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Finite generalized quadrangles as the union of few large subquadrangles

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Abstract

We study the question: what is the smallest number n of subquadrangles of order (s,t') of a finite generalized quadrangle Γ of order (s,t) such that the union of the point sets of all these subquadrangles is equal to the point set of Γ? It turns out that $n \geqslant s+1$ and if $n=s+1$, then except for a finite list of small examples, either all the subquadrangles are disjoint, or $\sqrt{t}=s=t'$ and all the subquadrangles meet pairwise in a common subquadrangle of order $(s,1)$. Examples exist in both cases and they show that a further classification is out of reach. A similar result holds for finite polar spaces.

Keywords: Finite generalized quadrangles; Subquadrangles; Polar spaces; Polar subspaces

1. Introduction, notation and statement of the results

A *finite generalized quadrangle of order* (s,t), $s,t \geqslant 1$, is a point-line geometry $\Gamma=(\mathscr{P},\mathscr{L},\mathbf{I})$ (where we treat the incidence relation $\mathbf{I}$ as a symmetric relation) satisfying the following axioms:

(GQ1) each point is incident with $1+t$ lines and two distinct points are incident with at most one line;

(GQ2) each line is incident with $1+s$ points and two distinct lines are incident with at most one point;

(GQ3) if x is a point and L is a line not incident with x, then there is a unique pair $(y,M) \in \mathscr{P} \times \mathscr{L}$ for which $x\mathbf{I}M\mathbf{I}y\mathbf{I}L$.

Generalized quadrangles were introduced by Tits [8]. The above definition is taken from Payne and Thas [4].

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A *subquadrangle* $\Gamma' = (\mathscr{P}', \mathscr{L}', \mathbf{I}')$ of a given generalized quadrangle $\Gamma = (\mathscr{P}, \mathscr{L}, \mathbf{I})$ is a generalized quadrangle for which $\mathscr{P}' \subseteq \mathscr{P}$, $\mathscr{L}' \subseteq \mathscr{L}$ and $\mathbf{I}'$ is the restriction of $\mathbf{I}$ to $(\mathscr{P}' \times \mathscr{L}') \cup (\mathscr{L}' \times \mathscr{P}')$. Let us define a *large subquadrangle* of a generalized quadrangle of order (s,t) as a subquadrangle of order (s,t') with $t' < t$, i.e., they are 'large' with respect to the point set (a large subquadrangle in this sense is often called a *full* subquadrangle). Natural questions are

(1) whether a given generalized quadrangle has a (large) subquadrangle;

(2) are there restrictions on the orders of a quadrangle and a (large) subquadrangle;

(3) how many large subquadrangles do we need to cover a generalized quadrangle?

Considerable attention is always given to the first question when a new class of quadrangles is discovered. The second question has been solved by Thas [5] and the answer is as follows, see also Payne and Thas [4].

Theorem 1 (Thas [5]). *Let Γ be a generalized quadrangle of order (s,t). If Γ contains a large subquadrangle Γ' of order (s,t'), then $t \geqslant st'$. If $t' > 1$, then $t \geqslant \sqrt{s^3}$. If $t = st'$, then every line of Γ not in Γ' is incident with a unique point of Γ'. If Γ' contains a large subquadrangle of order (s,t''), then $t'' = 1$, $t' = s$ and $t = s^2$.*

In the present paper, we give a fairly general answer to the third question. For short, we say that a generalized quadrangle *is the union of n large subquadrangles* if its point set is the union of the point sets of n large subquadrangles. Our main result is:

Theorem 2. *Let Γ be a generalized quadrangle of order (s,t) with $s,t > 1$. Then Γ cannot be the union of fewer than $s+1$ large subquadrangles. Also, if Γ is the union of $s+1$ subquadrangles, then, if $s > 2$, these subquadrangles all have the same order (s,t'), and one of the following holds (denoting by $\mathscr{S}$ the set of $s+1$ large subquadrangles):*

(i) *the point set of Γ is the disjoint union of the points sets of the members of $\mathscr{S}$, and $t' = (t-1)/(s+1)$;*

(ii) *there exists a large subquadrangle Γ^* of order $(s,1)$ such that every two members of $\mathscr{S}$ meet precisely in Γ^*. Every member of $\mathscr{S}$ has order (s,s), and $t = s^2$;*

(iii) $(t',s,t) = (2,4,8)$, *every two members of $\mathscr{S}$ meet in the nine points of an ovoid in both members, there are exactly* 30 *points of Γ which lie in at least two members of $\mathscr{S}$ and every such point lies in exactly* 3 *members, every member contains exactly* 18 *points which lie in three members of $\mathscr{S}$ and no line is contained in at least two members of $\mathscr{S}$;*

(iv) $(t',s,t) = (1,3,3)$ *and there are exactly two non-isomorphic examples, one with no line of Γ in at least two members of Γ, and the other with two unique concurrent lines contained in* 3 *members of Γ.*

(v) $(t',s,t) = (10,15,160)$ *and there exists a line L of Γ such that every two members of $\mathscr{S}$ meet precisely in L.*

There are plenty of examples for the first two cases. In fact, for case (i), every known generalized quadrangle Γ of order $(s,s+2)$ has at least $s+2$ different sets of

$s+1$ large subquadrangles of order $(s,1)$ whose union is Γ. Indeed, every known such quadrangle arises from a quadrangle Γ' of order $(s+1,s+1)$ by deleting a regular point p, all points collinear with p and all lines through p, and adding as new lines all *traces* containing p (a *trace* is the set of points collinear with two given non-collinear points). The set of points of Γ collinear in Γ' with a given point x of $\Gamma'\setminus\Gamma$, $x \neq p$, is easily seen to be the point set of a large subquadrangle of Γ. Varying x over some fixed line L of Γ' through p, we obtain a partition of the point set of Γ into large subquadrangles. Varying L, we obtain $s+2$ such partitions.

For case (ii), it is enough to have a regular line for which the corresponding dual net satisfies the axiom of Veblen, see Thas and Van Maldeghem [7]. Examples include the classical quadrangles $Q(5,q)$, the Tits quadrangles $T_3(O)$ (for O an ovoid in three-dimensional projective space), the generalized quadrangles discovered by Kantor [2], and the dual of the Roman generalized quadrangles discovered by Payne [3].

Concerning case (iii), an example exists which is the smallest case of a covering of $H(4,q^2)$ by a set of $2q^2-2q+1$ large subquadrangles isomorphic to $H(3,q^2)$. It is not known whether or not case (v) occurs.

