},\mathcal{L}) + u(Q,Q_2^*) = u(\mathcal{P},\hat{\mathcal{P}}_{\tau}) \qquad (6)$$

We eliminate the common terms in (5) and (6) and get

$$u(\mathcal{M}',\mathcal{M}^*) = u(\mathcal{P},\hat{\mathcal{P}}_{\tau}) + u(Q,Q_2) \geq n \qquad (7)$$

If $\mathcal{L} = [[k,b;k',b']]$ and $\mathcal{L}^* = [[k,b^*;k^{*\prime},b^{*\prime}]]$, then one sees easily that $u^*(\mathcal{M}',\mathcal{M}^*) = z(k',k^{*\prime})$ (with z the suitable v or w) since k' resp. $k^{*\prime}$ is the only coordinate of $\mathcal{M}'$ resp. $\mathcal{M}^*$ distinct from both 0 and ∞. Hence by (7), $2.u^*(\mathcal{M}',\mathcal{M}^*) = u(\mathcal{M}',\mathcal{M}^*) \geq n$. Similarly $u(Q_1,Q_1^*) = z(b,b^*)$. By proposition(2.2.1.10), $\mathcal{L}$ $\square_{\tau}$ $\mathcal{L}^*$, hence the result follows by putting $\hat{\mathcal{L}} = \mathcal{L}^*$.

(2). Suppose $\mathcal{P}(\square 2)_{\tau}\hat{\mathcal{P}}_{\tau}$.

We use the same notation as in **(1)**. The difference is, that in **(1)**, $\mathcal{P}$ and $\hat{\mathcal{P}}_{\tau}$ share the *first* coordinateline, while now, they have the same *second* coordinateline, say $\mathcal{K}'$. We apply the third main property in the quadrangle Q_1 I $\mathcal{M}$ I Q_1^* I $\mathcal{L}^*$ I $\hat{\mathcal{P}}_{\tau}$ $\perp$ $\mathcal{P}$ I $\mathcal{L}$ I Q_1 on the sides $\mathcal{M}$ and $\mathcal{L}$:

$$2.u(Q_1,Q_1^*) = 2.u(\mathcal{P},\hat{\mathcal{P}}_{\pi}) + u(\mathcal{L},\mathcal{K}') + u(\mathcal{L}^*,\mathcal{K}') \geq 2.n \qquad (1)$$

$$u(\mathcal{L},\mathcal{K}') = u(\mathcal{L}^*,\mathcal{K}') \qquad (2)$$

We apply the third main property in the quadrangle $\mathcal{K}\ I\ Q\ I\ \mathcal{L}\ I\ Q_1\ I\ \mathcal{M}\ I\ Q_1^*\ I\ \mathcal{L}^*\ I\ Q_2^*\ I\ \mathcal{K}$ on the sides $\mathcal{M}$ and $\mathcal{L}$:

$$2.u(Q_1,Q_1^*) = 2.u(Q,Q_2^*) + u(\mathcal{K},\mathcal{L}) + u(\mathcal{K},\mathcal{L}^*) \qquad (3)$$

$$u(\mathcal{K},\mathcal{L}^*) = 2.u(Q,Q_1) + u(\mathcal{K},\mathcal{L}) \qquad (4)$$

We apply the third main property in the quadrangle $\mathcal{K}'\ I\ \mathcal{P}\ I\ \mathcal{L}\ I\ Q\ I\ \mathcal{K}\ I\ Q_2^*\ I\ \mathcal{L}^*\ I\ \mathcal{P}^*\ I\ \mathcal{K}'$ on the side $\mathcal{L}$, taking (2) into account :

$$2.u(\hat{\mathcal{P}}_{\pi},Q_2^*) + u(\mathcal{K},\mathcal{L}^*) = 2.u(\mathcal{P},Q) + u(\mathcal{K},\mathcal{L}) \qquad (5)$$

Finally, we apply the third main property in the quadrangle $\mathcal{K}\ I\ Q\ I\ \mathcal{L}\ I\ Q_2\ I\ \mathcal{M}' \perp \mathcal{M}^*\ I\ Q_2^*\ I\ \mathcal{K}$ on the vertex Q :

$$u(\mathcal{M}',\mathcal{M}^*) = u(\mathcal{K},\mathcal{L}) + u(Q,Q_2) + u(Q,Q_2^*) \qquad (6)$$

Suppose for a moment $u(Q,Q_1) > 0$. Since $u(\mathcal{P},Q_1) = 0$ (same reason as above), $u(\mathcal{P},Q) = 0$. Eliminating $u(\mathcal{K},\mathcal{L})$ in (4) and (5), we get $0 = 2.u(Q,Q_1) + 2.u(\hat{\mathcal{P}}_{\pi},Q_2^*)$, contradicting $u(Q,Q_1) > 0$. Hence $u(Q,Q_1) = 0$. But (4) implies $u(\mathcal{K},\mathcal{L}) = u(\mathcal{L}^*,\mathcal{K})$ and by (3) :

$$u(\mathcal{K},\mathcal{L}) + u(Q,Q_2^*) = u(Q_1,Q_1^*) \qquad (7)$$

We eliminate the common terms of (5) and (6) and obtain :

$$u(\mathcal{M}',\mathcal{M}^*) = u(Q_1,Q_1^*) + u(Q,Q_2) \geq n \qquad (8)$$

By (1) and (8), we can put $\hat{\mathcal{L}} = \mathcal{L}^*$, similarly as in the first part of the proof.

(3). Suppose $\mathcal{P}(\square 3)_n \hat{\mathcal{P}}_n$

We keep the same notation as in the first two parts of this proof, except for the symbol $\mathcal{K}'$. We denote by $\mathcal{K}_0$ resp. $\mathcal{K}^*$ the first coordinate line of $\mathcal{P}$ resp. $\hat{\mathcal{P}}_n$ and we denote by $\mathcal{K}_0'$ resp. $\mathcal{K}^{*\prime}$ the second coordinate line of $\mathcal{P}$ resp. $\hat{\mathcal{P}}_n$. Note that $\mathcal{K}_0$ meets $\mathcal{K}^*$ and also $\mathcal{K}_0'$ meets $\mathcal{K}^{*\prime}$, namely in respectivily the first and second coordinatepoint of $\mathcal{P}$ and $\hat{\mathcal{P}}_n$. Let $(Q',\mathcal{K}')$ be such that $\hat{\mathcal{P}}_n \; I \; \mathcal{K}' \; I \; Q' \; I \; \mathcal{L}$ (indeed, we can assume that $\hat{\mathcal{P}}_n$ and $\mathcal{L}$ are not incident otherwise the statement is trivial). We will show again that $u(Q_1, Q_1^*) \geq n$ and $u(\mathcal{M}',\mathcal{M}^*) \geq n$. We apply the third main property in the quadrangle $\mathcal{P} \; I \; \mathcal{K}_0 \perp \mathcal{K}^* \; I \; \hat{\mathcal{P}}_n \; I \; \mathcal{K}^{*\prime} \perp \mathcal{K}_0' \; I \; \mathcal{P}$ on any side :

$$u(\mathcal{K}_0,\mathcal{K}^*) = u(\mathcal{K}_0',\mathcal{K}^{*\prime}) \tag{1}$$

We apply the third main property in the quadrangle $\mathcal{P} \; I \; \mathcal{K}_0 \perp \mathcal{K}^* \; I \; \hat{\mathcal{P}}_n \; I \; \mathcal{K}' \; I \; Q' \; I \; \mathcal{L} \; I \; \mathcal{P}$ on the vertices $\mathcal{P}$ and Q' :

$$u(\mathcal{P},Q') = u(\hat{\mathcal{P}}_n,Q') + u(\mathcal{K}',\mathcal{K}^*) \tag{2}$$

$$u(\mathcal{K}_0,\mathcal{K}^*) = u(\mathcal{L},\mathcal{K}') + u(\mathcal{P},Q') + u(\hat{\mathcal{P}}_n,Q') \tag{3}$$

We apply the third main property in the quadrangle $\mathcal{P} \; I \; \mathcal{K}_0' \perp \mathcal{K}^{*\prime} \; I \; \hat{\mathcal{P}}_n \; I \; \mathcal{K}' \; I \; Q' \; I \; \mathcal{L} \; I \; \mathcal{P}$ on the vertex $\mathcal{P}$:

$$u(\mathcal{P},Q') + u(\mathcal{L},\mathcal{K}_0') = u(\hat{\mathcal{P}}_n,Q') + u(\mathcal{K}',\mathcal{K}^{*\prime}) \tag{4}$$

If $u(\mathcal{K}',\mathcal{K}^*) > 0$, then by $u(\mathcal{K}^*,\mathcal{K}^{*\prime}) = 0$, we have $u(\mathcal{K}',\mathcal{K}^{*\prime}) = 0$. Eliminating $u(\mathcal{P},Q')$ in (2) and (4), we obtain $u(\mathcal{K}',\mathcal{K}^*) = 0$, a contradiction. Hence

$u(\mathcal{K}',\mathcal{K}^*) = 0$. Now (2) implies $u(\mathcal{P},Q') = u(\hat{\mathcal{P}}_{\gamma},Q')$; (3) implies :

$$u(\mathcal{K}_0,\mathcal{K}^*) = u(\mathcal{L},\mathcal{K}') + 2.u(\hat{\mathcal{P}}_{\gamma},Q') \tag{5}$$

We apply the third main property in the quadrangle $Q_1 \; I \; \mathcal{M} \; I \; Q_1^* \; I \; \mathcal{L}^* \; I \; \hat{\mathcal{P}}_{\gamma}$ $I \; \mathcal{K}' \; I \; Q' \; I \; \mathcal{L} \; I \; Q_1$ on the side $\mathcal{M}$:

$$2.u(Q_1,Q_1^*) = 2.u(\hat{\mathcal{P}}_{\gamma},Q') + u(\mathcal{L},\mathcal{K}') + u(\mathcal{L}^*,\mathcal{K}') \tag{6}$$

We remark that, since $u(\mathcal{P},\hat{\mathcal{P}}_{\gamma}) \geq n$, also $u^*(\mathcal{P},\hat{\mathcal{P}}_{\gamma}) \geq n$, because by the definition of u^*, $u \leq u^*$ for points. This implies $u^*(\mathcal{K}_0,\mathcal{K}^*) \geq n$ and hence by proposition(2.2.1.11), $u(\mathcal{K}_0,\mathcal{K}^*) \geq 2.n$. We substitute (5) into (6) :

$$2.u(Q_1,Q_1^*) = u(\mathcal{K}_0,\mathcal{K}^*) + u(\mathcal{L}^*,\mathcal{K}') \geq 2.n \tag{7}$$

Hence, again $u(Q_1,Q_1^*) \geq n$.

We apply the third main property in the quadrangle $Q_1 \; I \; \mathcal{M} \; I \; Q_1^* \; I \; \mathcal{L}^* \; I \; \hat{\mathcal{P}}_{\gamma}$ $I \; \mathcal{K}' \; I \; Q' \; I \; \mathcal{L} \; I \; Q_1$ on the side $\mathcal{L}$:

$$u(\mathcal{L},\mathcal{K}') + 2.u(Q_1,Q') = u(\mathcal{L}^*,\mathcal{K}') \tag{8}$$

We apply the third main property in the quadrangle $\mathcal{K} \; I \; Q \; I \; \mathcal{L} \; I \; Q_1 \; I \; \mathcal{M} \; I$ $Q_1^* \; I \; \mathcal{L}^* \; I \; Q_2^* \; I \; \mathcal{K}$ on the sides $\mathcal{M}$ and $\mathcal{L}$:

$$2.u(Q_1,Q_1^*) = 2.u(Q,Q_2^*) + u(\mathcal{K},\mathcal{L}) + u(\mathcal{K},\mathcal{L}^*) \tag{9}$$

$$u(\mathcal{K},\mathcal{L}^*) = 2.u(Q,Q_1) + u(\mathcal{K},\mathcal{L}) \tag{10}$$

We apply the main property in the quadrangle $\mathcal{K} \; I \; Q \; I \; \mathcal{L} \; I \; Q' \; I \; \mathcal{K}' \; I \; \hat{\mathcal{P}}_{\gamma} \; I$ $\mathcal{L}^* \; I \; Q_2^* \; I \; \mathcal{K}$ on the side $\mathcal{L}$:

$$2.u(\hat{\mathcal{P}}_{\gamma}, Q_2^*) + u(\mathcal{K},\mathcal{L}^*) + u(\mathcal{L}^*,\mathcal{K}') = 2.u(Q,Q') + u(\mathcal{K},\mathcal{L}) + u(\mathcal{L},\mathcal{K}') \qquad (11)$$

Finally, we apply the third main property in the quadrangle $\mathcal{K}\ I\ Q\ I\ \mathcal{L}\ I$ $Q_2\ I\ \mathcal{M}'\perp\mathcal{M}^*\ I\ Q_2^*\ I\ \mathcal{K}$ on the vertex Q :

$$u(\mathcal{M}',\mathcal{M}^*) = u(\mathcal{K},\mathcal{L}) + u(Q,Q_2) + u(Q,Q_2^*) \qquad (12)$$

In (8),(10) and (11), we eliminate $u(\mathcal{L},\mathcal{K}')$ and $u(\mathcal{K},\mathcal{L})$ and divide by 2 :

$$u(\hat{\mathcal{P}}_{\gamma}, Q_2^*) + u(Q,Q_1) + u(Q_1,Q') = u(Q,Q') \qquad (13)$$

Combining (10) and (9), we obtain :

$$u(Q_1,Q_1^*) = u(Q,Q_2^*) + u(Q,Q_1) + u(\mathcal{K},\mathcal{L}) \qquad (14)$$

Eliminating $u(\mathcal{K},\mathcal{L})$ in (12) and (14), we get :

$$u(\mathcal{M}',\mathcal{M}^*) = u(Q,Q_2) + u(Q_1,Q_1^*) - u(Q,Q_1) \qquad (15)$$

We now carry out the following operation : 2.(12)-2.(14)+(7)-(8) :

$$2.u(\mathcal{M}',\mathcal{M}^*) = u(\mathcal{K}_0,\mathcal{K}^*) + u(\mathcal{L},\mathcal{K}') + 2.u(Q,Q_2) + 2.u(Q_1,Q') - 2.u(Q,Q_1) \qquad (16)$$

If $u(Q,Q_1) = 0$, then by (7) and (15), $u(\mathcal{M}',\mathcal{M}^*) \geq n$. So suppose $u(Q,Q_1) >$ 0. If $u(Q_1,Q') \geq u(Q,Q_1)$, then (16) implies $2.u(\mathcal{M}',\mathcal{M}^*) \geq 2.n$, hence we can assume $u(Q_1,Q') \leq u(Q,Q_1)$. But by the triangle inequality, $u(Q,Q') =$ $u(Q_1,Q')$, together with (13) giving rise to the contradiction $u(Q,Q_1) =$ 0. Putting $\hat{\mathcal{L}}_{\gamma} = \mathcal{L}^*$, the result follows again.

This completes the proof of the proposition. **Q.E.D.**

The dual of this proposition is :

PROPOSITION(2.2.2.2). *If $\mathcal{P}_n$ I $\mathcal{L}_n$, then for every $\hat{\mathcal{L}}_n$, there exists a point $\mathcal{P}$ of $\mathcal{V}$ incident with $\hat{\mathcal{L}}_n$ in $\mathcal{V}$ and such that $2.u(\mathcal{P},\hat{\mathcal{P}}_n) \geq n$ for every $\hat{\mathcal{P}}_n$.*

PROPOSITION(2.2.2.3). *Suppose that $\hat{\mathcal{P}}_n, \hat{Q}_n, \hat{\mathcal{L}}_n, \hat{\mathcal{M}}_n$ exist such that $\hat{Q}_n$ I $\hat{\mathcal{L}}_n$ I $\hat{\mathcal{P}}_n$ I $\hat{\mathcal{M}}_n$ and*

$$2.u(\hat{\mathcal{P}}_n, \hat{Q}_n) + u(\hat{\mathcal{L}}_n, \hat{\mathcal{M}}_n) < n.$$

Then we have

(a) *There exists a point Q_n^* of $\mathcal{V}_n$ incident with $\mathcal{M}_n$ such that*

$$2.u(\hat{Q}_n, \hat{Q}_n^*) = 2.u(\hat{\mathcal{P}}_n, \hat{Q}_n) + u(\hat{\mathcal{L}}_n, \hat{\mathcal{M}}_n)$$

for all representatives $\hat{Q}_n^*$ of Q_n^*.

