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Classification of Embeddings of the Flag Geometries of Projective Planes in Finite Projective Spaces, Part 3

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The flag geometry $\Gamma = (\mathscr{P}, \mathscr{L}, \mathbf{I})$ of a finite projective plane Π of order s is the generalized hexagon of order $(s, 1)$ obtained from Π by putting $\mathscr{P}$ equal to the set of all flags of Π, by putting $\mathscr{L}$ equal to the set of all points and lines of Π, and where $\mathbf{I}$ is the natural incidence relation (inverse containment), i.e., Γ is the dual of the double of Π in the sense of H. Van Maldeghem (1998, "Generalized Polygons," Birkhäuser Verlag, Basel). Then we say that Γ is fully and weakly embedded in the finite projective space $\mathbf{PG}(d, q)$ if Γ is a subgeometry of the natural point-line geometry associated with $\mathbf{PG}(d, q)$, if $s = q$, if the set of points of Γ generates $\mathbf{PG}(d, q)$, and if the set of points of Γ not opposite any given point of Γ does not generate $\mathbf{PG}(d, q)$. In three earlier papers we have shown that the dimension d of the projective space belongs to $\{6, 7, 8\}$, that the projective plane Π is Desarguesian, and we have classified the full and weak embeddings of Γ (Γ as above) for $d = 6$ and for $d = 7$ in the case that there exists a line L of Γ and four distinct lines L_1, L_2, L_3, L_4 concurrent with Γ which generate a 4-dimensional space. In the present paper, we drop all these additional assumptions by completing the case $d = 7$ and handling the case $d = 8$. In particular, we find new examples for $d = 8$ (contrary to our original conjecture (J. A. Thas and H. Van Maldeghem, *Des. Codes Cryptogr.* **17** (1999), 97–104)). This means that we have now the complete classification of all fully and weakly embedded geometries Γ in $\mathbf{PG}(d, q)$, with Γ the flag geometry of a finite projective plane. © 2000 Academic Press

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1. DEFINITIONS AND STATEMENT OF THE MAIN RESULT

We finish our program of determining all full weak embeddings of generalized hexagons of order $(q, 1)$ in the projective space $\mathbf{PG}(d, q)$. Let

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us briefly recall that this is motivated by an attempt to characterize the "natural" embeddings of all finite Moufang classical hexagons. For more details, we refer to Parts 1 and 2 of this paper (see Thas and Van Maldeghem [6, 7]).

The problem we consider may be stated as follows. Let Π be a (finite) projective plane of order s. We define the *flag geometry* Γ of Π as follows. The points of Γ are the flags of Π (i.e., the incident point-line pairs of Π); the lines of Γ are the points and lines of Π. Incidence between points and lines of Γ is reverse containment. It follows that Γ is a (finite) generalized hexagon of order $(s, 1)$ (see (1.6) of Van Maldeghem [9]). The advantage of viewing Γ rather as a generalized hexagon than as a flag geometry of a projective plane is that one can apply results from the general theory of generalized hexagons. We will call Γ a *thin generalized hexagon* (since there are only 2 lines through every point of Γ).

Throughout, we assume that Γ is a thin generalized hexagon of order $(s, 1)$ with corresponding projective plane $\pi(\Gamma)=\Pi$. We introduce some further notation. For a point x of Γ, we denote by $x^{\perp}$ the set of points of Γ collinear with x (two points are *collinear* if they are incident with a common line); we denote by $x^{\perp\!\!\!\perp}$ the set of points of Γ not opposite x (i.e., not at distance 6 from x in the incidence graph of Γ). For a line L of Γ, we write $L^{\perp\!\!\!\perp}$ for the intersection of all sets $p^{\perp\!\!\!\perp}$ with p a point incident with L (in this notation we view L as the set of points incident with it). For an element x of Γ (point or line), we denote by $\Gamma_i(x)$ the set of elements of Γ at distance i from x in the incidence graph of Γ. In this notation, we have $p^{\perp}=\Gamma_0(p)\cup\Gamma_2(p)$, $p^{\perp\!\!\!\perp}=\Gamma_0(p)\cup\Gamma_2(p)\cup\Gamma_4(p)$ and $L^{\perp\!\!\!\perp}=\Gamma_1(L)\cup\Gamma_3(L)$, with p any point and L any line of Γ. Furthermore, an apartment of Γ is a thin subhexagon of order $(1, 1)$. It corresponds with a triangle in $\pi(\Gamma)$. Also, if x and y are two points of Γ at distance 4, then the unique point of $\Gamma_2(x)\cap\Gamma_2(y)$ will be denoted by $x \bowtie y$.

