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Classification of Embeddings of the Flag Geometries of Projective Planes in Finite Projective Spaces, Part 1

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The flag geometry $\Gamma = (\mathscr{P}, \mathscr{L}, \mathbf{I})$ of a finite projective plane Π of order s is the generalized hexagon of order $(s, 1)$ obtained from Π by putting $\mathscr{P}$ equal to the set of all flags of Π, by putting $\mathscr{L}$ equal to the set of all points and lines of Π, and where $\mathbf{I}$ is the natural incidence relation (inverse containment), i.e., Γ is the dual of the double of Π in the sense of H. Van Maldeghem (1998, "Generalized Polygons," Birkhäuser Verlag, Basel). Then we say that Γ is fully and weakly embedded in the finite projective space $\mathbf{PG}(d, q)$ if Γ is a subgeometry of the natural point-line geometry associated with $\mathbf{PG}(d, q)$, if $s = q$, if the set of points of Γ generates $\mathbf{PG}(d, q)$, and if the set of points of Γ not opposite any given point of Γ does not generate $\mathbf{PG}(d, q)$. In an earlier paper, we have shown that the dimension d of the projective space belongs to $\{6, 7, 8\}$, and that the projective plane Π is Desarguesian. Furthermore, we have given examples for $d = 6, 7$. In the present paper we show that for $d = 6$, only these examples exist, and we also partly handle the case $d = 7$. More precisely, we completely classify the full and weak embeddings of Γ (Γ as above) in the case that there are two opposite lines L, M of Γ with the property that the subspace of $\mathbf{PG}(d, q)$ generated by all lines of Γ meeting either L or M has dimension 6 (which is the case for all embeddings in $\mathbf{PG}(d, q)$, $d \in \{6, 7\}$). Together with Parts 2 and 3, this will provide the complete classification of all full and weak embeddings of Γ. © 2000 Academic Press

Key Words: generalized hexagons; projective planes; projective embeddings.

1. INTRODUCTION

The problem that we consider in this paper stems from an attempt to characterize the "natural" embeddings of all finite Moufang classical hexagons (these objects were first introduced by Tits [5]). In fact, it is well known that a finite Moufang hexagon of order (s, t) contains a subhexagon of

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order $(1, t)$ or $(s, 1)$ (or both). In order to distinguish these two (non-disjoint) cases, one sometimes calls a finite Moufang hexagon with a subhexagon of order $(1, t)$ *classical*, and one with a subhexagon of order $(s, 1)$ *dual classical*. The natural embeddings in $\mathbf{PG}(d, q)$ of all classical finite hexagons of order (q, t) have been characterized in several ways in Thas and Van Maldeghem [2, 3]. The main tool in these cases is the fact that all lines of Γ through a point of Γ belong to a plane of $\mathbf{PG}(d, q)$. The "natural" embeddings of the dual classical hexagons in general do not longer have that property. Hence one needs new techniques to handle these embeddings. In an earlier paper [4] and in the present paper (which is part of a series of papers), we introduce such a technique, namely, we look first at embeddings of hexagons of order $(q, 1)$ in $\mathbf{PG}(d, q)$. Part of our Main Result is that the embeddings of such geometries Γ of order $(q, 1)$ arising from the "natural" embeddings of the dual classical hexagons are characterized as follows: $d \neq 8$ and it must be a *weak embedding*, i.e., the points of Γ not opposite a given point of Γ do not generate the ambient projective space $\mathbf{PG}(d, q)$ (for precise definitions, see below). We have shown in [4] that the assumption of being weakly embedded implies that Γ must arise from a Desarguesian projective plane as described above, and that $d \in \{6, 7, 8\}$. The distinct cases $d = 6, 7, 8$ will be treated here and in two sequels, since they are quite involved.

2. PRELIMINARIES

2.1. *Definitions*

Let Π be a (finite) projective plane of order s. We define the *flag geometry* Γ of Π as follows. The points of Γ are the flags of Π (i.e., the incident point-line pairs of Π); the lines of Γ are the points and lines of Π. Incidence between points and lines of Γ is reverse containment. It follows that Γ is a (finite) generalized hexagon of order $(s, 1)$ (see (1.6) of Van Maldeghem [6]). The advantage of viewing Γ rather as a generalized hexagon than as a flag geometry of a projective plane is that one can apply results from the general theory of generalized hexagons. We will call Γ a *thin hexagon* (since there are only 2 lines through every point of Γ).

Throughout, we assume that Γ is a thin hexagon of order $(s, 1)$ with corresponding projective plane Π. We introduce some further notation. For a point x of Γ, we denote by $x^{\perp}$ the set of points of Γ collinear with x (two points are *collinear* if they are incident with a common line); we denote by $x^{\perp\!\!\!\perp}$ the set of points of Γ not opposite x (i.e., not at distance 6 from x in the incidence graph of Γ). For a line L of Γ, we write $L^{\perp\!\!\!\perp}$ for the intersection of all sets $p^{\perp\!\!\!\perp}$ with p a point incident with L (in this notation we view L as the set of points incident with it). For an element x of Γ (point or line), we denote by $\Gamma_i(x)$ the set of elements of Γ at distance i from x in the incidence graph of Γ. In this notation, we have $p^{\perp} = \Gamma_0(p) \cup \Gamma_2(p)$, $p^{\perp\!\!\perp} = \Gamma_0(p) \cup \Gamma_2(p) \cup \Gamma_4(p)$ and $L^{\perp\!\!\perp} = \Gamma_1(L) \cup \Gamma_3(L)$.

Let $\mathbf{PG}(d, q)$ be the d-dimensional projective space over the Galois field $\mathbf{GF}(q)$. We say that Γ is *weakly embedded in* $\mathbf{PG}(d, q)$ if the point set of Γ is a subset of the point set of $\mathbf{PG}(d, q)$ which generates $\mathbf{PG}(d, q)$, if the line set of Γ is a subset of the line set of $\mathbf{PG}(d, q)$, if the incidence relation in $\mathbf{PG}(d, q)$ restricted to Γ is the incidence relation in Γ, and if for every point of Γ, the set $x^{\perp\!\!\perp}$ does not generate $\mathbf{PG}(d, q)$. If moreover $s = q$, then we say that the weak embedding is also *full*.

For $d \neq 8$ the only known examples of weak full embeddings of finite thin hexagons in $\mathbf{PG}(d, q)$ arise from full embeddings of the dual classical generalized hexagons of order (q, q). In the next subsection we will give a brief independent description.

We can now state our Main Result.