(b) *For every $\hat{Q}_n^*$ such that Q_n^* I $\mathcal{M}_n$, there holds :*

$$2.u(\hat{Q}_n, \hat{Q}_n^*) \leq 2.u(\hat{\mathcal{P}}_n, \hat{Q}_n) + u(\hat{\mathcal{L}}_n, \hat{\mathcal{M}}_n).$$

PROOF. **(a)** We assume first $\nu(\hat{\mathcal{L}}_n) = \nu(\hat{\mathcal{M}}_n)$. Note that $u(\hat{\mathcal{L}}_n, \hat{\mathcal{M}}_n)$ is even by proposition(2.2.1.11). Now there exists a unique axis X such that all points incident with X have a shape which is incompatible with $\nu(\hat{\mathcal{L}}_n)$. Moreover, X is also an axis of $\hat{Q}_n$, say the first. Let $\mathcal{T}$ be the first coordinatepoint of $\hat{Q}_n$, then we define the point Q^* as the unique point on $\hat{\mathcal{M}}_n$ collinear with $\mathcal{T}$. Let $\mathcal{K}$ resp. $\mathcal{K}^*$ be the joining line of $\hat{Q}_n$ and $\mathcal{T}$ resp. Q^* and $\mathcal{T}$, let $\mathcal{N}$ be the second coordinateline of $\hat{Q}_n$, $\mathcal{S}$ the second coordinatepoint and $\mathcal{Y}$ the second axis of $\hat{Q}_n$. Let $(\mathcal{S}^*, \mathcal{N}^*)$ be the unique point-line pair such that $Q^* \mathrm{I} \mathcal{N}^* \mathrm{I} \mathcal{L}^* \mathrm{I} \mathcal{Y}$. If $Q^* \mathrm{I} \mathcal{N}$, then $\hat{\mathcal{P}}_{\gamma} = \hat{Q}_{\gamma} = Q^*$ and hence $u(\hat{\mathcal{P}}_{\gamma}, \hat{Q}_{\gamma}) = +\infty > n$, contradicting our assumptions. Hence Q^* is not incident with $\mathcal{N}$ and we can consider the pair $(\mathcal{R}, \mathcal{D})$ such that $Q^* \mathrm{I} \mathcal{D} \mathrm{I} \mathcal{R} \mathrm{I} \mathcal{N}$. Let $\mathcal{O}$ be the meeting point of X and $\mathcal{Y}$, then in the quadrangle $\mathcal{O} \mathrm{I} X \mathrm{I} \mathcal{T} \mathrm{I} \mathcal{K}^* \mathrm{I} Q^* \mathrm{I} \mathcal{N}^* \mathrm{I} \mathcal{L}^* \mathrm{I} \mathcal{Y} \mathrm{I} \mathcal{O}$, we have $u(X,\mathcal{Y}) = u(\mathcal{O},\mathcal{T}) = u(X,\mathcal{K}^*) = u(\mathcal{T},Q^*) = 0$. Hence, by the third main property, $u(\mathcal{O},\mathcal{L}^*) = u(\mathcal{Y},\mathcal{N}^*) = u(\mathcal{L},Q^*) = u(\mathcal{K}^*,\mathcal{N}^*) = 0$. In what follows, we keep this in mind. We apply the third main property in the quadrangle $\mathcal{T} \mathrm{I} \mathcal{K} \mathrm{I} \hat{Q}_{\gamma} \mathrm{I} \hat{\mathcal{L}}_{\gamma} \mathrm{I} \hat{\mathcal{P}}_{\gamma} \mathrm{I} \hat{\mathcal{M}}_{\gamma} \mathrm{I} Q^* \mathrm{I} \mathcal{K}^* \mathrm{I} \mathcal{T}$ on the side $\mathcal{K}^*$ and on the vertex Q^* :

$$u(\mathcal{K},\mathcal{K}^*) = u(\hat{\mathcal{L}}_{\gamma},\hat{\mathcal{M}}_{\gamma}) + 2.u(\hat{\mathcal{P}}_{\gamma},\hat{Q}_{\gamma}) \quad (1)$$

$$u(\hat{\mathcal{P}}_{\gamma},\hat{Q}_{\gamma}) = u(\hat{\mathcal{P}}_{\gamma},Q^*) \quad (2)$$

We apply the third main property in the quadrangle $\mathcal{Y} \mathrm{I} \mathcal{L} \mathrm{I} \mathcal{N} \mathrm{I} \mathcal{R} \mathrm{I} \mathcal{D} \mathrm{I} Q^* \mathrm{I} \mathcal{N}^* \mathrm{I} \mathcal{L} \mathrm{I} \mathcal{Y}$ on the side $\mathcal{Y}$:

$$2.u(\mathcal{L},\mathcal{L}^*) = 2.u(\mathcal{R},Q^*) + u(\mathcal{N},\mathcal{D}) + u(\mathcal{N}^*,\mathcal{D}) \quad (3)$$

We apply the third main property in the quadrangle $Q^* \mathrm{I} \mathcal{D} \mathrm{I} \mathcal{R} \mathrm{I} \mathcal{N} \mathrm{I} \hat{Q}_{\gamma} \mathrm{I} \hat{\mathcal{L}}_{\gamma} \mathrm{I} \hat{\mathcal{P}}_{\gamma} \mathrm{I} \hat{\mathcal{M}}_{\gamma} \mathrm{I} Q^*$ on the vertex Q^* and take (2) into account :

$$u(\mathcal{N},\hat{\mathcal{L}}_{\gamma}) + u(\mathcal{R},\hat{Q}_{\gamma}) = u(\mathcal{D},\hat{\mathcal{M}}_{\gamma}) + u(\mathcal{R},Q^*) \quad (4)$$

Finally, we apply the third main property in the quadrangle $\mathcal{T} \mathrm{I} \mathcal{K} \mathrm{I} \hat{Q}_{\gamma} \mathrm{I} \mathcal{N} \mathrm{I} \mathcal{R} \mathrm{I} \mathcal{D} \mathrm{I} Q^* \mathrm{I} \mathcal{K}^* \mathrm{I} \mathcal{T}$ on the vertices Q^* and $\mathcal{T}$:

$$u(\mathcal{K}^*,\mathcal{D}) + u(\mathcal{R},Q^*) = u(\mathcal{R},\hat{Q}_{\gamma}) \quad (5)$$

$$u(\mathcal{K},\mathcal{K}^*) = u(\mathcal{N},\mathcal{D}) + u(\mathcal{R},Q^*) + u(\mathcal{R},\hat{Q}_{\gamma}) \quad (6)$$

By (4) and (5), $u(\mathcal{N},\hat{\mathcal{L}}_n) + u(\mathcal{K}^*,\mathcal{D}) = u(\mathcal{D},\hat{\mathcal{M}}_n)$. By the triangle inequality, $0 = u(\mathcal{K}^*,\hat{\mathcal{M}}_n) \geq \inf\{u(\mathcal{K}^*,\mathcal{D}),u(\mathcal{D},\hat{\mathcal{M}}_n)\} = u(\mathcal{K}^*,\mathcal{D}) \geq 0$. Hence $u(\mathcal{K}^*,\mathcal{D}) = 0$ and by (5), $u(\mathcal{R},Q^*) = u(\mathcal{R},\hat{Q}_n)$. Combining (3) and (6), this implies :

$$2.u(\mathcal{L},\mathcal{L}^*) = u(\mathcal{K},\mathcal{K}^*) + u(\mathcal{N},\mathcal{D}) \qquad (7)$$

Since $\mathcal{T} \perp Q^*$, $\nu(Q^*) = \nu(\hat{Q}_n)$ (all points on $\hat{\mathcal{M}}_n$ posses X as one of their axes and the shape of a point is apparently completely determined by the sign of the valuation of the unique non-zero (old) coordinate of the projection of the point on any of its axes). Putting $2.j = 2.u(\hat{\mathcal{P}}_n,\hat{Q}_n) + u(\hat{\mathcal{L}}_n,\hat{\mathcal{M}}_n)$, proposition(2.2.1.9) and (1) and (7) imply $u(Q^*,\hat{Q}_n) \geq j$. If $u(Q^*,\hat{Q}_n) > j$, then $u(\mathcal{L},\mathcal{L}^*) \geq u(Q^*,\hat{Q}_n) > j$. Proposition(2.2.1.9) implies $u(Q^*,\hat{Q}_n) < j+1$ (because $u^*(\mathcal{K},\mathcal{K}^*) = j < j+1$ by (1)), a contradiction. Hence $u(Q^*,\hat{Q}_n) = j$. We denote by Q^*_n the n-equivalence class of Q^*. If $\hat{Q}^*_n$ is an arbitrary representative of Q^*_n, then $u(Q^*,\hat{Q}^*_n) \geq n$, hence by the triangle inequality, $u(\hat{Q}_n,\hat{Q}^*_n) = j$.

Suppose now $\nu(\hat{\mathcal{L}}_n) \neq \nu(\hat{\mathcal{M}}_n)$. The assertion is trivial for $\nu(\hat{\mathcal{P}}_n) \neq \nu(\hat{Q}_n)$, so suppose $\nu(\hat{\mathcal{P}}_n) = \nu(\hat{Q}_n)$. Let X be the unique axis of both $\hat{\mathcal{P}}_n$ and $\hat{Q}_n$ such that the shape of every point of X is incompatible with $\nu(\hat{\mathcal{M}}_n)$. The other axis of $\hat{\mathcal{P}}_n$ and $\hat{Q}_n$ has also this propperty, but with respect to $\hat{\mathcal{L}}_n$. Suppose without loss of generality that X is the first axis of $\hat{\mathcal{P}}_n$ and $\hat{Q}_n$. Let $\mathcal{K}$ resp. $\mathcal{N}$ be the first resp. second coordinateline of $\hat{Q}_n$, let $\mathcal{T}$ resp. $\mathcal{L}$ be the first resp. second coordinatepoint of $\hat{Q}_n$. Define $\mathcal{K}^*$ and Q^* such that $\mathcal{T}\ I\ \mathcal{K}^*\ I\ Q^*\ I\ \hat{\mathcal{M}}_n$. Similar as above, one proves $\nu(Q^*) = \nu(\hat{Q}_n)$. Let $\mathcal{N}^*$ resp. $\mathcal{L}^*$ be the second coordinateline resp. coordinate point of Q^*. Similar as above, Q^* is not incident with $\mathcal{N}$. Now let $\mathcal{R}$ and $\mathcal{D}$ be such that $Q^*\ I\ \mathcal{D}\ I\ \mathcal{R}\ I\ \mathcal{N}$. Similar as in the first part, one shows

$$u(\mathcal{K},\mathcal{K}^*) = 2.u(\hat{\mathcal{P}}_\gamma,\hat{Q}_\gamma) + u(\hat{\mathcal{L}}_\gamma,\mathcal{K}) \tag{8}$$

$$2.u(\mathcal{S},\mathcal{S}^*) = 2.u(\hat{\mathcal{P}}_\gamma,\hat{Q}_\gamma) + u(\mathcal{N},\mathcal{D}) \tag{9}$$

$$u(\mathcal{K}^*,\mathcal{D}) = u(\mathcal{K},\hat{\mathcal{L}}_\gamma) \tag{10}$$

If $u(\mathcal{N},\mathcal{D}) = 0$, then by (8),(9) and proposition(2.2.1.7), the result follows. If $u(\mathcal{N},\mathcal{D}) > 0$, then by the triangle inequality, $u(\mathcal{K}^*,\mathcal{D}) = 0$ and (8),(10) and again proposition(2.2.1.7) imply the assertion.

(b) We again suppose first $v(\hat{\mathcal{L}}_\gamma) = v(\hat{\mathcal{M}}_\gamma)$. We use the notation of the first part **(1)** of this proof. We show that for every point Q' I $\hat{\mathcal{M}}_\gamma$, there holds $2.u(\hat{Q}_\gamma,Q') \leq 2.j = 2.u(\hat{\mathcal{P}}_\gamma,\hat{Q}_\gamma) + u(\hat{\mathcal{L}}_\gamma,\hat{\mathcal{M}}_\gamma)$. Suppose that there is a point Q' I $\hat{\mathcal{M}}_\gamma$ such that $u(\hat{Q}_\gamma,Q') = k > j$. Then certainly $v(Q') = v(\hat{Q}_\gamma)$. Let $\mathcal{T}'$ be the first coordinatepoint of Q'. By proposition (2.2.1.11), $u(Q',Q^*) = u(\mathcal{T},\mathcal{T}')$ (or use theorem (2.2.1.12)!). But by definition $u(\mathcal{T}.\mathcal{T}') \geq u(\hat{Q}_\gamma,Q')$. Hence $inf\{u(Q',Q^*),u(\hat{Q}_\gamma,Q')\} = u(\hat{Q}_\gamma,Q') = k$. But by the triangle inequality, $u(Q^*.\hat{Q}_\gamma) \geq k$, which contradicts $j = u(Q^*,\hat{Q}_\gamma)$. Now let Q_γ' be any point of V_γ incident with $\mathcal{M}_\gamma$. Then there exists Q'' incident with $\hat{\mathcal{M}}_\gamma$ such that $2.u(Q'',\hat{Q}_\gamma') \geq n$ for all $\hat{Q}_\gamma'$. But by the preceding, $2.u(Q'',\hat{Q}_\gamma) \geq 2j$. Hence by the triangle inequality, $2.u(\hat{Q}_\gamma,\hat{Q}_\gamma') \geq inf\{2j,n\} = 2j$.

If $v(\hat{\mathcal{L}}_\gamma) \neq v(\hat{\mathcal{M}}_\gamma)$, then similar as above, one shows the assertion. **Q.E.D.**

And dually :

PROPOSITION(2.2.2.4). *Suppose* $\hat{\mathcal{P}}_n, \hat{Q}_n, \hat{\mathcal{L}}_n, \hat{\mathcal{M}}_n$ *is such that* $\hat{Q}_n \ I \ \hat{\mathcal{L}}_n \ I \ \hat{\mathcal{P}}_n \ I \ \hat{\mathcal{M}}_n$ *and*

$$u(\hat{\mathcal{P}}_n, \hat{Q}_n) + u(\hat{\mathcal{L}}_n, \hat{\mathcal{M}}_n) < n.$$

Then we have :

(a) *There exists a line* $\mathcal{M}_n^*$ *of* $\mathcal{V}_n$ *incident with* Q_n *such that*

$$u(\hat{\mathcal{M}}_n, \hat{\mathcal{M}}_n^*) = u(\hat{\mathcal{P}}_n, \hat{Q}_n) + u(\hat{\mathcal{L}}_n, \hat{\mathcal{M}}_n)$$

for all representatives $\hat{\mathcal{M}}_n^*$ *of* $\mathcal{M}_n^*$.

(b) *For every* $\hat{\mathcal{M}}_n^*$ *such that* $\mathcal{M}_n^* \ I \ Q_n$, *we have :*

$$u(\hat{\mathcal{M}}_n, \hat{\mathcal{M}}_n^*) \leq u(\hat{\mathcal{P}}_n, \hat{Q}_n) + u(\hat{\mathcal{L}}_n, \hat{\mathcal{M}}_n).$$

We define a valuation map in $\mathcal{V}_n$ as follows. The valuation of a pair of distinct points is by definition equal to the partial valuation of a pair of arbitrary respective represantatives of these points. Hence we can adopt the notation u for this valuation map in $\mathcal{V}_n$. Similarly for lines. Note that this is well defined (use the triangle inequality in order to show that the definition in independent of the choice of the respective representatives). This valuation map now defines $n+1$ partitions (amongst them two trivial ones) of each of the sets $\mathcal{P}(\mathcal{V}_n)$ and $\mathcal{L}(\mathcal{V}_n)$. We will now use the notation of section 1, paragraph 2. It is trivial to see that the geometries $(\mathbf{P}_i(\mathcal{V}_n), \mathbf{L}_i(\mathcal{V}_n), I)$ and $\mathcal{V}_{n-i}$, $0<i<n$ are canonical isomorphic. Hence we can identify them and so we can assume that the projection Π_j^n maps $\mathcal{V}_n$ (on)to $\mathcal{V}_j$. Furthermore, we define for every variety X of $\mathcal{V}$ the projection $\Pi_n(X)$ as the unique n-equivalence class containing X.

We recall that the valuation map is now also defined on point-line pairs of $\mathcal{V}_{\tau}$.

PROPOSITION(2.2.2.5). *Suppose* $\hat{\mathcal{P}}_{\tau}, \hat{Q}_{\tau}, \hat{\mathcal{L}}_{\tau}, \hat{\mathcal{M}}_{\tau}$ *are such that* $\hat{Q}_{\tau}\ I\ \hat{\mathcal{L}}_{\tau}\ I\ \hat{\mathcal{P}}_{\tau}\ I\ \hat{\mathcal{M}}_{\tau}$ *and*

$$2.u(\hat{\mathcal{P}}_{\tau}, \hat{Q}_{\tau}) + u(\hat{\mathcal{L}}_{\tau}, \hat{\mathcal{M}}_{\tau}) < n.$$

Then

$$u(Q_{\tau}, \mathcal{M}_{\tau}) = \left(u(\mathcal{P}_{\tau}, Q_{\tau}) + \frac{u(\mathcal{L}_{\tau}, \mathcal{M}_{\tau})}{2}, u(\mathcal{P}_{\tau}, Q_{\tau}) + u(\mathcal{L}_{\tau}, \mathcal{M}_{\tau})\right)$$

PROOF. Follows immediately from the two preceding propositions. **Q.E.D.**

PROPOSITION(2.2.2.6). *Suppose* $\mathcal{L}_{\tau}\ I\ \mathcal{P}_{\tau}$ *and* $\hat{\mathcal{L}}_{\tau}$ *is not incident with* $\hat{\mathcal{P}}_{\tau}$. *If* $\hat{\mathcal{L}}_{\tau}\ I\ Q\ I\ \mathcal{M}\ I\ \hat{\mathcal{P}}_{\tau}$, *then* $u(\hat{\mathcal{P}}_{\tau}, Q) + u(\hat{\mathcal{L}}_{\tau}, \mathcal{M}) \geq n$.