Let $\mathbf{PG}(d, q)$ be the d-dimensional projective space over the Galois field $\mathbf{GF}(q)$. We say that Γ is *weakly embedded in* $\mathbf{PG}(d, q)$ if the point set of Γ is a subset of the point set of $\mathbf{PG}(d, q)$ which generates $\mathbf{PG}(d, q)$, if the line set of Γ is a subset of the line set of $\mathbf{PG}(d, q)$, if the incidence relation in $\mathbf{PG}(d, q)$ restricted to Γ is the incidence relation in Γ, and if for every point of Γ, the set $x^{\perp\!\!\!\perp}$ does not generate $\mathbf{PG}(d, q)$. If moreover $s=q$, then we say that the weak embedding is also *full*.

The only previously known examples of weak full embeddings of finite thin hexagons in $\mathbf{PG}(d, q)$ arise from full embeddings of the dual classical generalized hexagons of order (q, q), and here $d=6$ or $d=7$; see Thas and Van Maldeghem [5]. Let us call these examples *classical*. In this paper, we will define a new class of fully weakly embedded finite thin hexagons (in $\mathbf{PG}(8, q)$), which we call *semi-classical*, and we will show that no more examples exist.

The following result is proved in [5–7].

THEOREM. *Let Γ be a thin generalized hexagon of order* $(q, 1)$ *weakly embedded in* $\mathbf{PG}(d, q)$, $d=6, 7$. *If for some* (*and then for every*) *line L of Γ there exist four distinct lines $L_1, L_2, L_3, L_4 \in \Gamma_2(L)$ such that the subspace $\langle L_1, L_2, L_3, L_4\rangle$ has dimension* 4, *then $L^{\perp\!\!\perp}$ is contained in a* 4-*dimensional space and the embedding is one of the classical examples.*

All classical examples arise in this way. In the present paper, we remove the extra condition in the theorem. More exactly, we will show:

MAIN RESULT. *Let Γ be a thin generalized hexagon of order* $(q, 1)$ *weakly embedded in* $\mathbf{PG}(d, q)$. *Then for every line L of Γ the subspace $\langle L^{\perp\!\!\perp}\rangle$ has dimension $\rho \leqslant 5$. There do not exist examples with $\rho=5$, $d=7$ and such that for some* (*and hence for every*) *line L of Γ, the space generated by four arbitrary distinct lines concurrent with L has dimension* 5. *If $d=8$, then only the semi-classical examples exist.*

CONCLUSION. *Let Γ be a thin generalized hexagon of order* $(q, 1)$ *weakly embedded in* $\mathbf{PG}(d, q)$. *Then it is one of the classical or semi-classical examples.*

