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SIMPLE GROUPS OF FINITE MORLEY RANK AND TITS BUILDINGS

BY

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ABSTRACT

THEOREM A: *If $\mathfrak{P}$ is an infinite Moufang polygon of finite Morley rank, then $\mathfrak{P}$ is either the projective plane, the symplectic quadrangle, or the split Cayley hexagon over some algebraically closed field. In particular, $\mathfrak{P}$ is an algebraic polygon.*

It follows that any infinite simple group of finite Morley rank with a spherical Moufang BN-pair of Tits rank 2 is either $\mathbf{PSL}_3(K), \mathbf{PSp}_4(K)$ or $\mathbf{G}_2(K)$ for some algebraically closed field K.

Spherical irreducible buildings of Tits rank ≥ 3 are uniquely determined by their rank 2 residues (i.e. polygons). Using Theorem A we show

THEOREM B: *If G is an infinite simple group of finite Morley rank with a spherical Moufang BN-pair of Tits rank ≥ 2, then G is (interpretably) isomorphic to a simple algebraic group over an algebraically closed field.*

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THEOREM C: *Let K be an infinite field, and let $G(K)$ denote the group of K-rational points of an isotropic adjoint absolutely simple K-algebraic group G of K-rank ≥ 2. Then $G(K)$ has finite Morley rank if and only if the field K is algebraically closed.*

We also obtain a result about BN-pairs in split K-algebraic groups: such a BN-pair always contains the root groups. Furthermore, we give a proof that the sets of points, lines and flags of any $\aleph_1$-categorical polygon have Morley degree 1.

Introduction

The Cherlin–Zil'ber Conjecture states that any infinite simple group of finite Morley rank is an algebraic group over an algebraically closed field. Even in the special case where $G(K)$ is the group of K-rational points of a simple algebraic group G defined over some infinite field K there is no general result stating that if $G(K)$ has finite Morley rank, then K has to be algebraically closed.* The problem stems not only from the fact that the Borel subgroups (i.e. the groups of rational points of the minimal K-parabolic subgroups) of $G(K)$ are not necessarily solvable. But even if they happen to be solvable and one could define some field K' like in [36, 3.20], it is not clear that K' has to be isomorphic to K.

Using Tits' theory of buildings we here answer this question for groups with spherical Moufang BN-pairs, in particular for all groups with spherical irreducible BN-pairs of Tits rank at least 3 (by the Tits rank we mean the rank of the associated Coxeter complex; we chose this terminology to avoid confusion with the different notions of rank occurring in this paper). Roughly speaking, a spherical building is uniquely determined by its rank 2 residues, i.e. by the polygons contained in it as subbuildings. Therefore most of the work is in fact in the rank 2 case. Our approach is rather geometric than group theoretic: the classification of the groups is a consequence of the classification of their underlying geometries.

Some of those who read an early version of this paper raised the question to what extent our results rely on the classification of Moufang polygons, and we would like to comment on that. We do use Tits' classification of the spherical Moufang buildings; for buildings of Tits rank at least three, a proof can be found in [50]. The complete classification of the Moufang polygons (Moufang buildings

* After this paper was accepted, this result was proved for isotropic groups in [60].

of Tits rank 2) has not yet appeared in print (see the remarks in Section 3), but the result is well-known (see the forthcoming book of Tits and Weiss [57] for a proof). However, we would like to stress the fact that the only place where this classification enters into our result is that we have indeed covered *all* Moufang polygons and their groups. The groups that occur (classical groups over fields or skew fields, algebraic groups, and what could vaguely be called Chevalley groups (e.g. twisted groups like ${}^2F_4(K)$, see [9]) are comprehensively covered by the present paper. In our context, the classification of Moufang polygons amounts to the statement that there is no other class of simple groups which act on Moufang polygons besides these. The proofs that we give for each class of polygons do *not* depend on the classification.

We would like to add some remarks on the proof. As mentioned above, one difficulty is that the Borel subgroups of the little projective group of a Moufang polygon need not be solvable. Also, the root groups will in general not be abelian (the quadrangles belonging to hermitian or pseudo-quadratic forms are good examples for these phenomena). These facts make it in some cases quite difficult to recover the underlying (skew) field from the polygon.

As the results of Baldwin [1], [2] and Tent [61] indicate, there is no hope to prove some kind of Feit–Higman Theorem [14] or other classification theorem for polygons of finite Morley rank without further assumptions.

Algebraic polygons and algebraic BN-pairs in characteristic 0 have been classified in [27, 28]. So our result for the rank 2 case may also be stated as follows (at least in characteristic 0): the algebraic polygons are precisely the infinite Moufang polygons of finite Morley rank.

We have organized the paper as follows. An introduction to the theory of polygons is given in Section 1 below. To keep the paper essentially self-contained and accessible we also give a short introduction to the modeltheoretic notions involved at the beginning of Section 2. For more details we refer the reader to [6], which also gives a brief exposition of Tits buildings and BN-pairs. That section contains also some general results about polygons of finite Morley rank. In Section 3 we classify the Moufang polygons of finite Morley rank. The section ends with a summary for the rank 2 case. Section 4 is a brief introduction to spherical buildings and the relevant notions, and this is applied in Section 5 to groups with spherical BN-pairs.

1. Basic facts about polygons

In this section we give the necessary background on (generalized) polygons or spherical Tits buildings of Tits rank 2. Some of the results are hard to find in the literature, even though most of them are known to geometers. Buildings of higher Tits rank are introduced in Section 4.

1.1 Incidence structures: An **incidence structure** is a triple $\mathfrak{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ consisting of a set $\mathcal{P}$ of **points**, a set $\mathcal{L}$ of **lines**, and a set $\mathcal{F} \subseteq \mathcal{P} \times \mathcal{L}$ of **flags**. We always assume that $\mathcal{P}$ and $\mathcal{L}$ are disjoint nonempty sets. If (a, ℓ) is a flag, then we say that the point a and the line ℓ are **incident**.

A k-**chain** is a sequence $(x_0, x_1, \ldots, x_k)$ of elements $x_i \in \mathcal{P} \cup \mathcal{L}$ with the property that x_i is incident with x_{i-1} for $i = 1, \ldots, k$. In this case, we say that the **distance** of x_0 and x_k is $d(x_0, x_k) \leq k$, and we say that $d(x_0, x_k) = k$ if there is no j-chain joining x_0 and x_k for $j < k$. Note that $d(x_0, x_k)$ is necessarily even if x_0 and x_k are of the same sort, i.e. if x_0 and x_k are both points or both lines. We call a set $\{x_0, x_1, \ldots, x_{2n-1}\}$ consisting of $2n$ distinct elements an **ordinary** n-**gon** if $(x_0, x_1, \ldots, x_{2n-1}, x_0)$ is a $2n$-chain.

For $x \in \mathcal{P} \cup \mathcal{L}$ we put $D_k(x) = \{y \in \mathcal{P} \cup \mathcal{L} | \ d(x, y) = k\}$. If a is a point, then $D_1(a)$ is called a **line pencil;** if ℓ is a line, then $D_1(\ell)$ is called a **point row**.

Note that if y is in $D_k(x)$, and if z is incident with y, then $d(x, z) = k \pm 1$.

1.2 Polygons: Let $n \geq 3$ be an integer. An incidence structure $\mathfrak{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ is called an n-**gon**, or **generalized** n-**gon**, if it satisfies the following three axioms:

(i) Every element $x \in \mathcal{P} \cup \mathcal{L}$ is incident with at least three other elements.

(ii) For all elements $x, y \in \mathcal{P} \cup \mathcal{L}$ we have $d(x, y) \leq n$.

(iii) If $d(x, y) = k < n$, then there is a unique k-chain $(x_0 = x, x_1, \ldots, x_k = y)$ joining x and y.

Note that the definition of a polygon is self-dual: if we replace $\mathfrak{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ by its dual $\mathfrak{P}^{\text{dual}} = (\mathcal{L}, \mathcal{P}, \mathcal{F}^{-1})$, where $\mathcal{F}^{-1} = \{(\ell, a) | \ (a, \ell) \in \mathcal{F}\}$, then $\mathfrak{P}$ is an n-gon if and only if $\mathfrak{P}^{\text{dual}}$ is an n-gon. *For that reason we will prove or state a result sometimes only for points; the corresponding statement for lines then follows by duality.*

1.3 Remark: What we have called an n-gon is often called a **thick generalized** n-**gon** in the literature [7, 40, 50]. Note that we obtain an **ordinary** n-**gon** if we require that every element is incident with exactly *two* elements (instead of axiom (i)). We have excluded digons, since they are trivial geometries $(\mathcal{P}, \mathcal{L}, \mathcal{P} \times \mathcal{L})$

where every point is incident with every line. Note also that the triangles (3-gons) are precisely the projective planes. As customary, we use the words quadrangle, hexagon and octagon for 4-, 6- and 8-gons, respectively.

1.4 Geometric operations: Let $\mathfrak{P}$ be an n-gon. Suppose that $d(x,y) = k < n$. By axiom (iii), there is a unique element $z \in D_{k-1}(x) \cap D_1(y)$, which we denote by $z = f_k(x,y)$ (in the terminology of buildings, $f_k(x,y)$ is essentially the same as $\mathrm{proj}_y x$, see [50]). The maps f_k are sometimes called the **geometric operations** of the polygon $\mathfrak{P}$.

We will make frequent use of the following fact: if $d(x,y) = k < n$, then $d(x,z) = k+1$ for all $z \in D_1(y)$ except $f_k(x,y)$; if $d(x,y) = n$, then $d(x,z) = n-1$ for all $z \in D_1(y)$. Thanks to condition (i), the set $D_n(x)$ is not empty. Similarly, if $(x_0, x_1, \ldots, x_k)$ is a k-chain with the property that $x_i \neq x_j$ for $0 \leq i < j \leq k$, for some $k \leq n$, then it can be completed into an ordinary n-gon.

1.5 Projectivities: If $d(x,y) = n$, then there is a bijection $[y,x] : D_1(x) \to D_1(y)$, given by $z \mapsto f_{n-1}(z,y)$, with inverse $[x,y]$. We call the map $[y,x]$ a **perspectivity** between x and y; a concatenation of perspectivities is called a **projectivity**, and we put $[x_3,x_2][x_2,x_1] = [x_3,x_2,x_1]$ etc. Thus the set Π of all projectivities of $\mathfrak{P}$ forms a groupoid. The group of all projectivities from x to x is denoted by $\Pi(x)$.

The next lemma basically says that many projectivities exist in an n-gon.