PROOF. Suppose $j = u(\hat{\mathcal{P}}_{\tau}, Q) + u(\hat{\mathcal{L}}_{\tau}, \mathcal{M}) < n$. Proposition(2.2.2.3) implies $u(\hat{\mathcal{L}}_{\tau}, \hat{\mathcal{L}}_{\tau}) \leq j$, a contradiction. **Q.E.D.**

PROPOSITION(2.2.2.7). *Suppose* $\mathcal{P}_{\tau}\ I\ \mathcal{L}_{\tau}\ I\ Q_{\tau}$ *with* $u(\mathcal{P}_{\tau}, Q_{\tau}) = 0$. *Let* $\hat{\mathcal{P}}_{\tau}$ *be arbitrary, then there exist* $\hat{Q}_{\tau}$ *and* $\hat{\mathcal{L}}_{\tau}$ *such that* $\hat{\mathcal{P}}_{\tau}\ I\ \hat{\mathcal{L}}_{\tau}\ I\ \hat{Q}_{\tau}$ *in* $\mathcal{V}$.

PROOF. Let $\hat{\mathcal{P}}_{\tau}$ be arbitrary. Let $\mathcal{L}$ be a representative of $\mathcal{L}_{\tau}$ incident with $\hat{\mathcal{P}}_{\tau}$ in $\mathcal{V}$ (cp. proposition(2.2.2.1)) and let Q be a representative of $\hat{Q}_{\tau}$ not incident with $\mathcal{L}$ (otherwise there is nothing to prove). Let $\mathcal{M}$ be the coordinateline of Q having a shape distinct from $v(\mathcal{L})$. Suppose without loss of generality that $\mathcal{M}$ is the first coordinateline of Q. Let $\mathcal{L}'$ and Q' be such that $\hat{\mathcal{P}}_{\tau}\ I\ \mathcal{L}'\ I\ Q'\ I\ \mathcal{M}$ (note that $\hat{\mathcal{P}}_{\tau}$ is not incident with $\mathcal{M}$ because this would contradict $u(\mathcal{P}_{\tau}, Q_{\tau}) = 0$). Let $\mathcal{T}$ and $\mathcal{N}$ be such that $Q\ I\ \mathcal{N}\ I\ \mathcal{T}\ I\ \mathcal{L}$. Denote by $\mathcal{S}'$ the first coordinatepoint of Q and by

$\mathcal{B}$ the coordinatepoint of $\hat{\mathcal{P}}_{\gamma}$ incident with the first axis of Q (well defined by the choice of $\mathcal{M}$ and the fact $\hat{\mathcal{P}}_{\gamma}$ I $\mathcal{L}$) and let $\mathcal{M}'$ be the corresponding coordinateline of $\hat{\mathcal{P}}_{\gamma}$. Note first $u(\mathcal{B},\mathcal{B}') = 0$, because either $\hat{\mathcal{P}}_{\gamma}$ and Q have distinct shape and then automatically $u(\mathcal{B},\mathcal{B}') = 0$ (follows from proposition (2.2.2.2) without running into details), or $\nu(\hat{\mathcal{P}}_{\gamma}) = \nu(Q)$ and then the first axis of Q determines $u(\hat{\mathcal{P}}_{\gamma},Q^*)$, with Q^* I $\mathcal{L}$ and $2.u(Q,Q^*) \geq n$ (cp. proposition(2.2.2.2)). By the triangle inequality, we have $u(\mathcal{B},\mathcal{B}') = 0$. Applying the third main property in the quadrangle $\mathcal{B}' \perp \mathcal{B} \perp \hat{\mathcal{P}}_{\gamma} \perp Q' \perp \mathcal{B}'$ on the axis containing $\mathcal{B}$ and $\mathcal{B}'$, we obtain $u(\mathcal{L}',\mathcal{M}) + u(\mathcal{L}',\mathcal{M}') + 2.u(\hat{\mathcal{P}}_{\gamma},Q) = 0$, hence all terms are zero. Hence applying the third main property in the quadrangle $\hat{\mathcal{P}}_{\gamma}$ I $\mathcal{L}'$ I Q' I $\mathcal{M}$ I Q I $\mathcal{N}$ I $\mathcal{T}$ I $\mathcal{L}$ I $\hat{\mathcal{P}}_{\gamma}$ on the side $\mathcal{L}$ and on the vertex Q', we get :

$$u(\mathcal{L},\mathcal{L}') = 2.u(Q,\mathcal{T}) + u(\mathcal{L},\mathcal{N}) + u(\mathcal{M},\mathcal{N}) \quad (1)$$

$$u(Q,Q') = u(\hat{\mathcal{P}}_{\gamma},\mathcal{T}) + u(Q,\mathcal{T}) + u(\mathcal{L},\mathcal{N}) \quad (2)$$

Proposition(2.2.2.6) implies $u(\hat{\mathcal{P}}_{\gamma},\mathcal{T}) + u(Q,\mathcal{T}) \geq n$, hence the right hand sides of (1) and (2) are not smaller than n. So we can put $Q' = \hat{Q}_{\gamma}$ and $\mathcal{L}' = \hat{\mathcal{L}}_{\gamma}$ and the result follows. **Q.E.D.**

Note that the axiom (GQ) is independent of the line-structure in a point neighbourhood. Hence the following theorem :

THEOREM(2.2.2.8). *The axiom* (GQ1) *is valid in* $\mathcal{V}_{\gamma}$.

PROOF. Suppose Q_{γ} I $\mathcal{L}_{\gamma}$ I $\mathcal{P}_{\gamma}$ I $\mathcal{M}_{\gamma}$ in $\mathcal{V}_{\gamma}$, $u(\mathcal{P}_{\gamma},Q_{\gamma}) = 0$ and $\mathcal{L}_{\gamma} \neq \mathcal{M}_{\gamma}$. By proposition(2.2.2.7), there exist $\hat{Q}_{\gamma},\hat{\mathcal{L}}_{\gamma},\hat{\mathcal{P}}_{\gamma}$ such that $\hat{Q}_{\gamma}$ I $\hat{\mathcal{L}}_{\gamma}$ I $\hat{\mathcal{P}}_{\gamma}$ in $\mathcal{V}$. By proposition(2.2.2.1), there exists $\hat{\mathcal{M}}_{\gamma}$ such that $\hat{\mathcal{P}}_{\gamma}$ I $\hat{\mathcal{M}}_{\gamma}$ in $\mathcal{V}$. Since

$\mathcal{L}_{\gamma} \neq \mathcal{M}_{\gamma}$, $u(\hat{\mathcal{L}}_{\gamma}.\hat{\mathcal{M}}_{\gamma}) < n$ and proposition(2.2.2.5) implies that, if $2.k = u(\mathcal{L}_{\gamma},\mathcal{M}_{\gamma})$,

$$u(Q_{\gamma},\mathcal{M}_{\gamma}) = (k,2k).$$

By the definitie of $u(Q_{\gamma},\mathcal{M}_{\gamma})$, the result follows. **Q.E.D.**

PROPOSITION(2.2.2.9). *Suppose* $\hat{\mathcal{P}}_{\gamma},\hat{Q}_{\gamma},\hat{\mathcal{L}}_{\gamma},\hat{\mathcal{M}}_{\gamma}$ *are such that* $\hat{Q}_{\gamma}$ *I* $\hat{\mathcal{L}}_{\gamma}$ *I* $\hat{\mathcal{P}}_{\gamma}$ *I* $\hat{\mathcal{M}}_{\gamma}$, *then*

$$u_1(Q_{\gamma},\mathcal{M}_{\gamma}) \geq u(\mathcal{P}_{\gamma},Q_{\gamma}) + \frac{u(\mathcal{L}_{\gamma},\mathcal{M}_{\gamma})}{2}$$

PROOF. To show **(a)** of proposition(2.2.2.3), we did not use the assumption $2.u(\hat{\mathcal{P}}_{\gamma},\hat{Q}_{\gamma}) + u(\hat{\mathcal{L}}_{\gamma},\hat{\mathcal{M}}_{\gamma}) < n$. Hence **(a)** is valid under the present assumptions. The result follows. **Q.E.D.**

PROPOSITION(2.2.2.10). *Let* $2.k < n$ *and suppose* $u(\mathcal{P}_{\gamma},\mathcal{L}_{\gamma}) = (k,2k)$. *Then there exists a line* $\mathcal{M}_{\gamma} \perp \mathcal{L}_{\gamma}$ *incident with* $\mathcal{P}_{\gamma}$ *and such that* $u(\mathcal{L}_{\gamma}\mathcal{M}_{\gamma}) = 2k$.

PROOF. Shoose $\hat{\mathcal{P}}_{\gamma}$ and $\hat{\mathcal{L}}_{\gamma}$ arbitrarily and let Q_{γ} and $\mathcal{M}_{\gamma}$ be determined by $\hat{\mathcal{P}}_{\gamma}$ *I* $\hat{\mathcal{M}}_{\gamma}$ *I* $\hat{Q}_{\gamma}$ *I* $\hat{\mathcal{L}}_{\gamma}$ ($\hat{\mathcal{P}}_{\gamma}$ is not incident with $\hat{\mathcal{L}}_{\gamma}$ since $2k < n$). By proposition(2.2.2.9), $2.u(\hat{\mathcal{P}}_{\gamma},\hat{Q}_{\gamma}) + u(\hat{\mathcal{L}}_{\gamma}\hat{\mathcal{M}}_{\gamma}) \leq 2.u_1(\mathcal{P}_{\gamma},\mathcal{L}_{\gamma}) = 2k < n$, hence by proposition(2.2.2.5) :

$$u(\mathcal{P}_{\gamma},Q_{\gamma}) + \frac{u(\mathcal{L}_{\gamma},\mathcal{M}_{\gamma})}{2} = k,$$

$$u(\mathcal{P}_{\gamma},Q_{\gamma}) + u(\mathcal{L}_{\gamma}\mathcal{M}_{\gamma}) = 2k.$$

Hence $u(\mathcal{L}_{\gamma},\mathcal{M}_{\gamma}) = 2k$. **Q.E.D.**

PROPOSITION(2.2.2.11). *Let $2.k < n$ and suppose $u(\mathcal{P}_\gamma,\mathcal{L}_\gamma) = (k,2k)$. If $\mathcal{P}_\gamma$ I $\mathcal{M}_\gamma$ I Q_γ I $\mathcal{L}_\gamma$, then $u(\mathcal{P}_\gamma,Q_\gamma) = 0$, $u(\mathcal{L}_\gamma,\mathcal{M}_\gamma) = 2k$ and there exist $\hat{\mathcal{P}}_\gamma,\hat{Q}_\gamma,\hat{\mathcal{L}}_\gamma$ and $\hat{\mathcal{M}}_\gamma$ such that $\hat{\mathcal{P}}_\gamma$ I $\hat{\mathcal{M}}_\gamma$ I $\hat{Q}_\gamma$ I $\hat{\mathcal{L}}_\gamma$.*

PROOF. Suppose $\mathcal{P}_\gamma$ I $\mathcal{M}_\gamma$ I Q_γ I $\mathcal{L}_\gamma$ and let $u(\mathcal{P}_\gamma,Q_\gamma) > 0$. We seek a contradiction. Choose $\hat{Q}_\gamma$ arbitrarily and let $\hat{\mathcal{L}}_\gamma$ and $\hat{\mathcal{M}}_\gamma$ be such that $\hat{\mathcal{L}}_\gamma$ I $\hat{Q}_\gamma$ I $\hat{\mathcal{M}}_\gamma$ (possible by proposition(2.2.2.1)); choose also $\hat{\mathcal{P}}_\gamma$ arbitrarily and let $\mathcal{M}^*$ and $\mathcal{P}^*$ be such that $\hat{\mathcal{P}}_\gamma$ I $\mathcal{M}^*$ I $\mathcal{P}^*$ I $\hat{\mathcal{M}}_\gamma$ (if $\hat{\mathcal{P}}_\gamma$ I $\hat{\mathcal{M}}_\gamma$, then the result follows similar to the proof of proposition(2.2.2.10)) ; finally let $\mathcal{L}^*$ and Q^* be such that $\hat{\mathcal{P}}_\gamma$ I $\mathcal{L}^*$ I Q^* I $\hat{\mathcal{L}}_\gamma$ (and $\hat{\mathcal{P}}_\gamma$ is not incident with $\hat{\mathcal{L}}_\gamma$ by assumption). Similar as in the proof of proposition (2.2.2.10), $u(\hat{\mathcal{P}}_\gamma,Q^*) = 0$ and $u(\hat{\mathcal{L}}_\gamma,\mathcal{L}^*) = 2k$. By the triangle inequality and our assumptions, $u(\hat{Q}_\gamma,Q^*) = 0$. We apply the third main property in the quadrangle Q^* I $\mathcal{L}^*$ I $\hat{\mathcal{P}}_\gamma$ I $\mathcal{M}^*$ I $\mathcal{P}^*$ I $\hat{\mathcal{M}}_\gamma$ I $\hat{Q}_\gamma$ I $\hat{\mathcal{L}}_\gamma$ I Q^* on the vertex Q^* :

$$u(\hat{\mathcal{L}}_\gamma,\mathcal{L}^*) = u(\hat{Q}_\gamma,\mathcal{P}^*) + u(\hat{\mathcal{P}}_\gamma,\mathcal{P}^*) + u(\hat{\mathcal{M}}_\gamma,\mathcal{M}^*) \qquad (1)$$

Since $\mathcal{P}_\gamma$ I $\mathcal{M}_\gamma$, we have by proposition(2.2.2.6), $u(\hat{\mathcal{P}}_\gamma,\mathcal{P}^*) + u(\hat{\mathcal{M}}_\gamma,\mathcal{M}^*) \geq n$, hence the right hand side of (1) is not smaller than n and consequently $2k = u(\hat{\mathcal{L}}_\gamma,\mathcal{L}^*) \geq n$, a contradiction. Hence $u(\mathcal{P}_\gamma,Q_\gamma) = 0$. But by propositions(2.2.2.1) and (2.2.2.7), we can re-choose $\hat{\mathcal{M}}_\gamma,\hat{Q}_\gamma$ and $\hat{\mathcal{L}}_\gamma$ such that $\hat{\mathcal{P}}_\gamma$ I $\hat{\mathcal{M}}_\gamma$ I $\hat{Q}_\gamma$ I $\hat{\mathcal{L}}_\gamma$. So the remainder of the proof is similar to the last part of the proof of proposition (2.2.2.10). **Q.E.D.**

PROPOSITION(2.2.2.12). *Suppose $\mathcal{P}_\gamma,Q_\gamma$ I $\mathcal{L}_\gamma,\mathcal{M}_\gamma$ and $u(\mathcal{P}_\gamma,Q_\gamma) = 0$. Then $\mathcal{L}_\gamma = \mathcal{M}_\gamma$ in $\mathcal{V}_\gamma$.*

PROOF. Let $\hat{\mathcal{P}}_\gamma \; I \; \hat{\mathcal{L}}_\gamma \; I \; \hat{\mathcal{Q}}_\gamma$ (possible by proposition(2.2.2.7)). Choose $\hat{\mathcal{M}}_\gamma$ arbitrarily, then by proposition(2.2.2.6), $u(\hat{\mathcal{P}}_\gamma, \hat{\mathcal{Q}}_\gamma) + u(\hat{\mathcal{L}}_\gamma, \hat{\mathcal{M}}_\gamma) \geq n$. But $u(\hat{\mathcal{P}}_\gamma, \hat{\mathcal{Q}}_\gamma) = 0$, so $u(\hat{\mathcal{L}}_\gamma, \hat{\mathcal{M}}_\gamma) \geq n$ hence $\mathcal{L}_\gamma = \mathcal{M}_\gamma$. **Q.E.D.**

PROPOSITION(2.2.2.13). *Suppose* $\mathcal{L}_\gamma \; I \; \mathcal{P}_\gamma \; I \; \mathcal{M}_\gamma$ *with* $u(\mathcal{L}_\gamma, \mathcal{M}_\gamma) = 0$. *Let* $\hat{\mathcal{L}}_\gamma$ *be arbitrary, then there exist* $\hat{\mathcal{P}}_\gamma$, $\hat{\mathcal{M}}_\gamma$ *and a point* $\mathcal{P}$ *such that* $\hat{\mathcal{L}}_\gamma \; I \; \mathcal{P} \; I \; \hat{\mathcal{M}}_\gamma$ *in* $\mathcal{V}$ *and* $u(\hat{\mathcal{P}}_\gamma, \mathcal{P}) \geq \frac{n}{2}$.