2. THE NEW EXAMPLES

Let V be a 3-dimensional vector space over $\mathbf{GF}(q)$, and let V^* be the dual space. We choose dual bases. Then the vector lines of the tensor product $V \otimes V^*$ can be seen as the point-line pairs of the projective plane $\mathbf{PG}(2, q)$. Indeed, it is easily calculated that the pair $\{(x_0, x_1, x_2), [a_0, a_1, a_2]\}$ (we use parentheses for the coordinates of points and brackets for those of lines) corresponds to the vector line generated by the vector $(a_0x_0, a_0x_1, a_0x_2, a_1x_0, a_1x_1, a_1x_2, a_2x_0, a_2x_1, a_2x_2)$. Hence we have a mapping θ of the point-line pairs of $\mathbf{PG}(2, q)$ into the set of points of $\mathbf{PG}(8, q)$ (and the image of θ is the Segre variety $\mathscr{S}_{2;2}$; see Hirschfeld and Thas [2, Sect. 25.5]). Let σ be any field automorphism of $\mathbf{GF}(q)$. We define a *twisted* version θ_σ of θ as follows. If p is a point of $\mathbf{PG}(2, q)$ and L a line of $\mathbf{PG}(2, q)$, then $\{p, L\}^{\theta_\sigma} = \{p^\sigma, L\}^\theta$, where p^σ is defined coordinatewise.

We denote coordinates in $\mathbf{PG}(8, q)$ by $X_{00}, X_{01}, X_{02}, X_{10}, ..., X_{22}$. It is then easy to calculate that the image under θ_σ of the set of flags of $\mathbf{PG}(2, q)$ is a set of points which generates $\mathbf{PG}(8, q)$ if and only if $\sigma \neq 1$. If $\sigma = 1$, then we will obtain the classical examples defined in [5]. So from now on we assume $\sigma \neq 1$.

Consider the flag $F=\{(x_0, x_1, x_2), [a_0, a_1, a_2]\}$ of $\mathbf{PG}(2, q)$. Any flag of $\mathbf{PG}(2, q)$ not opposite F (viewed as a point of the thin generalized hexagon

Γ corresponding with $\mathbf{PG}(2,q)$) has the form $\{(y_0, y_1, y_2), [b_0, b_1, b_2]\}$ with $b_0 y_0 + b_1 y_1 + b_2 y_2 = 0$ and either

$$b_0 x_0 + b_1 x_1 + b_2 x_2 = 0 \tag{1}$$

or

$$a_0 y_0 + a_1 y_1 + a_2 y_2 = 0. \tag{2}$$

Hence we see that, by multiplying Eq. (1) with y_0^σ, y_1^σ, y_2^σ, respectively, and first raising Eq. (2) to the power σ and then multiplying the result by b_0, b_1, b_2, respectively, the corresponding point $p = (b_i y_j^\sigma)_{i,\,j=0,\,1,\,2}$ of $\mathbf{PG}(8,q)$ satisfies either $x_0 X_{0j} + x_1 X_{1j} + x_2 X_{2j} = 0$, $j = 0, 1, 2$, or $a_0^\sigma X_{i0} + a_1^\sigma X_{i1} + a_2^\sigma X_{i2} = 0$, $i = 0, 1, 2$. Making the appropriate linear combinations (multiplying with a_j^σ and x_i, $i, j = 0, 1, 2$), we see that the point p satisfies the equation

$$\sum_{i,\,j=0}^{2} a_j^\sigma x_i X_{ij} = 0. \tag{3}$$

Remarking that the set of flags containing one fixed point (respectively line) of $\mathbf{PG}(2,q)$ is mapped under θ_σ onto the set of points of a line of $\mathbf{PG}(8,q)$—which is immediately checked with an elementary calculation—and identifying every flag of $\mathbf{PG}(2,q)$ with its image under θ_σ, we obtain a weak and full embedding of Γ in $\mathbf{PG}(8,q)$. We call this embedding (and every equivalent one with respect to the linear automorphism group of $\mathbf{PG}(8,q)$) a *semi-classical embedding of* Γ *in* $\mathbf{PG}(8,q)$ (*with respect to* σ).