1.6 Lemma: *Let h, ℓ be lines of the n-gon $(\mathcal{P}, \mathcal{L}, \mathcal{F})$. Then there exists an element $x \in D_n(h) \cap D_n(\ell)$ and therefore a projectivity $[h,x,\ell] : D_1(\ell) \to D_1(h)$. Hence the groupoid Π has two components if n is even, and one component if n is odd, that is, if n is odd, then there exist projectivities between any two elements $x, y \in \mathcal{P} \cup \mathcal{L}$, and if n is even, then such a projectivity exists if and only if x and y are of the same sort, i.e. if x and y are both points or both lines. In particular, the isomorphism type of the permutation group $\Pi(x)$ on $D_1(x)$ depends only on the sort of the element x.*

Proof: We have to show that $D_n(h) \cap D_n(\ell)$ is non-empty (cf. [50] 3.30). Choose $x \in D_n(h)$ such that $k = d(x,\ell)$ is maximal. We claim that $k = n$. Otherwise we would have $k \leq n-2$, and we could choose an element $y \in D_1(x)$ with $d(y,\ell) = k+1$. Now pick $z \in D_1(y) \setminus \{f_{n-1}(h,y), f_{k+1}(h,\ell)\}$. Thus $d(z,h) = n$, and $d(z,\ell) = k+2$, contradicting the choice of x. ∎

Suppose that $\{x_0, \ldots, x_{2n-1}\}$ is an ordinary n-gon. For

$$y \in D_1(x_n) \setminus \{x_{n-1}, x_{n+1}\}$$

we may put $\pi = [x_{2n-1}, y, x_1]$. This projectivity maps x_2 to x_{2n-2} and fixes x_0. Applying this construction twice we get the following lemma due to Knarr:

1.7 LEMMA ([25, 1.2]): *The stabilizer* $\Pi(x_1)_{x_0}$ *is transitive on* $D_1(x_1) \setminus \{x_0\}$ *and hence* $\Pi(x_1)$ *is 2-transitive on the set* $D_1(x_1)$. *See also 1.10 and 2.9.*

SCHUBERT CELLS, COORDINATES, AND ALGEBRAIC OPERATIONS. We have divided the point set of a polygon $\mathfrak{P}$ into sets $D_0(a) \cup D_2(a) \cup D_4(a) \cup \cdots$. We need to refine this partition a little more. For example, the point space of a projective plane can be partitioned into a point $\{(\infty)\}$, an affine line $\{(t) | \ t \in T\}$, and an affine plane $\{(s,t) | \ s,t \in T\}$ by means of coordinates; see [20]. To do this in general, we introduce **Schubert cells.**

1.8 Schubert cells: Let u, v be incident elements of the n-gon $\mathfrak{P}$. We put $D_k(u,v) = D_{k+1}(u) \cap D_k(v) = D_k(v) \setminus D_{k-1}(u)$, and we call this set a **Schubert cell.** Note that $D_k(v) = D_k(u,v) \cup D_{k-1}(v,u)$, and that $D_n(u) = D_{n-1}(u,v)$. If (a, ℓ) is a flag, then we have $\mathcal{P} = D_0(\ell, a) \cup D_1(a, \ell) \cup D_2(\ell, a) \cup \cdots$, and thus $\mathcal{P}$ and $\mathcal{L}$ are each partitioned into n Schubert cells. The set $D_{n-1}(u,v)$ is called a **big cell.**

For the remainder of this section, we fix an ordinary n-gon $\{x_0, x_1, \ldots, x_{2n-1}\}$ in the n-gon $\mathfrak{P}$. Note that in the terminology of buildings, the Schubert cell $D_k(x_0, x_1)$ is the preimage of x_k under the retraction of $\mathfrak{P}$ onto the apartment $\{x_0, \ldots, x_{2n-1}\}$ based at the flag $\{x_0, x_1\}$; see [50] 3.3. The set $\bigcup_{i=0}^{2n-1} D_1(x_i)$ is called a **hat-rack.**

1.9 Coordinatization: Consider an element $x \in D_k(x_{2n-1}, x_0)$, and let

$$(x_{2n-1}, x_0, x'_1, x'_2, \ldots, x'_k = x)$$

denote the corresponding $(k+1)$-chain. Note that $d(x'_i, x_{n+i}) = n$, for $i = 1, \ldots, k$, so we may put $t_i(x) = f_{n-1}(x'_i, x_{n+i-1}) \in T_i$, where

$$T_i = D_1(x_{n+i}, x_{n+i-1}).$$

In this way we have attached **coordinates**

$$(t_1(x), t_2(x), \ldots, t_k(x)) \in T_1 \times T_2 \times \cdots \times T_k$$

to the element x. Note that we can recover x from these coordinates: $x'_1 = f_{n-1}(t_1(x), x_0)$, $x'_2 = f_{n-1}(t_2(x), x'_1)$, and so on. Clearly, we may coordinatize the Schubert cells $D_k(x_0, x_{2n-1})$ in a perfectly similar way. Note also that we have a bijection

$$\begin{array}{ccc} D_k(x_{2n-1}, x_0) \times D_1(x_{n+k}) & \to & \bigcup\{\{x\} \times D_1(x) \mid x \in D_k(x_{2n-1}, x_0)\} \\ (x, y) & \mapsto & (x, f_{n-1}(y, x)) \end{array}$$

hence we get a decomposition of $\mathcal{F}$ into n sets of the form $C \times D_1(a)$, where C is a Schubert cell contained in $\mathcal{P}$ and where a is a point.

In modeltheoretic terminology, this amounts to saying that the **definable closure** (see Section 2) of a hat-rack is the whole polygon $\mathfrak{P}$. ∎

This coordinatization uses $2n-2$ 'parameter sets', the sets $T_1, \ldots, T_{n-1}$, and their counterparts for the Schubert cells $D_k(x_0, x_{2n-1})$. Of course, we could choose fixed projectivities between T_i and T_{i-2} and thus reduce the number of parameter sets to two if n is even, and to one if n is odd. Also, in a projective plane one can define a planar ternary ring (and a double loop), and similar results hold for polygons. This leads to quadratic quaternary rings etc.; see [18], [19], [58], [23]. We will use this fact in the classification of Moufang quadrangles of finite Morley rank.

The additive right loop will be important later:

1.10 Addition: Fix an ordinary n-gon $(x_0, \ldots, x_{2n-1})$. Put $D = D_1(x_1, x_0) = D_1(x_0) \setminus \{x_1\}$, and put $0 = x_{2n-1}$. Next choose an element $a \in D_1(x_1) \setminus \{x_2, x_0\}$. For $y \in D$ put $a_y = f_{n-1}(a, [x_n, x_0](y))$ and consider the projectivity

$$\pi_y = [x_0, a_y, x_2, a_0, x_0].$$

It fixes x_1 and maps 0 to y. Note also that $\pi_0 = \mathrm{id}$. Hence we may define maps $\pm : D \times D \to D$ by putting $x + y = \pi_y(x)$ and $x - y = \pi_y^{-1}(x)$. The structure $(D, +)$ is a **right loop,** i.e. satisfies the following identities: $(x+y)-y = (x-y)+y = x$, $0 + x = x + 0 = x - 0 = x$. ∎

REGULARITY AND PAPPIAN POLYGONS. Regularity is an important geometric concept in quadrangles and hexagons. Let $\mathfrak{P}$ be an n-gon, and let $a \in \mathcal{P}$ be a point. For $x \in D_n(a)$ and $0 < i \leq n/2$ put $a^x_{[i]} = D_i(a) \cap D_{n-i}(x)$. The point a is called **distance-i-regular** if for all $x, y \in D_n(a)$ the sets $a^x_{[i]}$ and $a^y_{[i]}$ are either equal or have at most one element in common. Clearly, automorphisms (see below) preserve regularity. A point is called **regular** if it is distance-j-regular, for all $1 < j \leq n/2$. Regular lines are defined dually.

The point a is called **projective** if it is distance-2-regular, and if in addition for all $x, y \in D_n(a)$ the intersection $a^x_{[2]} \cap a^y_{[2]}$ contains a unique element whenever

$a^x_{[2]} \neq a^y_{[2]}$. It is easy to see that the **derived incidence structure**

$$\mathfrak{P}_a^{\triangle} = \Big(\{a\} \cup D_2(a), \{D_1(\ell)|\ \ell \in D_1(a)\} \cup \{a^x_{[2]}|\ x \in D_n(a)\}, \in\Big)$$

is a projective plane if and only if a is projective.

One of the main results of [43, 44] is that if $n = 4, 6$, and if every point of $\mathfrak{P}$ is projective, then $\mathfrak{P}$ is either the symplectic quadrangle or the split Cayley hexagon over some field K, and the derived projective planes are in fact Pappian over K. Therefore, these two types of polygons are called **Pappian** (in addition to the Pappian projective planes).

ROOT GROUPS AND MOUFANG POLYGONS.

1.11 Group actions: Suppose a group G acts on a set X. The action is called **effective** (or **faithful**) if only 1_G fixes all elements of X. It is called **free** if 1_G is the only group element fixing any element of X, and it is called **regular** if it is free and effective (i.e. if G acts sharply transitively).

1.12 Automorphisms: An **automorphism** of a polygon $\mathfrak{P}$ is a permutation of $\mathcal{P} \cup \mathcal{L}$ that maps points onto points, lines onto lines, and flags onto flags. We denote the group of all automorphisms of $\mathfrak{P}$ by $\mathrm{Aut}(\mathfrak{P})$. Note that this coincides precisely with the model theoretic notion of an automorphism of the structure $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ (see Section 2 below).

Suppose that g is an automorphism of the polygon $\mathfrak{P}$ and that $\pi = [x_1, \ldots, x_k]$ is a projectivity from x_k to x_1. Then clearly $\pi^g = [g(x_1), \ldots, g(x_k)]$ is a projectivity from $g(x_k)$ to $g(x_1)$, and $g\pi = \pi^g g|_{D_1(x_1)}$. This simple fact leads to the following lemma.

1.13 LEMMA: *Let g be an automorphism of the n-gon $\mathfrak{P}$, and suppose that g fixes the point row $D_1(\ell)$ elementwise. Let h be any line, and let $h' = g(h)$. Then the restriction $g|_{D_1(h)} : D_1(h) \to D_1(h')$ can be written as a projectivity $\pi = [h', x', \ell, x, h]$ of length at most 4.*

Proof: Choose $x \in D_n(\ell) \cap D_n(h)$, and put $x' = g(x)$. ∎

1.14 Root groups: A **root** of an n-gon $\mathfrak{P}$ is an n-chain $\alpha = (x_0, \ldots, x_n)$ consisting of $(n + 1)$ distinct elements. The **root group** U_α is the group of all automorphisms that fix the set $X = D_1(x_1) \cup D_1(x_2) \cup \cdots \cup D_1(x_{n-1})$ elementwise (note that we disregard the elements x_0, x_n). The group U_α fixes x_0 and x_1, and hence acts on the set $D = D_1(x_0) \setminus \{x_1\}$. The elements of U_α are called **root elations**.

The group $\Sigma = \langle U_\alpha | \ \alpha$ a root$\rangle$ generated by all root groups is called the **little projective group** of the polygon $\mathfrak{P}$. It is a normal subgroup of $\mathrm{Aut}(\mathfrak{P})$.

There are strong connections between root elations and projectivities: let $\{x_0, \ldots, x_{2n-1}\}$ be an ordinary n-gon, and consider the root $\alpha = (x_0, \ldots, x_n)$. Consider also the addition defined in 1.10 (with respect to some $a \in D_1(x_1) \setminus \{x_2, x_0\}$). We use the notation of 1.10. Let $g \in U_\alpha$ be a root elation, with $g(0) = c$. Because of $g(a_0) = g(a_c)$, it follows from 1.13 that $g(x) = [x_0, a_c, x_2, a_0, x_0]x = x + c$ for all $x \in D$, and, similarly, that $g^{-1}(x) = x - c$.

1.15 Lemma: *The action of the root group U_α on*

$$D = D_1(x_1, x_0) = D_1(x_0) \setminus \{x_1\}$$

is free.

Proof: Suppose that $g \in U_\alpha$ fixes $x \in D$. Then it fixes also the ordinary n-gon $\{x_0, \ldots, x_{2n-1} = x\}$ determined by this element. Pick $a \in D_1(x_1) \setminus \{x_0, x_2\}$ and consider the projectivity $\pi = [x_0, f_{n-1}(a, x_{n+1}), x_2]$. Since $\pi = \pi^g$, and since g fixes $D_1(x_2)$ elementwise, it fixes $D_1(x_0)$ elementwise. It follows that g fixes the hat-rack $\bigcup_{j=0}^{2n-1} D_1(x_j)$ elementwise, and hence, by the coordinatization, fixes $\mathfrak{P}$ elementwise. ∎

In particular, we have:

1.16 Lemma: *The map $g \mapsto g^{-1}(0)$ is an imbedding of U_α into $(D, +)$.*

Proof: Clearly, we have $(hg)^{-1}(0) = g^{-1}(h^{-1}(0)) = h^{-1}(0) + g^{-1}(0)$. It follows from the definition of the subtraction map that $(hg^{-1})^{-1}(0) = g(h^{-1}(0)) = h^{-1}(0) - g^{-1}(0)$. Since the action of U_α is free, the map is injective. ∎

1.17 Definition: The root α is called **Moufang** if the group U_α acts transitively (and thus regularly) on the set D or, equivalently, on the set of all ordinary n-gons containing $x_0, \ldots, x_n$. The polygon is called **Moufang** if every root is Moufang.