PROOF. Let $\hat{\mathcal{L}}_\gamma$ and $\hat{\mathcal{P}}_\gamma \; I \; \hat{\mathcal{M}}'_\gamma$ be arbitrary (with $\mathcal{L}_\gamma = \mathcal{L}'_\gamma$). Let $\mathcal{T}$ be the coordinatepoint of $\hat{\mathcal{M}}'_\gamma$ satisfying $\nu(\mathcal{T}) \neq \nu(\hat{\mathcal{P}}_\gamma)$ (possible since the coordinatepoints of a line have distinct shape) and let $\mathcal{N}$ be the coordinateline of $\hat{\mathcal{L}}_\gamma$ such that $\nu(\mathcal{N})$ is incompatible with $\nu(\hat{\mathcal{P}}'_\gamma)$ (this is possible because every shape of points is incompatible with at least one of the two shapes of coordinatelines of an arbitrary line). Denote by $\mathcal{S}$ the coordinatepoint of $\hat{\mathcal{P}}'_\gamma$ such that $\nu(\mathcal{S})$ is incompatible with $\nu(\hat{\mathcal{M}}'_\gamma)$. Define the points $\mathcal{P}, \mathcal{R}, \mathcal{P}'$ and the lines $\mathcal{M}, \mathcal{K}, \mathcal{L}'$ by $\mathcal{T} \; I \; \mathcal{M} \; I \; \mathcal{P} \; I \; \hat{\mathcal{L}}_\gamma$, $\hat{\mathcal{L}}_\gamma \; I \; \mathcal{R} \; I \; \mathcal{K} \; I \; \hat{\mathcal{P}}_\gamma$ and $\mathcal{S} \; I \; \mathcal{L}' \; I \; \mathcal{P}' \; I \; \mathcal{M}$. One can check that all these elements are well defined. Now note the following. If $\nu(\hat{\mathcal{L}}_\gamma) \neq \nu(\hat{\mathcal{M}}'_\gamma)$, then $\nu(\hat{\mathcal{L}}_\gamma) \neq \nu(\mathcal{M})$ (since $\nu(\mathcal{M})$ is not incompatible with $\nu(\mathcal{T})$) and so $u(\hat{\mathcal{L}}_\gamma, \mathcal{M}) = 0$. Also $\nu(\mathcal{P}) \neq \nu(\mathcal{T})$ and hence $u(\mathcal{P}, \mathcal{T}) = 0$. If $\nu(\hat{\mathcal{L}}_\gamma) = \nu(\hat{\mathcal{M}}'_\gamma)$, then we choose $\hat{\mathcal{L}}'_\gamma \; I \; \hat{\mathcal{P}}_\gamma$ (with $\mathcal{L}_\gamma = \mathcal{L}'_\gamma$), denote by $\mathcal{N}'$ the coordinateline of $\hat{\mathcal{L}}'_\gamma$ concurrent with $\mathcal{N}$ and proposition(2.2.1.11) implies $u(\hat{\mathcal{L}}'_\gamma, \hat{\mathcal{M}}'_\gamma) = u^\star(\hat{\mathcal{L}}'_\gamma, \hat{\mathcal{M}}'_\gamma) = u(\mathcal{N}^\star, \mathcal{N}') = 0$, where $\mathcal{N}^\star$ denotes the coordinateline of $\hat{\mathcal{M}}'_\gamma$ corresponding to $\mathcal{T}$. Hence by the triangle inequality, $u(\mathcal{N}, \mathcal{N}^\star) = 0$, implying with the third main property applied in the quadrangle with sides $\mathcal{N}, \mathcal{N}^\star, \mathcal{M}$ and $\hat{\mathcal{L}}_\gamma$ that $u(\hat{\mathcal{L}}_\gamma, \mathcal{M}) = 0 = u(\mathcal{P}, \mathcal{T})$.

By proposition(2.2.2.6),

$$u(\mathcal{R},\hat{\mathcal{P}}_{\gamma}) + u(\mathcal{K},\hat{\mathcal{L}}_{\gamma}) \geq n \tag{1}$$

Finally, we remark that if in the next formulas, there is a term missing,

this means that the term in question is zero in an obvious way, e.g. the two elements in the argument have distinct shape (e.g. $u(\mathcal{L},\hat{\mathcal{P}}_{\gamma}) = 0$).

We apply the third main property in the quadrangle $\mathcal{T}\ I\ \mathcal{M}\ I\ \mathcal{P}\ I\ \hat{\mathcal{L}}_{\gamma}\ I\ \mathcal{R}\ I\ \mathcal{K}\ I\ \hat{\mathcal{P}}_{\gamma}\ I\ \hat{\mathcal{M}}'_{\gamma}\ I\ \mathcal{T}$ on the vertex $\mathcal{T}$:

$$u(\mathcal{M},\hat{\mathcal{M}}'_{\gamma}) = u(\mathcal{R},\hat{\mathcal{P}}_{\gamma}) + u(\mathcal{K},\hat{\mathcal{L}}_{\gamma}) + u(\mathcal{R},\mathcal{P}) \tag{2}$$

By (1), the right hand side of (2) is not smaller than n and hence $\mathcal{M}$ is a representative of $\mathcal{M}_{\gamma}$. Now observe the following. We dualize in the proof of proposition(2.2.2.1) the case $(\square 1)_{\gamma}$. So in fact, this is a piece of the proof of proposition(2.2.2.2). But in this piece, we show that, with the present notation, $2.u(\hat{\mathcal{P}}_{\gamma},\mathcal{P}') \geq n$, since $\mathcal{M}$ is another representative of $\mathcal{M}_{\gamma}$ having a common coordinatepoint. Let $\frac{n}{2} \leq \ell \leq \frac{n+1}{2}$ and ℓ an integer, then $\Pi_{\ell}(\hat{\mathcal{P}}_{\gamma}) = \Pi_{\ell}(\mathcal{P}')$ and hence $\Pi_{\ell}(\mathcal{P}')\ I\ \Pi_{\ell}(\hat{\mathcal{L}}_{\gamma})$. proposition(2.2.2.6) and the facts $\mathcal{P}'\ I\ \mathcal{M}\ I\ \mathcal{P}\ I\ \hat{\mathcal{L}}_{\gamma}$ and $u(\hat{\mathcal{L}}_{\gamma},\mathcal{M}) = 0$ imply $u(\mathcal{P},\mathcal{P}') \geq \ell$. By the triangle inequality, there follows $2.u(\hat{\mathcal{P}}_{\gamma},\mathcal{P}) \geq n$. Putting $\hat{\mathcal{M}}_{\gamma} = \mathcal{M}$, the assertion follows. **Q.E.D.**

PROPOSITION(2.2.2.14). *Suppose $\mathcal{P}_{\gamma}, Q_{\gamma}\ I\ \mathcal{L}_{\gamma}, \mathcal{M}_{\gamma}$ and $u(\mathcal{L}_{\gamma},\mathcal{M}_{\gamma}) = 0$. Then $u(\mathcal{P}_{\gamma}, Q_{\gamma}) \geq \frac{n}{2}$.*

PROOF. The proof is dual to the proof of proposition(2.2.2.12), now using proposition(2.2.2.13) in stead of proposition(2.2.2.7). **Q.E.D.**

PROPOSITION(2.2.2.15). *The axiom* (GQ2) *for* $k=0$ *is valid in* $\mathcal{V}_\gamma$.

PROOF. Suppose $u(\mathcal{P}_\gamma,\mathcal{L}_\gamma) = (0,0)$. By proposition(2.2.2.10), there exists a line in $\mathcal{V}_\gamma$ incident with $\mathcal{P}_\gamma$ and concurrent with $\mathcal{L}_\gamma$. Suppose now there are two such lines, i.e. let $\mathcal{P}_\gamma\ I\ \mathcal{M}^1_\gamma\ I\ Q^1_\gamma\ I\ \mathcal{L}_\gamma$ and $\mathcal{P}_\gamma\ I\ \mathcal{M}^2_\gamma\ I\ Q^2_\gamma\ I\ \mathcal{L}_\gamma$. Choose $\hat{\mathcal{P}}_\gamma$ arbirarily and by propositions(2.2.2.1) and (2.2.2.7), there exist chains $\hat{\mathcal{P}}_\gamma\ I\ \hat{\mathcal{M}}^1_\gamma\ I\ \hat{Q}^1_\gamma\ I\ \hat{\mathcal{L}}_\gamma$ and $\hat{\mathcal{P}}_\gamma\ I\ \hat{\mathcal{M}}^2_\gamma\ I\ \hat{Q}^2_\gamma$. If $\hat{Q}^2_\gamma\ I\ \hat{\mathcal{L}}_\gamma$, then $\hat{\mathcal{M}}^1_\gamma = \hat{\mathcal{M}}^2_\gamma$ and hence $\mathcal{M}^1 = \mathcal{M}^2$. So suppose now that $\hat{Q}^2_\gamma$ is not incident with $\hat{\mathcal{L}}_\gamma$. Put $\hat{Q}^2_\gamma\ I\ \mathcal{L}^*\ I\ Q^*\ I\ \hat{\mathcal{L}}_\gamma$. Since $Q^2_\gamma\ I\ \mathcal{L}_\gamma$, we have by proposition(2.2.2.6),

$$u(\hat{Q}^2_\gamma, Q^*) + u(\hat{\mathcal{L}}_\gamma, \mathcal{L}^*) \geq n \tag{1}$$

We apply the third main property in the quadrangle $\hat{\mathcal{P}}_\gamma\ I\ \hat{\mathcal{M}}^1_\gamma\ I\ \hat{Q}^1_\gamma\ I\ \hat{\mathcal{L}}_\gamma\ I\ Q^*\ I\ \mathcal{L}^*\ I\ \hat{Q}^2_\gamma\ I\ \hat{\mathcal{M}}^2_\gamma\ I\ \hat{\mathcal{P}}_\gamma$ on the vertex $\hat{\mathcal{P}}_\gamma$ and we use the fact that $u(\mathcal{P}_\gamma, Q^1_\gamma) = u(\mathcal{P}_\gamma, Q^2_\gamma) = 0$ (by proposition(2.2.2.11)) :

$$u(\hat{\mathcal{M}}^1_\gamma, \hat{\mathcal{M}}^2_\gamma) = u(\hat{Q}^1_\gamma, Q^*) + u(\hat{Q}^2_\gamma, Q^*) + u(\hat{\mathcal{L}}_\gamma, \mathcal{L}^*) \geq n$$

by (1). Hence $\mathcal{M}^1_\gamma = \mathcal{M}^2_\gamma$ always. The fact that $u(Q^1_\gamma, Q^2_\gamma) \geq \frac{n}{2}$ follows directly from $u(\mathcal{L}_\gamma, \mathcal{M}^1_\gamma) = 0$ (by proposition(2.2.2.11)) and proposition (2.2.2.14). This completes the proof of the proposition.

Q.E.D.

PROPOSITION(2.2.2.16). *The axiom* (GQ2) *is valid in* $\mathcal{V}_\gamma$.

PROOF. By propositions(2.2.11) and (2.2.15), it suffices to show that, if $u(\mathcal{P}_{\gamma},\mathcal{L}_{\gamma}) = (k,2k)$ with $2k < n$, and $\mathcal{P}_{\gamma}\ I\ \mathcal{M}^{i}_{\gamma} \perp \mathcal{L}_{\gamma}$, $i=1,2$, then $\mathcal{M}^{1}_{\gamma} = \mathcal{M}^{2}_{\gamma}$. So let $\mathcal{P}_{\gamma}\ I\ \mathcal{M}^{i}_{\gamma}\ I\ Q^{i}_{\gamma}\ I\ \mathcal{L}_{\gamma}$. Similar as above, there exist a chain $\hat{\mathcal{L}}_{\gamma}\ I\ \hat{Q}^{1}_{\gamma}\ I\ \hat{\mathcal{M}}^{1}_{\gamma}\ I\ \hat{\mathcal{P}}_{\gamma}\ I\ \hat{\mathcal{M}}^{2}_{\gamma}\ I\ \hat{Q}^{2}_{\gamma}$ with $\mathcal{L}_{\gamma}=\mathcal{L}^{1}_{\gamma}$. If $\hat{Q}^{2}_{\gamma}\ I\ \hat{\mathcal{L}}_{\gamma}$, then $\hat{\mathcal{M}}^{1}_{\gamma} = \hat{\mathcal{M}}^{2}_{\gamma}$ and also $\mathcal{M}^{1} = \mathcal{M}^{2}$. Suppose now that $\hat{Q}^{2}_{\gamma}$ is not incident with $\hat{\mathcal{L}}_{\gamma}$. Then put $\hat{Q}^{2}_{\gamma}\ I\ \mathcal{L}^{*}\ I\ Q^{*}\ I\ \hat{\mathcal{L}}_{\gamma}$. Since $Q^{2}_{\gamma}\ I\ \mathcal{L}_{\gamma}$, we have by proposition(2.2.2.6) :

$$u(\hat{Q}^{2}_{\gamma}, Q^{*}) + u(\hat{\mathcal{L}}_{\gamma}, \mathcal{L}^{*}) \geq n \tag{1}$$

We apply the third main property in the quadrangle $\hat{\mathcal{P}}_{\gamma}\ I\ \hat{\mathcal{M}}^{1}_{\gamma}\ I\ \hat{Q}^{1}_{\gamma}\ I\ \hat{\mathcal{L}}_{\gamma}\ I\ Q^{*}\ I\ \mathcal{L}^{*}\ I\ \hat{Q}^{2}_{\gamma}\ I\ \hat{\mathcal{M}}^{2}_{\gamma}\ I\ \hat{\mathcal{P}}_{\gamma}$ on the vertex $\hat{\mathcal{P}}_{\gamma}$ and use the fact that $u(\mathcal{P}_{\gamma}, Q^{1}_{\gamma}) = u(\mathcal{P}_{\gamma}, Q^{2}_{\gamma}) = 0$ (by proposition(2.2.2.11)) :

$$u(\hat{\mathcal{M}}^{1}_{\gamma}, \hat{\mathcal{M}}^{2}_{\gamma}) = u(\hat{Q}^{1}_{\gamma}, Q^{*}) + u(\hat{Q}^{2}_{\gamma}, Q^{*}) + u(\hat{\mathcal{L}}_{\gamma}, \mathcal{L}^{*}) \geq n$$

by (1). Hence $\mathcal{M}^{1}_{\gamma} = \mathcal{M}^{2}_{\gamma}$. **Q.E.D.**

THEOREM(2.2.2.17). *The geometry $\mathcal{V}_1$ is a generalized quadrangle.*

PROOF. We check the axioms (QQ1) up to (QQ4).

(QQ1). The point (∞) is incident with the lines $[\infty]$, $[0]$ and $[1]$. The new coordinates of these lines are respectivily $[[\infty,\infty,\infty]]$, $[[0,\infty,\infty]]$ and $[[1,\infty,\infty]]$. Apparently, they are mapped onto three different lines of $\mathcal{V}_1$ by the projection Π_1. Furthermore, the coordinatelines of any point $\mathcal{P}$ in $\mathcal{V}$ have distinct shape, hence they are mapped onto two distinct lines of $\mathcal{V}_1$ both incident with $\Pi_1(\mathcal{P})$.

(QQ2). Dual to (QQ1).

(QQ3). Trivial since e.g. $u((0),[0,0]) = (0,0)$.

(QQ4). Suppose $\mathcal{P}$ is not incident with $\mathcal{L}$ in $\mathcal{V}_1$, then the only possibility is $u(\mathcal{P},\mathcal{L}) = (0,0)$. the result follows from (GQ2).

Q.E.D.

In the next paragraph, we will define the linestructure in a point-neighbourhood and we will check the strip-axiom (NP). But first we still need to check the conditions (PS3) and (PS4).

LEMMA(2.2.2.18). *For every a in $\mathcal{R}_1$, there exists an element b in $\mathcal{R}_1$ such that $v(a,b) = j$ for any pre-assigned $j \in \mathbf{Z}$. If $v(a) \geq 0$ and $j \geq 0$, then $v(b) \geq 0$. For every a^- in $\mathcal{R}_1$, there exists an element b^- in $\mathcal{R}_1 \cup \{\infty\}$ such that $w(a,b) = n$ for any pre-assigned $n \in \mathbf{N}^*$.*

PROOF. Let $a \in \mathcal{R}_1$. By (v3), there exist $x, y \in \mathcal{R}_1$ such that $v(x,y) = j$. Consider $(0,x)$ and $[1,a,0]$. They are not incident with one another and hence there exist elements k',c,ℓ such that $(0,x)$ I $[0,x,k']$ I (c,ℓ,x) I $[1,a,0]$. One has $Q_1(1,c,\ell,x) = a$. Put $Q_1(1,c,\ell,y) = b$, then by (v6), $v(a,b) = j$. If $v(a) \geq 0$ and $j \geq 0$, then by (v2), $v(b) \geq 0$. Now let $a^- \in \mathcal{R}_1$. If $v(a) = -n$, then we take $b=\infty$ nemen. If $v(a) < -n$, then we construct b such that $v(b) = -n$ (this is possible by the first part of the proof), and hence $v(a,b) = v(a)$ by (v2), so $w(a,b) = n$. If $v(a) > -n$, then we construct b such that $v(a,b) = n + 2v(a) > v(a)$. Then $v(b) = v(a)$ and $w(a,b) = v(a,b) - v(a) - v(b) = n$. **Q.E.D.**

Dually, we obtain :

LEMMA(2.2.2.19). *For every k in $\mathcal{R}_2$, there exists an element of $\mathcal{R}_2$ such that $v(k,\ell) = j$ for any pre-assigned $j \in \mathbf{Z}$. If $v(k) \geq 0$ and $j \geq 0$, then $v(\ell) \geq 0$. For every k^- in $\mathcal{R}_2$, there exists an element ℓ^- in $\mathcal{R}_2 \cup \{\infty\}$ such that $w(k,\ell) = n$ for every pre-assigned $n \in \mathbf{N}^*$.*

PROOF. Dual to lemma(2.2.2.18). **Q.E.D.**

THEOREM(2.2.2.20). *The conditions* (PS1),(PS2),(PS3) *and* (PS4) *are satisfied in $\mathcal{V}_n$.*

PROOF. Let $\mathcal{P} = ((a,\ell,a'))$ be a point of $\mathcal{V}$ and suppose $v(a') \geq 0$. By lemma(2.2.2.17), there exists b such that $v(a',b) = n$. Put $Q = ((a,\ell,b))$ and consequently $u(\mathcal{P},Q) = n$ by definition. Hence (PS3). Dually for (PS4). Finally, (PS1) and (PS2) are trivial. **Q.E.D.**

2.2.3. The strip-axiom.

In the next four propositions, we will discuss certain classes of coordinate transformations. The goal is to reduce the strip-axiom to the neighbourhoods of axispoints.