It is easily seen that the group $\mathbf{PGL}_3(q)$ acts in a natural way as an automorphism group and as a subgroup of $\mathbf{PGL}_9(q)$ on the embedding. Hence every two pairs of opposite lines of Γ are projectively equivalent. Consider the point $(1,0,0)$ and the line $[1,0,0]$ of $\mathbf{PG}(2,q)$. The corresponding (opposite) lines of Γ are respectively given by

$$L := \langle (0,0,0,1,0,0,0,0,0), (0,0,0,0,0,0,1,0,0) \rangle$$

and

$$M := \langle (0,1,0,0,0,0,0,0,0), (0,0,1,0,0,0,0,0,0) \rangle.$$

The set of points $\Gamma_3(L) \cap \Gamma_3(M)$ corresponds to the set of flags of $\mathbf{PG}(2,q)$ with a point on $[1,0,0]$ and a line through $(1,0,0)$; hence flags of the form $\{(0, y, 1), [0, 1, -y]\}$, with $y \in \mathbf{GF}(q)$, or $\{(0,1,0), [0,0,1]\}$. The corresponding set of points of Γ is given by

$$\{(0,0,0,0,y^\sigma,1,0,-y^{\sigma+1},-y) \mid y \in \mathbf{GF}(q)\} \cup \{0,0,0,0,0,0,0,1,0\}.$$

We call this set a σ-curve. It lies in a 3-dimensional projective space.

We now look at some properties of σ-curves.

LEMMA 1. *Let $\mathscr{A}$ be a σ-curve in* $\mathbf{PG}(3, q)$, *for some automorphism σ of* $\mathbf{GF}(q)$.

(i) *Then the stabilizer G of $\mathscr{A}$ in $\mathbf{PGL}_4(q)$ contains a subgroup G_0 which induces a sharply 3-transitive group on $\mathscr{A}$ isomorphic to $\mathbf{PGL}_2(q)$. If σ is not involutive, then $G = G_0$; if $\sigma^2 = 1$, then G is isomorphic to the subgroup of* $\mathrm{Aut}(\mathbf{PGL}_2(q))$ *containing $\mathbf{PGL}_2(q)$ and all semi-linear maps with corresponding field automorphism σ. It follows that each plane containing three distinct points of $\mathscr{A}$, contains exactly $q' + 1$ points of $\mathscr{A}$, with $\mathbf{GF}(q')$ the field of fixed elements of σ.*

(ii) *If σ is not involutive, then for every $x \in \mathscr{A}$, the stabilizer G_x fixes exactly two lines T and T' through x. If $\sigma^2 = 1$, then for every $x \in \mathscr{A}$, the stabilizer G_x has a unique orbit $\{T, T'\}$ of length 2 on the set of lines of $\mathbf{PG}(3, q)$ through x.*

(iii) *Let $x \in \mathscr{A}$. Assume that for some point $y \in \mathscr{A} \setminus \{x\}$, there exists a line T_x through x and a line T_y through y such that each plane through T_x (respectively T_y) contains at most two points of $\mathscr{A}$; hence there is a unique plane π_x (respectively π_y) through T_x (respectively T_y) meeting $\mathscr{A}$ in a unique point. Further, assume that the mapping $(T_x, z) \mapsto \langle T_y, z\rangle$, $\pi_x \mapsto \langle T_y, x\rangle$ and $\langle T_x, y\rangle \mapsto \pi_y$, for $z \in \mathscr{A} \setminus \{x, y\}$, is a (linear) projectivity and that each line of the unique hyperbolic quadric containing the intersections $\langle T_x, z\rangle \cap \langle T_y, z\rangle$, for all $z \in \mathscr{A} \setminus \{x, y\}$, contains exactly one point of $\mathscr{A}$. Then $T_x = T$ or $T_x = T'$.*

Proof. Let Γ be the thin generalized hexagon arising from the Desarguesian plane $\mathbf{PG}(2, q)$, and consider a semi-classical embedding with corresponding σ. Let l be any point of $\mathbf{PG}(2, q)$ and m be any line of $\mathbf{PG}(2, q)$ not incident with l. Let L and M be the respective corresponding (opposite) lines of Γ. Then the first assertion of (i) follows from the following facts: (a) $\Gamma_3(L) \cap \Gamma_3(M)$ is a σ-curve, (b) the automorphism group induced by $\mathbf{PGL}_9(q)$ on Γ contains $\mathbf{PGL}_3(q)$, (c) the stabilizer in $\mathbf{PGL}_3(q)$ (viewed as a permutation group on $\mathbf{PG}(2, q)$) of $\{l, m\}$ acts sharply 3-transitive on the set of flags $\{x, u\}$, where x is a point incident with m and u is a line incident with both x and l, (d) this stabilizer induces on $\langle \Gamma_3(L) \cap \Gamma_3(M)\rangle$ a permutation group G_0 isomorphic to $\mathbf{PGL}_2(q)$, acting sharply 3-transitive on the σ-curve.