MAIN RESULT. *If Γ is a thin generalized hexagon weakly and fully embedded in some projective space* $\mathbf{PG}(d, q)$, *and if $L^{\perp\!\!\perp}$ is contained in a 4-dimensional subspace of* $\mathbf{PG}(d, q)$, *for some line L of Γ, then the embedding is one of the examples described below.*

Parts 2 *and* 3. Adding an additional example (for $d = 8$), we will show that the condition "$L^{\perp\!\!\perp}$ is contained in a 4-dimensional subspace of $\mathbf{PG}(d, q)$, for some line L of Γ" can be dropped.

2.2. *The Examples in* $\mathbf{PG}(d, q)$, *with* $d = 6, 7$

Let V be a 3-dimensional vector space over $\mathbf{GF}(q)$, and let V^* be the dual space. We choose dual bases. Then the vector lines of the tensor product $V \otimes V^*$ can be seen as the point-line pairs of the projective plane $\mathbf{PG}(2, q)$. Indeed, it is easily calculated that the pair $\{(x_0, x_1, x_2), [a_0, a_1, a_2]\}$ (we use parentheses for the coordinates of points and brackets for those of lines) corresponds to the vector line generated by the vector $(a_0x_0, a_0x_1, a_0x_2, a_1x_0, a_1x_1, a_1x_2, a_2x_0, a_2x_1, a_2x_2)$. In fact, the point-line pairs of $\mathbf{PG}(2, q)$ are bijectively mapped (and we denote this bijection by θ) onto the Segre variety $\mathcal{S}_{2;2}$ in $\mathbf{PG}(8, q)$; see Hirschfeld and Thas [1, Sect. 25.5]. We denote coordinates in $\mathbf{PG}(8, q)$ by $X_{00}, X_{01}, X_{02}, X_{10}, ..., X_{22}$. It then is easily seen that the incident point-line pairs of $\mathbf{PG}(2, q)$ are mapped into the hyperplane $\mathbf{PG}(7, q)$ of $\mathbf{PG}(8, q)$ with equation $X_{00} + X_{11} + X_{22} = 0$, and that the image under θ of the set of flags of $\mathbf{PG}(2, q)$ is a set of points which generates $\mathbf{PG}(7, q)$ (this follows from the fact that $\mathcal{S}_{2;2}$ generates $\mathbf{PG}(8, q)$). It is shown in [4] that this set defines a weak and full embedding of the thin generalized hexagon Γ associated with $\mathbf{PG}(2, q)$. We call this embedding (and every equivalent one with respect to the linear automorphism group of $\mathbf{PG}(7, q)$) a *natural embedding of Γ in* $\mathbf{PG}(7, q)$.

It is shown in [4] that the intersection of all hyperplanes spanned by $x^{\perp\!\!\perp}$, with x running through the set of points of Γ, is a point k with coordinates $x_{ii}=1$, $x_{ij}=0$, $i, j \in \{0, 1, 2\}$, $j \neq i$. This point lies in $\mathbf{PG}(7, q)$ if and only if the characteristic of $\mathbf{GF}(q)$ is equal to 3. Hence, in this case, we can project the weakly embedded thin hexagon Γ from k onto some hyperplane $\mathbf{PG}(6, q)$ of $\mathbf{PG}(7, q)$ not containing k to obtain a weak and full embedding of Γ in the 6-dimensional projective space $\mathbf{PG}(6, q)$. We call this embedding also a *natural embedding of* Γ.

The exceptional behaviour over fields with characteristic 3 is in conformity with the special behaviour of classical generalized hexagons over such fields (the hexagons related to Dickson's group $G_2(q)$, $q=3^e$, are at the same time classical and dual classical).

2.3. *Some Known Results*

Standing Hypotheses. *From now on we suppose that* $\Gamma=(\mathscr{P}, \mathscr{L}, \mathbf{I})$ *is a generalized hexagon of order* $(q, 1)$ *weakly embedded in* $\mathbf{PG}(d, q)$. *We denote by* $\pi(\Gamma)$ *the projective plane for which the dual of the double is isomorphic to* Γ.

We now recall some facts and definitions from [4].

Let $x \in \mathscr{P}$. The set $x^{\perp\!\!\perp}$ does not generate $\mathbf{PG}(d, q)$; hence it generates some (proper) subspace of $\mathbf{PG}(d, q)$ which we will denote by ζ_x. For any line L of Γ, we denote by ξ_L the subspace of $\mathbf{PG}(d, q)$ generated by $\Gamma_3(L)$.

Lemma 1. *For every* $x \in \mathscr{P}$, *the space* $\zeta_x = \langle x^{\perp\!\!\perp} \rangle$ *is a hyperplane which does not contain any point of* $\Gamma_6(x)$. *In particular,* $\zeta_x \neq \zeta_y$ *for* $x, y \in \mathscr{P}$ *with* $x \neq y$. *Also, there is a unique* $(d-2)$*-space* $\tilde{\xi}_L$ *contained in all* ζ_x, $L \in \mathscr{L}$ *and* $x\mathbf{I}L$.

Lemma 2. *For every line* $L \in \mathscr{L}$, *the space* $\xi_L = \langle L^{\perp\!\!\perp} \rangle$ *has dimension either* $d-3$ *or* $d-2$, *and it contains no point of* $\Gamma_5(L)$.

Lemma 3. *Every apartment* Σ *of* Γ *generates a* 5*-dimensional subspace of* $\mathbf{PG}(d, q)$.

Lemma 4. *Let* Γ *be weakly embedded in* $\mathbf{PG}(d, q)$. *Let* U *be any subspace of* $\mathbf{PG}(d, q)$ *containing an apartment* Σ *of* Γ. *Then the points* x *of* Γ *in* U *for which* $\Gamma_1(x) \subseteq U$ *together with the lines of* Γ *in* U *form a (weak) subhexagon* Γ' *of* Γ. *Let* L, M *be two concurrent lines of* Σ *and let* x, y *be two points not contained in* Σ *but incident with respectively* L *and* M. *If* U *contains* $\Gamma_1(x)$ *and* $\Gamma_1(y)$, *then* Γ' *has some order* $(s, 1)$, $1 < s \leqslant q$.

Lemma 5. *Let* Γ *be weakly embedded in* $\mathbf{PG}(d, q)$. *Then* $6 \leqslant d \leqslant 8$.

Lemma 6. *The projective plane* $\pi(\Gamma)$ *is isomorphic to* $\mathbf{PG}(2, q)$.