PROPOSITION(2.2.3.1). *We recoordinatize the generalized quadrangle $\mathcal{V}$ by means of $(\mathcal{R}_1,\mathcal{R}_2,Q_1^*,Q_2^*)$ with respect to the new quadrangle* (∞) *I* $[\infty]$ *I* (0) *I* $[0,0]$ *I* $(0,0,x)$ *I* $[0,x,0]$ *I* $(0,x)$ *I* $[0]$ *I* (∞) *(old coordinates of* *course !), with* $x \in \mathcal{R}_1$ *arbitrary but fixed and* $v(x) \geq 0$. *We denote the new coordinates between two stars. We determine the coordinatization by putting*

$$*(a)* = (a) \quad \forall a \in \mathcal{R}_1,$$

$$*[k]* = [k] \quad \forall k \in \mathcal{R}_2.$$

Then $(\mathcal{R}_1, \mathcal{R}_2, Q_1^*, Q_2^*, v)$ *is a V-KQR. The shapes of the points and lines are preserved and moreover, the induced new partial valuation coincides with the old one.*

PROOF. We define the bijections π and λ_a, $a \in \mathcal{R}_1$, as follows :

$$\pi : \mathcal{R}_1 \to \mathcal{R}_2 : a \to a^\pi \text{ waarbij } *(0,a)* = (0,a^\pi),$$

$$\lambda_a : \mathcal{R}_2 \to \mathcal{R}_2 : k \to k^{\lambda_a} \text{ waarbij } *[a,k]* = [a,k^{\lambda_a}].$$

We show that $v(a,b) = v(a^\pi, b^\pi)$ and $v(k,\ell) = v(k^{\lambda_a}, \ell^{\lambda_a})$ for all a,b,k,ℓ. First note that $*(0,0,a)* = (0,0,a^\pi)$ and $*(a,0,0)* = (a,0^{\lambda_a},x)$. We now apply the first main property in the quadrangle $[1]\ I\ (1,a^\pi) \perp (a,0^{\lambda_a},x)\ I\ [0,x,0]\ I\ (b,0^{\lambda_b},x) \perp (1,b^\pi)\ I\ [1]$ on the side $[1]$:

$$v(a^\pi, b^\pi) = v(a,b) + 2.v(1) = v(a,b).$$

Note also $*[0,0,k]* = [0,x,k^{\lambda_0}]$. By the construction of the new coordinates, we have :

$$[k] \perp *[k,0,0]* \perp *[1,k]* \perp *[0,0,k]* \perp *[a,k]*.$$

Hence

$$[k] \perp [k,x,0] \perp [1,Q_2(1,k,x,0)] \perp [0,x,k^{\lambda_0}] \perp [a,k^{\lambda_a}].$$

A similar expression holds for ℓ. We apply the first main property in the quadrangle $(0,x)\ I\ [0,x,k^{\lambda 0}]\ I\ (1,Q_2(1,k,x,0),x)\ I\ [1,Q_2(1,k,x,0)]\ I$ $(1)\ I\ [1,Q_2(1,\ell,x,0)]\ I\ (1,Q_2(1,\ell,x,0),x)\ I\ [0,x,\ell^{\lambda 0}]\ I\ (0,x)$ on the vertex $(0,x)$:

$$v(k^{\lambda 0},\ell^{\lambda 0}) = v(Q_2(1,k,x,0),Q_2(1,\ell,x,0)) \tag{1}$$

We apply the first main property in the quadrangle $(0,0,x)\ I\ [k,x,0] \perp [1,Q_2(1,k,x,0)]\ I\ (1)\ I\ [1,Q_2(1,\ell,x,0)] \perp [\ell,x,0]\ I\ (0,0,x)$ on the vertex (1) :

$$v(Q_2(1,k,x,0),Q_2(1,\ell,x,0)) = v(k,\ell)+v(1) = v(k,\ell) \tag{2}$$

We apply the first main property in the quadrangle $(0,x)\ I\ [0,x,k^{\lambda 0}] \perp [a,k^{\lambda a}]\ I\ (a)\ I\ [a,\ell^{\lambda a}] \perp [0,x,\ell^{\lambda 0}]\ I\ (0,x)$ on the vertex $(0,x)$:

$$v(k^{\lambda 0},\ell^{\lambda 0}) = v(k^{\lambda a},\ell^{\lambda a}) \tag{3}$$

By (1),(2),(3), $v(k,\ell) = v(k^{\lambda a},\ell^{\lambda a})$.

Now we have $*(a,\ell,a')* = (a,\ell^{\lambda a},a'^{\pi})$ and $*[k,b,k']* = [k,b^{\pi},k'^{\lambda 0}]$, hence

$$Q_1^*(k,a,\ell,a')^{\pi} = Q_1(k,a,\ell^{\lambda a},a'^{\pi}) \tag{5}$$

$$Q_2^*(a,k,b,k')^{\lambda a} = Q_2(a,k,b^{\pi},k'^{\lambda 0}) \tag{6}$$

The axioms $(v1)$,(v2) and (v3) are trivially satisfied for the QQR $(\mathcal{R}_1,\mathcal{R}_2,Q_1^*,Q_2^*,v)$. We now check (v4).

Let there be given :

$$Q_1^*(k_1,a_1,\ell_1,a_1') = Q_1^*(k_1,a_2,\ell_2,a_2') = b_1,$$

$$Q_2^*(a_1,k_1,b_1,k_1') = Q_2^*(a_1,k_2,b_2,k_2') = \ell_1,$$

$$Q_1^*(k_2,a_1,\ell_1,a_1') = Q_1^*(k_2,a_3,\ell_3,a_3') = b_2,$$

$$Q_2^*(a_3,k_2,b_2,k_2') = Q_2^*(a_3,k_3,b_3,k_3') = \ell_3,$$
$$Q_1^*(k_3,a_3,\ell_3,a_3') = Q_1^*(k_3,a_2,\ell_2,a_2') = b_3,$$
$$Q_2^*(a_2,k_3,b_3,k_3') = Q_2^*(a_2,k_1,b_1,k_4') = \ell_2.$$

By (5),(6), this is equivalent to

$$Q_1(k_1,a_1,\ell_1^{\lambda a_1},a_1'^{\pi}) = Q_1(k_1,a_2,\ell_2^{\lambda a_2},a_2'^{\pi}) = b_1^{\pi},$$
$$Q_2(a_1,k_1,b_1^{\pi},k_1'^{\lambda 0}) = Q_2(a_1,k_2,b_2^{\pi},k_2'^{\lambda 0}) = \ell_1^{\lambda a_1},$$
$$Q_1(k_2,a_1,\ell_1^{\lambda a_1},a_1'^{\pi}) = Q_1(k_2,a_3,\ell_3^{\lambda a_3},a_3'^{\pi}) = b_2^{\pi},$$
$$Q_2(a_3,k_2,b_2^{\pi},k_2'^{\lambda 0}) = Q_2(a_3,k_3,b_3^{\pi},k_3'^{\lambda 0}) = \ell_3^{\lambda a_3},$$
$$Q_1(k_3,a_3,\ell_3^{\lambda a_3},a_3'^{\pi}) = Q_1(k_3,a_2,\ell_2^{\lambda a_2},a_2'^{\pi}) = b_3^{\pi},$$
$$Q_2(a_2,k_3,b_3^{\pi},k_3'^{\lambda 0}) = Q_2(a_2,k_1,b_1^{\pi},k_4'^{\lambda 0}) = \ell_2^{\lambda a_2}.$$

By (v4) applied on $(\mathcal{R}_1,\mathcal{R}_2,Q_1,Q_2,v)$, we have :

$$v(k_1,k_2) + v(k_1'^{\lambda 0},k_4'^{\lambda 0}) = v(k_1,k_3) + v(k_2,k_3) + v(a_2,a_3)$$

Since $v(k_1'^{\lambda 0},k_4'^{\lambda 0}) = v(k_1',k_4')$, (v4) follows for $(\mathcal{R}_1,\mathcal{R}_2,Q_1^*,Q_2^*,v)$. Similar for the other axioms. Hence $(\mathcal{R}_1,\mathcal{R}_2,Q_1,Q_2,v)$ is a V-QQR.

Now let for arbitrary $a^+ \in \mathcal{R}_1$, k_a and b_a be such that $[k_a,b_a,0]\ I\ (a,0^{\lambda a},0)$. By the first main property in the quadrangle $(0,0)\ I\ [0,0,0^{\lambda a}]\ I\ (a,0^{\lambda a},0)\ I\ [k_a,b_a,0]\ I\ (0,0,b_a)\ I\ [0,0]\ I\ (0,0,0)\ I\ [0,0,0]$ applied on the vertex $(0,0)$, we have :

$$2.v(0^{\lambda a}) = v(b_a) + v(a) \tag{7}$$

We apply the first main property in the quadrangle $[0,0]\ I\ (0,0,b_a)\ I\ [k_a,b_a,0]\ I\ (a,0^{\lambda a},0)\ I\ [a,0^{\lambda a}]\ I\ (a,0^{\lambda a},x)\ I\ [0,x,0]\ I\ (0,0,x)\ I\ [0,0]$

on the side [0,0] :

$$v(x, b_a) = v(x)$$

Hence $v(b_a) \geq 0$ and by (7), also $v(0^{\lambda a}) \geq 0$. Hence $v(k) = v(k,0) = v(k^{\lambda a}, 0^{\lambda a}) \geq 0 \iff v(k^{\lambda a}) \geq 0$ (by the triangle inequality and still assuming $v(a) \geq 0$). Now let a be again arbitrary in $\mathcal{R}_1$. Since $0^{\pi} = x$ and $v(x) \geq 0$, we also have $v(a) \geq 0 \iff v(a^{\pi}) \geq 0$. Hence, one can check easily that the shapes (+,+,+) and [+,+,+] are invariant under our coordinate transformation. Now we also have $*((a^-, \ell^+, a'^+))* = ((a, \ell, a'^{\pi}))$ and hence also (-,+,+) is invariant. Suppose now that the point $\mathcal{P}$ of shape (+,-,-) is transformed into a point of shape *(-,-,-)*. Let e.g. $\mathcal{P} = ((a^+, \ell^-, a'^-)) = *((c^-, n^-, c'^-))*$. Then $\mathcal{P}$ is collinear with $*(c,0,0)*$ $= (c, 0^{\lambda c}, x) = ((c^-, 0, x^+))$. But this contadicts the fact that no line is incident with points of the shape (-,+,+) and at the same time with points of the shape (+,-,-). Similarly, one shows that points of shape (-,-,-) are transformed into points of shape (-,-,-). We conclude that the shape of points is an invariant. Since the shape of any line is determined by the shape of the points incident with the line, the shape of lines is also an invariant.

Now note that $w(a^-, b^-) = v(a,b) - v(a) - v(b) = v(a^{\pi}, b^{\pi}) - v(a^{\pi}, 0^{\pi}) - v(b^{\pi}, 0^{\pi}) = v(a^{\pi}, b^{\pi}) - v(a^{\pi}, x) - v(b^{\pi}, x) = v(a^{\pi}, b^{\pi}) - v(a^{\pi}) - v(b^{\pi}) = w(a^{\pi}, b^{\pi})$. Similarly $w(k^-, \ell^-) = w(k^{\lambda a}, \ell^{\lambda a})$, for all a^+.

To show that the partial valuation is invariant, we remark first that there is a certain symmetry in the problem. Indeed, if $(0,0) = *(0,y)*$, then $v(y) \geq 0$ since the shape of points is preserved under our transformation. Hence everything we prove for this coordinate transformation is also valid for the inverse coordinate transformation. Hence it suffices to show that every n-equivalenceclass w.r.t. $(\mathcal{R}_1, \mathcal{R}_2, Q_1^*, Q_2^*, v)$ is a subset of a certain n-equivalenceclass w.r.t. $(\mathcal{R}_1, \mathcal{R}_2, Q_1, Q_2, v)$ to prove that they are actually equal. If we denote the partial valuation w.r.t. $(\mathcal{R}_1, \mathcal{R}_2, Q_1^*, Q_2^*, v)$ by u', then we show $u'(X_1, X_2) \geq n \Longrightarrow u(X_1, X_2) \geq n$ and we will have $u'(X_1, X_2) = u(X_1, X_2)$, for all X_1, X_2 both points or both lines. We show this for the most difficult case, namely the case where X_1 and X_2 are lines of shape $[-,-,-]$. The other cases are similar, but easier.

So let $\mathcal{L}_1, \mathcal{L}_2$ be such that $u'(\mathcal{L}_1, \mathcal{L}_2) \geq n$. There are three steps in the proof.

First step : $\mathcal{L}_1 [\square 1]_n^* \mathcal{L}_2$.

The notation $[\square 1]_n^*$ is self-explaining : it is the equivalent of $[\square 1]_n$ for $(\mathcal{R}_1, \mathcal{R}_2, Q_1^*, Q_2^*, v)$. Denote by $\mathcal{L}$ the first coordinateline of both $\mathcal{L}_1$ and $\mathcal{L}_2$, and by $\mathcal{L}_i'$ the second coordinateline of $\mathcal{L}_i$, $i=1,2$. Then $(0,x)$ I $\mathcal{L}_i'$, $i=1,2$. Now denote by $\mathcal{M}_i$ the unique line incident with (0) and meeting $\mathcal{L}_i'$, $i=1,2$. Then $u'(\mathcal{M}_1, \mathcal{M}_2) = u'(\mathcal{L}_1', \mathcal{L}_2') = u'(\mathcal{L}_1, \mathcal{L}_2)$ almost by definition. But also $u(\mathcal{M}_1, \mathcal{M}_2) = u(\mathcal{L}_1', \mathcal{L}_2') = u(\mathcal{L}_1, \mathcal{L}_2)$ (apply e.g. (GQ2) on $\mathcal{V}_k$, $k=u(\mathcal{M}_1, \mathcal{M}_2)$). Now $u'(\mathcal{M}_1, \mathcal{M}_2) = w(k_1, k_2)$, where $\mathcal{M}_i = *[0, k_i]*$. By $w(k_1, k_2) = w(k_1^{\lambda}0, k_2^{\lambda}0)$ (cp. above). But $\mathcal{M}_i = [0, k_i^{\lambda}0]$, hence $w(k_1^{\lambda}0, k_2^{\lambda}0) = u(\mathcal{M}_1, \mathcal{M}_2)$, hence the assertion.

Second step : $\mathcal{L}_1 [\square 2]_n^* \mathcal{L}_2$.

Completely similar to the first step.

Third step : $\mathcal{L}_1\,[\square 3]^{*}_{n}\mathcal{L}_2$.

Denote by $\mathcal{P}_i$ resp. $\mathcal{P}_i'$ the first resp. second coordinatepoint of $\mathcal{L}_i$, $i=1,2$. Then $u'(\mathcal{P}_1,\mathcal{P}_2) \geq n$. Put $\mathcal{P}_i = (a_i,\ldots)$, then again $u'(\mathcal{P}_1,\mathcal{P}_2) = w(a_1,a_2) = w(a_1^{\pi},a_2^{\pi}) = u(\mathcal{P}_1,\mathcal{P}_2)$ (because, after all, $\mathcal{P}_1$ and $\mathcal{P}_2$ are incident with a common line of shape [-,+,+] and they have both shape (+,+,-); so apply proposition(2.2.1.11)). Similarly $u(\mathcal{P}_1',\mathcal{P}_2') = u'(\mathcal{P}_1',\mathcal{P}_2')$. By proposition(2.2.2.12), $\Pi_n(\mathcal{L}_1) = \Pi_n(\mathcal{L}_2)$ and hence $u(\mathcal{L}_1,\mathcal{L}_2) \geq n$ (because $u(\mathcal{P}_i,\mathcal{P}_i') = 0$ since $v(\mathcal{P}_i) \neq v(\mathcal{P}_i')$, $i=1,2$). This completes this third step.

Concerning the other cases, we have the following remark for the reader who wants to reconstruct those proofs. Note that we did ***not*** prove that $w(k^{\lambda}a,\ell^{\lambda}a) = w(k^{-},\ell^{-})$ for a^{-}. But we can get round this by using homogeneous coordinates at the right time and then work with those coordinate belonging to $\mathcal{R}_2$ which does not give rise to a coordinateline meeting $[\infty]$. A postiori, we then have $w(k^{\lambda}a,\ell^{\lambda}a) = w(k^{-},\ell^{-})$ for all a^{-}.

Q.E.D.