Let

$$\mathscr{A} = \{(1, r, r^\sigma, r^{\sigma+1}) \mid r \in \mathbf{GF}(q)\} \cup \{(0, 0, 0, 1)\}.$$

The plane $\langle (1, 0, 0, 0), (0, 0, 0, 1), (1, 1, 1, 1)\rangle$ contains exactly $q' + 1$ points of $\mathscr{A}$, with $\mathbf{GF}(q')$ the field of fixed elements of σ. By the 3-transitivity it follows that each plane containing three distinct points of $\mathscr{A}$ contains exactly $q'+1$ points of $\mathscr{A}$.

Let x be the point $(0, 0, 0, 1) \in \mathscr{A}$. A generic element of G_0 fixing only x has matrix

$$A_c := \begin{pmatrix} 1 & 0 & 0 & 0 \\ c & 1 & 0 & 0 \\ c^\sigma & 0 & 1 & 0 \\ c^{\sigma+1} & c^\sigma & c & 1 \end{pmatrix},$$

$c \in \mathbf{GF}(q)$, while a generic element of G_0 fixing the points x and $y := (1, 0, 0, 0)$ has matrix

$$B_d := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d^\sigma & 0 \\ 0 & 0 & 0 & d^{\sigma+1} \end{pmatrix},$$

with $d \in \mathbf{GF}(q)^\times$. It is now easy to check that the orbit of the line S through x containing the point $(1, 1, 0, 0)$ under the group $\langle A_c, B_d \mid c \in \mathbf{GF}(q), d \in \mathbf{GF}(q)^\times \rangle$ is the set of lines through x not lying in the plane π_x with equation $X_0 = 0$. One can easily see that there are planes through S meeting $\mathscr{A}$ in at least three points (indeed, if ℓ is any element of $\mathbf{GF}(q) \setminus \mathbf{GF}(q')$ and if r is defined as

$$r^\sigma = \frac{\ell^\sigma(\ell+1)}{\ell^\sigma - \ell},$$

then one checks that the plane through S and the point $(1, r, r^\sigma, r^{\sigma+1})$ also contains the point $(1, s, s^\sigma, s^{\sigma+1})$, with $s = \frac{r+\ell}{1+\ell}$). On the other hand, all planes through the lines $T := \langle x, (0, 1, 0, 0) \rangle$ and $T' := \langle x, (0, 0, 1, 0) \rangle$ contain at most two points of $\mathscr{A}$. Hence G_x preserves π_x. Similarly $H := G_{x,y,z}$ preserves π_x, π_y and π_z, with $y := (1, 0, 0, 0)$ and $z := (1, 1, 1, 1)$. Note that π_y has equation $X_3 = 0$ and π_z has equation $X_0 + X_3 = X_1 + X_2$. It is easy to calculate that a generic element θ_a of H (as a linear automorphism of $\mathbf{PG}(3, q)$) has matrix

$$M := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 1-a & 0 \\ 0 & 1-a & a & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

for some $a \in \mathbf{GF}(q)$. Expressing that θ_a maps a point with coordinates $(1, x, x^\sigma, x^{\sigma+1})$ onto a point of $\mathscr{A}$, we obtain the conditions

$$(ax+(1-a)x^\sigma)^\sigma = (1-a)x + ax^\sigma,$$
$$x^{\sigma+1} = (ax+(1-a)x^\sigma)((1-a)x+ax^\sigma),$$

for all $x \in \mathbf{GF}(q)$. The last condition is, after an elementary computation, equivalent with $a^2 - a = 0$ (taking into account that $x - x^\sigma$ is not identical zero). Hence $a = 0$ or $a = 1$. Now the first condition says that either $a = 1$, or $a = 0$ and σ is involutive. This proves (i) completely.