So in a Moufang polygon we may identify the root group U_α with the additive loop $(D, +)$. Moreover, there is the following result due to Knarr [25]:

1.18 Proposition: *Let $\Pi^+(x)$ denote the group of all projectivities which can be written as a concatenation of an* **even** *number of projectivities. Clearly, $\Pi^+(x)$ is a normal subgroup of index at most 2 in $\Pi(x)$. If $\mathfrak{P}$ is Moufang, then $\Sigma_x|_{D_1(x)} = \Pi^+(x)$ [25, 2.3]. If n is even, and if x is distance-$n/2$-regular, then* *the identity can be written as a projectivity of length* 3, *so* $\Pi(x) = \Pi^+(x)$; *see Knarr [25, 1.5].*

It has been proved by Tits [53] and Weiss [59] that Moufang n-gons exist only for $n = 3, 4, 6, 8$. All Moufang polygons have been classified by Tits although a complete proof has not yet been published [51]. We will inspect the Moufang polygons in Section 3.

2. Polygons of finite Morley rank

SOME MODEL THEORY. Let T be a countable first-order theory in some fixed language L, and let M be a model of T, which for technical reasons we assume to be saturated (see [10] or [38]). An **automorphism** of M is a permutation of the elements in M which preserves all relations in the language L. The automorphism group of a structure therefore depends on and, as we will see below, determines the underlying language. We will usually consider the language $L_{pol} = \{\mathcal{P}, \mathcal{L}, \mathcal{F}\}$ consisting of unary predicates $\mathcal{P}$ and $\mathcal{L}$ and a binary predicate $\mathcal{F}$, for which the modeltheoretic notion of automorphism coincides with the definition given in 1.12.

We denote the group of automorphisms of M fixing a set $A \subset M$ pointwise by $\mathrm{Aut}(M/A)$ and write $\mathrm{Aut}(M)$ for $\mathrm{Aut}(M/\emptyset)$. If $\bar{a} \in M^n, B \subseteq M$, then the **type** of $\bar{a}$ over B is defined as $\mathrm{tp}(\bar{a}/B) = \{\phi(\bar{x}, \bar{b}) | \ \bar{b} \subseteq B, \phi(\bar{a}, \bar{b}) \text{ holds}\}$ and $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{a}/\emptyset)$.

The set $A \subseteq M^n$ is **definable** (over $B \subseteq M$) if there is some L-formula $\phi(\bar{x}, \bar{b})$ (with parameters $\bar{b} \subseteq B$) such that $\phi(\bar{x}, \bar{b})$ is satisfied exactly by the elements of A. A set is called **0-definable** if it is definable over the empty set. If there is no danger of confusion, we usually identify A with its defining formula $\phi(\bar{x}, \bar{b})$ (or even, e.g., the predicate $\mathcal{P}$ with the set of points in a polygon). In the special case where $A = \{a_1, \ldots, a_n\}$ is a finite set, the elements $a_i, i = 1, \ldots, n$, are **algebraic** over B, and if $\phi(\bar{x}, \bar{b})$ is satisfied by a single element a_1, then a_1 is said to be **definable** over B. For $B \subseteq M$, $\mathrm{acl}(B)$ and $\mathrm{dcl}(B)$ denote the set of elements of M algebraic (resp. definable) over B.

An **interpretable** set is a set of the form A/E, where $A \subseteq M^n$ is a definable set and E a definable n-ary equivalence relation on A. It is clear that the interpretable sets comprise the definable ones by taking E to be equality. To say that a structure (M', L') is interpretable in some model M means that its underlying set M', the **universe**, as well as all relations in the language L' are interpretable in M, e.g. a polygon, group, etc. with L' the corresponding language.

To facilitate working with interpretable sets we extend L into a many-sorted language L^{eq} by adding for any 0-definable (in L) equivalence relation E on M^n a sort S_E and an n-ary function symbol f_E. Any L-structure M can be canonically embedded into an L^{eq}-structure M^{eq}: The elements of sort S_E in M^{eq} are the equivalence classes modulo E in M^n. We identify $S_=$ with M. The interpretation of f_E is a function from $S_=^n$ to S_E taking any n-tuple to its equivalence class modulo E. Any automorphism of M uniquely determines an automorphism of M^{eq}. There is a unique way of extending T to a theory T^{eq} in this language and for any model N of T there is a unique extension N^{eq} to a model of T^{eq}. If M' is interpretable in M, the **induced structure** of M on M' consists of the traces of the 0-definable sets in M^{eq} on M'. We refer the reader to [38] for details of this construction.

2.1 Fact: If M is a saturated L_1-structure and $L_2 \subseteq L_1$ is a sublanguage with the property that $\mathrm{Aut}_{L_1}(M) = \mathrm{Aut}_{L_2}(M)$, then $L_1 = L_2$ in the sense that there are exactly the same 0-definable sets. (See e.g. [38], chapter 9.)

2.2 Morley Rank: The **Morley rank** $\mathrm{RM}(A)$ of an interpretable set A is defined inductively as follows:

(i) $\mathrm{RM}(A) \geq 0$ if $A \neq \emptyset$.

(ii) $\mathrm{RM}(A) \geq \alpha + 1$ for an ordinal α if there are pairwise disjoint interpretable sets $A_i \subseteq A$ for $i < \omega$ with $\mathrm{RM}(A_i) \geq \alpha$.

(iii) $\mathrm{RM}(A) \geq \delta$ for a limit ordinal δ if $\mathrm{RM}(A) \geq \alpha$ for all $\alpha < \delta$.

We say $\mathrm{RM}(A) = \alpha$ if $\mathrm{RM}(A) \geq \alpha$ and not $\mathrm{RM}(A) \geq \alpha + 1$. In that case we define the **Morley degree** $\deg(A)$ to be the biggest n such that there are pairwise disjoint $A_1, \ldots, A_n \subseteq A$ such that $\mathrm{RM}(A_i) = \alpha$ for all $i = 1, \ldots, n$.

Note that $\mathrm{RM}(A) = 0$ if and only if A is finite. Thus the Morley rank and Morley degree measure the complexity of the definable (or interpretable) subsets of a given set. Note that in general $\mathrm{RM}(A)$ need not be finite or even ordinal valued. We say that a structure interpretable in M has finite Morley rank if its universe (as an interpretable set) has finite Morley rank. If M itself has finite Morley rank, then all structures interpretable in M have finite Morley rank as well, and in that case we say that the theory T has finite Morley rank. For $\bar{a} \in M^n, B \subseteq M$, we define

$$\mathrm{RM}(\bar{a}/B) = \mathrm{RM}(\mathrm{tp}(\bar{a}/B)) = \min\{\mathrm{RM}(\phi(\bar{x}, \bar{b}))|\ \phi(\bar{x}, \bar{b}) \in \mathrm{tp}(\bar{a}/B)\}.$$

If M has finite Morley rank, we can make the following definitions: For $\bar{a} \in M^n$ and $B, C \subseteq M$, we say that $\bar{a}$ is independent from B over C if $\mathrm{RM}(\bar{a}/C) = \mathrm{RM}(\bar{a}/C \cup B)$ (see e.g. [35] for more details).

If A is definable (or interpretable) over B and $a \in A$ is such that $\mathrm{RM}(a/B) = \mathrm{RM}(A)$, we say that a is a **generic** of A over B. Note that if $\deg(A) = 1$, then all generics have the same type as there are no disjoint definable subsets having the same rank as A.

If T is $\aleph_1$-categorical (i.e. there is a unique model of T of cardinality $\aleph_1$ (up to isomorphism)), then the Morley rank is finite and has the useful property of being additive: For all $a, b, A \subset M^{eq}$ it satisfies

$$\mathrm{RM}(ab/A) = \mathrm{RM}(a/Ab) + \mathrm{RM}(b/A).$$

POLYGONS OF FINITE MORLEY RANK. We now prove some general model theoretic results about polygons. For simplicity we assume that our language contains predicates for points, lines and flags. But everything we say remains true also if the polygon and these predicates are merely interpretable in some other structure.

Our first statement is true without the assumption of finite Morley rank:

2.3 PROPOSITION: *Let $\mathfrak{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ be a generalized n-gon. Let $\{x_0, \ldots, x_{2n-1}\}$ be an ordinary n-gon and set*

$$X = D_1(x_0) \cup D_1(x_1) \cup \{x_0, \ldots, x_{2n-1}\}.$$

Then $\mathcal{P} \cup \mathcal{L} \subseteq \mathrm{dcl}(X)$.

Proof: Since for x, y with $d(x, y) = n$ the projectivity $[x, y]$ defines a bijection between $D_1(x)$ and $D_1(y)$, it follows easily that $D_1(x_i) \subseteq \mathrm{dcl}(X)$ for $i = 0, \ldots 2n-1$. The rest follows from 1.9. ∎

If there is a definable surjection from $D_1(x_0)$ to $D_1(x_1)$, which is the case, e.g., if n is odd, or if the polygon has a definable polarity, one can drop $D_1(x_1)$ at the expense of adding the necessary parameters for the surjection in order to get that the polygon is in the definable closure of a point row and a finite set (see also [50] Chap. 4, Theorem 1.1).

Next, we will show that the set of points in an infinite $\aleph_1$-categorical polygon is necessarily of degree 1. We should point out that the Moufang polygons of finite Morley rank all turn out to be $\aleph_1$-categorical.

2.4 Definition: Let $\mathfrak{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ be a generalized n-gon of finite Morley rank. Let a be a point and let ℓ be a line. We put $m = \mathrm{RM}(D_1(\ell))$ and $m' = \mathrm{RM}(D_1(a))$, and we call the numbers (m, m') the **parameters** of $\mathfrak{P}$.

The parameters (m, m') do not depend on the specific elements a, ℓ, and $m = m'$ if n is odd: If a, b are points, then there exists a projectivity $[b, x, a]$, hence the sets $D_1(a)$ and $D_1(b)$ have the same Morley rank; if n is odd, then projectivities exist also between points and lines.

2.5 Example: Let $\mathfrak{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ be a polygon and let (a, ℓ) be a flag. Suppose that a group $G \subseteq \mathrm{Aut}(\mathfrak{P})$ acts transitively on the set $\mathcal{F}$ of flags. Then $\mathfrak{P}$ is isomorphic to the coset geometry $(G/G_a, G/G_\ell, \{(ga, g\ell)|\ g \in G\})$. If the structure $(G, G_a, G_\ell, 1, \cdot)$ has finite Morley rank, then the resulting polygon $\mathfrak{P}$ is interpretable in this structure and has finite Morley rank.

In particular, the Pappian polygons associated to the simple algebraic groups $\mathbf{PSL}_3(K)$, $\mathbf{PSp}_4(K)$, $\mathbf{G}_2(K)$ over some algebraically closed field K (i.e. the projective plane, the symplectic quadrangle and the split Cayley hexagon over K) have finite Morley rank (and parameters $(1,1)$, as we will see). The full automorphism groups of these polygons are semidirect products of the groups themselves extended by the group of all field automorphisms of K [50] 5.10.

2.6 Remark: Since the Pappian polygons have projective points, there is a definable bijection between the point rows and the line pencils even if n is even — the lines in their derived projective plane are either ordinary point rows, or sets of the form $a^x_{[2]}$, which are in definable bijection with line pencils. So it follows from 2.3 that the Pappian polygons are contained in the definable closure of one of their point rows and a finite collection of points.

2.7 Definition: Let $\mathfrak{P}$ be an n-gon of finite Morley rank, with parameters (m, m'). It is convenient to make the following definitions: $m_k = m + m' + m + m' + \cdots$ (k summands) and $m'_k = m' + m + m' + \cdots$ (k summands).

We will need the following formulas for the Morley rank of various sets.