PROPOSITION(2.2.3.2). *We recoordinatize the generalized quadrangle $\mathcal{V}$ by means of $(\mathcal{R}_1,\mathcal{R}_2,Q_1^{*},Q_2^{*})$ with respect to the new quadrangle (∞) I $[\infty]$ I (x) I $[x,0]$ I $(x,0,0)$ I $[0,0,0]$ I $(0,0)$ I $[0]$ I (∞) (old coordinates of course !), with $x \in \mathcal{R}_1$ arbitrary but fixed and $v(x) \geq 0$. We denote the new coordinates between stars. We determine the new coordinates by putting :*

$$*(0,a)* = (0,a) \quad \forall a \in \mathcal{R}_1,$$

$$*[k]* = [k] \quad \forall k \in \mathcal{R}_2.$$

Then $(\mathcal{R}_1, \mathcal{R}_2, Q_1^\star, Q_2^\star, v)$ *is a V-QQR. The shapes of points and lines are preserved and moreover, the induced new partial valuation coincides with the old one.*

PROOF. Similar to the previous proposition. **Q.E.D.**

PROPOSITION(2.2.3.3). *We recoordinatize the generalized quadrangle* $\mathcal{V}$ *by means of* $(\mathcal{R}_1, \mathcal{R}_2, Q_1^\star, Q_2^\star)$ *with respect to the new quadrangle* (∞) *I* $[\infty]$ *I* (0) *I* $[0,\mu]$ *I* $(0,\mu,0)$ *I* $[0,0,\mu]$ *I* $(0,0)$ *I* $[0]$ *I* (∞) *(old coordinates again), with* $\mu \in \mathcal{R}_2$ *arbitrary but fixed and* $v(\mu) \geq 0$. *We denote the new coordinates between stars and determine them completely by putting :*

$$\star(a)\star = (a) \quad \forall a \in \mathcal{R}_1,$$
$$\star[k]\star = [k] \quad \forall k \in \mathcal{R}_2.$$

Then $(\mathcal{R}_1, \mathcal{R}_2, Q_1^\star, Q_2^\star, v)$ *is a V-QQR. The shapes of points and lines are preserved and moreover the induced new partial valuation coincides with the old one.*

PROOF. Dual to proposition(2.2.3.1). **Q.E.D.**

PROPOSITION(2.2.3.4). *We recoordinatize the generalized quadrangle* $\mathcal{V}$ *by means of* $(\mathcal{R}_1, \mathcal{R}_2, Q_1^\star, Q_2^\star)$ *with respect to the new quadrangle* (∞) *I* $[\infty]$ *I* (0) *I* $[0,0]$ *I* $(0,0,0)$ *I* $[y,0,0]$ *I* $(y,0)$ *I* $[y]$ *I* (∞) *(old coordinates), with* $\mu \in \mathcal{R}_2$ *arbitrary but fixed and* $v(\mu) \geq 0$. *We denote the new coordinates between stars and determine them completely by putting :*

$$*(a)* = (a) \quad \forall a \in \mathcal{R}_1,$$

$$*[0,k]* = [0,k] \quad \forall k \in \mathcal{R}_2.$$

Then $(\mathcal{R}_1, \mathcal{R}_2, Q_1^*, Q_2^*, v)$ *is a V-KQR. The shapes of points and lines are preserved and moreover, the induced new partial valuation coincides with the old one.*

PROOF. Dual to proposition(2.2.3.2). **Q.E.D.**

Now let $\mathcal{C}$ be an $(n-1)$-point-neighbourhood in $\mathcal{V}_n$. In what follows, we are going to define the level $n-1$ Hjelmslev quadrangle $\mathcal{W}_{n-1}(\mathcal{V}_n, \mathcal{C})$ and also the $(n-2)$-point-neighbourhood $\mathcal{N}_{\mathcal{C}}$ it contains. We start with the neighbourhoods only containing points having representatives of shape $(+,+,+)$.

Let $\mathcal{P}_n$ be an arbitrary point of $\mathcal{C}$. Let $\hat{\mathcal{P}}_n = ((a,\ell;a',\ell'))$ be arbitrary. We recoordinatize $\mathcal{V}$ w.r.t. the new quadrangle (∞) I $[\infty]$ I (a) I $[a,\ell]$ I (a,ℓ,a') I $[0,a',\ell']$ I $(0,a')$ I $[0]$ I (∞). By propositions(2.2.3.1), (2.2.3.2) and (2.2.3.3), the new QQR can be denoted by $(\mathcal{R}_1, \mathcal{R}_2, Q_1^*, Q_2^*, v)$ and it is a V-QQR. Moreover, the corresponding geometry $\mathcal{V}_n^*$ is canonical isomorphic to $\mathcal{V}_n$, i.e. there exists an isomorfism $\psi : \mathcal{V}_n \to \mathcal{V}_n^*$ acting as follows. Let X_n be a variety of $\mathcal{V}_n$, take $\hat{X}_n$ arbitrarily and define $\psi(X_n) = \Pi_n^*(\hat{X}_n)$ (where Π_n^* is the corresponding projection map from $\mathcal{V}$ onto $\mathcal{V}_n^*$). Consider any element x of $\mathcal{R}_1$ such that $v(x) = 1$. It exists by lemma (2.2.2.18). We will now recoordinatize $\mathcal{V}$ once again and we forget the previous recoordinatization. We denote the new coordinates between two stars and all objects defined via the new coordinates are also denoted with a star. To avoid exceptions, we mention here explicitly that u^*

does not longer denotes the partial *-valuation (we do not use it anymore anyway).

We keep the same quadrangle (∞) I $[\infty]$ I (0) I $[0,0]$ I $(0,0,0)$ I $[0,0,0]$ I $(0,0)$ I $[0]$ I (∞) with the same coordinates. We also keep $\mathcal{R}_2$ and put $*[k]* = [k]$ for all $k \in \mathcal{R}_2$. We choose a new arbitrary $\mathcal{R}_1^*$ (with the suitable cardinality) and only require that $*(1)* = (x)$. Hence we have two bijections

$$\pi : \mathcal{R}_1 \to \mathcal{R}_1^* : a \to a^\pi \quad \text{where } (a) = *(a^\pi)*.$$
$$\lambda : \mathcal{R}_2 \to \mathcal{R}_2 : k \to k^\lambda \quad \text{where } [0,k] = *[0,k^\lambda]$$

Since $*[1]* = [1]$, we have $(a,0,0) = *(a^\pi,0,0)*$, $(0,0,a) = *(0,0,a^\pi)*$ and $(k,a) = *(k,a^\pi)*$ for all $k \in \mathcal{R}_2$. Since $(0,0) = *(0,0)*$, also $[0,0,k] = *[0,0,k^\lambda]*$ and $[a,k] = *[a^\pi,k^\lambda]*$ for all $a \in \mathcal{R}_1$. Furthermore, note that $[k,0,0] = *[k,0,0]*$. From this discussion follows immediately

$$(a,\ell,a') = *(a^\pi,\ell^\lambda,a'^\pi)*$$
$$[k,b,k'] = *[k,b^\pi,k'^\lambda]*$$

and also

$$Q_1(k,a,\ell,a')^\pi = Q_1^*(k,a^\pi,\ell^\lambda,a'^\pi) \qquad (1)$$
$$Q_2(a,k,b,k')^\lambda = Q_2^*(a^\pi,k,b^\pi,k'^\lambda) \qquad (2)$$

We define now $v^*(a^\pi,b^\pi) = v(a,b) - 1$ and $v^*(k,\ell) = v(k,\ell)$.

LEMMA(2.2.3.5). *With the above notation,* $(\mathcal{R}_1^*,\mathcal{R}_2,Q_1^*,Q_2^*,v^*)$ *is a V-QQR. Moreover,* $\Pi_\lambda(\mathcal{P}) \in \mathcal{C} \iff \mathrm{v}^*(\mathcal{P}) = *(+,+,+)*$ *and for every* $\mathcal{P},Q \in \Pi_\lambda^{-1}(\mathcal{C})$ *:* $u^*(\mathcal{P},Q) = u(\mathcal{P},Q) - 1$.

PROOF. The axioms (v1),(v2) and (v3) are trivially satisfied. We now show (v4). Since π and λ are bijections, we can change a_i by a_i^{π}, ℓ_i by ℓ_i^{λ}, b_i by b_i^{π}, a_i' by $a_i'^{\pi}$ and k_i' by $k_i'^{\lambda}$ in (v4). Hence, there is given :

$$Q_1^*(k_1, a_1^{\pi}, \ell_1^{\lambda}, a_1'^{\pi}) = Q_1^*(k_1, a_2^{\pi}, \ell_2^{\lambda}, a_2'^{\pi}) = b_1^{\pi},$$
$$Q_2^*(a_1^{\pi}, k_1, b_1^{\pi}, k_1'^{\lambda}) = Q_2^*(a_1^{\pi}, k_2, b_2^{\pi}, k_2'^{\lambda}) = \ell_1^{\lambda},$$
$$Q_1^*(k_2, a_1^{\pi}, \ell_1^{\lambda}, a_1'^{\pi}) = Q_1^*(k_2, a_3^{\pi}, \ell_3^{\lambda}, a_3'^{\pi}) = b_2^{\pi},$$
$$Q_2^*(a_3^{\pi}, k_2, b_2^{\pi}, k_2'^{\lambda}) = Q_2^*(a_3^{\pi}, k_3, b_3^{\pi}, k_3'^{\lambda}) = \ell_3^{\lambda},$$
$$Q_1^*(k_3, a_3^{\pi}, \ell_3^{\lambda}, a_3'^{\pi}) = Q_1^*(k_3, a_2^{\pi}, \ell_2^{\lambda}, a_2'^{\pi}) = b_3^{\pi},$$
$$Q_2^*(a_2^{\pi}, k_3, b_3^{\pi}, k_3'^{\lambda}) = Q_2^*(a_2^{\pi}, k_1, b_1^{\pi}, k_4'^{\lambda}) = \ell_2^{\lambda}.$$

By (1) and (2), we then have

$$Q_1(k_1, a_1, \ell_1, a_1') = Q_1(k_1, a_2, \ell_2, a_2') = b_1,$$
$$Q_2(a_1, k_1, b_1, k_1') = Q_2(a_1, k_2, b_2, k_2') = \ell_1,$$
$$Q_1(k_2, a_1, \ell_1, a_1') = Q_1(k_2, a_3, \ell_3, a_3') = b_2,$$
$$Q_2(a_3, k_2, b_2, k_2') = Q_2(a_3, k_3, b_3, k_3') = \ell_3,$$
$$Q_1(k_3, a_3, \ell_3, a_3') = Q_1(k_3, a_2, \ell_2, a_2') = b_3,$$
$$Q_2(a_2, k_3, b_3, k_3') = Q_2(a_2, k_1, b_1, k_4') = \ell_2.$$

By (v4), applied in $(\mathcal{R}_1, \mathcal{R}_2, Q_1, Q_2, v)$, we have

$$v(k_1, k_2) + v(k_1', k_4') = v(k_1, k_3) + v(k_2, k_3) + v(a_2, a_3).$$

By the definitie of v^*, it follows from this

$$v^*(k_1, k_2) + v(k_1', k_4') = v^*(k_1, k_3) + v^*(k_2, k_3) + v^*(a_2^{\pi}, a_3^{\pi}) + 1.$$

We must show that $v(k_1', k_4') - 1 = v^*(k_1'^{\lambda}, k_4'^{\lambda})$. We prove in general that for all $k, \ell \in \mathcal{R}_2$, $v(k,\ell) - 1 = v(k^\lambda, \ell^\lambda)$. By definition, we have : $*[k^\lambda,0,0]* \perp *[1,k^\lambda]*$, hence $[k^\lambda,0,0] \perp [x,k]$ and also $[\ell^\lambda,0,0] \perp [x,\ell]$. We apply the first main property in the quadrangle $(0,0,0)\ I\ [k^\lambda,0,0] \perp [x,k]\ I\ (x)\ I\ [x,\ell]\ I\ [\ell^\lambda,0,0]\ I\ (0,0,0)$ on the vertex $(0,0,0)$:

$$v(k^\lambda, \ell^\lambda) + v(x) = v(k,\ell)$$

Since $v(x) = 1$, the result follows. Similarly, one shows the other axioms of valuation.

Now let $\Pi_\chi(\mathcal{P}) \in \mathcal{C}$ and put $\mathcal{P} = ((a,\ell,a'))$, then $v(a), v(\ell), v(a') > 0$ since $u(\mathcal{P},(0,0,0)) > 0$. Hence $v^*(a^\pi), v^*(\ell^\lambda), v^*(a'^\pi) \geq 0$ (since $0^\lambda = 0$ and $0^\pi = 0$) and so $\mathcal{P} = ((a^\pi, \ell^\lambda, a'^\pi))$ and $\nu^*(\mathcal{P}) = *(+,+,+)*$. For the converse, one simply reverses this proof.

Let $\Pi_\chi(\mathcal{P}), \Pi_\chi(\mathcal{Q}) \in \mathcal{C}$, then we proof $u^*(\mathcal{P},\mathcal{Q}) = u(\mathcal{P},\mathcal{Q}) - 1$ again in three steps.

First step : $\mathcal{P}(\square 1)_j \mathcal{Q}$, $j \geq 1$.

Put $\mathcal{P} = ((a,\ell,a'))$ and $\mathcal{Q} = ((b,k,b'))$, then $a=b$, $\ell=k$ and $v(a',b') \geq j$. Hence $v(a'^\pi, b'^\pi) \geq j-1$. Consequently $\mathcal{P}(\square 1)^*_{j-1}\mathcal{Q}$.

Second step : $\mathcal{P}(\square 2)_j \mathcal{Q}$, $j \geq 1$.

Similarly as above, we conclude $\mathcal{P}(\square 2)^*_{j-1}\mathcal{Q}$.

Third step : $\mathcal{P}(\square 3)_j \mathcal{Q}$, $j \geq 1$.

Using $v(k,\ell) - 1 = v(k^\lambda, \ell^\lambda)$, one shows again similarly that $\mathcal{P}(\square 3)^*_{j-1}\mathcal{Q}$.

Reversing these arguments, one shows that $\mathcal{P}(\square i)^{*}_{j-1}Q$ implies $\mathcal{P}(\square i)_{j-1}Q$, $i=1,2,3$. Hence the assertion. Q.E.D.

We now structure $\mathcal{V}_n$ to a level n Hjelmslev quadrangle by induction on n, valid for all V-QQRs as follows :

If $n=1$, the generalized quadrangle $\mathcal{V}_1$ can be transformed in a trivial standard way to a level 1 Hjelmslev quadrangle.

We define now $\mathcal{W}_{n-1}(\mathcal{V}_n, \mathcal{C}) = \mathcal{V}_{n-1}$ and $\mathcal{N}_{\mathcal{C}}$ is by definition the $(n-2)$-point-neighbourhood containing $\Pi^{*}_{n-1}((\infty))$. Note that $\mathcal{N}_{\mathcal{C}}$ consists of all points of $\mathcal{V}^{*}_{n-1}$ having a representative of shape [-,-,-] and hence every representative has this shape. We now show that this definition is independent from the choice of the point $\mathcal{P}_n \in \mathcal{C}$, the choice of $\hat{\mathcal{P}}_n$ and the choice of the element x in $\mathcal{R}_1$.

Let $x' \in \mathcal{R}_1$ with $v(x')=1$. Then $v^{*}(x'^{\pi})=0$. The coordinate transformation turning (x') into the unit point boils down to the coordinate transformation we just described composed with a coordinate transformation turning (x'^{π}) into the unit point, with this time $v^{*}(x'^{\pi}) = 0$. But the latter preserves the valuation and hence it preserves the corresponding quotient-geometries. Hence the definition is independent from the choice of x.

Another choice for $\mathcal{P}_n$ and $\hat{\mathcal{P}}_n$ boils down to a coordinate transformation κ which is the composition of coordinate transformations as in propositions(2.3.1),(2.3.2) and (2.3.3), but with the restriction that, with the notation of these propositions, $v(x) > 0$ ((2.3.1) and (2.3.2)) and $v(y) > 0$ ((2.3.3)). Denoting the coordinate transformation of lemma (2.2.3.5) by ω and the analogue applied after κ by μ, one can check that

$$\mu \circ \kappa \circ \omega^{-1}$$

is the composition of coordinate transformations as in propositions (2.2.3.1),(2.2.3.2) and (2.3.3). From these propositions, our assertion follows.

So at this point, we are left to prove the axioms (IS) and (NP). We will do this by induction.

PROPOSITION(2.2.3.6). *The axiom* (NP1) *is valid in* $\mathcal{V}_n$ *for* $\mathcal{C}$.

PROOF. This follows from lemma(2.2.3.5) by the general fact that a point $\mathcal{P}$ of $\mathcal{V}$ is collinear with some point of shape (-,-,-) if and only if $v(\mathcal{P}) \neq (+,+,+)$. This is quite obvious and left to the reader. **Q.E.D.**

PROPOSITION(2.2.3.7). *The axiom* (NP2) *for* $j=1$ *is valid in* $\mathcal{V}_n$ *for* $\mathcal{C}$.

PROOF. We show this proposition in six steps.

(1) Let $\mathcal{L}_{n-1}$ be a line of $\mathcal{V}^*_{n-1}$ and Q_{n-1} an affine point of $(\mathcal{V}^*_{n-1}, \mathcal{N}_{\mathcal{C}})$ incident with $\mathcal{L}_{n-1}$. By (NP1), Q_{n-1} is also a point of $\mathcal{V}_n$. We show that for every $\hat{\mathcal{L}}_{n-1}$, the point Q_{n-1} is incident with $\Pi_{n-1}(\hat{\mathcal{L}}_{n-1})$ in $\mathcal{V}_n$. Therefore, choose $\hat{Q}_{n-1}\ I\ \mathcal{L}$ with $\mathcal{L}$ any representative of $\mathcal{L}_{n-1}$. We have $u^*(\mathcal{L},\hat{\mathcal{L}}_{n-1}) \geq n-1$. Hence it suffices to show the assertion for the distinct "*"-i-equivalences, i=1,2,3. We will only deal with the case

$\nu(\mathcal{L}) = [+,+,+]$. The case $\nu(\mathcal{L}) = [-,+,+]$ is completely similar.

First case : $\mathcal{L}[\square 1]^{*}_{n-1}\hat{\mathcal{L}}_{n-1}$.

Let $\mathcal{L} = *[k,b,k']*$ and $\hat{\mathcal{L}}_{n-1} = *[k,b,k'']*$ with $2.v^{*}(k',k'') \geq n-1$. Then $\mathcal{L} = [k,b\pi^{-1},k'\lambda^{-1}]$ and $\hat{\mathcal{L}}_{n-1} = [k,b\pi^{-1},k''\lambda^{-1}]$ with $2.v(k'\lambda^{-1},k''\lambda^{-1}) = 2.v(k',k'')+2 \geq n+1$. The assertion is trivial since $\Pi_n(\mathcal{L}) = \Pi_n(\hat{\mathcal{L}}_{n-1})$.