LEMMA 7. *Let L and M be two arbitrary opposite lines of Γ. Let L_0, L_1, ..., L_k be $k+1$ distinct elements of $\Gamma_2(L)$, $1 \leqslant k \leqslant q$, and put $\Gamma_2(M) \cap \Gamma_2(L_i) = \{M_i\}$, $0 \leqslant i \leqslant k$. Then the dimension of the subspace U of* $\mathbf{PG}(d, q)$ *generated by $L_0, L_1, ..., L_k$ is equal to the dimension of the subspace V generated by $M_0, M_1, ..., M_k$.*

LEMMA 8. *Let L_0, L_1, L_2 be three distinct lines of Γ concurrent with some line $L \in \mathscr{L}$. Then $U := \langle L_0, L_1, L_2 \rangle$ has dimension* 4.

2.4. *Case Distinction*

Suppose that $\langle L^{\perp\!\!\!\perp} \rangle$ is 4-dimensional for some line L of Γ. Then, for any line M of Γ opposite L, the subspace $U = \langle L^{\perp\!\!\!\perp} \cup M^{\perp\!\!\!\perp} \rangle$ has dimension 6 by Lemma 2. Also by Lemma 2 we have either $d=6$ or $d=7$. Further, by Lemma 7, we have that $\langle N^{\perp\!\!\!\perp} \rangle$ is 4-dimensional for any line N of Γ. We handle the cases $d=6$ and $d=7$ separately in the next two sections.

3. THE CASE $d=6$

Let L and M be two opposite lines of Γ. From the previous section we know that ξ_L and ξ_M are 4-dimensional spaces and that $\langle \xi_L, \xi_M \rangle = \mathbf{PG}(6, q)$. Hence the intersection $\pi_{L,M} := \xi_L \cap \xi_M$ is a plane. By Lemma 8, the $q+1$ points of $\Gamma_3(L) \cap \Gamma_3(M)$ form an oval $\mathcal{O}_{L,M}$ in $\pi_{L,M}$. Clearly, for M' opposite L, $M' \neq M$, the ovals $\mathcal{O}_{L,M}$ and $\mathcal{O}_{L,M'}$ meet in the unique point of $\Gamma_3(M) \cap \Gamma_3(M') \cap \Gamma_3(L)$.

3.1. *The Case q Even*

Suppose that q is even. Remark that, with the above notation, for any point $x \in \mathcal{O}_{L,M}$, the subspace $\zeta_x \cap \pi_{L,M}$ is a line tangent to $\mathcal{O}_{L,M}$. Now all tangent lines of $\mathcal{O}_{L,M}$ are incident with a common point $n_{L,M}$ (the nucleus of the oval).

Let L_0 and L_1 be distinct lines of Γ concurrent with L. Since $\langle \xi_{L_0}, \xi_{L_1} \rangle$ contains an apartment, it must coincide with $\mathbf{PG}(6, q)$ by Lemma 4. Hence $\xi_{L_0} \cap \xi_{L_1}$ is a plane containing L. As ζ_{y_i}, with $\{y_i\} = \Gamma_1(L_i) \cap \Gamma_1(L)$, as well as ζ_{x_i}, with $\{x_i\} = \Gamma_1(L_i) \cap \Gamma_3(M)$, contains $n_{L,M}$, $i = 0, 1$, we have that $\xi_{L_0} \cap \xi_{L_1} = \langle L, n_{L,M} \rangle$. The plane $\xi_{L_0} \cap \xi_{L_1}$ apparently is independent of M. So all ovals $\mathcal{O}_{L,M'}$, M' opposite L, have their nucleus in the plane $\pi_L := \langle L, n_{L,M} \rangle$. Notice that the planes $\pi_{L,M}$ and $\pi_{L,M'}$, for $M \neq M'$, meet in exactly one point. Indeed, otherwise they define a 3-space containing L, hence ξ_L, a contradiction. It follows that distinct ovals $\mathcal{O}_{L,M}$ have distinct nuclei in π_L, L fixed.

Since $\langle \xi_L, \xi_{L_0} \rangle = \zeta_x$, with $\{x\} = \Gamma_1(L) \cap \Gamma_1(L_0)$, the space $\xi_L \cap \xi_{L_0}$ is 3-dimensional. Since both π_L and π_{L_0} are contained in both ξ_L and ξ_{L_0}, the planes π_L and π_{L_0} meet in a line (as $L_0 \not\subseteq \pi_L$ we have $\pi_L \neq \pi_{L_0}$). It follows that $\pi_L \cap \pi_{L_0} \cap \pi_{L_1}$ is a point r not lying on L. If $\pi_{L_0} \cap \pi_{L_1}$ is a line T, then $T \subseteq \xi_{L_0} \cap \xi_{L_1} = \pi_L$, and so $T \subseteq \pi_L \cap \pi_{L_0} \cap \pi_{L_1}$, a contradiction. So $\pi_{L_0} \cap \pi_{L_1} = \{r\}$. Now we may choose M opposite L_0 but concurrent with L_1 such that $n_{L_0, M} \neq r$. But π_M has a line R of $\mathbf{PG}(6, q)$ in common with π_{L_1}, and this line does not contain $n_{L_0, M}$, which lies in π_{L_0}. Consequently π_M is spanned by R and $n_{L_0, M}$, implying that π_M, and hence M, is contained in $\langle L_0, L_1, r \rangle \subseteq \xi_L$. This contradicts Lemma 2. Hence q cannot be even if $d = 6$.