2.8 PROPOSITION: *Let $\mathfrak{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ be an $\aleph_1$-categorical n-gon with parameters (m, m'). Let $a \in \mathcal{P}$ be a point. Then $\mathrm{RM}(D_k(a)) = m_k$ for $k < n$, and $\mathrm{RM}(D_n(a)) = m'_{n-1}$. Moreover $\mathrm{RM}(\mathcal{P}) = m_{n-1}$, $\mathrm{RM}(\mathcal{L}) = m'_{n-1}$, and $\mathrm{RM}(\mathcal{F}) = n(m + m')/2 = m_n = m'_n$. In particular, $\mathfrak{P}$ is finite if and only if $m = m' = 0$.*

Proof: This is clear from the coordinatization in 1.9. ∎

2.9 PROPOSITION: *Let $\mathfrak{P}$ be an $\aleph_1$-categorical n-gon and suppose that $m > 0$. Let $\ell \in \mathcal{L}$ be a line. If $a, b, c, d \in D_1(\ell)$ are independent generics, we can find a* *finite set A and an A-definable projectivity $\pi \in \Pi(\ell)$ with $\pi(a) = c$ and $\pi(b) = d$ such that a, b are independent of A over ℓ.*

Proof: We first construct a projectivity $\pi_1 = [\ell, y, h, x, \ell] \in \Pi(\ell)$ with $\pi_1(a) = c$ with $\{a, b\}$ generic over $\{y, h, x, \ell\}$.

Choose $h \in D_2(\ell)$ independent of $\{a, b, c, d\}$ over ℓ (note that this implies that $a, b, c, d \notin D_1(h)$). Next choose $x \in D_n(h) \cap D_n(\ell)$ independent of $\{h, \ell, a, b, c, d\}$. Put $s = [h, x, \ell](a)$ and $o = f_2(h, \ell)$. Next choose an ordinary n-gon

$$\{x_0 = o, x_1 = h, x_2 = s, \ldots, x_{2n-2} = c, x_{2n-1} = \ell\}$$

and let $y \in D_1(x_n) \setminus \{x_{n-1}, x_{n+1}\}$. Now $\pi_1 = [\ell, y, h, x, \ell]$ takes a to c fixing o and it is left to show that we can choose y such that $\{x, h, y\}$ is independent of $\{a, b\}$ over ℓ.

Since the ordinary n-gon $\{x_0, \ldots, x_{2n-1}\}$ is uniquely determined by the element $x_{n-1} \in D_{n-3}(h, s)$ and since every $y \in D_1(x_n) \setminus \{x_{n-1}, x_{n+1}\}$ works for π_1, the set $Y = \{y \in D_n(h) \cap D_n(\ell) | \ [\ell, y, h](s) = c\}$ has Morley rank $m'_{n-3} + m'$ (note that $m = m'$ if n is odd).

If $y \in Y$ is generic over $\{h, \ell, x, a, b, c\}$, we have the following rank inequalities:

$$\begin{aligned}
m'_{n-1} &\geq \mathrm{RM}(y)\\
&\geq \mathrm{RM}(y/h, \ell, x)\\
&\geq \mathrm{RM}(y/h, \ell, x, a, b)\\
&= \mathrm{RM}(y, c/h, \ell, x, a, b) \qquad (\text{since } c = \pi_1(a))\\
&= \mathrm{RM}(y/h, \ell, x, a, b, c) + \mathrm{RM}(c/h, \ell, x, a, b)\\
&= (m'_{n-3} + m') + m = m'_{n-1},
\end{aligned}$$

showing that y is indeed independent of $\{a, b\}$ over h, ℓ, x. Since $\{x, h\}$ is, by choice, independent of $\{a, b\}$ over ℓ, it follows that $\{x, h, y\}$ is independent of $\{a, b\}$ over ℓ, as desired.

We repeat this construction to find a projectivity $\pi_2 = [\ell, y', h', x', \ell]$ fixing c and taking $\pi_1(b)$ to d, where we have to choose $h' \in D_1(c)$ independent of $\{a, b, d\}$ over ℓ. As above, we can find x' and y' with the required properties. ∎

2.10 THEOREM: *Let $\mathfrak{P} = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ be an $\aleph_1$-categorical polygon with $m > 0$. Let $\ell \in \mathcal{L}$ be a line. Then the point row $D_1(\ell)$ has Morley degree 1.*

Proof: Let $\phi(x) \subset D_1(\ell)$ be a formula (possibly with parameters) of maximal rank and degree 1. If $D_1(\ell)$ does not have degree 1, we can find generics $a, b, c, d \in D_1(\ell)$ independent over ℓ with $a, b, c \in \phi(x)$ and $d \in \neg\phi(x)$.

By the previous corollary there is some A-definable projectivity π with $\pi(a) = c, \pi(b) = d$ and a, b generic over A. But now a satisfies $\phi(x) \wedge \phi(\pi(x))$, whereas b satisfies $\phi(x) \wedge \neg\phi(\pi(x))$, contradicting the assumption that ϕ has degree 1. ∎

2.11 COROLLARY: *If $\mathfrak{P}$ is an $\aleph_1$-categorical polygon with $m, m' > 0$, then $\mathcal{P}, \mathcal{L}$ and $\mathcal{F}$ all have Morley degree 1.*

Proof: This is immediate from 2.10 and 1.9. ∎

The previous corollary was proved by Hrushovski for the case $n = 3$ and generalized by Nesin to odd n (see Theorem 12.7 in [6] and [32]). However, the proof in [6] is wrong for $n \geq 5$ (p. 262, last paragraph). The formulas 2.8 for the Morley rank of the Schubert varieties are also proved in [32].

3. The classification

By the results of Tits and Weiss [53, 59] Moufang n-gons exist only for $n = 3, 4, 6, 8$. Now we are going to consider each class of Moufang n-gons individually. We will assume the classification of the Moufang polygons as stated by Tits and Weiss in [57] (this classification includes one class of Moufang quadrangles overlooked by Tits in [51]).

The classification of Moufang projective planes amounts to the classification of alternative fields; see [34]. For the Moufang octagons the classification may be found in [54]. For the Moufang hexagons and quadrangles see [51] and [57]. A discussion of the classification without proof may be found in [58].

The following lemmata will be used in our classification of Moufang polygons of finite Morley rank.

3.1 LEMMA: *Let K be a field. In the group $\mathrm{PSL}_2(K)$, a copy K' of the field K is definable.*

Proof: Consider the standard action of $\mathbf{PSL}_2(K)$ on the projective line $\mathbb{P}^1_K = K \cup \{\infty\}$. Let $u_b \in \mathbf{PSL}_2(K)$ denote the transvection $x \mapsto x + b$. The centralizer of $u_1 = (x \mapsto x+1)$ in $\mathbf{PSL}_2(K)$ is $U = \{u_b = (x \mapsto x+b) | \, b \in K\}$, and this group is isomorphic to the additive group $(K, +)$. The stabilizer of ∞ is the normalizer B of U; thus $B = \mathbf{PSL}_2(K)_\infty = \{g | \, [g^{-1}u_1 g, u_1] = 1\}$ is definable as well, and so is the torus $T = B \cap B^g = \{(x \mapsto a^2 x) | a \in K^2\}$, where $g \in \mathbf{PSL}_2(K) \setminus B$. The torus T acts on U by $t_{a^2} : u_b \mapsto u_{a^2 b}$ and we may identify it with the subset

$V = \{u_{a^2} | \ a \in K\} \subseteq U$. We transfer the multiplication from T to V and denote it by .

If $\mathrm{char}(K) = 2$, then $(V, +, \cdot)$ is isomorphic to the subfield $K^2 \subseteq K$. If $\mathrm{char}(K) \neq 2$, then $4a = (1+a)^2 - (1-a)^2$, whence $V - V = U$, and we can extend the multiplication $\cdot$ from the set V of squares to all of U. This yields in either case $(K, +, \cdot) \cong (U, +, \cdot)$. ∎

Next we observe the following (using the same notation as in 1.10):

3.2 LEMMA: *Let $\mathfrak{P}$ be a polygon. The additive right loop $(D, +)$ is definable. If the root $\alpha = (x_0, \ldots, x_n)$ is Moufang, then the root group U_α is isomorphic to $(D, +)$ (and therefore definable), and acts definably on the polygon.*

Proof: The first part follows from the remarks after 1.17: we may identify U_α with $(D, +)$. But it is clear from the coordinatization how to extend the action of $(D, +)$ from D to a definable action on $\mathcal{P} \cup \mathcal{L}$, and the Moufang condition guarantees that this extension is an automorphism. ∎

The following lemma will be also important for the classification of Moufang polygons of finite Morley rank.

3.3 LEMMA: *Let $\mathfrak{P}$ be a Moufang polygon (not necessarily of finite Morley rank). Suppose that one of the groups of projectivities $\Pi(x)$ is (as a permutation group) isomorphic to a subgroup of $\mathbf{PGL}_2(K)$, for some field K. Then $\mathbf{PSL}_2(K)$ is a definable subgroup of $\Pi(x)$.*

Proof: Since the group $\Pi(x)$ is 2-transitive, it contains $\mathbf{PSL}_2(K)$ by Lemma 4.9 below. Let $o \in D_1(x)$. The stabilizer $\Pi(x)_o$ contains a unique normal abelian subgroup $U_o \cong (K, +)$ which acts regularly on $D_1(x) \setminus \{o\}$ (namely the group of strictly upper triangular matrices). On the other hand, the additive right loop $(D, +)$ (which is in the case of a Moufang polygon the same as a root group by the lemma above) is a set of fixed-point free projectivities that acts transitively on $D_1(x) \setminus \{o\}$. Therefore the group $U_o \cong (D, +)$ is definable. Pick another element $\infty \in D_1(x) \setminus \{o\}$ and consider the corresponding group U_∞ (which we may identify with the group of all strictly lower triangular matrices). It follows readily from the Gaussian algorithm that $\mathbf{PSL}_2(K) = U_o U_\infty U_o U_\infty U_o$, hence this group is definable. ∎

If the polygon $\mathfrak{P}$ in the lemma above is infinite and of finite Morley rank, then it follows from 3.1 that the field K is algebraically closed. The same proof shows also the following:

3.4 LEMMA: *Let $\mathfrak{P}$ be an infinite Moufang polygon of finite Morley rank, and let ℓ be a line. If the restriction $\Sigma_\ell|_{D_1(\ell)}$ is contained in $\mathbf{PGL}_2(K)$, for some field K, then K is algebraically closed, and $\Sigma_\ell|_{D_1(\ell)} \cong \mathbf{PSL}_2(K)$ is a definable subgroup of $\Pi(x)$.*

MOUFANG PROJECTIVE PLANES. The triangles are precisely the projective planes. A Moufang projective plane is coordinatized by an alternative field, see [20] — this fact was proved first by Moufang [30], and that's where the terminology comes from. It is clear from the definition that the coordinatizing alternative field of a Moufang projective plane of finite Morley rank is definable and hence has finite Morley rank. Thus the classification is reduced to the classification of alternative fields of finite Morley rank. By the results of Macintyre [29], Cherlin [11], and Rose [41] an infinite alternative field of finite Morley rank is an algebraically closed field. Thus an infinite projective Moufang plane of finite Morley rank is Pappian over some algebraically closed field.

MOUFANG QUADRANGLES. According to the classification of Moufang quadrangles by Tits and Weiss [57], there are (up to duality) the following types of Moufang quadrangles:

(1) *Orthogonal quadrangles.* These belong to quadratic forms of Witt index 2 over fields. They live in projective spaces of dimension ≥ 4. The orthogonal quadrangle in $\mathbb{P}^4$ is — via the Klein correspondence — the dual of the symplectic quadrangle over the same field (the symplectic quadrangle consists of the totally isotropic subspaces of the symplectic form in a four dimensional vectorspace).

(2) *Hermitian quadrangles.* These belong to hermitian or pseudo-quadratic forms over (skew) fields which admit an involutive antiautomorphism [8]. They live in projective spaces of dimension ≥ 3.