Second case : $\mathcal{L}[\square 2]^{*}_{n-1}\hat{\mathcal{L}}_{n-1}$.

Let $\mathcal{L} = *[k,b,k']*$ and $\hat{\mathcal{L}}_{n-1} = *[k'',b',k']*$ with $2.v^{*}(k,k'') \geq n-1$. Then $\mathcal{L} = [k,b\pi^{-1},k'\lambda^{-1}]$ and $\hat{\mathcal{L}}_{n-1} = [k'',b'\pi^{-1},k'\lambda^{-1}]$. Let $\hat{Q}_{n-1} = (a,\ldots)$ and note that $v(a) > 0$ since $u(\hat{Q}_{n-1},(0,0,0)) > 0$. We construct the unique line $M = [k'',d,\ell]$ incident with $\hat{Q}_{n-1}$ and meeting $[k'']$. Then we apply the first main property in the quadrangle $\hat{Q}_{n-1}\ I\ \mathcal{L} \perp [0,k'\lambda^{-1}]\ I\ (\hat{0})\ I\ [0,\ell] \perp M\ I\ \hat{Q}_{n-1}$ on the vertex (0) :

$$2.v(k'\lambda^{-1},\ell) = 2.v(k,k'') + 2.v(a) \geq n+1 \qquad (1)$$

Next we apply the first main property in the quadrangle $\hat{Q}_{n-1}\ I\ \mathcal{L} \perp \hat{\mathcal{L}}_{n-1}\ I\ (k'',b\pi^{-1})\ I\ [k'']\ I\ (k'',d)\ I\ M\ I\ \hat{Q}_{n-1}$ on the side $[k'']$:

$$v(b\pi^{-1},d) = 2.v(k,k'') + v(a) \geq n \qquad (2)$$

By (1),(2) and proposition(2.2.1.10), it follows that $u(M,\hat{\mathcal{L}}_{n-1}) \geq n$ and hence the result.

Third case : $\mathcal{L}[\square 3]^{*}_{n-1}\hat{\mathcal{L}}_{n-1}$.

Let $\mathcal{L} = [k,b,k']$ and $\hat{\mathcal{L}}_{n-1} = [k,b',k']$ with $v^{*}(b,b') \geq n-1$. Then $\mathcal{L} = [k,b\pi^{-1},k'\lambda^{-1}]$; $\hat{\mathcal{L}}_{n-1} = [k,b'\pi^{-1},k\lambda^{-1}]$ and $v(b\pi^{-1},b'\pi^{-1}) = v^{*}(b,b') + 1 \geq n$. So $u(\mathcal{L},\hat{\mathcal{L}}_{n-1}) \geq n$ and $Q_{n-1}\ I\ \Pi_n(\hat{\mathcal{L}}_{n-1})$ trivially.

(2) Let $\mathcal{L}_r$ be a line of $\mathcal{V}_r$ meeting $\mathcal{C}$ and let Q_r I $\mathcal{L}_r$ in $\mathcal{V}_r$ with $Q_r \in \mathcal{C}$. Choose $\hat{\mathcal{L}}_r$ I $\hat{Q}_r$ arbitrarily and let $\mathcal{M}_{r-1} = \Pi^*_{r-1}(\hat{\mathcal{L}}_r)$. By part **(1)** of the proof, every affine point of $\mathcal{M}_{r-1}$ is also incident with $\mathcal{L}_r$ and moreover, Q_r, viewed as a point of $\mathcal{V}^*_{r-1}$, is incident with $\mathcal{M}_{r-1}$. Hence every point of $\mathcal{C}$ incident with $\mathcal{L}_r$ is incident with a "component of" $\mathcal{L}_r$.

(3) Let $\mathcal{P}^1_r$, $\mathcal{P}^2_r$ I $\mathcal{L}_r$ in $\mathcal{V}_r$ and $\mathcal{P}^i_r \in \mathcal{C}$, i=1,2. Let $\mathcal{M}^i_{r-1}$ be this line of $\mathcal{V}^*_{r-1}$ incident with $\mathcal{P}^i_r$, for which all its affine points lie on $\mathcal{L}_r$ and which is constructed as in part **(2)** of this proof. Also, put $\mathcal{M}^i_{r-1} = \Pi^*_{r-1}(\mathcal{L}^i)$ with $\mathcal{L}^i$ a representative of $\mathcal{L}_r$, i=1,2. Suppose $\nu(\mathcal{L}^i) = [+,+,+]$ (the case $\nu(\mathcal{L}^i) = [-,+,+]$ is completely similar). By the construction procedure of $\mathcal{L}^2$, we can in fact choose $\mathcal{L}^2$ such that $\mathcal{L}^1$ and $\mathcal{L}^2$ meet the same line through (∞). Let $\mathcal{L}^i = [k, b_i, k'_i] = *[k, b^{\pi}_i, k'^{\lambda}_i]*$. We have : $2.v(k'_1, k'_2) \geq n$ (since $u(\mathcal{L}^1, \mathcal{L}^2) \geq n$) and by proposition(2.2.1.10), $v(b_1, b_2) \geq n$. Hence $2.v^*(k'^{\lambda}_1, k'^{\lambda}_2) \geq (n-1)-1$ and $v^*(b^{\pi}_1, b^{\pi}_2) \geq n-1$. We conclude :

(3.1) $u^*(\mathcal{M}^1_{r-1}, \mathcal{M}^2_{r-1}) \geq (n-1)-1$,

(3.2) $\mathcal{M}^i_{r-1}$ I $\Pi^*_{r-1}((k, b_1))$.

Fixing $\mathcal{P}^1_r$ and $\mathcal{M}^1_{r-1}$ and varying $\mathcal{P}^2_r$ and $\mathcal{M}^2_{r-1}$, **(3.1)** and **(3.2)** together tell us that $\mathcal{M}^2_{r-1}$ belongs to a strip $\mathfrak{S}$ of width 1 with base point $\Pi^*_{r-1}((k, b_1))$.

(4) We use the notation of **(3)**. Denote $\Pi^*_{r-1}((k, b_1))$ briefly by $\mathcal{T}$. We now show that every affine point of every line $\mathcal{M}_{r-1}$ of $\mathfrak{S}$ is incident (in $\mathcal{V}_r$) with $\mathcal{L}_r$. Therefore, choose $\hat{\mathcal{M}}_{r-1}$ I (k, b_1) (this is possible by proposition(2.2.2.1)), so $\hat{\mathcal{M}}_{r-1} = *[k, b^{\pi}_1, k'^{\lambda}]* = [k, b_1, k']$ with $2.u^*(k'^{\lambda}_1, k'^{\lambda}) \geq (n-1)-1$ ($\mathcal{M}_{r-1}$ belongs to $\mathfrak{S}$!), consequently $2.u(k'_1, k') \geq$

n, hence by definition $\Pi_n(\hat{\mathcal{M}}_{n-1}) = \mathcal{L}_n$. Now if Q_{n-1} I $\mathcal{M}_{n-1}$ and Q_{n-1} is affine, then there exists a representative $\mathcal{M}$ of $\mathcal{M}_{n-1}$ incident with any pre-assigned $\hat{Q}_{n-1}$ and meeting $\Pi^*_{n-1}((k))$ (follows also from (GQ2) by projecting onto $\mathcal{V}^*_{n-1}$ and applied on $\Pi^*_{n-1}((k))$ and Q_{n-1}). Putting $\mathcal{M}$ = $*[k,d^{\pi},\ell^{\lambda}]*$, we must have $v^*(b_1^{\pi},d^{\pi}) \geq n-1$ (by proposition(2.2.2.6) and the fact that $\mathcal{T}$ I $\mathcal{M}_{n-1}$) and so $v(b_1,d) \geq n$. Similarly as above and using proposition(2.2.1.10), $2.u(k_1',\ell) \geq n$, hence $\Pi_n(\mathcal{M}) = \mathcal{L}_n$ and so it is clear that Q_{n-1} lies on $\mathcal{L}_n$.

(5) We summarize the first four parts (with the above notation). Every point of $\mathcal{C} \cap \sigma(\mathcal{L}_n)$ is incident with at leat one component of $\mathfrak{S}$ and all components of $\mathfrak{S}$ are covered this way. But by proposition(2.2.2.12), no two components of $\mathfrak{S}$ can meet in a point Q_n of $\mathcal{C} \cap \sigma(\mathcal{L}_n)$, after all, $u^*(Q_n,\mathcal{T}) = 0$. In fact, we do not need proposition(2.2.2.12) and we could apply the induction hypothesis and properties(2.11) and (2.14) of [7].

(6) There remains to show that every strip of width 1 arises in the above way. Note first that by the induction hypothesis $\mathcal{V}^*_{n-1}$ is a level n-1 Hjelmslev quadrangle and so by property (2.37) of [7], every point at infinity of every component of any strip of width 1 is a base point. So let $\mathfrak{S}$ be a strip of width 1 in $\mathcal{C}$ and let $\mathcal{M}_{n-1} \in \mathfrak{S}$. Let $\hat{\mathcal{M}}_{n-1}$ be arbitrary and suppose $\hat{\mathcal{M}}_{n-1} = [k,b,\ldots]$ with $v(k) \geq 0$ (the case $v(k) < 0$ is similar). So $\Pi^*_{n-1}((k,b))$ is a base point of $\mathfrak{S}$ and by the previous parts of this proof, $\mathfrak{S}$ arises by intersecting $\Pi_n(\hat{\mathcal{M}}_{n-1})$ with $\mathcal{C}$. Q.E.D.

LEMMA(2.2.3.8). *Let $\mathfrak{S}$ be a strip of width j in $\mathcal{V}_n$ and suppose that at least one component (and hence all) of $\mathfrak{S}$ meets the point-neighbourhood $\mathcal{C}$*

and that at least one base point $\mathcal{P}_{r}$ of $\mathfrak{S}$ does not belong to $\mathcal{C}$. Then $\mathfrak{S}\cap\mathcal{C}$ corresponds to a strip of width $j+1$ in $\mathcal{V}^{\star}_{r-1}$.

PROOF. If $\mathcal{L}_{r}$ is a component of $\mathfrak{S}$, then $\sigma(\mathcal{L}_{r})\cap\mathcal{C}$ corresponds to a strip $\mathfrak{S}_{\mathcal{L}_{r}}$ of width 1 in $\mathcal{V}^{\star}_{r-1}$. If $\hat{\mathcal{P}}_{r}$ is arbitrary, then, by an argument in the preceding proof, $\Pi^{\star}_{r-1}(\hat{\mathcal{P}}_{r}) = \mathcal{P}_{r-1}$ is a base point of $\mathfrak{S}_{\mathcal{L}_{r}}$. Now let $\mathcal{L}'_{r}$ be a second arbitrary component of $\mathfrak{S}$. Choose $\hat{\mathcal{L}}_{r}\ I\ \hat{\mathcal{P}}_{r}\ I\ \hat{\mathcal{L}}'_{r}$ and consider $\mathcal{M}_{r-1} = \Pi^{\star}_{r-1}(\hat{\mathcal{L}}_{r})$ and $\mathcal{M}'_{r-1} = \Pi^{\star}_{r-1}(\mathcal{L}'_{r})$. We assume again $\nu(\hat{\mathcal{L}}_{r}) = [+,+,+]$ (the case $\nu(\hat{\mathcal{L}}_{r}) = [-,+,+]$ is again completely similar). Hence also $\nu^{\star}(\hat{\mathcal{L}}_{r}) = {}^{\star}[+,+,+]^{\star} = \nu^{\star}(\hat{\mathcal{L}}'_{r})$. Note that $\nu^{\star}(\hat{\mathcal{P}}_{r}) = (-,+,+)$ since $\mathcal{P}_{r}$ does not belong to $\mathcal{C}$. Putting $\hat{\mathcal{L}}_{r} = [\ldots,k]$ and $\hat{\mathcal{L}}'_{r} = [\ldots,k']$, we have $2.v(k,k') \geq n-j$ since $u(\hat{\mathcal{L}}_{r},\hat{\mathcal{L}}'_{r}) \geq n-j$. Hence $2.v^{\star}(k^{\lambda},k'^{\lambda}) \geq n-j-2 = (n-1)-(j+1)$ and $\mathcal{M}'_{r-1}$ belongs to the strip $\mathfrak{S}^{\star}$ of width $j+1$ with base point $\Pi^{\star}_{r-1}(\hat{\mathcal{P}}_{r}) = \mathcal{P}_{r-1}$ containing $\mathcal{M}_{r-1}$.

Now let $\mathcal{L}'_{r-1}$ be any component of $\mathfrak{S}^{\star}$. Choose $\hat{\mathcal{L}}'_{r-1}\ I\ \hat{\mathcal{P}}_{r}$ and let $\hat{\mathcal{L}}_{r} = {}^{\star}[k,b^{\pi},k'^{\lambda}]^{\star}$, $\hat{\mathcal{L}}'_{r-1} = {}^{\star}[\ell,c^{\pi},\ell'^{\lambda}]^{\star}$ and $\hat{\mathcal{P}}_{r} = {}^{\star}(a^{-},\ldots)^{\star}$. We apply the first main property in the quadrangle ${}^{\star}0,0)^{\star}\ I\ {}^{\star}[0,k'^{\lambda}]^{\star} \perp \hat{\mathcal{L}}_{r}\ I\ \hat{\mathcal{P}}_{r}\ I\ \hat{\mathcal{L}}'_{r-1} \perp {}^{\star}[0,\ell'^{\lambda}]^{\star}\ I\ {}^{\star}(0,0)^{\star}$ on the vertex ${}^{\star}(0,0)^{\star}$:

$$2.v^{\star}(k,\ell) = 2.v^{\star}(k'^{\lambda},\ell'^{\lambda}) - 2.v^{\star}(a)$$

$$\geq (n-1)-(j+1) - 2.(-1) = n-j \qquad (1)$$

We also have :

$$2.v(k',\ell') = 2.v^{\star}(k'^{\lambda},\ell'^{\lambda}) + 2$$

$$\geq (n-1)-(j+1) + 2 = n-j \qquad (2)$$

Since $\hat{\mathcal{L}}_{r}$ and $\hat{\mathcal{L}}'_{r-1}$ meet, $u(\hat{\mathcal{L}}_{r},\hat{\mathcal{L}}'_{r-1})$ equals either $2.v(k,\ell)$ or $2.v(k',\ell')$. But by (1) and (2), both numbers are not smaller than $n-j$, hence $\mathcal{L}'_{r} =$

$\hat{\Pi}_n(\mathcal{L}'_{n-1})$ belongs to $\mathfrak{S}$. But by the preceding proposition, $\mathcal{L}'_{n-1}$ is a component of $\mathfrak{S}_{\mathcal{L}'_n}$.

So we have showed that the union of all $\mathfrak{S}_{\mathcal{L}_n}$ for $\mathcal{L}_n$ varying over the set of all components of $\mathfrak{S}$ is a strip of width $j+1$. Now, there are no further lines of $\mathcal{V}^*_{n-1}$ for which the affine shadow is part of $\mathfrak{S}\cap\mathcal{C}$ (recapture the argument of [8],§4.3.2(3)). The result follows. Q.E.D.

PROPOSITION(2.2.3.9). *Let $\mathcal{C}_{n-j}$ be an $(n-j)$-point-neighbourhood in $\mathcal{V}_n$ only containing points with representatives of shape* (+,+,+), $1\leq j\leq n-1$. *Then the axiom* (NP2) *voor $\mathcal{C}_{n-j}$ is valid.*

PROOF. By proposition(2.2.3.7), (NP2) holds for $j=1$ and the unique $(n-1)$-point-neighbourhood $\mathcal{C}_{n-1}$ containing $\mathcal{C}_{n-j}$. Let $\mathcal{C}_{n-i}$ denote the unique $(n-i)$-point-neighbourhood containing $\mathcal{C}_{n-j}$, $1\leq i\leq j$. Let $\mathcal{L}_n$ be any line meeting $\mathcal{C}_{n-j}$. Then $\mathcal{L}_n$ determines in $\mathcal{W}_{n-1}(\mathcal{V}_n,\mathcal{C}_{n-1})$ a strip of width 1. The latter defines on his turn in $\mathcal{W}_{n-2}(\mathcal{V}_n,\mathcal{C}_{n-2})$ a strip of width 2 by lemma(2.2.3.8). But the latter defines in $\mathcal{W}_{n-3}(\mathcal{V}_n,\mathcal{C}_{n-3})$ a strip of width 3, etc... Eventually, $\mathcal{L}_n$ will determine in $\mathcal{W}_{n-j}(\mathcal{V}_n,\mathcal{C}_{n-j})$ a strip of width j as required in (NP2). Q.E.D.

One can now show similar results for point-neighbourhoods containing points having representatives of shape (-,+,+), (+,-,-) or (-,-,-). The skeleton of the proofs of these cases is contained in [20] (e.g. the explicit form of the coordinate transformations to axis points). Anyway, even without this information, the interested reader can reconstruct those proofs by "translating" the case (+,+,+) above. Hence the next theorem :

THEOREM(2.2.3.10). *The axiom* (NP2) *is valid in* $\mathcal{V}_n$.

By the induction hypothesis, the definition of $\mathcal{W}_{n-1}(\mathcal{V}_n,\mathcal{C})$ implies :

THEOREM(2.2.3.11). *The axiom* (IS) *is valid in* $\mathcal{V}_n$.