Applied to the classical quadrangles $Q(5,q)$ and generalized to finite polar spaces of arbitrary (finite) rank, we obtain (with similar definitions for polar spaces as for quadrangles above):

Theorem 3. *Let Γ be a finite polar space of rank r naturally embedded in* $\mathrm{PG}(d,q)$. *Suppose that Γ is the union of $k \leqslant q+1$ large polar subspaces of rank r, and that $q>2$ if $r=2$. Then $k=q+1$ and either $r=2$ and one of the cases* (iii) *or* (iv) *of Theorem* 2 *holds* (*where for case* (iii) *the quadrangle Γ is isomorphic to $H(4,4)$*), *or Γ is an elliptic quadric and there exist $q+1$ hyperplanes of* $\mathrm{PG}(d,q)$ *containing a $(d-2)$-dimensional space U such that each hyperplane meets Γ precisely in a large polar subspace* (*which is a parabolic quadric*). *Also, U meets Γ in a large polar subspace of rank r* (*which is a hyperbolic quadric*).

Hence one can see that the fact that makes it possible to write an elliptic quadric in d-dimensional projective space as the union of $(q+1)$ subquadrics is strongly related to the fact that there exist (hyperbolic) quadrics of the same rank in $(d-2)$-dimensional projective space.

Let us mention here that Peter Johnson (unpublished) proves related results, allowing also infinite polar spaces of possibly infinite rank.

Finally, we mention a corollary, which gives a characterization of the quadrangles of Kantor mentioned above. For the definition of flock quadrangle, we refer to e.g. Thas [6].

Corollary. *Let Γ be a flock quadrangle of order (q^2,q), q odd, with elation point (∞). Then Γ is isomorphic to the flock quadrangle of Kantor, or to the classical quadrangle $H(3,q^2)$ if and only if the dual of Γ is the union of $q+1$ large subquadrangles all containing (∞).*

2. Proof of Theorem 2

Let Γ be a finite generalized quadrangle of order (s,t), $s,t \geqslant 2$. Suppose that $\mathscr{S}$ is a set of n large subquadrangles whose union is Γ.

Lemma 4. *We have* $n \geqslant s+1$.

Proof. Suppose by way of contradiction that $n \leqslant s$. Let L be any line of Γ. Since there are $s+1$ points incident with L, there must be at least two points of the same member of $\mathscr{S}$ on L; hence, L belongs to at least one member of $\mathscr{S}$. So we have the inequality

$$s(1+t')(1+st') \geqslant (1+t)(1+st).$$

Since $t \geqslant st'$, this implies $s+t \geqslant 1+st \geqslant 1+2t$, hence $s>t$, in contradiction with $t \geqslant st' \geqslant s$. □

From now on we assume that $n=s+1$.

Lemma 5. *If a point of* Γ *is contained in at least two members of* $\mathscr{S}$, *then every line of* Γ *incident with* x *is a line of some member of* $\mathscr{S}$.

Proof. Let $\mathscr{S}' \subseteq \mathscr{S}$ be defined such that x is contained in every member of $\mathscr{S}'$ and in no member of $\mathscr{S} \setminus \mathscr{S}'$ and suppose that $\mathscr{S}'$ has cardinality $\ell > 1$. Let M be a line through x not belonging to one of the members of $\mathscr{S}$. Then the $s+1-\ell$ elements of $\mathscr{S} \setminus \mathscr{S}'$ have to cover the s points on M distinct from x. This is only possible if at least one member covers at least two points, hence M is contained in some member Γ_M of $\mathscr{S} \setminus \mathscr{S}'$. □

The following lemma is crucial.

Lemma 6. *If every point of some line* L *of* Γ *is contained in at least two members of* $\mathscr{S}$, *then either* $s=2$, *or* L *is contained in at least* s *members of* $\mathscr{S}$.

Proof. Suppose that the line L of Γ is contained in $\ell \geqslant 1$ members of $\mathscr{S}$, which we gather in the set $\mathscr{S}' \subseteq \mathscr{S}$ (note that indeed $\ell \geqslant 1$ by the previous lemma). Let x be any point on L. There are at most $\ell t'$ lines through x distinct from L and belonging to one of the members of $\mathscr{S}'$, where

$$t' = \max\{t^* \mid \text{some member of } \mathscr{S} \text{ has order} (s,t^*)\}.$$

Let M be a line through x not belonging to one of the members of $\mathscr{S}'$. Then by Lemma 5 M is contained in some member Γ_M of $\mathscr{S} \setminus \mathscr{S}'$. Suppose some other line M' concurrent with L is also contained in Γ_M. If M' is not incident with x, then this implies that L is in Γ_M, a contradiction to our assumptions. Therefore, M' is incident with x. Since there are at least $t-\ell t'$ lines through x not contained in any member of $\mathscr{S}'$, there are at least $(t-\ell t')/(t'+1)$ members of $\mathscr{S}\backslash\mathscr{S}'$ containing x. Varying x, this gives us a total of at least

$$\ell+\frac{(t-\ell t')(s+1)}{t'+1}$$

elements of $\mathscr{S}$. Expressing that this is at most equal to $s+1$, we obtain after a short calculation

$$l\geqslant\frac{(t-t'-1)(s+1)}{st'-1},$$

which, using $t\geqslant st'$, simplifies to

$$\ell\geqslant s-\frac{t'+1}{st'-1}.$$

Noting that $t'+1\geqslant st'-1$ (which is equivalent with $(s-1)t'\leqslant 2$) if and only if $s=2$ or 3, we are done if $s>3$. Suppose now that $s=3$ and $\ell<3$. Then $t'+1\geqslant 3t'-1$, hence $t'=1$ and $\ell=2$. Consequently, equality holds in the above expressions, implying first that $t=st'=4$, and second that each x is contained in exactly $(t-\ell t')/(t'+1)=\frac{1}{2}$ members of $\mathscr{S}\backslash\mathscr{S}'$, a contradiction. □

We now treat some special cases.

Lemma 7. *If all members of* $\mathscr{S}$ *have order* $(s,1)$, *then either* $s=t=2$, *or* $s=t=3$, *or* $t=s+2$ *and* $\mathscr{S}$ *forms a partition of the point set of* Γ.

Proof. Since all points of Γ must be covered, we have

$$(s+1)^3\geqslant(s+1)(1+st).$$

This implies $2s+s^2\geqslant st$, hence $t\leqslant s+2$. By the divisibility condition $s+t\,|\,(1+st)st$ (see Payne and Thas [4, 1.2.2]), $t\neq s+1$. Hence $t=s+2$ or $t=s$. If $t=s+2$, then the assertion follows from the equality $(s+1)^3=(s+1)(1+st)$. So we may suppose that $s=t$. Note that every line of Γ meets every member of $\mathscr{S}$ in exactly one point if it is not contained in it (this follows from Theorem 1).

(i) First suppose that some line L of Γ is contained in $\ell>1$ members of $\mathscr{S}$. Since this implies that all points of L are contained in at least two members of $\mathscr{S}$, we conclude with Lemma 6 that L is contained in at least s members of $\mathscr{S}$. Let $\mathscr{S}'$ be the set of elements of $\mathscr{S}$ containing L.