(3) *Mixed quadrangles.* These are certain subquadrangles of symplectic quadrangles over fields of characteristic 2, containing orthogonal quadrangles over a subfield [52, 51].

(4) *Exceptional quadrangles of type* $\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$ associated to forms of algebraic groups of type $\mathbf{E}_6$, $\mathbf{E}_7$, and $\mathbf{E}_8$ [56].

(5) *Exceptional quadrangles of type* $\mathbf{F}_4$, discovered recently by Weiss. By an unpublished result of Mühlherr and Van Maldeghem, they are associated to certain involutions in mixed buildings of type $\mathbf{F}_4$. The name of this class is not standard and has been given by Van Maldeghem in [58] in connection with the association with these $\mathbf{F}_4$-buildings of mixed type.

The underlying vectorspaces in the first two cases need not be finite dimensional.

First we treat some special cases.

3.5 PROPOSITION: *An infinite orthogonal quadrangle $\mathfrak{P}$ has finite Morley rank if and only if its underlying field K is algebraically closed. Since a quadratic form over an algebraically closed field has always maximal Witt index, this implies that the corresponding projective space is four dimensional (so the quadrangle is dual to the symplectic quadrangle over K).*

Proof: Let ℓ be a line. The little projective group induces a subgroup of $\mathbf{PGL}_2(K)$ on $D_1(\ell)$, because every root elation of $\mathfrak{P}$ is induced by a linear map of the ambient vectorspace (so $\Sigma \subseteq \mathbf{PGL}(V)$). By 3.4 the field K is algebraically closed. ∎

Projective planes are coordinatized by planar ternary rings. Similarly, generalized quadrangles are coordinatized by quadratic quaternary rings; see [18], [19]. We use this coordinatization method to get rid of some more quadrangles.

3.6 PROPOSITION: *An infinite mixed quadrangle $\mathfrak{P}$ has finite Morley rank if and only if the underlying field K is algebraicaly closed (in which case it is in fact an orthogonal quadrangle).*

Proof: We will first give an explicit description of the quadrangle: there is a field K of characteristic 2 and a subfield K' containing the subfield K^2 of all squares of K. Furthermore, there are sub-vectorspaces L and L' of K and K' respectively, viewed as vectorspaces over K' and K^2 respectively. The subsets L and L' generate K and K', respectively (the latter viewed as a ring).

Let ∞ be an element not contained in K. The points of $\mathfrak{P}$ can be presented as (∞), (a), (k,b), (a,l,a') with $a,a',b \in L$ and $k,l \in L'$. The lines of $\mathfrak{P}$ can be presented as $[\infty]$,[k], $[a,l]$, $[k,b,k']$ with $a,b \in L$ and $k,k',l \in L'$. Incidence is given by

$$[k,b,k'] \text{ I } [k, ak+a', a^2k+l] \text{ I } (a,l,a') \text{ I } [a,l]$$
$$\text{I } (a) \text{ I } [\infty] \text{ I } (\infty) \text{ I } [k] \text{ I } (k,b) \text{ I } [k,b,k'],$$

where 'I' stands for 'is incident with'. The elements (∞), $[\infty]$, (0), $[0,0]$, $(0,0,0)$, $[0,0,0]$, $(0,0)$, $[0]$ form an ordinary quadrangle. For this description, see Hanssens –Van Maldeghem [18].

From this description it is clear that the abelian groups L and L' act as root groups with roots $([0,0,0],(0,0),[0],(\infty),[\infty])$ and $((0,0,0),[0,0],(0),[\infty],(\infty))$, respectively, and hence $(L,+)$ and $(L',+)$ are definable. Unfortunately, there is no obvious way to define K from $\mathfrak{P}$. Instead we will define an orthogonal quadrangle contained in it and show that it coincides with $\mathfrak{P}$.

We define an inclusion $L' \subseteq L$ by identifying $k \in L'$ with the unique $b \in L$ such that the point (k, b) is collinear with $(1, 0, 0)$. Note that the point $(1, 0, 0)$ plays no special role, as the automorphism group of the symplectic quadrangle containing $\mathfrak{P}$ acts transitively on the ordinary pentagons contained in it; see e.g. Joswig [22]. The multiplication $a \cdot k$, with $a \in L$ and $k \in L'$, can be recovered as follows: the point $(k, a \cdot k)$ is given by $f_3((a, 0, 0), [k])$. The set K'' of all $k \in L'$ such that $L' \cdot k = L'$ is clearly a subfield of K and it contains K^2. Moreover, it is definable. Restricting coordinates to K'', we obtain a definable orthogonal quadrangle [58]. By 3.5 the field K'' is algebraically closed. This implies however $K''^2 = K''$, hence $K^2 = K$ and $\mathfrak{P}$ is an orthogonal quadrangle. ∎

3.7 PROPOSITION: *Let K be an infinite proper skew field with an involutive antiautomorphism σ. The σ-hermitian quadrangle $\mathfrak{P}$ in $\mathbb{P}^3_K$ does not have finite Morley rank.*

Proof: For the proof it will suffice to show that the skew field K (and the involution σ) is definable in $\mathfrak{P}$ as a proper skew field does not have finite Morley rank.

We use the following description of $\mathfrak{P}$ (see [58]):

The points are the elements (∞), (a), (k, b) and (a, l, a') with $a, a', b \in K$ and $k, l \in K_\sigma = \{t^\sigma - t \mid t \in K\}$. The lines are the elements $[\infty]$, $[k]$, $[a, l]$ and $[k, b, k']$ with $a, b \in K$ and $k, k', l \in K_\sigma$. Incidence is given by

$$(a, l, a') \ \mathrm{I} \ [a, l] \ \mathrm{I} \ (a) \ \mathrm{I} \ [\infty] \ \mathrm{I} \ (\infty) \ \mathrm{I} \ [k] \ \mathrm{I} \ (k, b) \ \mathrm{I} \ [k, b, k'],$$

and (a, l, a') is incident with $[k, b, k']$ if and only if

$$\begin{cases} k' = l + a^\sigma k a + a a'^\sigma - a' a^\sigma, \\ b = a' - ak. \end{cases}$$

In fact, this description of $\mathfrak{P}$ comes from the standard embedding of $\mathfrak{P}$ in a 3-dimensional projective space $\mathbb{P}^3_K$ over K. For the proof it will be useful to have the coordinates of the points and lines of $\mathfrak{P}$ in $\mathbb{P}^3_K$. This is given in the following table, where we denote the line incident with two points x and y by $\langle x, y\rangle$:

POINTS	
Coordinates in $\mathfrak{P}$	Points in $\mathbb{P}^3_K$
(∞)	$(1,0,0,0)$
(a)	$(a,0,1,0)$
(k,b)	$(-b,1,k,0)$
(a,l,a')	$(l+aa'^{\sigma},-a^{\sigma},a'^{\sigma},1)$

LINES	
Coordinates in $\mathfrak{P}$	Lines in $\mathbb{P}^3_K$
$[\infty]$	$\langle(1,0,0,0),(0,0,1,0)\rangle$
$[k]$	$\langle(1,0,0,0),(0,1,k,0)\rangle$
$[a,l]$	$\langle(a,0,1,0),(l,-a^{\sigma},0,1)\rangle$
$[k,b,k']$	$\langle(-b,1,k,0),(k',0,b^{\sigma},1)\rangle$

Let B be the set of points

$$B=\{(\infty),(0),(0,0),(0,0,0),(1),(0,1),(1,0,0),(0,0,1)\}.$$

Notice that the line $[\infty]$ is definable from B, and the point row defined by $[\infty]$ is the projective line $K\cup\{\infty\}$.

Now consider $\mathfrak{P}$ in the language of polygons extended by parameters for B, a predicate $\hat{K}$ for the point row defined by $[\infty]$ minus the point (∞), and let $+,\cdot,\sigma$ be function symbols with the obvious interpretation on $\hat{K}$.

The proposition will now follow from 2.1 and the following lemma:

3.8 Lemma: $\mathrm{Aut}(\mathfrak{P},B)=\mathrm{Aut}(\mathfrak{P},B,\hat{K},+,\cdot,\sigma)$.

Proof: Note that clearly $\mathrm{Aut}(K,+,\cdot,\sigma)=\{\phi\in\mathrm{Aut}(K)|\ \phi\sigma=\sigma\phi\}$, and also $\mathrm{Aut}(\mathfrak{P},B,\hat{K},+,\cdot,\sigma)\subseteq\mathrm{Aut}(\mathfrak{P},B)$. So suppose that $\phi\in\mathrm{Aut}(\mathfrak{P})$ fixes B elementwise. By Tits [50] 8.6.II, every automorphism of $\mathfrak{P}$ is induced by a semi-linear mapping of $\mathbb{P}^3_K$. Let us denote by ϕ^* the extension of ϕ to $\mathbb{P}^3_K$. Since ϕ fixes B elementwise, ϕ^* fixes the points $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$, $(1,0,1,0)$, $(-1,1,0,0)$, $(0,-1,0,1)$ and $(0,0,1,1)$. So, by the fundamental theorem of projective geometry, ϕ is (induced by) a field automorphism. It is obvious that the point (a) is mapped onto the point (a^{ϕ}) (because $(a,0,1,0)$ in $\mathbb{P}^3_K$ is mapped onto $(a^{\phi},0,1,0)$ by ϕ^*). Hence $(a,0,0)$ in $\mathfrak{P}$ is mapped onto $(a^{\phi},0,0)$. This means that ϕ^* maps $(0,-a^{\sigma},0,1)$ onto $(0,(-a^{\phi})^{\sigma},0,1)$, but clearly $(0,-a^{\sigma},0,1)$ is mapped onto $(0,(-a^{\sigma})^{\phi},0,1)$. Hence $\sigma\phi=\phi\sigma$, so $\phi\in\mathrm{Aut}(K,+,\cdot,\sigma)$, and in fact $\phi\in\mathrm{Aut}(\mathfrak{P},B,\hat{K},+,\cdot,\sigma)$. ∎

It follows that $(\mathfrak{P}, B, \hat{K}, +, \cdot, \sigma)$ and $(\mathfrak{P}, B)$ have the same 0-definable sets; i.e. the field structure of K is definable (in the polygon language) in $\mathfrak{P}$ and hence K is commutative, a contradiction.

3.9 COROLLARY: *Let $\mathfrak{P}$ be an infinite Moufang quadrangle of finite Morley rank. If $\mathfrak{P}$ has a regular line (cf. Section 1) and if $\mathfrak{P}$ is not dual to a hermitian quadrangle over a skew field of characteristic 2 in k-dimensional space, with $k > 3$, then $\mathfrak{P}$ is the orthogonal quadrangle over some algebraically closed field K (and hence the dual of the symplectic quadrangle over K).*

Proof: If $\mathfrak{P}$ is a Moufang quadrangle with regular lines, then $\mathfrak{P}$ is either an orthogonal quadrangle, or a quadrangle of mixed type, or the dual of a hermitian quadrangle over a proper skew field in three dimensional projective space, or the dual of a hermitian quadrangle over a proper skew field of characteristic 2 in k-dimensional space, with $k > 3$. ∎

3.10 THEOREM: *Let $\mathfrak{P}$ be an infinite Moufang quadrangle of finite Morley rank. Then $\mathfrak{P}$ is either the orthogonal quadrangle, or its dual, the symplectic quadrangle, over some algebraically closed field K.*

Proof: Suppose that $\mathfrak{P}$ has no regular line and no regular point. Let U_i, $i \in \{1, 2, 3, 4, 5, 6, 7, 8\}$, be the set of root groups corresponding to the eight roots in an ordinary quadrangle, ordered as in Tits [55]. Then, according to the classification of Moufang quadrangles (see also Tits [55]), we may assume that U_2 is commutative. According to Tits [56], the restriction of U_{2i+1} to $[U_{2i}, U_{2i+2}]$ defines a subquadrangle $\mathfrak{Q}$ with either regular lines or regular points, and such that $\mathfrak{Q}$ is not isomorphic or dual to a hermitan quadrangle over some skew field of characteristic 2 in k-dimensional space, $k > 3$. This subquadrangle is definable by the coordinatization process — we merely restrict the coordinates from U_{2i+1} to the definable subsets $[U_{2i}, U_{2i+2}]$. By the previous theorem $\mathfrak{Q}$ is isomorphic to the orthogonal quadrangle over an algebraically closed field, or to the symplectic quadrangle over an algebraically closed field. The only Moufang quadrangles extending orthogonal quadrangles are the exceptional ones, but in that case either $\mathfrak{Q}$ lives in a 9, 11 or 15 dimensional projective space (exceptional quadrangles of type $\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$), or the underlying field is not perfect (exceptional quadrangles of type $\mathbf{F}_4$) which is impossible over an algebraically closed field. ∎

MOUFANG HEXAGONS.