3.2. *The Case q Odd*

In this case, all ovals $\mathcal{O}_{L,M}$ are conics. We fix L and we put $\{L_0, L_1, ..., L_q\} = \Gamma_2(L)$. We let x be a point on L_0 not on L, and we let $M_1, M_2, ..., M_q$ be the elements of $\Gamma_6(L) \cap \Gamma_3(x)$. We now project $(\Gamma_1(L) \cup \Gamma_3(L)) \setminus \{x\}$ from x onto a $\mathbf{PG}(3, q) \subseteq \xi_L$ not containing x. Let L' be the projection of L, let R'_i be the line containing the projection of $\mathcal{O}_{L, M_i} \setminus \{x\}$, $i = 1, 2, ..., q$, and let L'_i be the projection of L_i, $i = 1, 2, ..., q$. As $\langle L_i, L_j \rangle$, $i \neq j$ and $i, j \in \{1, 2, ..., q\}$, does not contain x, we have $L'_i \cap L'_j = \emptyset$. Each line R'_i, as well as L', meets each L'_j, $i, j \in \{1, 2, ..., q\}$. Hence we obtain a hyperbolic quadric $\mathcal{H}$ with one set of generators $\{L', R'_1, R'_2, ..., R'_q\}$, and the second set of generators is $\{L'_1, L'_2, ..., L'_q, L'_x\}$, where L'_x necessarily consists of the intersections of $\mathbf{PG}(3, q)$ with L_0 and the tangent lines at x of $\mathcal{O}_{L, M_j}$, $j = 1, 2, ..., q$; hence all these tangent lines are distinct and lie in a unique plane $\theta_{L,x}$. Consider now four distinct points r_1, r_2, r_3, r_4 on L; let m_1, m_2, m_3, m_4 be the unique points of Γ collinear with r_1, r_2, r_3, r_4, respectively, and lying on the conic $\mathcal{O}_{L, M_i}$, for arbitrary $i \in \{1, 2, ..., q\}$. Let r'_j and m'_j be the projections of respectively r_j and m_j, $j = 1, 2, 3, 4$, where by definition m'_j is the point $L'_x \cap R'_i$ if m_j is on L_0. It is clear that the cross-ratio $(r'_1, r'_2; r'_3, r'_4)$ is equal to the cross-ratio $(m'_1, m'_2; m'_3, m'_4)$ (property of $\mathcal{H}$). Since projection preserves cross-ratios, we conclude that $(r_1, r_2; r_3, r_4) = (m_1, m_2; m_3, m_4)$ (the latter is the cross-ratio of four points on a conic). Hence the map from $\Gamma_1(L)$ to $\mathcal{O}_{L, M_i}$ which maps a point y onto the unique collinear (in Γ) point of $\mathcal{O}_{L, M_i}$ is a linear automorphism. Hence $L_0 \cup L_1 \cup \cdots \cup L_q$ is a rational normal cubic scroll in ξ_L, which we denote by $\mathcal{S}_L$.

Now we choose a point $y \in \Gamma_1(L_0) \cap \Gamma_3(L)$, $y \neq x$, and we assume that $\theta_{L,y} = \theta_{L,x}$. Then the tangent line at y of any conic $\mathcal{O}_{L,M}$, with M opposite L and at distance 3 from y, meets the tangent line at x of every conic $\mathcal{O}_{L, M_j}$, $j = 1, 2, ..., q$. But $\mathcal{O}_{L,M}$ meets $\mathcal{O}_{L, M_1}$ also in some point z of $\Gamma_3(L)$; hence the planes of $\mathcal{O}_{L,M}$ and $\mathcal{O}_{L, M_1}$ have a line K in common, which cannot be the tangent line at z of one of these conics since the tangent lines at a common point of different conics on $\mathcal{S}_L$ are distinct. Hence K meets both $\mathcal{O}_{L,M}$ and $\mathcal{O}_{L,M_1}$ in a second point of $\Gamma_3(L)$, and so $|\mathcal{O}_{L,M} \cap \mathcal{O}_{L,M_1}| \geqslant 2$, a contradiction. Furthermore, by projection from x, it is easily seen that

$$\left(\bigcup_{y \in L_0 \setminus L} \theta_{L,y} \right) \cup \langle L_0, L \rangle$$

is a 3-dimensional space.

Now let $\Gamma_1(x) = \{L_0, N\}$ and let N_0, N_1 be two distinct elements of $\Gamma_2(N) \setminus \{L_0\}$. Put $\{u_i\} = \Gamma_1(N) \cap \Gamma_1(N_i)$, $i = 0, 1$. The 4-spaces ξ_N and ξ_{N_0} are contained in the 5-space ζ_{u_0}, hence they intersect in some 3-space η_{u_0}. Since η_{u_0} and the plane of $\mathcal{O}_{L,N_0}$ are both contained in ξ_{N_0}, there is at least one line K in their intersection. But K cannot be incident with a point of $\mathcal{O}_{L,N_0} \setminus \{x\}$ since such a point would then have to belong to ξ_N, a contradiction (it lies at distance 5 from N). So K is the unique tangent line at x of $\mathcal{O}_{L,N_0}$ (and note this implies that ξ_N contains $\theta_{L,x}$). Since η_{u_0} does not contain L_0 and since $\theta_{L,x}$ contains L_0, it follows that K is also the intersection of η_{u_0} with $\theta_{L,x}$. Hence, if $\mathcal{S}_N$, $\mathcal{S}_{N_0}$, $\mathcal{S}_{N_1}$ and $\mathcal{O}_{L,N_1}$ are given, then the plane $\theta_{L,x}$ is known and so is K, and hence so is $\mathcal{O}_{L,N_0}$ (this can easily be seen by projecting $\mathcal{S}_{N_0}$ from x as above: the plane of $\mathcal{O}_{L,N_0}$ is determined by the unique generator of the hyperbolic quadric $\mathcal{H}$—distinct from the projection of the plane $\langle N, K \rangle$—through the projection of the tangent line K). Can we also recover L from these data?

First we recall that $\theta_{L,x}$ is contained in ξ_N. Since it is also contained in ξ_L, and since ξ_L and ξ_N generate $\mathbf{PG}(6, q)$ (otherwise Γ is induced in a hyperplane), we have $\theta_{L,x} = \xi_L \cap \xi_N$. Symmetrically, if $\{z\} = \Gamma_1(L_0) \cap \Gamma_1(L)$, then $\theta_{N,z} = \xi_N \cap \xi_L$. But $\theta_{L,x}$ is determined, hence $\theta_{N,z}$ and consequently, as $\mathcal{S}_N$ is known, also z is known. So we can already recover the intersection of L and L_0.

We now coordinatize the situation in ξ_L. There are given two conics $\mathcal{C}_i = \mathcal{O}_{L,N_i}$, $i = 0, 1$, in two different planes (meeting in exactly one point x) which we may take as having equations $X_3 = X_4 = 0$ and $X_1 = X_2 = 0$. The point x has then coordinates $(1, 0, 0, 0, 0)$. The equations of $\mathcal{C}_0$ respectively $\mathcal{C}_1$ can be chosen as

$$\mathcal{C}_0 \leftrightarrow X_2^2 - X_0 X_1 = X_3 = X_4 = 0,$$

$$\mathcal{C}_1 \leftrightarrow X_4^2 - X_0 X_3 = X_1 = X_2 = 0.$$

The line L_0 lies in the plane $\theta_{L,x}$ spanned by the tangent lines of $\mathcal{C}_0$ and $\mathcal{C}_1$ at x, and hence can be chosen to contain the points x and $y = (0, 0, 1, 0, 1)$.