Assume first that $s\geqslant 4$ and $\ell=s$. If every line of Γ concurrent with L is contained in a member of $\mathscr{S}'$, then every point is in a member of $\mathscr{S}'$, contradicting Lemma 4. So there exists a line N meeting L not contained in a member of $\mathscr{S}'$. But that means that two members of $\mathscr{S}'$ share a line $L'\neq L$ incident with the meeting point y of L and N. Again, L' is contained in at least s members of $\mathscr{S}$. Suppose first that it is contained in precisely s members, which we gather in $\mathscr{S}''$. Then clearly $|\mathscr{S}'\cap\mathscr{S}''|$ is either s or $s-1$. In the first case, the $s-1$ lines through y distinct from L and L' must lie in the unique element of $\mathscr{S}\backslash\mathscr{S}'$; in the second case these $s-1$ lines must lie together with L and L' in one of the members of $\mathscr{S}\setminus(\mathscr{S}'\cap\mathscr{S}'')$. In either case we deduce $s-1\leqslant 2$, or $s=3$, a contradiction. Now, suppose that L' is contained in all members of $\mathscr{S}$, then we interchange the roles of L and L' in the next paragraph.

So assume now that $s\geqslant 4$ and $\ell=s+1$. Let x be any point on L. Then some line $K\neq L$ through x is also contained in at least s members of $\mathscr{S}$. At most one member remains to cover the points of Γ collinear with x and not incident with L or K, and that member also contains L. Hence $s=2$.

(ii) Now, suppose that no line of Γ lies in two distinct members of $\mathscr{S}$. It follows readily from Lemma 5 that any point x which is contained in at least two members of $\mathscr{S}$, lies in exactly $(s+1)/2$ members of $\mathscr{S}$.

Now, consider any line M contained in some member of $\mathscr{S}$, say, Γ'. Every member of $\mathscr{S}\setminus\{\Gamma'\}$ meets M in exactly one point. But every point defines exactly $(s-1)/2$ members of $\mathscr{S}\setminus\{\Gamma'\}$. Hence s is odd and $2s$ must be divisible by $s-1$, which implies that $s=3$.

This completes the proof of the lemma. □

Lemma 8. *If at least one member of $\mathscr{S}$ has order (s,t') with $t'>1$, and if two collinear points x,y of Γ lie each in at least two members of $\mathscr{S}$, then all points of the line joining x and y do, or $s=2$, or $(t',s,t)=(2,4,8)$.*

Proof. Suppose z is a point of the line L of Γ incident with both x and y, with the property that it lies in a unique member Γ' of $\mathscr{S}$, and suppose that x, respectively, y is contained in two members of $\mathscr{S}$, say Γ_1 and Γ_2, respectively, Γ_3 and Γ_4. By Lemma 5, we may assume that L belongs to Γ_1, and similarly to Γ_3 as well. This implies that $\Gamma_1=\Gamma_3=\Gamma'$ (otherwise z is contained in at least two members of $\mathscr{S}$). Let M be a line through x not belonging to Γ_1. Remark that by Lemma 5 every line through x lies in some member of $\mathscr{S}$.

Let t' be the largest number such that $\mathscr{S}$ contains a member of order (s,t'). Then it is clear that x lies in at least

$$\frac{t-t'}{t'+1}$$

members of $\mathscr{S}\setminus\{\Gamma'\}$. We now show that, provided $s>2$ and $s\neq 4$, this is more than half of the members of $\mathscr{S}\setminus\{\Gamma'\}$, i.e., we show that

$$\frac{t-t'}{t'+1}>\frac{s}{2}.$$

Suppose on the contrary that

$$\frac{t-t'}{t'+1}\leqslant\frac{s}{2}.$$

Then $2t-2t'\leqslant st'+s$. From Theorem 1, we infer $t\geqslant st'$, hence $2t-2t'\leqslant t+s$. Multiplying with s, we obtain $st-2t\leqslant s^2$. Since we may suppose that $t'>1$ and $s>2$, we use $t\geqslant\sqrt{s^3}$ (see Theorem 1) to obtain $s-2\leqslant\sqrt{s}$, which implies after a short calculation $(s-4)(s-1)\leqslant 0$. Hence $s=3,4$, disregarding the case $s=2$. If $s=3$, then automatically

$t'=3$ and hence, since $t \geqslant st'$, $t=9$. But in this case there are at least $(t-t')/(t'+1)=\frac{3}{2}$, hence 2 members of $\mathscr{S}\backslash\{\Gamma'\}$ containing x.

If $s=4$, then since $(s-4)(s-1)=0$, equality holds in every equation above, so $(t',s,t)=(2,4,8)$.

If $s>2$ and $(t',s,t)\neq(2,4,8)$, then we similarly deduce that y is contained in more than half of the members of $\mathscr{S}\backslash\{\Gamma'\}$. Hence at least one member of $\mathscr{S}\backslash\{\Gamma'\}$ contains both x and y and hence L is contained in at least two members of $\mathscr{S}$, therefore also z is. $\square$

Lemma 9. *Suppose that at least one member of* $\mathscr{S}$ *has order* (s,t') *with* $t'>1$ *and that* $st>9$. If $(t',s,t)\neq(2,4,8)$, *then one of the following holds*:

(i) *no line of* Γ *is contained in at least two members of* $\mathscr{S}$;

(ii) *all members of* $\mathscr{S}$ *have the same order*, $(t',s,t)=(s,s,s^2)$ *and there exists a large subquadrangle* Γ^* *of order* $(s,1)$ *such that the intersection of any two members of* $\mathscr{S}$ *is exactly* Γ^*;

(iii) *there is a unique line* L *of* Γ *belonging to at least two members of* $\mathscr{S}$. *In this case* L *belongs to all members of* $\mathscr{S}$ *and* $(t',s,t)=(10,15,160)$.

Proof. We may assume that there exists a line L contained in at least two members of $\mathscr{S}$. By Lemma 6, L is contained in $\ell\geqslant s$ members of $\mathscr{S}$. Suppose first that $\ell=s$. Let Γ' be the unique element of $\mathscr{S}$ not containing L. Let u be any point of Γ not on L, and not contained in any member of $\mathscr{S}\backslash\{\Gamma'\}$ (u exists by Lemma 4). Let L_u be the unique line of Γ through u meeting L. Then the s points of L_u not on L all belong to Γ' (since if one such point belongs to a member Γ'' of $\mathscr{S}\backslash\{\Gamma'\}$, the line L_u and hence the point u also belongs to Γ''), and hence so does L_u. So there exists a point x on L (namely, the intersection of L and L_u) contained in Γ'. It is easily seen that there is at least one other point z in Γ' not collinear with x. Let M be the line of Γ through z and meeting L. Since M is not incident with x, the line M belongs to a member of $\mathscr{S}\backslash\{\Gamma'\}$. But that means that z is contained in at least two members of $\mathscr{S}$. By Lemmas 8 and 6, the line M, and hence the point z belongs to at least s members of $\mathscr{S}$. We can do this reasoning with every point of Γ' collinear with z, but not collinear with x. But by Lemma 8, this property also holds for all points of Γ' collinear with x. Hence all points of Γ' are contained in at least s members of $\mathscr{S}$. Deleting Γ' from $\mathscr{S}$, we obtain a contradiction to Lemma 4.