3.11 PROPOSITION: *Let $\mathfrak{P}$ be an infinite Moufang hexagon of finite Morley rank. Then $\mathfrak{P}$ is isomorphic to the split Cayley hexagon over some algebraically closed field K.*

Proof: We may assume that every point of the hexagon $\mathfrak{P}$ is regular; see [39]. Let x be a point and ℓ_1, ℓ_2 two distinct lines incident with x. Let G_i, $i = 1, 2$, be the definable group acting on the set of lines through x obtained from the action of any root group fixing all elements incident with ℓ_i and acting faithfully on the lines through x. Then G_1 and G_2 generate $\mathbf{PSL}_2(K)$, for some infinite field K. Hence by the Gaussian algorithm, $\mathbf{PSL}_2(K)$ is definable, hence K is algebraically closed.

More geometrically, one could alternatively argue as follows; see [58]. For two opposite points x, y, let $x^y = x^y_{[2]}$ denote the set of points collinear with x and not opposite y, cf. Section 1. Now fix x and y. The set of points

$$\bigcup\{v^u | u \in x^y \text{ and } v \in y^x, d(u,v) = 6\}$$

forms the point set of a (definable) projective plane $\mathfrak{R}$ the lines of which are the sets a^b for b a point of $\mathfrak{R}$ and a an element of the set

$$\bigcup\{u^v | u \in x^y \text{ and } v \in y^x, d(u,v) = 6\}.$$

The plane $\mathfrak{R}$ is Pappian over some field K, hence by the previous section, K is algebraically closed.

The Moufang hexagons besides the split Cayley hexagon are either associated to field extensions of degree 3 over K, to proper subfields of characteristic 3 of K containing K^3, or to certain simple Jordan division algebras (including some skew fields) over K. Since K is algebraically closed, these do not exist over K. ∎

MOUFANG OCTAGONS.

3.12 LEMMA: *Let $\mathfrak{P}$ be a Moufang octagon and let x be a point of $\mathfrak{P}$. Up to duality, we may assume that the group $\Pi(x)$ is isomorphic to a general Suzuki group $\mathrm{GSz}(K,\sigma)$, where σ is an endomorphism of K whose square is the Frobenius endomorphism $x \mapsto x^2$. Then every projectivity in $\Pi(x)$ that fixes two elements (lines) of $D_1(x)$ has length ≤ 4.*

Proof: Using the coordinatization of $\mathfrak{P}$ as in [24], we can parametrize $D_1(x)$ by the set $K^{(2)}_\sigma \cup \{(\infty)\}$ (see also [54]) where $K^{(2)}_\sigma = \{(a,b) \in K^2\}$ is a group with operation $(a,b) \oplus (c,d) = (a+c, b+d+ca^\sigma)$. For $k \in K$, one defines $k \otimes (a,b) = (ka, k^{1+\sigma}b)$. The stabilizer in $\Pi(x)$ of $(0,0)$ and (∞) is the group $\{\varphi_k : (a,b) \mapsto k \otimes (a,b) | \; k \in K \setminus \{0\}\}$ (according to [24], Theorem F). By the same reference, there is a generalized homology θ fixing all lines through a point y collinear with x and inducing φ_k on $D_1(x)$. If z is a point with $d(x,z) = d(y,z) = n$, then the projectivity $[x, \theta(z), y, z, x]$ induces φ on $D_1(x)$. ∎

3.13 PROPOSITION: *There exists no infinite Moufang octagon of finite Morley rank.*

Proof: Let x be a point of the octagon. It follows from the previous lemma that every projectivity fixing two elements in $D_1(x)$ has lenght ≤ 4. Thus $\Pi(x)$ is bounded and hence definable. The commutator group of $\Pi(x)$ is an infinite Suzuki group, hence a 2-transitive Zassenhaus group of finite Morley rank in which the stabilizer of a point splits as $H \rtimes T$ with H containing a central involution (see e.g. [46] 12.52 – 12.54). But this contradicts the classification of such groups by De Bonis–Nesin (see [6] 11.90). ∎

SUMMARY: MOUFANG POLYGONS OF FINITE MORLEY RANK. Putting together the information about Moufang polygons of finite Morley rank, we have the following results:

3.14 THEOREM: *Let $\mathfrak{P}$ be an infinite Moufang polygon of finite Morley rank. Then the little projective group Σ is definably isomorphic to a simple linear K-algebraic group of K-rank 2 where K is an algebraically closed field definable in $\mathfrak{P}$, and precisely one of the following cases occurs:*

(i) *$\mathfrak{P}$ is definably isomorphic to the projective plane over K with $\Sigma \cong$ $\mathbf{PSL}_3(K)$.*

(ii) *$\mathfrak{P}$ is definably isomorphic to the symplectic quadrangle over K with $\Sigma \cong$ $\mathbf{PSp}_4(K)$.*

(iii) *$\mathfrak{P}$ is definably isomorphic to the split Cayley hexagon over K with $\Sigma \cong$ $\mathbf{G}_2(K)$.*

In particular, each projectivity group in $\mathfrak{P}$ is (as a permutation group) definably isomorphic to the action of $\mathbf{PSL}_2(K)$ on the projective line $\mathbb{P}^1_K$. Thus $\mathfrak{P}$ has parameters $(1,1)$.

Moreover, the group Σ is the unique maximal definable subgroup of $\mathrm{Aut}(\mathfrak{P}) \cong$ $\Sigma \rtimes \mathrm{Aut}(K)$.

Proof: We have shown that $\mathfrak{P}$ is abstractly isomorphic to one of the three polygons above.

First we prove that the little projective groups are definable. The full automorphism groups of these three Moufang polygons are semidirect products of their little projective groups Σ and the group $\mathrm{Aut}(K)$ of all field automorphisms; see [50] 5.10. The group Σ is generated by all root groups; it follows from [4, I.2.2] that there is a finite sequence of roots $(\alpha_1, \ldots, \alpha_k)$ such that $\Sigma = U_{\alpha_1} \cdots U_{\alpha_k}$. This implies that Σ is a definable group.

Now we want to show that the isomorphisms are definable. It is proved in [25] that in each of the three cases the point rows and line pencils are projective lines with the groups of projectivities being isomorphic to $\mathbf{PSL}_2(K)$ (in [25] this fact is stated only for finite polygons, but the proofs given for the symplectic quadrangle and the split Cayley hexagon go through over infinite fields). By 3.3, all the groups of projectivities are definable and we can define an algebraically closed field K on the point rows of $\mathfrak{P}$ by 3.1. Clearly, once we have defined K, we can also define $\mathbf{PSL}_2(K)$ and the corresponding isomorphic Pappian polygon $\mathfrak{P}'$, say, inside $\mathfrak{P}$.

Let $\Pi(x)$ be a group of projectivities in $\mathfrak{P}$. We identify the field K with $D_1(x) \setminus \{x_0\}$ for some $x_0 \in D_1(x)$. Since $\Pi(x) \cong \mathbf{PSL}_2(K)$ is sharply 3-transitive on $D_1(x)$, it is clear that we can definably identify $\Pi(x)$ with $\mathbf{PSL}_2(K)$.

Since $\mathfrak{P}$ is in the definable closure of $D_1(x) \cup X$ for some finite set X by 2.5, it is enough to show that we can find an isomorphism between $\mathfrak{P}$ and $\mathfrak{P}'$ whose restriciton to $D_1(x)$ is definable. Let ψ be any isomorphism from $\mathfrak{P}$ to $\mathfrak{P}'$ (not necessarily definable). Then there is a K-algebraic (and hence definable) isomorphism ϕ from $D_1(x)$, which we have identified with $\mathbb{P}^1_K$, to $D_1(\psi(x))$.

It follows from [15] that the set of isomorphisms between the permutation groups $(\Pi(x), D_1(x))$ and $(\Pi(\psi(x)), D_1(\psi(x))$ is in one-to-one correspondence with $\mathbf{PGL}_2(K) \rtimes \mathrm{Aut}(K)$ (by composing one fixed isomorphism with all automorphisms of the projective line). Since $\mathrm{Aut}(\mathfrak{P}) = \Sigma \rtimes \mathrm{Aut}(K)$, it follows readily that $\mathrm{Aut}(\mathfrak{P}/x)|_{D_1(x)} \cong \mathbf{PGL}_2(K) \rtimes \mathrm{Aut}(K)$, so there exists an extension of ϕ to $\mathfrak{P}$.

Finally, if $\Sigma' \leq \mathrm{Aut}(\mathfrak{P})$ is definable, then $\Sigma' \cap \Sigma$ is normal in Σ' and $\Sigma'/(\Sigma \cap \Sigma')$ is a definable group of field automorphisms, hence reduced to the identity by [21, Prop. 3]. ∎

Thus, we have established Theorem A.

4. Spherical buildings

In this section we give a short exposition of spherical buildings. There are several excellent references on this subject: the books by Ronan [40], Brown [7], and Taylor [46], the chapter on buildings in the book by Suzuki [45], the (very readable) overview article by Scharlau [42], and finally Tits' monograph [50].

Buildings may be introduced in different ways, e.g. as complexes, as diagram geometries, or as chamber systems. We will consider spherical buildings as complexes, i.e. as posets with certain geometric properties.

4.1 Complexes: A k-**simplex** is a poset (partially ordered set) $(S, \subseteq)$ which is as a poset isomorphic to the powerset 2^I of a finite set I of cardinality k. The number k is called the **rank** of the simplex. A **complex** is a poset $(\Delta, \subseteq)$ with the following two properties:

Cplx$_1$: for every $A \in \Delta$, the poset $\{B \in \Delta |\ B \subseteq A\}$ is a simplex.

Cplx$_2$: any two elements $A, B \in \Delta$ have a unique greatest lower bound $A \sqcap B$.

A complex has a unique minimal element $\emptyset$. If the elements A and B have a common upper bound, then their least upper bound is denoted by $A \sqcup B$, and we say that $A \sqcup B$ **exists.**

A **morphism** between two complexes is an order-preserving map whose restriction to each simplex is an isomorphism. A subset of a complex is called a **subcomplex** if the inclusion is a morphism.

A **chamber** in a complex Δ is a maximal element. If A is contained in a chamber C, and if $\operatorname{rank} C - \operatorname{rank} A = 1$, then A is called a **panel of** C. Two chambers are called **adjacent** if they have a panel in common. A complex is called a **chamber complex** if

ChCplx$_1$: every element is contained in some chamber, and

ChCplx$_2$: given two chambers C, C' there exists a **gallery**, i.e. a sequence $C = C_0, C_1, \ldots, C_k = C'$ of chambers such that C_i is adjacent with C_{i-1} for $i = 1, \ldots, k$.

The **rank** of a chamber complex is the rank of its chambers; by **ChCplx$_2$**, this is well-defined. To avoid confusion with the Morley rank, we will call this the **Tits rank**. A chamber complex is called **thin** if every panel is contained in precisely two chambers; it is called **thick** if every panel is contained in at least 3 chambers.

A **numbering** or **type function** of a chamber complex $(\Delta, \subseteq)$ is a surjective morphism $type : \Delta \to 2^I$ where the **type set** I is some finite set (this should not be confused with the modeltheoretic notion of a type given in Section 2).