Now note that the group of collineations $\sigma_{a,b}$, $a, b \in \mathbf{GF}(q)$, of ξ_L defined by

$$\begin{cases} x_0 \mapsto x_0 + a^2 x_1 + 2ax_2 + b^2 x_3 + 2bx_4, \\ x_1 \mapsto x_1, \\ x_2 \mapsto x_2 + ax_1, \\ x_3 \mapsto x_3, \\ x_4 \mapsto x_4 + bx_3 \end{cases}$$

stabilizes both $\mathscr{C}_0$ and $\mathscr{C}_1$, and acts transitively on the points of L_0 distinct from x. Hence we may assume that L contains y (which explains our notation for the point y; see above). Now, a generic point on $\mathscr{C}_0$, respectively $\mathscr{C}_1$, has coordinates $(l^2, m^2, lm, 0, 0)$, respectively $(s^2, 0, 0, t^2, st)$. Hence $\mathscr{S}_L \backslash L_0$ is contained in the point set

$$\{(\alpha l^2 + \beta s^2, \alpha m^2, \alpha lm, \beta t^2, \beta st) \mid \alpha, \beta, l, m, s, t \in \mathbf{GF}(q)\}.$$

One can easily verify that all these points, and also the line L_0, belong to the hypersurface $\mathscr{K}$ with equation

$$X_0 X_1 X_3 = X_1 X_4^2 + X_3 X_2^2.$$

So the cubic scroll $\mathscr{S}_L$ is contained in $\mathscr{K}$, and so is L. Suppose L contains the point r with coordinates $(x_0, x_1, ..., x_4)$, which we assume not to be incident with L_0. Then a generic point r_ℓ, $\ell \in \mathbf{GF}(q)$, of L distinct from y has coordinates $(x_0, x_1, x_2 + \ell, x_3, x_4 + \ell)$. One easily calculates that this point lies on $\mathscr{K}$ if and only if (taking into account that also r belongs to $\mathscr{K}$!) $\ell^2 x_1 + 2\ell x_1 x_4 + \ell^2 x_3 + 2\ell x_3 x_2 = 0$. Since this must be the case for every $\ell \in \mathbf{GF}(q)$, we deduce that $x_1 + x_3 = 0$ and $x_1(x_4 - x_2) = 0$. However, if $x_1 = x_3 = 0$, then the point r, and hence the line L, is contained in $\theta_{L,x}$, a contradiction. Hence $x_1 = -x_3$ and $x_2 = x_4$. This implies that $x_0 x_1 x_3 = x_1 x_4^2 + x_3 x_2^2 = 0$ and so $x_0 = 0$. The line L is now completely and uniquely determined and has equations $X_1 + X_3 = X_2 - X_4 = X_0 = 0$. Notice that $\{\sigma_{a,-a} \mid a \in \mathbf{GF}(q)\}$ acts transitively on the points of L distinct from y, hence the point $(0, 1, 0, -1, 0)$ can be considered as an arbitrary point of L distinct from y. If we join $(0, 1, 0, -1, 0)$ to a generic point $(\ell^2, 1, \ell, 0, 0)$ of $\mathscr{C}_0$ distinct from x, then the line we obtain contains a point of $\mathscr{C}_1$ if and only if $\ell = 0$. Hence all elements of $\Gamma_2(L)$ are also uniquely determined.

So if $\mathscr{S}_N$, $\mathscr{S}_{N_0}$ and $\mathscr{S}_{N_1}$ are given, then all lines of Γ opposite both N_0 and N_1 are determined, and so are all lines at distance 4 from both N_0 and N_1. Since all lines at distance at most 2 from one of N_0 or N_1 are given, we conclude that Γ is completely determined by $\mathscr{S}_N$, $\mathscr{S}_{N_0}$ and $\mathscr{S}_{N_1}$.

Now we show that the configuration formed by $\mathscr{S}_N \cup \mathscr{S}_{N_0} \cup \mathscr{S}_{N_1}$ is projectively unique. We prove this by coordinatizing this configuration inside

$\mathbf{PG}(6, q)$. We put e_i equal to the point with coordinates $(0, ..., 0, 1, 0, ..., 0)$, with the 1 appearing in the $(i+1)$st position, $i=0, 1, ..., 6$. We choose a reference system in $\mathbf{PG}(6, q)$ as follows. The line N is e_3e_4, and a conic $\mathcal{O}_{N,K}$, with K opposite N lies in the plane $e_0e_2e_5$ and has equations $X_0^2 - X_2X_5 = X_1 = X_3 = X_4 = X_6 = 0$. Without loss of generality, we may assume that N_0 is the line e_2e_3 and N_1 is the line e_4e_5. Furthermore, we may choose e_1e_2 and e_5e_6 to be two lines of Γ, which we denote K_0 and K_1 respectively. We may also assume that, the lines K_0 and K_1 being at distance 4, the line $K := e_1e_6$ belongs to Γ.

We now first claim that the conics $\mathcal{O}_{N_0, K_1}$ and $\mathcal{O}_{N_1, K_0}$ lie in the planes $e_0e_1e_4$ and $e_0e_3e_6$, respectively. Indeed, since $\xi_N \cap \xi_K$ contains e_0, the four hyperplanes ζ_{e_i}, $i=1, 3, 4, 6$, contain e_0. But the tangent lines of $\mathcal{O}_{N,K}$ at e_2 and at e_5 also contain e_0. Hence $\theta_{N, e_2} = \theta_{K_0, e_3} = \langle N_0, e_0, e_2 \rangle$ and so $\langle N_0, e_0, e_2 \rangle \subseteq \xi_{K_0} \subseteq \xi_{e_2}$. Consequently e_0 is in ζ_{e_2}; analogously e_0 is in ζ_{e_5}. Hence e_0 is contained in all ζ_{e_i}, $i=1, 2, ..., 6$, and hence also in ξ_X for $X = N_0, N_1, K_0, K_1$. The claim follows.

In fact, since the tangent line at the point e_i of the conic $\mathcal{O}_{V,W}$ (with $\mathcal{O}_{V,W} \in \{\mathcal{O}_{N,K}, \mathcal{O}_{N_0,K_1}, \mathcal{O}_{N_1,K_0}\}$) through e_i and e_{i+3}, $i=1, 2, ..., 6$ and subscripts to be taken modulo 6, is given by the intersection of ζ_{e_i}, $\xi_{e_{i-1}e_{i-2}}$ and $\xi_{e_{i+1}e_{i+2}}$, we immediately see that this tangent line contains e_0, for all $i \in \{1, 2, ..., 6\}$. We have already chosen the point $(1, 0, 1, 0, 0, 1, 0)$ to be contained in Γ. Now we may choose the points $(1, 1, 0, 0, 1, 0, 0)$ and $(1, 0, 0, 1, 0, 0, 1)$ also such that they belong to Γ. Note that any mapping $\sigma_{a,b,c}$ defined by