Now, suppose that L is contained in exactly $s+1$ members of $\mathscr{S}$. By the previous paragraph, we may assume that every line which is contained in at least two members of $\mathscr{S}$, is contained in all members of $\mathscr{S}$. Let x be any point on L. Let C be the number of lines through x contained in all members of $\mathscr{S}$. Then, since by Lemma 5 every line through x is contained in either 1 or all members of $\mathscr{S}$, C satisfies the equation $(s+1)C+(s+1-C)=\tau$, where τ is the sum of all t^*+1 such that (s,t^*) is the order of a member of $\mathscr{S}$. Hence C is a constant. If $C>1$, then the set of all points lying in all members of $\mathscr{S}$ forms a large subquadrangle Γ^*, which is also a large subquadrangle of any member of $\mathscr{S}$. Now (ii) follows from Theorem 1.

So we may assume that $C=1$. Then, clearly, there is a unique line L contained in all members of $\mathscr{S}$ and every point of Γ contained in at least two members of $\mathscr{S}$ is incident with L. Let $\Gamma' \in \mathscr{S}$ have order (s,t'). Let $\mathscr{R}$ be the set of all lines of Γ not contained in any member of $\mathscr{S}$. Then every element of $\mathscr{R}$ is incident with a unique point of every element of $\mathscr{S}$, hence with a unique point of Γ'. Conversely, every point of Γ' not incident with L is incident with exactly $t-t'$ elements of $\mathscr{R}$. So the set $\mathscr{R}$ has size $(1+s)st'(t-t')$. Similarly, if $\Gamma'' \in \mathscr{S}$ has order (s,t''), then $\mathscr{R}$ has size $(1+s)st''(t-t'')$. It follows that $t'(t-t')=t''(t-t'')$, therefore either $t'=t''$ or $t'+t''=t$. In the latter case, we consider a point of L and deduce from $C=1$ and Lemma 5 that $\mathscr{S}=\{\Gamma',\Gamma''\}$, so $s=1$, a contradiction. We conclude that $t'=t''$, so all members of $\mathscr{S}$ have the same order (s,t'). If x is incident with L, then every line through x distinct from L belongs to exactly one member of $\mathscr{S}$, so we deduce from this that $(1+s)t'=t$.

We now show that the parameter set $(t',s,t)=(t',s,t'+st')$ is never feasible, except for $(t',s,t)=(10,15,160)$. Indeed, we must have $s+t \mid (1+st)st$, which is readily seen to be equivalent with $s+t \mid (s^2-1)s^2$. Let k be the greatest common divisor of s^2 and $s+t$. Let p^i divide k, with p prime and i maximal. Let p^j divide s, with $j \leqslant i$ maximal. Then p^j divides $t=(1+s)t'$, and so p^j divides t'. It follows that p^{2j} divides st', so p^i divides $s+t-st'=s+t'$ (because $i \leqslant 2j$). We conclude that k divides $s+t'$. Suppose first that $k \leqslant (s+t')/3$.

Note that $s+t=s+(1+s)t'$ and $s+1$ are relatively prime. Hence $(s+t)/k$ must divide $s-1$. However, the greatest common divisor of $s-1$ and $s+t'+st'$ is a divisor of $(s-1)+1+t'+(s-1)t'+t'$, hence of $1+2t'$. Consequently, $(s+t)/k$ must divide $1+2t'$. So we obtain

$$\begin{aligned} 1+2t' &\geqslant \frac{s+t'+st'}{k} \\ &\geqslant \frac{s+t'+st'}{\frac{s+t'}{3}} \\ &\geqslant 3+3\frac{st'}{s+t'}, \end{aligned}$$

which implies $2t'^2 \geqslant 2s+2t'+st'$. This is only possible if $2t' \geqslant 2+s$.

On the other hand, we also have

$$\begin{aligned} s-1 &\geqslant \frac{s+t'+st'}{k} \\ &\geqslant 3+3\frac{st'}{s+t'}, \end{aligned}$$

which implies that $s^2 \geqslant 2st'+4s+4t'$. Using the inequality $2t' \geqslant 2+s$, this means that $s^2 \geqslant s^2+8s+4$, a contradiction. Hence we have shown that $k=s+t'$ or $k=(s+t')/2$. Moreover, we have shown that, if $T=[(1+2t')(s+t')]/(s+t'+st') \geqslant 3$, then $S=[(s-1)(s+t')]/(s+t'+st')<3$.

First, let $k=s+t'$. Then $(s+t'+st')/(s+t')$ divides both $s-1$ and $1+2t'$. Hence both S and T (defined above) are positive integers. We first show that $T \geqslant 3$.

Indeed, the only other possibilities are $T=1$ and 2. If $T=1$, then one calculates that $st'+2t'^2=0$, a contradiction. If $T=2$, then one computes that $2t'^2=s+t'$, and since $s\leqslant t'^2$, this implies $t'^2\leqslant t'$, so $t'=1=s$, a contradiction. So we must have $S=1$ or 2. If $S=1$, then an elementary calculation shows $s^2=2s+2t'$. Since $t'<s$ (indeed, $t'=s$ implies $t>s^2$), we have $s^2<4s$, so $s=2$ (because s must clearly be even). But now $t'=0$ follows, a contradiction. Suppose now $S=2$. Then $s^2=3s+3t'+st'$. Hence the quadratic equation $s^2-(3+t')s-3t'=0$ in s has an integer solution. The discriminant is, however, $t'^2+18t'+9=(t'+9)^2-72$. So the square root d of the discriminant satisfies $t'+3<d<t'+9$. If $d=t'+i$, with $i=4,6,8$, then t' is not an integer. If $d=t'+5$, then $t'=2$ and $s=6$, a contradiction. If $d=t'+7$, then $t'=10$, $s=15$ and $t=160$ and all divisibility conditions are satisfied.

Now, suppose that $k=(s+t')/2$. Then both S and T must be even integers. But $T\neq 2$ as above, hence $S=2$. This again implies $(t',s,t)=(10,15,160)$, except that this cannot happen now since it means that $s+t'$ is odd. □

Lemma 10. *Suppose that at least one member of* $\mathscr{S}$ *has order* (s,t') *with* $t'>1$, *that* $st>9$ *and that* $(t',s,t)\neq(2,4,8)$. *If no line of* Γ *is contained in at least two members of* $\mathscr{S}$, *then* (*the point set of*) Γ *is the disjoint union of* (*the point sets of*) *the members of* $\mathscr{S}$ *and* $(t',s,t)=(t',s,st'+t'+1)$.