Hence we showed that, par abus de langage, $\mathcal{V}_n$ is a level n Hjelmslev quadrangle and $\mathcal{H} = (\mathcal{V}_n, \Pi_n^{n+1})_{n\in\mathbf{N}}$ is an HQ-Artmann-sequence.

REMARK(2.2.3.12). Now one can see where the structure of the tables 1 and 2 comes from : $\mathcal{V}$ is projected onto a generalized quadrangle $\mathcal{V}_1$ and if the shape of a variety X has i plus-signs, then X is mapped onto a variety with i coordinates (coordinates w.r.t. the projected quadrangle), i=0,1, 2,3. This completes remark(2.1.3.14).

Denote by $\mathcal{V}_\infty$ the inverse limit of the base sequence of $\mathcal{H} = (\mathcal{V}_n, \Pi_n^{n+1})_{n\in\mathbf{N}}$

THEOREM(2.2.3.13). $\mathcal{V}$ *is isomorphic to* $\mathcal{V}_\infty$, *provided* $(\mathcal{R}_1, \mathcal{R}_2, Q_1, Q_2, v)$ *is complete.*

PROOF. We define a map

$$\Psi : \mathcal{P}(\mathcal{V}) \to \mathcal{P}(\mathcal{V}_\infty) \;:\; \mathcal{P} \to (\Pi_n(\mathcal{P}))_{n\in\mathbf{N}}$$
$$\mathcal{L}(\mathcal{V}) \to \mathcal{L}(\mathcal{V}_\infty) \;:\; \mathcal{L} \to (\Pi_n(\mathcal{L}))_{n\in\mathbf{N}}.$$

Then Ψ is certainly injective because distinct points (or lines) of $\mathcal{V}$

have a distinct image under Π_n for n bigger than their mutual partial valuation. If $\mathcal{P}\ I\ \mathcal{L}$ in $\mathcal{V}$ then by definition $\Pi_n(\mathcal{P})\ I\ \Pi_n(\mathcal{L})$ for all n, hence Ψ preserves incidence. Now let $(\mathcal{P}_n)_{n\in\mathbf{N}}$ be a point of $\mathcal{V}_\infty$. Let $\hat{\mathcal{P}}_n$ be arbitrary (for every n). Suppose e.g. (the other cases are copies) $\nu(\hat{\mathcal{P}}_1) = (-,+,+)$. Then $\nu(\hat{\mathcal{P}}_n) = (-,+,+)$ and if we set $\hat{\mathcal{P}}_n = ((a_n, \ell_n, a'_n))$, then $w(a_i, a_j) \geq \inf\{i,j\}$, $v(a'_i, a'_j) \geq \inf\{i,j\}$ and $2.v(\ell_i, \ell_j) \geq \inf\{i,j\}$. The latter follows from [7],property(2.11). Hence $(a'_n)_{n\in\mathbf{N}}$ and $(\ell_n)_{n\in\mathbf{N}}$ are Cauchy-sequences with a respective limit a' and ℓ. If for certain n, $v(a_n) > -n$, then for all $k > n$, $n \leq w(a_k, a_n) = v(a_k, a_n) - v(a_k) - v(a_n) <$ $v(a_k, a_n) - v(a_k) + n$, so $v(a_k) < v(a_k, a_n)$ and hence $v(a_k) = v(a_n)$. Consequently $(a_n)_{n\in\mathbf{N}}$ is a Cauchy-sequence with a limit a and $w(a, a_k) \geq k$ for all k (because $v(a) = v(a_n)$). If for all n, $v(a_n) \leq -n$, then $w(\infty, a_n)$ $\geq n$ by definition and we put $a=\infty$. From these arguments follows that $((a, \ell, a')) = \mathcal{P}$ is a point such that $\Pi_n(\mathcal{P}) = \mathcal{P}_n$. Hence $\Psi(\mathcal{P}) = (\mathcal{P}_n)_{n\in\mathbf{N}}$. Similarly for lines. Hence Ψ is surjective. Suppose now $\mathcal{P}$ is not incident with $\mathcal{L}$ in $\mathcal{V}$ and let $i = u_2(\mathcal{P}, \mathcal{L})$. Then $i \in \mathbf{N}$. If $\Pi_{i+1}(\mathcal{P})\ I$ $\Pi_{i+1}(\mathcal{L})$, then there exists a representative $\mathcal{L}'$ of $\Pi_{i+1}(\mathcal{L})$ such that $\mathcal{P}\ I$ $\mathcal{L}'$. But then $u_2(\mathcal{P}, \mathcal{L}) \geq i+1$, a contradiction. Hence Ψ^{-1} also preserves incidence and Ψ is an isomorphism of generalized quadrangles. **Q.E.D.**

In view of [8],corollary(5.3) ; the fact that, if $(\mathcal{R}_1, \mathcal{R}_2, Q_1, Q_2, v)$ is not (necessarily) complete, then the map Ψ in the above proof is a monomorphism and the comment in §1.3, we have proved the main theorem (see §1.3).

In the next section, we will construct examples of V-QQRs. To obtain non-classical buildings of type $\tilde{C}_2$, it suffices to check that the given V-QQR coordinatizes a non-classical generalized quadrangle.

3. EXAMPLES EN REMARKS.

3.1. Examples.

3.1.1. Example 1.

Let $\mathcal{R}_1 = \mathcal{R}_2 = GF(q)((t))$ with $q = 2^h$, $h > 1$. Let h_1 and h_2 be positive integers such that q-1 and $-1+2^{1+h_1+h_2}$ are relatively prime (e.g. h=3, h_1=1, h_2=0 ; h=4, h_1=1, h_2=1 etc...) with $(h_1,h_2) \neq (0,0)$. We first define a finite QQR $(GF(q),GF(q),Q_1,Q_2)$ as follows.

Put $\theta_i = 2^{h_i}$ and define

$$Q_1(k,a,\ell,a') = k^{2\theta_1}.a + a',$$
$$Q_2(a,k,b,k') = a^{\theta_2}.k + k'.$$

One can easily check that this defines a QQR which is moreover doubly regular. The reason why this works is essentially the fact that $2^{1+h_1+h_2}-1 = 2\theta_1\theta_2-1$ is a permutation of $GF(q)$. We now extend θ_i to $\mathcal{R}_i$ by putting

$$(\Sigma x_r t^r)^{\theta_i} = \Sigma x_r^{\theta_i} t^r \ , \quad i=1,2.$$

Note that θ_i is a fieldautomorphism of $\mathcal{R}_i$. We define two new multiplications as follows.

$$x \otimes_1 y = x^{2\theta_1}.y$$
$$x \otimes_2 y = x^{\theta_2}.y$$

for all $x,y \in GF(q)((t))$. Let v be the standard valuation on $GF(q)((t))$, then we prove that $(\mathcal{R}_1, \mathcal{R}_2, +, +, \otimes_1, \otimes_2)$ gives rise to a CV-QQR. Besides trivial and obvious observations, this amounts to show that the following system of equations with unknowns x and y has a unique solution, for every $a \neq 0$ and every $b \neq 0$.

$$x \otimes_1 y = x^{2\theta_1} . y = a \tag{1}$$

$$y \otimes_2 x = y^{\theta_2} . x = b \tag{2}$$

Well, we apply θ_2 to (1) and eliminate y^{θ_2} in $(1)^{\theta_2}$ and (2) :

$$x^{2\theta_1\theta_2} = x . c \tag{3}$$

with $c = a^{\theta_1}/b$. Put $\theta_1\theta_2 = \theta$. Since $v(x) = v(x^{\theta})$, $v(x) = v(c)$. The coefficients of $t^{v(c)}$ in both sides of (3) have to be equal, that means if $c = \sum c_r t^r$ and $x = \sum x_r t^r$, then $x_{v(c)}^{2\theta-1} = c_{v(c)}$ and hence $x_{v(c)}$ follows since $2\theta-1$ is a permutation of GF(q). The other coefficients of x follow from simple linear equations of the form $c_{v(c)} . x_r =$ $f(x_{v(c)}, \ldots, x_{r-1}, c_{v(c)}, \ldots, c_r)$ and this has a unique solution.

In order to show that v is a valuation on the given QQR, it suffices to show, besides trivial and easy verifications, that for all k and all ℓ :

$$v(\ell^{2\theta} . k - k^{2\theta} . \ell) = v(k) + v(\ell) + v(k, \ell) \tag{4}$$

Suppose first that $v(k) + 2.v(\ell) \neq v(\ell) + 2.v(k)$. Then $v(k) \neq v(\ell)$ and the result follows from the triangle inequality. So we may assume that $v(k) + 2.v(\ell) = v(\ell) + 2.v(k)$, so $v(k) = v(\ell) = n$. There are two cases.

First case *:* $v(k,\ell)=n$.

In this case $k=k_r t^r$, $\ell=\ell_r t^r$ and $k_r\neq\ell_r$. The term in t^{3n} of $\ell^{2\theta}k-k^{2\theta}\ell$ becomes here $\ell_r^{2\theta}k_r-k_r^{2\theta}\ell_r$. If this were zero, then $\ell_r^{2\theta-1}=k^{2\theta-1}$, and since $2\theta-1$ is a permutation, this would imply $\ell_r=k_r$, a contradiction. Hence $v(\ell^{2\theta}k-k^{2\theta}\ell)=3n=v(k)+v(\ell)+v(k,\ell)$.

Second case *:* $v(k,\ell)>n$.

In this case, we put $v(k,\ell)=n+j$ and

$$k=k_r t^r+\ldots+k_{r+j-1}t^{r+j-1}+k_{r+j}t^{r+1}+\ldots$$
$$\ell=k_r t^r+\ldots+k_{r+j-1}t^{r+j-1}+\ell_{r+j}t^{r+1}+\ldots$$

By simple calculation, one sees that the terms in t^i for $3n\leq i<3n+j$ in $\ell^{2\theta}k-k^{2\theta}\ell$ disappear. A similar calculation shows us that the term in t^{3n+j} in $\ell^{2\theta}k-k^{2\theta}\ell$ equals $k_{r+j}k_r^{2\theta}-\ell_{r+j}k_r^{2\theta}\neq 0$. Hence the result.

Hence the above mentioned QQR is a CV-QQR (complete since $GF(q)((t))$ is complete and the completeness only depends on the valuation and not on the operations). It defines a non-classical $\widetilde{C}_2$-building Δ (since the generalized quadrangle coordinatized by $(\mathcal{R}_1,\mathcal{R}_2,+,+,\otimes_1,\otimes_2)$ is not classical). We can see that as follows. The line $[\infty]$ is regular. But we now show that the line $[0,0,0]$ is not regular. Consider an arbitrary element $y\in GF(q)((t))$. One can check that $[0,0,0]$ and $[t,t,0]$ are concurrent with $[0,0,1]$, $[1,1,1]$ and $[0,0]$. But the line $[y^{\theta^2}t,y^{2\theta-1}t,0]$ meets $[0,0]$ and $[0,0,1]$, and it meets $[1,1,1]$ $\Longleftrightarrow y^{\theta}=y$, but we can choose y and θ such that $y^{\theta}\neq y$, hence the generalized quadrangle is not Moufang and hence it is not classical. One residue in Δ is certainly the generalized quadrangle coordinatized by the QQR $(GF(q),GF(q),Q_1,Q_2)$, as above. It is a non-classical residue (namely of type $T_2(O)$ (cp. [13])) (another reason why Δ is non-classical).

Note that this example provides the first explicitly defined affine buildings of type $\tilde{C}_2$ which are non-classical and have non-classical finite residues.

3.1.2. Example 2.

Let $\mathcal{R}_1 = \mathcal{R}_2 = GF(q)((t))$ with $q = 2^h$, $h > 1$. We define $(\mathcal{R}_1, \mathcal{R}_2, Q_1, Q_2)$ as follows.

$$Q_1(k,a,\ell,a') = k^2 a + a'$$
$$Q_2(a,k,b,k') = a^\theta k + k'$$

where $(\sum x_i t^i)^\theta = \sum x_i (\frac{t}{1+t})^i$. Again, θ is a fieldautomorphism and similar as in example 1, one shows that $(\mathcal{R}_1, \mathcal{R}_2, Q_1, Q_2, v)$, with v as above, is a CV-QQR. This time, we only know that at least one residue in the corresponding affine building Δ is isomorphic to the symplectic generalized quadrangle $W(q)$. But similar as above, one can show that Δ is non-classical.

3.1.3. Example 3.

Let $\mathbf{F}$ be a local field, v the valuation and suppose that $\mathbf{F}$ is complete w.r.t. v. Let θ be any element of $\mathbf{F}$ with valuation 1. Let ψ be any fieldautomorphism preserving the valuation. We put $\mathcal{R}_1 = \mathbf{F} \times \mathbf{F}$, $\mathcal{R}_2 = \mathbf{F}$ and (with $x = (x_1, x_2) \in \mathcal{R}_1$) :

$$Q_1(k,a,\ell,a') = (a_1 k + a_1' \ , \ a_2 k^\psi + a_2')$$
$$Q_2(a,k,b,k') = a_1^2 k + \theta a_2^2 k^\psi + k' - 2a_1 b_1 - 2\theta a_2 b_2$$

One can check that this defines a dually regular QQR. The proof amounts to solve the equation $a_1^2 p + \theta a_2^2 p^{\psi} = \ell$ for p. But by the completeness of **F**, this can always be done (similar in fact to [17],§4.6.3).

We now extend v to $\mathcal{R}_1$ by defining :

$$v(a) = v(a_1^2 + \theta a_2^2).$$

In view of the observations that $v(k) = v(k^{\psi})$ and $inf\{2v(a_1), 2v(a_2)+1\} = v(a)$, one can easily check that this provides a CV-QQR.

One of the residues of the corresponding affine building for the case $\mathbf{F} = \mathrm{GF}(q)((t))$ is the symplectic generalized quadrangle $W(q)$, q any prime power.

If q is even, then the above QQR is doubly regular and not isomorphic to one of the previous examples.

Still in the case $\mathbf{F} = \mathrm{GF}(q)((t))$, one can omit the condition $v(\theta) = 1$ for suitable ψ and suitable θ itself. Some (if not all) residues of the corresponding affine building will then be isomorphic to Kantors' "bad eggs" (cp. [13]).

3.1.4. Example 4.

Let $\mathcal{K}$ be any field of characteristic 2 and set $\mathbf{F} = \mathcal{K}((t))$. Let θ_i, i=1, 2,3, be any element of $\mathbf{F}((t))$ satisfying $v(\theta_i) = 1$ and $v(\theta_i + t) > 1$ (v is again the natural valuation on $\mathbf{F}((t))$). Define the automorphism

$$\psi_i : \mathbf{F} \to \mathbf{F} : (\sum x_r t^r)^{\psi_i} = \sum x_r \theta_i^r , \quad i=1,2,3.$$

Next, we define three new multiplications. For every $a, b \in \mathbf{F}^2$, we set

$$a \otimes_i b = (a_1, a_2) \otimes_i (b_1, b_2) = (a_1 . b_1 + t . a_2^{\psi_i} . b_2^{\psi_i} \;,\; a_1 . b_2 + a_2 . b_1).$$

Finally, we define the quaternary operations :

$$Q_1 (k, a, \ell, a') = k \otimes_1 a + a',$$

$$Q_2 (a, k, b, k') = (a \otimes_2 a) \otimes_3 k + k'.$$

One can show (although it is not a trivial proof !) that this defines a CV-QQR. Moreover, we have here a new infinite generalized quadrangle $\mathcal{V}$ with the following special property. As in the finite case, one can define the geometry related to the regular point (∞) (in the standard way : points are the points collinear with (∞) ; lines are the lines through (∞) and the "hyperbolic" lines related to (∞), see [13]). In our case, this geometry is a projective plane coordinatized by the planar ternary ring $T(k, a, a') = Q_1 (k, a, 0, a') = k \otimes_1 a + a'$ (see [5]), and if $\theta_1 \neq t$, then this projective plane is a proper division ring plane, hence not classical. It would be very interesting to have such analogue for the finite case.

3.2. Miscellaneous remarks.

3.2.1. Remark 1.

The second and third main property in section 2 show that the generalized quadrangle $\mathcal{V}$ is a *generalized quadrangle with valuation* in the sense of [19].

3.2.2. Remark 2.

No set of axioms in this paper is chosen such that there is no overlapping between the axioms. We have not tried to find the most beautiful, or the smallest possible set of axioms, but we think we have written down sets of axioms which are easiest to check in the examples and which do not make the theory heavier. For example, surjectivity of the map v is requiring to much : only $1 \in \mathrm{Im}(v)$ would be enough, but we consider this fact as a minor detail.

3.2.3. Remark 3.

In his work on Klingenberg-geometries, D.Keppens [12] defined amongst other things, Klingenberg-quadrangles as a generalization of generalized quadrangles. He also gives a description of the classical cases. But our definition and his definition do not suite in the sense of "a Hjelmslev-structure is in general a special case of the analoguous Klingenberg-structure" (cp. projective and affine Hjelmslev- and Klingenberg-planes). But here is the connection. Let $\mathcal{V}_{2n}$ be a level $2n$ Hjelmslev-quadrangle with canonical n-image $\mathcal{V}_n$. Define a new set of lines in $\mathcal{V}_n$ as the set of pointwise projections of the lines in $\mathcal{V}_{2n}$, then we obtain a Klingenberg-quadrangle in the sense of Keppens [12]. The n^{th} floor of a classical building of type $\tilde{C}_2$ gives rise to a classical Klingenberg quadrangle as described in [12].

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