$$(X_0, X_1, X_2, X_3, X_4, X_5, X_6) \mapsto (X_0, aX_1, bX_2, cX_3, a^{-1}X_4, b^{-1}X_5, c^{-1}X_6),$$

with $a, b, c \in \mathbf{GF}(q)^\times$, preserves the point e_i and the conic $\mathcal{O}_{V,W}$ containing e_i and e_{i+3}, $i=1, 2, ..., 6$ and the subscripts taken modulo 6. Moreover, for $b=1$, this map preserves $\mathcal{O}_{N,K}$ pointwise, while the image of $(0, 0, 0, 1, 1, 0, 0)$ is $(0, 0, 0, c, a^{-1}, 0, 0)$, which is an arbitrary point on N distinct from e_3 and e_4. So, without loss of generality, the projectivity between N and $\mathcal{O}_{N,K}$ may be chosen as

$$(0, 0, 0, x_3, x_4, 0, 0) \mapsto (\alpha x_3 x_4, 0, x_3^2, 0, 0, \alpha^2 x_4^2, 0),$$

with α any chosen element of $\mathbf{GF}(q)^\times$. The reason not to put α equal to 1 (which would be allowed) will become clear later. The projectivity from N_0 to $\mathcal{O}_{N_0,K_1}$ that defines $\mathcal{S}_{N_0}$ is of the type

$$(0, 0, x_2, x_3, 0, 0, 0) \mapsto (\beta x_2 x_3, x_2^2, 0, 0, \beta^2 x_3^2, 0, 0),$$

with $\beta \in \mathbf{GF}(q)^\times$. We claim that β is determined by α.

Therefore, we first remark that a generic point of $\mathcal{S}_{N_0}$ (not on N_0) has coordinates $(\beta x_2 x_3, x_2^2, \rho x_2, \rho x_3, \beta^2 x_3^2, 0, 0)$, $x_2, x_3, \rho \in \mathbf{GF}(q)$. We want to determine the unique conic $\mathcal{C}$ on $\mathcal{S}_{N_0}$ through the points e_1 and $e_{34} := (0, 0, 0, 1, 1, 0, 0)$. The tangent line at e_1 of $\mathcal{C}$ lies in the plane $e_0 e_1 e_2$, hence we may assume that this tangent line contains the point $(1, 0, u, 0, 0, 0, 0)$ for some $u \in \mathbf{GF}(q)$. Since the plane of the conic $\mathcal{C}$ intersects the line joining $(0, 0, x_2, x_3, 0, 0, 0)$ and $(\beta x_2 x_3, x_2^2, 0, 0, \beta^2 x_3^2, 0, 0)$ in some point r_{x_2, x_3}, for all $x_2, x_3 \in \mathbf{GF}(q)$, with $(x_2, x_3) \neq (0, 0)$, we must have

$$\begin{vmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & u & 0 & 0 \\ 0 & 0 & x_2 & x_3 & 0 \\ \beta x_2 x_3 & x_2^2 & 0 & 0 & \beta^2 x_3^2 \end{vmatrix} = 0.$$

It is easily calculated that this is equivalent with $u = \beta$. Also, it follows that the point r_{x_2, x_3} has coordinates $(\beta x_2 x_3, x_2^2, \beta^2 x_2 x_3, \beta^2 x_3^2, \beta^2 x_3^2, 0, 0)$. Hence the tangent line at e_{34} of $\mathcal{C}$ is generated by e_{34} and $(\beta, 0, \beta^2, 0, 0, 0, 0)$. By an above argument, the tangent of $\mathcal{C}$ at e_{34} contains a point of the tangent at $(\alpha, 0, 1, 0, 0, \alpha^2, 0)$ of the conic $\mathcal{O}_{N,K}$ (consider the apartment containing e_1, e_2, e_3, e_{34} and $(\alpha, 0, 1, 0, 0, \alpha^2, 0)$). Hence the points $(\alpha, 0, 1, 0, 0, \alpha^2, 0)$, $(0, 0, 0, 1, 1, 0, 0)$, $(1, 0, 0, 0, 0, 2\alpha, 0)$ and $(\beta, 0, \beta^2, 0, 0, 0, 0)$ are coplanar. We easily deduce that $\alpha\beta = 2$. This shows our claim.

We now interchange the roles of K and K_1, of N and N_0, and of K_0 and N_1. This boils down to interchanging α and β^{-1}. Indeed, we can perform the following coordinate change: $X_1 \leftrightarrow X_5$, $X_2 \leftrightarrow X_4$, $X_0 \leftrightarrow X_0$, $X_3 \leftrightarrow X_3$, $X_6 \leftrightarrow X_6$. The projectivity between N and $\mathcal{O}_{N,K}$ now becomes

$$\begin{aligned}(0, 0, x_4, x_3, 0, 0, 0) \mapsto\; & (\alpha x_3 x_4, \alpha^2 x_4^2, 0, 0, x_3^2, 0, 0) \\ & = (\alpha^{-1} x_3 x_4, x_4^2, 0, 0, (\alpha^{-1})^2\, x_3^2, 0, 0),\end{aligned}$$

while the projectivity from N_0 to $\mathcal{O}_{N_0, K_1}$ becomes

$$\begin{aligned}(0, 0, 0, x_3, x_2, 0, 0) \mapsto\; & (\beta x_2 x_3, 0, \beta^2 x_3^2, 0, 0, x_2^2, 0) \\ & = (\beta^{-1} x_2 x_3, 0, x_3^2, 0, 0, (\beta^{-1})^2\, x_2^2, 0).\end{aligned}$$

Hence the old relation $\alpha\beta = 2$ becomes $\alpha^{-1}\beta^{-1} = 2$. This implies that $1 = 4$. Hence q is a power of 3. Since β is uniquely determined by α, we conclude that the configuration $\mathcal{S}_N \cup \mathcal{S}_{N_0}$ is unique. Similarly $\mathcal{S}_N \cup \mathcal{S}_{N_0} \cup \mathcal{S}_{N_1}$ is unique. Consequently, if Γ exists, then q is a power of 3 and up to a projectivity Γ is unique. Hence we obtain the example of Subsection 2.2.

4. THE CASE $d=7$

In this section, we investigate the case $d=7$ under the assumption stated in our Main Result. Hence $\langle L^{\perp\!\!\perp}\rangle$ has dimension 4 for any line L of Γ.

Let L and M be two opposite lines of Γ. Put $U=\langle L^{\perp\!\!\perp}\cup M^{\perp\!\!\perp}\rangle$. Since the dimension of U is equal to 6, we deduce that the dimension of $\pi_{L,M}:=\xi_L\cap\xi_M$ is equal to 2.