Proof. Let $\mathscr{R}$ be the set of points of Γ contained in at least two members of $\mathscr{S}$. Let $x\in\mathscr{R}$. Assume that x is not contained in all members of $\mathscr{S}$ and let Γ' be a member of $\mathscr{S}$ not containing x. Let y be a point of Γ' collinear with x. By Lemma 5, the line xy belongs to some member of $\mathscr{S}$. Clearly, xy does not belong to Γ' since otherwise x would belong to Γ'. So xy belongs to another member, which implies $y\in\mathscr{R}$. By Lemmas 8 and 6, the line xy belongs to at least two members of $\mathscr{S}$, a contradiction. Hence x belongs to all members of $\mathscr{S}$.

Now, let z be a point of Γ not belonging to $\mathscr{R}$. Then z is contained in a unique member Γ_1 of $\mathscr{S}$. Consider two members Γ_2 and Γ_3 of $\mathscr{S}$ with $\Gamma_1\neq\Gamma_2\neq\Gamma_3\neq\Gamma_1$. Let Γ_i have order (s,t_i), $i=1,2,3$. The number of points of Γ_j, $j=2,3$, collinear with z is $1+st_j$ (since this set forms an ovoid in Γ_j). Every line through z not in Γ_1 meets Γ_j, $j=2,3$, because every such line is contained in no member of $\mathscr{S}$ and therefore cannot contain two points of the same member. Hence there are precisely $1+st_j-(t-t_1)$ lines of Γ_1 through z meeting Γ_j, or in other words, there are precisely $1+st_j-t+t_1$ elements of $\mathscr{R}$ collinear with z. Since this number should be independent of j, we conclude $t_2=t_3$. It is now easy to see that all members of $\mathscr{S}$ have the same order (s,t').

There remains to show that $\mathscr{R}$ is empty. Suppose it is not empty. Then by Lemma 5 we have $(s+1)(t'+1)=t+1$. Let z, Γ_1 and Γ_2 be as above. Remember that there are precisely $1+(s+1)t'-t$ elements of $\mathscr{R}$ collinear with z. In view of the equality $(s+1)(t'+1)=t+1$, this number becomes $1-s$, a contradiction.

The lemma is proved. □

All we still have to consider are the small cases, i.e., the cases $st < 9$ and $(t', s, t) = (2, 4, 8)$.

Lemma 11. *If* $(s, t) = (4, 8)$, *then every member of* $\mathcal{S}$ *has order* $(4, 2)$, *every two members of* $\mathcal{S}$ *meet in the nine points of an ovoid in both members, there are exactly* 30 *points of* Γ *which lie in at least two members of* $\mathcal{S}$ *and every such point lies in exactly* 3 *members, every member contains exactly* 18 *points which lie in three members of* $\mathcal{S}$ *and no line is contained in at least two members of* $\mathcal{S}$.

Proof. Note that by counting the points, there are at least 2 members of $\mathcal{S}$ of order $(4, 2)$. Since every possible subquadrangle of Γ has either order $(4, 2)$ or order $(4, 1)$, and since $45+45+25+25+25=165$, we see that, if $\mathcal{S}$ contains exactly two members of order $(4, 2)$, then the point set of Γ is the disjoint union of the point sets of the elements of $\mathcal{S}$. But similarly as before, we count in two ways the number of lines not belonging to any member of $\mathcal{S}$. Starting with the points of a member of $\mathcal{S}$ of order $(4, 2)$, we obtain $45 \times 6 = 270$; starting with the points of a member of $\mathcal{S}$ of order $(4, 1)$, we obtain $25 \times 7 = 175$, a contradiction. Hence $\mathcal{S}$ contains at least three members of order $(4, 2)$ and there exists at least one point of Γ lying in at least two members of $\mathcal{S}$. Let $\mathcal{D}$ be the set of all such points.

(i) First suppose that no line of Γ is contained in at least two members of $\mathcal{S}$. We showed that $\mathcal{D}$ is non-empty. By Lemma 5, we can now write 9 as the sum of a number of 3's and at most two 2's. So clearly only 3's are possible, hence no element of $\mathcal{D}$ belongs to a member of $\mathcal{S}$ of order $(4, 1)$. Hence, again, the number of lines of Γ which do not belong to any member of $\mathcal{S}$ is equal to $25 \times 7 = 175$, provided $\mathcal{S}$ contains an element of order $(4, 1)$. So $175 = d \times 6$, where d is de number of points of a member of $\mathcal{S}$ of order $(4, 2)$ not belonging to any other member of $\mathcal{S}$. Since 6 does not divide 175, this leads to a contradiction. Therefore, all members of $\mathcal{S}$ have order $(4, 2)$. Also, the number of points of such a member of $\mathcal{S}$ not belonging to any other member of $\mathcal{S}$ must be a constant d. And so there are exactly $6d$ lines of Γ not contained in any member of $\mathcal{S}$. Counting all lines of Γ, we obtain

$$297 = 6d + 5 \times 27,$$

hence $d = 27$. Counting the number of pairs (x, Γ'), where x is a point of $\Gamma' \in \mathcal{S}$ and x lies in at least two (and hence exactly in three) elements of $\mathcal{S}$, we obtain that the number of points contained in three members of $\mathcal{S}$ is equal to

$$\frac{5 \times (45 - d)}{3} = 30.$$

Note that, since $t = st'$, with $(t', s, t) = (2, 4, 8)$, every line of any member of $\mathcal{S}$ meets every other member of $\mathcal{S}$ in a point. Hence two members meet in an ovoid of both members. Since $d = 27$, there are $45 - d = 18$ points of each member of $\mathcal{S}$ belonging to three members of $\mathcal{S}$.

(ii) Now, suppose that there exists a line L of Γ belonging to at least two members Γ_1 and Γ_2 of $\mathcal{S} = \{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5\}$. Since every line through every point of L

must belong to some member of $\mathscr{S}$ (by Lemma 5), and these members have either order $(4,2)$ or $(4,1)$, we deduce that every point of L is in at least two members of $\{\Gamma_3,\Gamma_4,\Gamma_5\}$. Since L is incident with 5 points, at least one pair must appear twice, so we have shown that L lies in at least 4 elements of $\mathscr{S}$, say, $\Gamma_1,\Gamma_2,\Gamma_3$ and Γ_4.

(a) First, suppose that L does not lie in Γ_5. If no line of Γ_5 meets L, then we consider any pair of collinear points x,y with xy concurrent with L, and such that L is incident with neither x nor y. If both x and y belong to Γ_5, then xy belongs to Γ_5, a contradiction. Hence we may assume that x does not belong to Γ_5. Consequently, x belongs to, say, Γ_1. But then also the line xy and the point y belong to Γ_1. We conclude that in this case Γ is the union of $\mathscr{S}\setminus\{\Gamma_5\}$, contradicting Lemma 4. So there is a unique point x on L incident with some lines of Γ_5. We claim that every line M meeting L not in x is contained in a unique element of $\mathscr{S}\setminus\{\Gamma_5\}$. Indeed, M is contained in at least one such element and the order $(4,t')$ of Γ_i, $i\in\{1,2,3,4\}$, satisfies $t'\leqslant 2$. So $2\times 4=8$ implies that Γ_i has order $(4,2)$, for all $i\in\{1,2,3,4\}$, and our claim follows.