Before we come to the notion of a building, we have to consider a special class of thin chamber complexes, the Coxeter complexes. The relation between Coxeter complexes and buildings is very much the same as the relation between ordinary polygons and polygons.

4.2 Coxeter complexes: Let I be a finite set. A **Coxeter matrix** over I is a symmetric matrix $(m_{ij})_{i,j\in I}$ with integral entries, subject to the conditions $m_{ii}=1$ and $m_{ij}\geq 2$ for $j\neq i$. We assign to it a **Coxeter diagram:** it is the graph that has the elements of I as nodes, where the nodes i and j are joined by $m_{ij}-2$ strokes, or by a single stroke labeled m_{ij} if $m_{ij}>3$. Finally, we assign to each Coxeter matrix the **Coxeter system** (W,I), where the **Coxeter group** W is presented as $W=\langle I|\ (ij)^{m_{ij}}=1\rangle$. The subgroups $P_J=\langle J\rangle$, where $J\subseteq I$, are called the **parabolic subgroups;** they are again Coxeter groups. The set $\mathbf{A}=\{wP_Jw^{-1}|\ w\in W \text{ and } \emptyset\neq J\subseteq I\}$, ordered by the reversal of the inclusion, is a thin numbered chamber complex (with minimal element $P_I=W$) of Tits rank $|J|$, the **Coxeter complex**. Its type function is given by $gP_Jg^{-1}\mapsto I\setminus J$.

The Coxeter diagram of a Coxeter complex is uniquely determined by its isomorphism class.

A Coxeter complex is called **spherical** if it is finite or equivalently if its Coxeter group W is finite. It is called **irreducible** if its diagram is connected, or equivalently if it is not the join of two Coxeter complexes.

Finally, we can give the definition of a building.

4.3 Buildings: Let $(\Delta,\subseteq)$ be a chamber complex. A subcomplex $\mathbf{A}\subseteq\Delta$ is called an **apartment** if $\mathbf{A}$ is a Coxeter complex. A thick chamber complex $\mathfrak{B}=(\Delta,\subseteq)$ is called a **building** if

Bldg$_1$ Given two chambers $C,C'\in\Delta$, there exists an apartment $\mathbf{A}$ containing both chambers.

Bldg$_2$ Given two apartments $\mathbf{A},\mathbf{B}\subseteq\Delta$ there exists an isomorphism $\mathbf{A}\to\mathbf{B}$ fixing the intersection $\mathbf{A}\cap\mathbf{B}$ elementwise.

By **Bldg$_2$** the isomorphism type of the apartments is uniquely determined, and we may assign a Coxeter diagram to the building $\mathfrak{B}$. Moreover, the numbering of the apartments extends to a numbering $type:\Delta\to 2^I$.

We call the building $\mathfrak{B}$ **spherical** or **irreducible** if its Coxeter diagram has the corresponding properties. A reducible building is the join of two buildings.

The generalized polygons are precisely the spherical irreducible buildings of Tits rank 2: given an n-gon $\mathfrak{P}=(\mathcal{P},\mathcal{L},\mathcal{F})$, put

$$\mathfrak{B}=(\{\emptyset\}\cup\mathcal{P}\cup\mathcal{L}\cup\{\{a,\ell\}|\ (a,\ell)\in\mathcal{F}\},\subseteq).$$

The apartments in a polygon are the ordinary polygons contained in it, and its Coxeter group is the dihedral group of order $2n$.

Another important class of buildings are the finite dimensional projective spaces: the chamber complex associated to such a projective space is the set of all flags, ordered by inclusion. The type of a subspace is its (projective) dimension.

4.4 Roots and the Moufang condition: Let $\mathbf{A}$ be a spherical Coxeter complex. A **folding** is an idempotent homomorphism $\phi : \mathbf{A} \to \mathbf{A}$ which is two-to-one on the chambers in $\mathbf{A}$. The image $\alpha = \phi(\mathbf{A})$ of a folding is called a **root**. Each panel in $\mathbf{A}$ determines a root: given two distinct adjacent chambers $C, C' \in \mathbf{A}$, there exists precisely one root containing C and not containing C'. Every root α has an **opposite root** $-\alpha$, obtained by switching the roles of C and C'. Their common **wall** $\partial\alpha$ is given by $\partial\alpha = (-\alpha) \cap \alpha$.

Let $\mathfrak{B}$ be a spherical building. We let $\mathrm{Aut}(\mathfrak{B})$ denote the group of all type-preserving automorphisms of the poset $\mathfrak{B}$ (this group is denoted by $\mathrm{Spe}(\mathfrak{B})$ in [50], and $\mathrm{Aut}(\mathfrak{B})$ is reserved there for the group of all order-preserving, but not necessarily type-preserving automorphisms. The latter kind of automorphisms is not important here, so to keep our notation consistent with the case of polygons, we deviate from [50] at this point). Now let $\mathbf{A}$ be an apartment in $\mathfrak{B}$, and let $\alpha \subseteq \mathbf{A}$ be a root. The **root group** U_α is the subgroup of automorphisms in $Aut(\mathfrak{B})$ fixing all chambers that have a panel in $\alpha \setminus \partial\alpha$. The root α is called **Moufang** if the group U_α acts transitively on the set of all apartments containing α. The building is called a **Moufang building** if every root is Moufang. For polygons, this is precisely the Moufang condition stated in Section 1.

One of the main results in [50] is that every spherical irreducible building of Tits rank ≥ 3 is Moufang. In [50] Tits also classified the irreducible spherical buildings of Tits rank ≥ 3 explicitly. This is a far-reaching generalization of the well-known fact that every projective space of dimension ≥ 3 is desarguesian.

4.5 Residues: Let $\mathfrak{B} = (\Delta, \subseteq)$ be a building, and let $A \in \Delta$. The **residue** or **star** of A in Δ is defined as $\mathrm{Res}_\Delta A = \{B \in \Delta \mid B \supseteq A\}$. Endowed with the induced ordering this is again a building of Tits rank $(\mathrm{rank}\,\Delta - \mathrm{rank}\,A)$. Its Coxeter diagram is the restriction of the Coxeter diagram of Δ to the set $I \setminus type(A)$. If $\mathfrak{B}$ is an irreducible spherical building of Tits rank ≥ 3, then its irreducible residues of Tits rank 2 are either Moufang projective planes or Moufang quadrangles. If A is a panel and if $C \supseteq$ is a chamber, then we put $S_i(C) = \mathrm{Res}_\Delta A$, where $\{i\} = I \setminus type(A)$. If $I = \{1, \ldots, k\}$, then each chamber is contained in precisely k rank one residues $S_1(C), \ldots, S_k(C)$, and the intersection of these residues is precisely $\{C\}$. The union $S_1(C) \cup \cdots \cup S_k(C)$ is denoted by $E_1(C)$ in [50].

4.6 Strongly transitive groups and BN-pairs: Let $\mathfrak{B}$ be a spherical building. If a group $G \subseteq \mathrm{Aut}(\mathfrak{B})$ acts transitively on the pairs $(\mathbf{A}, C)$, where $\mathbf{A}$ is an apartment containing and C a chamber contained in $\mathbf{A}$, then the group G has a (saturated) BN-pair or **Tits system** (G, B, N, S): fix such a pair $(\mathbf{A}, C)$, put $B = G_C$, and let N denote the group of all elements of G fixing $\mathbf{A}$ setwise. The group $W = N/(N \cap B)$ may be identified with the Coxeter group of the building. It has a distinguished set $\{s_i | \, i \in I\}$ of involutive generators. The subgroups of G containing B are called the **parabolic subgroups**; they correspond precisely to the stabilizers G_A, where $A \subseteq C$. Such a group G is called **strongly transitive**. We refer the reader to [7], [40], [45], [46], [50] for more details about BN-pairs.

If $\mathfrak{B}$ is a spherical Moufang building, and if Σ is the group generated by all root groups, then Σ acts strongly transitively on $\mathfrak{B}$. As in the case of polygons, we call Σ the **little projective group** of the building. If G acts strongly transitively on $\mathfrak{B}$, and if G induces the little projective group Σ on $\mathfrak{B}$, then the BN-pair of G will be called **Moufang**.

4.7 Some examples: Spherical buildings of rank ≥ 3 are typically associated to the following classes of groups:

(1) **Algebraic groups.** Let K be an infinite field, and let G be an isotropic adjoint absolutely simple algebraic K-group (an algebraic K-group G is called isotropic if it contains a K-split torus, and it is called absolutely simple if $G(L)$ is simple for every algebraically closed field extension L/K. 'Adjoint' means that it is in a certain sense minimal among all K-groups which are isogeneous to G, or, in other words, that it acts effectively on its building. See also [5, 49]). If G has a K-split torus of K-rank ≥ 2, then the groups of the K-rational points of the K-parabolic subgroups of G give rise to a Moufang building whose little projective group is given by the group of K-rational points $G(K)$. See [5, 4, 42] and in particular [50] Chap. 5 for more details.

(2) **Classical groups.** Let G be a classical group over a skew field D acting on a projective space. By a classical group we mean the unimodular group $\mathbf{PSL}_k D$, or orthogonal resp. unitary groups belonging to (pseudo-) quadratic forms of finite Witt index ≥ 2 in (possibly infinite-dimensional) D-vectorspaces; see [8], [50], [12], [17], [46].

(3) **(Twisted) Chevalley groups**. These groups are twisted by some field and diagram automorphism; these need not be algebraic groups (an example in rank 2 are the twisted Ree groups ${}^2F_4(K)$). See the book by Carter [9].

The Borel (i.e. minimal parabolic) subgroups of these buildings need **not** be solvable.

UNIQUENESS OF THE BN-PAIR IN PAPPIAN POLYGONS AND BUILDINGS. We call a spherical irreducible building **Pappian** if its irreducible residues of rank 2 are Pappian polygons.

4.8 THEOREM: *A spherical building is Pappian if and only if it arises from an isotropic K-split absolutely simple adjoint K-algebraic group. Thus, the Pappian buildings of Tits rank ≥ 3 are the following:*

- $\mathbf{A_k}$ *the Pappian projective spaces over fields.*
- $\mathbf{B_k}$ *the polar spaces belonging to orthogonal groups of maximal Witt index in even-dimensional projective spaces over fields.*
- $\mathbf{C_k}$ *the polar spaces belonging to symplectic groups in odd-dimensional projective spaces over fields.*
- $\mathbf{D_k}$ *the oriflame geometries belonging to orthogonal groups of maximal Witt index in odd-dimensional projective spaces over fields.*
- $\mathbf{E_k}$ $(k = 6, 7, 8)$ *the unique E_k-buildings over fields.*
- $\mathbf{F_4}$ *the metasymplectic spaces belonging to K-split groups of type F_4 over fields K.*

Proof: This is immediate from Tits' classification [50]. ∎

The automorphism group of a Moufang or even Pappian polygon can have several quite different BN-pairs; see e.g. [50, 11.14] for a particularly bad example. However, the little projective group Σ of a Pappian polygon or building does not contain any proper strongly transitive subgroup, as we will see. First we need a lemma about the smallest case, i.e. $\mathbf{PGL}_2(K)$.

4.9 LEMMA ([48, 3.2]): *Let K be a field and let $G \subseteq \mathbf{PGL}_2(K)$ be a two-transitive subgroup (with respect to the usual action on the projective line $\mathbb{P}^1_K = K \cup \{\infty\}$). Then $\mathbf{PSL}_2(K) \subseteq G$; in particular, $\mathbf{PSL}_2(K)$ has no proper two-transitive subgroups.*

Proof: Consider the stabilizer G_∞ acting on K. Since G is two-transitive, the action of G_∞ on K is still transitive; hence for each $b \in K$ there exists an $g_b \in G_\infty$ with $g_b(0) = b$. We want to show that G_∞ contains the maps $u_b : x \mapsto x + b$ for all $b \in K$. If $g_b = (x \mapsto ax + b) \neq u_b$, then $a \neq 1$, and g_b has the unique fixed point $c = b/(1-a)$ in K. Then $g_c^{-1} g_b g_c$ fixes 0, hence $g_c^{-1} g_b g_c = (x \mapsto ax)$, and thus $u_b = g_b g_c^{-1} g_b^{-1} g_c \in G_\infty$. Now let $g \in G \setminus G_\infty$. Since the transvections $\{u_b, g^{-1} u_b g \mid b \in K\}$ generate $\mathbf{PSL}_2(K)$, the claim follows. ∎

The Pappian polygons are the polygons associated to the absolutely simple K-algebraic groups of K-rank 2 which are K-split.