By Lemma 8, the $q+1$ points of $\Gamma_3(L)\cap\Gamma_3(M)$ form an oval $\mathcal{O}_{L,M}$ in $\pi_{L,M}$. Now we can copy the first paragraph of Subsection 3.2 word by word, except that, since we do not know yet that the oval $\mathcal{O}_{L,M_i}$ is a conic in π_{L,M_i}, we have to substitute the cross-ratio $(xm_1, xm_2; xm_3, xm_4)$ to $(m_1, m_2; m_3, m_4)$ (where xm_j means the tangent line of $\mathcal{O}_{L,M_i}$ at m_j if $x=m_j$, $1\leqslant j\leqslant 4$). So we have the equality $(r_1, r_2; r_3, r_4)=(xm_1, xm_2; xm_3, xm_4)$. Choosing another arbitrary point y on $\mathcal{O}_{L,M_i}$, we obtain the equality $(xm_1, xm_2; xm_3, xm_4)=(ym_1, ym_2; ym_3, ym_4)$. This shows that $\mathcal{O}_{L,M_i}$ is a conic. Now it is clear that $\mathcal{O}_{L,M}$ is a conic for any two opposite lines L, M of Γ.

Hence we may again conclude that $\Gamma_2(L)$ constitutes a rational normal cubic scroll in ξ_L, and as before we denote it by $\mathcal{S}_L$.

Now we choose an apartment Σ of Γ. Let $\{e_i \mid i\in\{1, 2, 3, 4, 5, 6\}\}$ be the set of points of Σ with e_i collinear with e_{i+1}, $1\leqslant i\leqslant 5$ and e_1 collinear with e_6 (in Γ). We put $e_6e_1=K$, $e_1e_2=K_0$, $e_5e_6=K_1$, $e_3e_4=N$, $e_2e_3=N_0$ and $e_4e_5=N_1$ (this is consistent with our notation in the previous section). We define the point e_0 (respectively e_7) to be the intersection of the tangent lines of $\mathcal{O}_{K,N}$ (respectively $\mathcal{O}_{K_0,N_1}$) at the points e_2 and e_5 (respectively e_3 and e_6).

We claim that $U:=\langle\{e_0, e_1, ..., e_7\}\rangle=\mathbf{PG}(7, q)$. Clearly we have $\xi_K\cup\xi_N\cup\xi_{K_0}\cup\xi_{N_1}\subseteq U$, hence the claim follows from Lemma 4.

So we choose coordinates as follows: $e_i=(0, ..., 0, 1, 0, ..., 0)$, where the number of 0s preceding the 1 is exactly i (as before).

Let f be the intersection of the tangent lines of $\mathcal{O}_{K_1,N_0}$ at the points e_1 and e_4. We claim that f belongs to the line e_0e_7. Note that e_0 belongs to ζ_{e_i}, for $i=1, 3, 4, 6$, since it is contained in both $\xi_K\subseteq\zeta_{e_1}\cap\zeta_{e_6}$ and $\xi_N\subseteq\zeta_{e_3}\cap\zeta_{e_4}$. It is also clear that the hyperplane ζ_{e_2} meets the plane $\pi_{K,N}$ in the tangent line of $\mathcal{O}_{K,N}$ at e_2 (otherwise ζ_{e_2} contains a point of Γ opposite e_2 in Γ, contradicting Lemma 1). Hence e_0 is also contained in ζ_{e_2} and, similarly, in ζ_{e_5}. So we conclude that e_0 is contained in $W:=\zeta_{e_1}\cap\zeta_{e_2}\cap\cdots\cap\zeta_{e_6}$. Similarly, also e_7 and f are contained in W. If the dimension of W were strictly larger than 2, then there would be $i\in\{1, 2, ..., 6\}$ such that ζ_{e_i} contains $W_i:=\zeta_{e_1}\cap\cdots\cap\zeta_{e_{i-1}}\cap\zeta_{e_{i+1}}\cap\cdots\cap\zeta_{e_6}$. But this is impossible, because W_i contains the point $e_{i\pm 3}$ (choose the sign such that $1\leqslant i\pm 3\leqslant 6$), and ζ_{e_i} does not. Hence W is the line e_0e_7. We may choose coordinates in such a way that

$f=(1,0,0,0,0,0,0,1)$. Furthermore, we may assume that the points $(1,0,1,0,0,1,0,0)$ and $(0,0,0,1,0,0,1,1)$ belong to $\mathcal{O}_{K,N}$ and $\mathcal{O}_{K_0,N_1}$, respectively. Also, we may now assume that the point $(1,1,0,0,1,0,0,1)$ is the unique point of $\mathcal{O}_{K_1,N_0}$ collinear in Γ with the point $(0,0,1,1,0,0,0,0)$ of e_2e_3.

We now consider the cubic scrolls $\mathcal{S}_L$, for $L\in\{K,K_0,K_1,N,N_0,N_1\}$. Each such scroll defines a unique projectivity ρ_L from the set of points of L onto the conic $\mathcal{O}_{L,M}$, with $M\in\{K,K_0,K_1,N,N_0,N_1\}$ opposite L, and this projectivity maps a point p onto the unique point p' with the property that the line pp' is entirely contained in $\mathcal{S}_L$. Since each such projectivity preserves the cross-ratio, it is determined by the image of three distinct points. Hence there exist nonzero elements $\alpha,\alpha_0,\alpha_1,\beta,\beta_0,\beta_1\in\mathbf{GF}(q)$ such that

$$\rho_K\colon K\to\mathcal{O}_{K,N}\colon (0,b,0,0,0,0,a,0)\mapsto(\alpha ab,0,\alpha^2b^2,0,0,a^2,0,0),$$

$$\rho_{K_0}\colon K_0\to\mathcal{O}_{K_0,N_1}\colon (0,a,b,0,0,0,0,0)\mapsto(0,0,0,\alpha_0^2b^2,0,0,a^2,\alpha_0ab),$$

$$\rho_{K_1}\colon K_1\to\mathcal{O}_{K_1,N_0}\colon (0,0,0,0,0,a,b,0)\mapsto(\alpha_1ab,\alpha_1^2b^2,0,0,a^2,0,0,\alpha_1ab),$$

$$\rho_N\colon N\to\mathcal{O}_{K,N}\colon (0,0,0,a,b,0,0,0)\mapsto(\beta ab,0,a^2,0,0,\beta^2b^2,0,0),$$

$$\rho_{N_0}\colon N_0\to\mathcal{O}_{K_1,N_0}\colon (0,0,a,b,0,0,0,0)\mapsto(\beta_0ab,a^2,0,0,\beta_0^2b^2,0,0,\beta_0ab),$$

$$\rho_{N_1}\colon N_1\to\mathcal{O}_{K_0,N_1}\colon (0,0,0,0,a,b,0,0)\mapsto(0,0,0,a^2,0,0,\beta_1^2b^2,\beta_1ab).$$

Note that by earlier choices, we have $\beta_0=1$. But for reasons of symmetry, we will only use that fact at the end.