Now, consider a point z of Γ_5 not collinear with x. Then z is incident with a line N meeting L, and N belongs to, say, Γ_1, but not to Γ_2,Γ_3 or Γ_4. But z is incident with Γ_1 and with Γ_5, hence it is incident with at least one other element of $\mathscr{S}$ (indeed, if not, then by Lemma 5, the order (s,t'') of Γ_5 satisfies $3+t''+1\geqslant 9$, since Γ_1 has order $(4,2)$, and this contradicts Theorem 1), say, Γ_2. But then N belongs to Γ_2 as well, a contradiction.

(b) So we may suppose that L belongs to all members of $\mathscr{S}$. In fact, by the foregoing, we may assume that every line of Γ which belongs to at least two members of $\mathscr{S}$, belongs to all members of $\mathscr{S}$. Suppose now that a line M meeting L belongs to at least two members of $\mathscr{S}$. Each line through the meeting point x of L and M must belong to a member of $\mathscr{S}$. But every member of $\mathscr{S}$ has at most 3 lines through x, two of which are L and M. This leads to a contradiction. So every line of Γ meeting L belongs to a unique element of $\mathscr{S}$. It follows that three elements of $\mathscr{S}$, say $\Gamma_1,\Gamma_2,\Gamma_3$, have order $(4,2)$, and two of them, say Γ_4,Γ_5, have order $(4,1)$. Since Γ_1,Γ_2 and Γ_3 each have 40 points off L, and Γ_4 and Γ_5 each have 20 points off L, and since $165=3\cdot 40+2\cdot 20+5$, no point off L belongs to at least two members of $\mathscr{S}$. Counting in two ways (as above) the number of lines not belonging to any member of $\mathscr{S}$, we obtain $40\cdot 6=20\cdot 7$, a contradiction.

The lemma is proved. □

Example. Let Γ be the unitary quadrangle $H(4,q^2)$ embedded in a standard way in $\mathrm{PG}(4,q^2)$. Let π be a plane of $\mathrm{PG}(4,q^2)$ meeting $H(4,q^2)$ in a non-degenerate hermitian curve $\mathscr{C}$. Let L be the polar line of π. Then L meets $H(4,q^2)$ in $q+1$ points $x_0,x_1,\dots x_q$. Let $x_{q+1},\dots,x_{q^2}$ be the remaining points on L. The hyperplane determined by π and x_i, $i\in\{q+1,q+2,\dots,q^2\}$, meets $H(4,q^2)$ in a non-degenerate hermitian variety $H(3,q^2)$, which is a subquadrangle of order (q^2,q). These q^2-q subquadrangles cover already all points of Γ, except for the points off $\mathscr{C}$ and collinear with one of the x_i, $i\in\{0,1,\dots,q\}$. Let π' be a plane containing $q+1$ lines of Γ through x_0. Then the hyperplane generated by π' and L meets $H(4,q^2)$ in a subquadrangle of order (q^2,q). The lines of Γ through x_0 form a hermitian curve in the residue of x and the tangent hyperplane of $H(4,q^2)$ in x. The point set of a hermitian curve $\mathscr{U}$ can be partitioned into q^2-q+1 intersections with $(q+1)$-secants. Indeed, it suffices to consider a point off $\mathscr{U}$ in a projective plane where $\mathscr{U}$ lives, and the $(q+1)$-secants through x together with the polar line of x with respect to $\mathscr{U}$ do the job. Hence we can find q^2-q+1 additional subquadrangles containing $\{x_0,x_1,\dots,x_q\}$ and covering all points on all lines of Γ through x_i, $i\in\{0,1,\dots,q\}$. So we have covered the point set of Γ by $2q^2-2q+1$ subquadrangles of order (q^2,q). For $q=2$, this number equals exactly $5=q^2+1$. My conjecture is that $2q^2-2q+1$ is the least possible number to cover $H(4,q^2)$ with subquadrangles of order (q^2,q), and the proof probably will not be too difficult at all.

Lemma 12. *If $(s,t)=(3,3)$, then there are exactly two non-isomorphic examples, one with no line of Γ in at least two members of Γ, and the other with a unique pair of concurrent lines contained in 3 members of Γ.*

Proof. We distinguish two cases.

(i) Suppose first that there is some line L, which is contained in at least two members of $\mathscr{S}=\{\Gamma_1,\Gamma_2,\Gamma_3,\Gamma_4\}$, say, Γ_1 and Γ_2. By Lemma 4, L is contained in at least 3 members of $\mathscr{S}$, say L is also contained in Γ_3. If L is, moreover, contained in Γ_4, then through every point x of L, there is a line $L_x\neq L$ contained in at least 2, and hence in at least 3 members of $\mathscr{S}$. Taking $x\neq y$, both incident with L, we see that L_x and L_y are contained in at least two members of $\mathscr{S}$, a contradiction because two opposite lines determine a subquadrangle completely. Hence L is not contained in Γ_4. But L must be incident with a unique point z of Γ_4 (by Theorem 1). Through z, there must be a line $M\neq L$ contained in at least 2, and hence again 3 members of $\mathscr{S}$. Suppose these members are $\Gamma_2,\Gamma_3,\Gamma_4$. Let M' be the unique line of Γ_1 through z distinct from L. No line concurrent with L and not incident with z can belong to Γ_4. On the other hand, whenever a point u not collinear with z belongs to Γ_i, for some $i\in\{1,2,3\}$, then the unique line through u concurrent with L belongs to Γ_i. Hence we deduce easily that every line concurrent with L and not incident with z must belong to some Γ_i, $i=1,2,3$. Now, notice that L is a regular line (since Γ contains subquadrangles of order $(3,1)$, Γ is isomorphic to $Q(4,3)$). It is now easily seen using the projective plane corresponding to L that Γ_1 and Γ_2 share a line N concurrent with L and not through z. But then Γ_3 should contain three lines through the meeting point of L and N, a contradiction. So M belongs to Γ_1,Γ_2 and Γ_3. It follows that Γ_4 contains the two lines through z which are distinct from L and M. There are actually three choices for Γ_4 at this stage, but they are easily seen to be equivalent under the automorphism group of Γ (which is a classical quadrangle) fixing the line L pointwise, and fixing all lines meeting L (root elations).