The point rows and the line pencils of these polygons are projective lines $\mathbb{P}^1_K$, and the group induced by Σ on such a point row or line pencil is $\mathbf{PGL}_2(K)$.

4.10 PROPOSITION: *Let Σ be the little projective group of a Pappian n-gon $\mathfrak{P}$ over a field K with at least four elements. If $G \subseteq \Sigma$ is strongly transitive on $\mathfrak{P}$, then $G = \Sigma$.*

Proof: Let x_0 be any vertex in $\mathfrak{P}$. Since $\mathfrak{P}$ is Pappian, the induced group Σ_x on $D_1(x_0)$ is $\Sigma_x|_{D_1(x_0)} = \mathbf{PGL}_2(K)$, except if $\mathfrak{P}$ is the symplectic quadrangle and x_0 is a point, in which case $\Sigma_x|_{D_1(x_0)} = \mathbf{PSL}_2(K)$. Let x_n be a vertex opposite to x_0. Since G acts strongly transitively on $\mathfrak{G}$, the group G_{x_0,x_n} acts still two-transitively on $D_1(x_0)$. By Lemma 4.9 above, $\mathbf{PSL}_2(K) \subseteq G_{x_0,x_n}|_{D_1(x_0)}$.

Let $\{x_0, \ldots, x_{2n-1}\}$ be an ordinary n-gon, and let $\alpha = (x_0, \ldots, x_n)$. We want to show that $U_\alpha \subseteq G_{x_0,x_1,\ldots,x_n}$. We may identify $D_1(x_0) \setminus \{x_1\} = D$ with K. Then $G_{x_0,\ldots,x_n} = G_{x_0,x_1,x_n}$ induces the group $\{(x \mapsto a^2x + b) | \ a, b \in K\}$ on D.

Now let $a \in K$ be an element with $a^2 \neq 1$, and let $t \in G_{x_0,x_1,\ldots,x_{2n-1}} = G_{x_{2n-1},x_0,x_1,x_n}$ be an element which induces the map $x \mapsto a^2x$ on D. Note that t is contained in the torus $T = \Sigma_{x_0,\ldots,x_{2n-1}} \cong K^* \times K^*$ of Σ. For every $b \in K$ we can find an element $g \in G_{x_0,\ldots,x_n}$ which induces the map $x \mapsto x + b/(1-a^2)$ on D. The element g can be written as $g = us$, where $s \in T$ is a torus element, and $u \in U_\alpha$ is a root elation. The commutator $[t, g]$ induces the map $x \mapsto x + b$ on D, and we claim that $[t, g] \in U_\alpha$:

$$[t, g] = [t, us] = tust^{-1}s^{-1}u^{-1} = (tut^{-1})u^{-1} \in U_\alpha,$$

since $tut^{-1} \in U_\alpha$. Therefore G contains all root groups, and thus $G = \Sigma$. ∎

Proposition 4.10 holds also over the fields $\mathbb{F}_2$ and $\mathbb{F}_3$, as one can check directly.

4.11 THEOREM: *Let K be a field, and let G be an absolutely simple isotropic K-split K-algebraic group; let $G^+ \subseteq G(K)$ denote the normal subgroup generated by all root groups $U(K)$ (thus G^+ is the little projective group of the corresponding building). Let $H \subseteq G(K)$ be an abstract subgroup which acts strongly transitively in the building of $G(K)$, i.e. a subgroup with $H \cdot T(K) = G(K)$. Then $G^+ \subseteq H$. Thus, if K is algebraically closed, then $H = G(K)$.*

Proof: If G has K-rank 1, then this is Lemma 4.9. The case of K-rank 2 is 4.10. The proof for K-rank ≥ 3 is literally the same as in 4.10—the ordinary n-gon has to be replaced by an apartment $\mathbf{A}$, and x_0, x_1 by a pair of opposite panels. ∎

A special case of Theorem 4.11 is proved in [6, pp. 282–285]: there the assumptions are that K is algebraically closed and that H is a definable subgroup. The proof is quite long and uses some non-trivial modeltheoretic results.

5. Application: simple groups of finite Morley rank

5.1 THEOREM: *Let $\mathfrak{B} = (\Delta, \subseteq)$ be an infinite irreducible spherical building of Tits rank ≥ 3 and of finite Morley rank. Then $\mathfrak{B}$ is the building associated to a simple linear algebraic group G over some algebraically closed field K. The field K is definable in $\mathfrak{B}$, and the little projective group Σ of $\mathfrak{B}$ is definably isomorphic to the algebraic group G.*

Proof: This follows directly from 3.14 and 4.8. For the diagrams which have only single strokes (type A_k, D_k, E_6, E_7, E_8) there is a shortcut, since the residues of rank 2 are projective planes over the corresponding field K, hence K is directly interpretable in the building. The case of polar spaces (type B_k) and metasymplectic spaces (type F_4) requires the classification 3.14 of Moufang quadrangles of finite Morley rank. (This case is missing in [6], Fact 12.38.)

The proof that Σ is a definable group which is definably isomorphic to G is more or less the same as the one given in 3.14 for the polygons. To see that the isomorphism is definable, one can use [50] 4.1.1. to reduce to a problem about definable isomorphisms between copies of $\mathbf{PSL}_2(K)$. ∎

We have classified the infinite spherical irreducible Moufang buildings of finite Morley rank and Tits rank ≥ 2. Now we want to apply this result to simple groups of finite Morley rank. Of course we want to consider these groups in the pure group language $\{e, \cdot, ^{-1}\}$, so we have to show that a spherical Moufang building is interpretable in its little projective group Σ. The following proposition, which is the main step towards this, is due to B. Mühlherr.

5.2 PROPOSITION: *Let $\mathfrak{B}$ be an irreducible spherical building of Tits rank $k \geq 2$, let $G \subseteq \mathrm{Aut}(\mathfrak{B})$ be a group containing the little projective group, and let $B = G_C$ be the stabilizer of a chamber C. Then there is an infinite subgroup Z of B which is definable in the pure group G.*

Proof: For $i = 1, \ldots k$, let $\alpha_i \subseteq \mathbf{A}$ denote the root which is determined as follows: the wall of α_i is given by the reflection on the i-panel of C, and $-\alpha_i$ contains

C. Note that the root group U_α fixes C. Pick an element $u_i \in U_{\alpha_i} \setminus \{\mathrm{id}\}$. Then $\mathrm{Fix}(u_i) \cap S_i(C) = \{C\}$; see [42, 5.3.12.(b)].

Now let H denote the group generated by $\{u_1, \ldots, u_k\}$. Our next claim is that H fixes no chamber except for C: suppose that H fixes a chamber $D \neq C$. Let $C = C_0, \ldots, C_r = D$ be a minimal gallery. Since H fixes C and D, it fixes C_1 as well. But $C_1 \in S_i(C) \setminus \{C\}$ for some i, and so u_i has to fix C_1, a contradiction.

Now consider the centralizer $Z = \mathcal{Z}_G(H)$. This is a definable group, and since $\mathrm{Fix}(H) \cap \mathrm{Cham}\,\mathfrak{B} = \{C\}$, the group Z is contained in B. It remains to show that Z is infinite.

Let us first consider the case that $\mathfrak{B}$ is not an octagon. Then there is a root system $\Phi \subseteq \mathbb{R}^k$ (which may have multiple root vectors). The roots of $\mathbf{A}$ correspond precisely to the half-rays $\{t\rho |\ t > 0\}$ where $\rho \in \Phi$ is a root vector, and the commutators of these groups and the root groups can be determined in terms of the root vectors, i.e. $[U_\rho, U_\sigma] \subseteq U_{(\rho,\sigma)}$, cf. [5] 2.5.

Let $\alpha_1, \ldots, \alpha_k$ be simple roots corresponding to the reflections on the walls of the Weyl chamber $C \subseteq \mathbf{A}$, and let β be a **maximal** root vector. Then we have $[U_\alpha, U_\beta] \subseteq U_{(\alpha,\beta)} = 0$ for all $\alpha \in \{t\alpha_i |\ t > 0, i = 1, \ldots, k\}$, so the infinite group U_β is contained in Z.

In the case of octagons the same argument goes through if the axioms of a root system are slightly changed, cf. [55]. ∎

5.3 THEOREM: *Let G be a group of finite Morley rank. Suppose that G acts effectively and strongly transitively on an infinite irreducible Moufang building $\mathfrak{B}$ of Tits rank ≥ 2. If either G is Moufang (i.e. if G contains the little projective group Σ of $\mathfrak{B}$), or if the parabolic subgroups of G are definable in G, then G is simple and definably isomorphic to a simple K-algebraic group over some algebraically closed field K, and $\mathfrak{B}$ is the standard building of G defined in terms of its Borel subgroups.*

Proof: We show first that the building is interpretable in G, provided that $\Sigma \subseteq G$. Let C be a chamber and put $B = G_C$. We know from 5.2 that an infinite group $Z \subseteq B$ is definable in G. Now let $P \supseteq B$ be a maximal parabolic subgroup. We claim that P is definable. Let Z° denote the connected component of Z; since Z is infinite, Z° is infinite as well. The group $\langle gZ^\circ g^{-1} |\ g \in P\rangle \subseteq P$ is definable by Zil'ber's Indecomposability Theorem and infinite. Hence the normalizer of this group is also definable and contains P. Since the maximal parabolic subgroups are maximal subgroups, and since G is simple, we conclude that P is the normalizer of $\langle gZ^\circ g^{-1} |\ g \in P\rangle$, and hence definable.

Every parabolic subgroup of G is a finite intersection of maximal parabolic subgroups. Hence every parabolic subgroup is definable in G. Since we may identify $\mathfrak{B}$ with the cosets of the parabolic subgroups containing B, the building $\mathfrak{B}$ is interpretable in G (and has finite Morley rank).

It follows from 5.1 that $\mathfrak{B}$ is Pappian over an algebraically closed field K. It remains to show that $G = \Sigma$. Since Σ is interpretable in G, so is their intersection $\Sigma \cap G$. This is a normal subgroup of G, and $G/G \cap \Sigma$ is a definable group of field automorphisms (note that both K and the action of $G/G \cap \Sigma$ on K are interpretable in G), whence $G = G \cap \Sigma = \Sigma$ by 4.11. ∎

This establishes Theorem B.

In the case of Tits rank at least 3, a similar result to Theorem 5.3 is stated without proof in Borovik–Nesin [6, 12.39], based on [Fact 12.38, loc. cit.]. However, Fact 12.38, which is attributed there to Tits [50], is not correct (and not stated in [50])—the polar and the metasymplectic spaces yield counterexamples.

5.4 COROLLARY: *Let G be an adjoint absolutely simple isotropic K-group of K-rank ≥ 2, where K is an infinite field. Then the group $G(K)$ of K-rational points has finite Morley rank if and only if K is algebraically closed. (See also [60].)*

In particular, for $K = \mathbb{R}$, we obtain:

5.5 COROLLARY: *Let G be a simple noncompact real Lie group of $\mathbb{R}$-rank ≥ 2. Then G has finite Morley rank if and only if G is a complex Lie group.*

This is a complement to a result of A. Pillay and A. Nesin who showed that a compact connected simple Lie group does not have finite Morley rank [33]. See 4.7 for other (non-algebraic) groups covered by our classification.

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