Note also that, if $q=2$, then $\alpha=\alpha_0=\alpha_1=\beta=\beta_0=\beta_1=1$ and so $\mathcal{S}_K\cup\mathcal{S}_{K_0}\cup\mathcal{S}_{K_1}\cup\mathcal{S}_N\cup\mathcal{S}_{N_0}\cup\mathcal{S}_{N_1}$ is uniquely determined. Now we assume $q>2$.

Consider the point $g=(0,1,0,0,0,0,1,0)$, which belongs to $\mathcal{S}_{K_0}$. Our first aim is to calculate the coordinates of the points of the conic $\mathcal{C}_g$ on $\mathcal{S}_{K_0}$ through the points e_3 and g. Put $h=(0,1,1,0,0,0,0,0)$ and $h'=\rho_{K_0}(h)$ $=(0,0,0,\alpha_0^2,0,0,1,\alpha_0)$. Then it is clear that each point of $\mathcal{C}_g$ lies in the 3-space Z generated by e_3, g, h and h'. Moreover, the unique point of $\mathcal{C}_g$ collinear on $\mathcal{S}_{K_0}$ with the generic point $(0,x,y,0,0,0,0,0)$ of K_0 lies on the line Z' generated by $(0,x,y,0,0,0,0,0)$ and $(0,0,0,\alpha_0^2y^2,0,0,x^2,\alpha_0xy)$. It is easily calculated that $Z\cap Z'=\{(0,x^2,xy,\alpha_0^2y^2,0,0,x^2,\alpha_0xy)\}=:$ $\{p_{x,y}\}$, if $x\neq y$. Interchanging the roles of $(0,x,y,0,0,0,0,0)$ and h (for some fixed $x,y\in\mathbf{GF}(q)$, $x\neq y$), we see that also $p_{1,1}$ belongs to $\mathcal{C}_g$.

Similarly, a generic point $p'_{x,y}$ of the unique conic $\mathcal{C}'_g$ on $\mathcal{S}_{K_1}$ through the points e_4 and g has coordinates $(\alpha_1xy,\alpha_1^2y^2,0,0,x^2,\alpha_1^2xy,\alpha_1^2y^2,\alpha_1xy)$. From this, one can determine the line tangent to $\mathcal{C}_g$ (respectively $\mathcal{C}'_g$) at the point $g = p_{1,0} = p'_{0,1}$. This line contains g and the point $r = (\partial p_{1,y}/\partial y)|_{y=0} = (0, 0, 1, 0, 0, 0, 0, \alpha_0)$ (respectively $r' = (\partial p'_{x,1}/\partial x)|_{x=0} = (\alpha_1, 0, 0, 0, 0, \alpha_1^2, 0, \alpha_1))$.

Now let g' be the unique point of $\Gamma_4(g)$ on N, and let g'' be the unique point of Γ collinear with both g and g'. The coordinates of g'' are $(\alpha, 0, \alpha^2, 0, 0, 1, 0, 0)$. Note that both the conics $\mathscr{C}_g$ and $\mathscr{C}'_g$ are contained in $\mathscr{S}_{g'g''}$, hence the points r, r', g, g'' all lie in the plane tangent to $\mathscr{S}_{g'g''}$ at g. One easily computes that this implies that $\alpha_1 = \alpha^{-1}$ and $\alpha_0 = -\alpha^{-1}$.

Interchanging the roles of K and N, of K_0 and N_1 and of K_1 and N_0 (which boils down to interchanging α and β, α_0 and β_1, and α_1 and β_0), we also obtain $\beta_0 = \beta^{-1}$ and $\beta_1 = -\beta^{-1}$. We can also interchange the roles of K and K_0, of K_1 and N_0, and of N and N_1 (which boils down to interchanging α and α_0^{-1}, α_1 and β_0^{-1}, and β and β_1^{-1}). This gives us the additional relation $\alpha_0 = \beta_0^{-1}$. Noticing that we can choose without loss of generality $\beta_0 = 1$, we obtain $(\alpha, \alpha_0, \alpha_1, \beta, \beta_0, \beta_1) = (-1, 1, -1, 1, 1, -1)$.

We conclude that the configuration $\mathscr{S}_K \cup \mathscr{S}_{K_0} \cup \mathscr{S}_{K_1} \cup \mathscr{S}_N \cup \mathscr{S}_{N_0} \cup \mathscr{S}_{N_1}$ is projectively unique. We now claim that this determines the embedding completely. It is clear that $\Gamma_2(K)$ is uniquely determined. Now let $L \in \Gamma_4(K)$. Put $\{L'\} = \Gamma_2(K) \cap \Gamma_2(L)$. If $L' \in \{K_0, K_1\}$, then L is a line of either $\mathscr{S}_{K_0}$ or $\mathscr{S}_{K_1}$, and hence determined. So suppose $L' \notin \{K_0, K_1\}$. Let $\{M\} = \Gamma_2(L') \cap \Gamma_2(N)$. Then L belongs to the rational normal cubic scroll $\mathscr{S}_{L'}$, which is determined by the line L', the unique conic $\mathscr{C}$ of the scroll $\mathscr{S}_{N_1}$ through the points e_6 and $M \cap N$ (for $q = 2$, one has to add the condition that $\mathscr{C} \cap \mathscr{O}_{K_0, N_1} = \{e_6\}$), and a projectivity between L' and the conic $\mathscr{C}$. But interchanging the roles of N_0 and M, the roles of K_0 and L', also noticing that the conic $\mathscr{O}_{K_1, M}$ on $\mathscr{S}_{K_1}$ is uniquely determined, and applying the arguments of the previous paragraphs, we see that this projectivity between L' and $\mathscr{C}$ is uniquely determined by the projectivity from K to $\mathscr{O}_{K,N}$ associated to $\mathscr{S}_K$. It follows that all lines of $\mathscr{S}_{L'}$ are determined. Since every line of Γ is not opposite either K or N, our claim follows.

We have shown that the embedding, if it exists, is unique. Hence every weak full embedding of Γ in $\mathbf{PG}(7, q)$ is isomorphic to the example of Subsection 2.2. The proof of our Main Result is complete.

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