(ii) Now, suppose that no line of Γ is contained in at least 2 members of $\mathscr{S}$. Notice that the lines of Γ can be viewed as the non-isotropic points of a unitary polarity in $\mathrm{PG}(3,4)$. Let $\mathscr{Q}$ be the corresponding hermitian variety. The points of Γ are then the sets of 'polar quadrangles', i.e., sets of 4 pairwise conjugate (w.r.t. the unitary polarity) points of PG(3,4). It is readily seen that the lines of a subquadrangle of order (3,1) of Γ correspond to the points off $\mathscr{Q}$ but in a tangent plane of $\mathscr{Q}$. And two subquadrangles meet in 4 non-collinear points if and only if the corresponding points (the intersections of $\mathscr{Q}$ with the tangent planes) on $\mathscr{Q}$ are on a line contained in $\mathscr{Q}$. Hence $\mathscr{S}$ defines a set of 4 points on a line T of $\mathscr{Q}$. Now, let θ be an element of order 5 of the automorphism group of $\mathscr{Q}$, preserving T. Then the corresponding set $\mathscr{S}'$ of 5 subquadrangles of Γ covers Γ and no line of Γ belongs to two members of $\mathscr{S}'$. Since every point of Γ is in at most two members of $\mathscr{S}'$, the number of pairs (x,Γ'), where x is a point in a member Γ' of $\mathscr{S}'$, is at most 80, if we first count the points x. But it is exactly 80 is we first count the members of $\mathscr{S}'$. Hence every point lies in exactly 2 members of $\mathscr{S}'$.

Now, the members of $\mathscr{S}$ correspond to 4 collinear points on $\mathscr{Q}$. Hence these points are contained in a line of $\mathscr{Q}$ and so $\mathscr{S}$ arises from $\mathscr{S}'$ by deleting one member. Any members gives rise to an isomorphic set of 4 subquadrangles because of the transitive group of order 5 acting on it. Since every point is covered twice by the $\mathscr{S}'$, deleting a member does give rise to a set $\mathscr{S}$ of 4 subquadrangles whose union is Γ.

This completes the proof of the lemma. □

The case $s=2$ will not be treated here. It is an easy case. Indeed, if $t=4$, then there are coverings with 3 subquadrangles of order (2,1). Extending any number of them to a subquadrangle of order (2,2) gives an example of a covering with $s+1$ subquadrangles for which the order is not necessarily a constant pair. If $t=2$, then all coverings with 3 subquadrangles of order (2,1) can be found as an easy exercise.

We now turn our attention to finite polar spaces, in order to show Theorem 3.

3. Proof of Theorem 3

Let Γ be a finite non-degenerate classical polar space of rank $\ell \geqslant 2$, viewed as a geometry over the diagram of type B_ℓ. This just means that we consider quadrics and hermitian varieties together with their totally singular subspaces. We assume $\ell > 2$, since otherwise the result follows readily from Theorem 2. Indeed, this is clear if we show that Case (i) of Theorem 2 never occurs with classical quadrangles of order (s,t) with $t \geqslant s > 2$. The only possibilities are $(s,t) \in \{(q,q),(q,q^2),(q^2,q^3)\}$, for some prime power q. Then $t' \in \{(q-1)/(q+1), q-1, (q^3-1)/(q^2+1)$ and this leads to a contradiction (every subquadrangle of a classical quadrangle must again be a classical quadrangle). Denote by PG(m,q), q a power of a prime, the ambient projective space (and in characteristic 2 we consider a symplectic polar space embedded as a quadric). We suppose that the point set of Γ is covered by the point sets of $k \leqslant q+1$ polar subspaces of rank ℓ and each of these polar subspaces has also $q+1$ points on a line. Let $\mathscr{P}$ be the set of these polar subspaces. First, we want to show that $k=q+1$. To that end, we prove a lemma, which is well-known (it is a special case of the main result of Bose and Burton [1]), but we include a proof for the sake of completeness.

Table 1

Polar space	Number of points	Number of maximal subspaces
$Q^+(2\ell-1,q)$	$\frac{(q^\ell-1)(q^{\ell-1}+1)}{q-1}$	$2(q+1)(q^2+1)\dots(q^{\ell-1}+1)$
$Q(2\ell,q)$	$\frac{q^{2\ell}-1}{q-1}$	$(q+1)(q^2+1)\dots(q^\ell+1)$
$Q^-(2\ell+1,q)$	$\frac{(q^{\ell+1}+1)(q^\ell-1)}{q-1}$	$(q^2+1)(q^3+1)\dots(q^{\ell+1}+1)$
$H(2n-1,q^2)$	$\frac{(q^{2n}-1)(q^{2n-1}+1)}{q^2-1}$	$(q+1)(q^3+1)\dots(q^{2n-1}+1)$
$H(2n,q^2)$	$\frac{(q^{2n+1}+1)(q^{2n}-1)}{q^2-1}$	$(q^3+1)(q^5+1)\dots(q^{2n+1}+1)$

Lemma 13. *Let the point set of* PG(d,q), $d\geqslant 2$, *be the union of* $q+1$ *hyperplanes. Then all these hyperplanes have a* $(d-2)$*-dimensional subspace in common and hence every point of* PG(d,q) *either belongs to all these hyperlanes, or to exactly one. Also, the point set of* PG(d,q) *cannot be the union of* q *hyperplanes.*

Proof. Let $\mathscr{S}$ be the set of these $q+1$ hyperplanes. Let H_1 and H_2 be two of them, and suppose they meet in the $(d-2)$-dimensional subspace U. Suppose that H_3 is a member of $\mathscr{S}$ not containing U. Then there is some hyperplane H of **PG**(d,q) containing U, but not belonging to $\mathscr{S}$. If we intersect every member of $\mathscr{S}$ with H, then we obtain a set of at most q different $(d-2)$-dimensional subspaces of H covering all points of H. Hence

$$q^{d-1}+q^{d-2}+\cdots+q+1\leqslant q(q^{d-2}+\cdots+q+1),$$

a contradiction. The result follows, noting that a similar counting argument proves that q hyperplanes cannot cover all points of PG(d,q). □

Now, we list the number of points and the number of maximal singular subspaces of the various finite polar spaces of rank ℓ. We use the following notation: $Q^-(2\ell+1,q)$ for the elliptic quadric, $Q(2\ell,q)$ for the parabolic quadric, $Q^+(2\ell-1,q)$ for the hyperbolic quadric, $H(n,q)$ for the hermitian variety in PG(n,q) (see Table 1). Note that we do not have to consider symplectic polar spaces since they are either isomorphic to a quadric (in characteristic 2), or they do not have proper large polar subspaces of the same rank (odd characteristic).

Note that, if $\Gamma\cong Q^-(2\ell+1,q)$, then every member of $\mathscr{P}$ is isomorphic to $Q(2\ell,q)$ or $Q^+(2\ell-1,q)$; if $\Gamma\cong Q(2\ell,q)$, then every member of $\mathscr{P}$ is isomorphic to $Q^+(2\ell-1,q)$; if $\Gamma\cong H(2\ell,q)$, then every member of $\mathscr{P}$ is isomorphic to $H(2\ell-1,q)$; finally, Γ cannot be isomorphic to either $Q^+(2\ell-1,q)$ or $H(2\ell-1,q)$.

Lemma 14. *With the above notation, we must have* $k=q+1$.

Proof. Suppose that $k \leqslant q$. Consider a maximal singular subspace U of Γ and suppose that U does not belong to any member of $\mathscr{P}$. Note that U has dimension $\ell - 1$. Then every member of $\mathscr{P}$ can have at most an $(\ell - 2)$-dimensional subspace in common with U. That implies that U must be the union of at most q subspaces of dimension at most $\ell - 2$, contradicting Lemma 13. Hence every maximal singular subspace U belongs to a member of $\mathscr{P}$. So the number of maximal singular subspaces of Γ must be at most q times the number of maximal singular subspaces of any element of $\mathscr{P}$ having a maximum number of maximal singular subspaces, contradicting the number of maximal singular subspaces given above. □

Lemma 15. *Each point of Γ belongs to a maximal singular subspace which does not belong to any member of $\mathscr{P}$.*

Proof. Suppose by way of contradiction that a point x exists such that every maximal singular subspace of Γ through x belongs to some member of $\mathscr{P}$. Then every line xy on Γ is contained in some member of $\mathscr{P}$. Projecting the whole situation on a hyperplane of $\mathrm{PG}(m,q)$ (the space of Γ) not containing x, we obtain a covering of a polar space Γ' of rank $\ell - 1$ by at most $q+1$ proper polar subspaces of the same rank such that every maximal singular subspace of Γ' is contained in one of the polar subspaces. The same counting argument as in the previous proof leads to a contradiction (now considering $q+1$ polar subspaces instead of q, but the contradiction remains). □

Lemma 16. *Every maximal singular subspace U of Γ which does not belong to any member of $\mathscr{P}$ contains a unique $(\ell-3)$-dimensional subspace V such that every point of V belongs to every member of $\mathscr{P}$, and every other point of U belongs to exactly one member of $\mathscr{P}$.*

Proof. It is easily seen that $\mathscr{P}$ induces a covering of the point set of U consisting of at most $q+1$ proper projective subspaces of U. Counting the points, one immediately finds that there must be exactly $q+1$ proper subspaces of dimension $\ell - 2$ and hence the result follows directly from Lemma 13. □

The last two lemmata imply:

Lemma 17. *Every point of Γ is contained in either every member of $\mathscr{P}$, or in exactly one. Also, if two points x and y belong to all members of $\mathscr{P}$ and x and y are collinear in Γ, then all points of the line xy belong to all members of $\mathscr{P}$.* □

So the geometry Γ' having as point set the set of all points of Γ which belong to all members of $\mathscr{P}$ (with lines and other subspaces induced by Γ) satisfies the one-or-all axiom of polar spaces; hence it is a polar space of rank ℓ provided we prove that it contains at least one singular subspace of dimension $\ell - 1$, and that no point of it is collinear in Γ with all other points of Γ'.

We know by Lemma 16 that there is at least one singular subspace V of dimension $\ell-3$ contained in all members of $\mathscr{P}$. We project from V onto a subspace of dimension $m-\ell+2$, skew to V. The projection of Γ is a generalized quadrangle Γ^*, and the projections of the members of $\mathscr{P}$ induce a covering $\mathscr{P}^*$ of Γ^* of $q+1$ large subquadrangles such that each point of Γ^* is in either a unique member of $\mathscr{P}^*$, or in all members of $\mathscr{P}^*$. From Theorem 2, it readily follows that either Γ^* is isomorphic to the elliptic quadric $Q^-(5,q)$, all members of $\mathscr{P}^*$ are isomorphic to $Q(4,q)$, and the intersection of all members is isomorphic to $Q^+(3,q)$, or $q=2$. In the first case, it follows that there are plenty of maximal singular subspaces in Γ'. Now, suppose $q=2$. We may assume that Γ' does not contain a singular subspace of dimension $\ell-1$, hence that no line of Γ^* belongs to all members of $\mathscr{P}^*$. If Γ^* is isomorphic to $Q(4,2)$, then it is readily seen that exactly 6 points of Γ^* are contained in each member of $\mathscr{P}^*$ (which has order $(2,1)$), contradicting the fact that no two such points can be collinear in Γ^*. Now, suppose that Γ^* is isomorphic to $Q^-(5,2)$. Let the three members of $\mathscr{P}^*$ have respective orders $(2,t_1)$, $(2,t_2)$ and $(2,t_3)$. If there is a point of Γ^* in all members of $\mathscr{P}^*$, then by Lemma 5, $3+t_1+t_2+t_3=5$, a contradiction. Hence the point set of Γ^* is the disjoint union of the point sets of the members of $\mathscr{P}^*$. This implies that all members of $\mathscr{P}^*$ are isomorphic to $Q^-(3,2)$. Consequently, every element of $\mathscr{P}$ is isomorphic to $Q^+(2\ell-1,2)$ and Γ itself is isomorphic to $Q^-(2\ell+1,2)$, $n\geqslant 3$. Counting the number of points, we must have $3(2^{\ell-1}+1)\geqslant 2^{\ell+1}+1$, implying $\ell\leqslant 2$, a contradiction.

Hence we have shown that there is a maximal singular subspace contained in all members of $\mathscr{P}$. Moreover, our arguments show that Γ is isomorphic to $Q^-(2\ell+1,q)$ and every member of $\mathscr{P}$ is isomorphic to $Q(2\ell,q)$.

Now, suppose that there exists a point x of Γ' such that all points which belong to Γ' are collinear in Γ with x. The number of points of Γ not collinear with x is $q^{2\ell}$. The number of points in each member of $\mathscr{P}$ not collinear with x is $q^{2\ell-1}$. Since each point must occur exactly once, this implies $(q+1)q^{2\ell-1}=q^{2\ell}$, a contradiction.

Hence we have shown that the intersection of all members of $\mathscr{P}$ is a polar subspace of rank ℓ. And it is clear that it must be isomorphic to $Q^+(2\ell-1,q)$. Theorem 3 is proved.

4. Proof of the Corollary

For the notions below not defined in this paper, we refer to Payne and Thas [4] or Thas [6].

Let Γ be a flock quadrangle of order (t^2,t), t odd, covered (as set of lines!) by a set $\mathscr{S}$ of $t+1$ subquadrangles of order (t,t), all containing the point (∞). Then all these subquadrangles meet in a subquadrangle Γ' of order $(1,t)$, by Theorem 2. According to Theorem 7.2 of Thas and Van Maldeghem [7], we have to show that the net corresponding with the point (∞) satisfies the axiom of Veblen. By Theorem 8.1 of *loc.cit.*, this is equivalent to showing that every two non-collinear points x,y, with

x collinear with (∞) and y not collinear with (∞), are contained in a subquadrangle of order (t,t). Since Γ is an elation generalized quadrangle, we may assume that y belongs to Γ' (because there is an automorphism group acting regularly on the points of Γ not collinear with (∞)). It is now easy to see that exactly one member of $\mathscr{S}$ contains x, namely the unique member containing all lines of Γ through x. □

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