It is now easy to prove (ii) with the explicit form of the elements of G_x above.

We finally prove (iii). Any quadric $\mathscr{H}$ containing $\mathscr{A}$ has equation

$$\sum_{0 \leqslant i,\, j \leqslant 3} a_{ij} X_i X_j = 0,$$

with $a_{00} = a_{33} = 0$ and

$$a_{01} + a_{02} r^{\sigma-1} + a_{03} r^\sigma + a_{11} r + a_{12} r^\sigma$$
$$+ a_{13} r^{\sigma+1} + a_{22} r^{2\sigma-1} + a_{23} r^{2\sigma} = 0,$$

for all $r \in \mathbf{GF}(q)^\times$. If $\sigma \neq 2$ and $\sigma^{-1} \neq 2$, then this implies $a_{01} = a_{02} = a_{03} + a_{12} = a_{11} = a_{13} = a_{22} = a_{23} = 0$. So in such a case the quadric $\mathscr{H}$ has equation $X_0 X_3 = X_1 X_2$. As T_x is a generator of $\mathscr{H}$ we necessarily have either T_x: $X_0 = X_1 = 0$ or T_x: $X_0 = X_2 = 0$, that is, $T_x \in \{T, T'\}$. Next let either $\sigma = 2$ or $\sigma^{-1} = 2$. Then $\mathscr{A}$ is a twisted cubic. In such a case T_x, T, T' are special unisecants of $\mathscr{A}$ at x (see Hirschfeld [1]) and so $T_x \in \{T, T'\}$. For the last assertion, we remark that in [7, Lemma 16], we have derived the equation above of $\mathscr{A}$ precisely under the assumptions of the lemma, and the result was that the lines T_x and T_y are two generators of the hyperbolic quadric in question, which has equation $X_0 X_3 = X_1 X_2$. It is now easy to see that the generators of this quadric through x are precisely T and T'.

The lemma is proved. ∎

We will call the lines T and T' the *tangent lines* or *special unisecants of* $\mathscr{A}$ *at* x, and the plane $\langle T, T' \rangle$ will be called the *osculating plane of* $\mathscr{A}$ *at* x (unlike in [7], where we called it a tangent plane; the reason for this change is the fact that for q even and σ a generating automorphism, we obtain precisely the osculating plane of a $(q+1)$-arc, see property ARC2 in Section 4 below).

We have the following corollary.

COROLLARY 2. *Let σ and σ' be two automorphisms of* **GF**(q). *Then a σ-curve is isomorphic to a σ'-curve, for the group* $\mathbf{PGL}_3(q)$, *if and only if* $\sigma' \in \{\sigma, \sigma^{-1}\}$.

Proof. Let $\mathscr{A}$ be the σ-curve with points

$$\mathscr{A} = \{(1, r, r^{\sigma}, r^{\sigma+1}) \mid r \in \mathbf{GF}(q)\} \cup \{(0, 0, 0, 1)\},$$

and let $\mathscr{A}'$ be the σ'-curve with points

$$\mathscr{A}' = \{(1, r, r^{\sigma'}, r^{\sigma'+1}) \mid r \in \mathbf{GF}(q)\} \cup \{(0, 0, 0, 1)\}.$$

Suppose there is a collineation θ mapping $\mathscr{A}$ to $\mathscr{A}'$. Then by the previous lemma, we may assume that θ fixes $(1, 0, 0, 0)$, $(0, 0, 0, 1)$ and $(1, 1, 1, 1)$. Moreover, the pair of intersection points $(0, 1, 0, 0)$ and $(0, 0, 1, 0)$ of the coplanar common tangents of $\mathscr{A}$ and $\mathscr{A}'$ at the respective points $(1, 0, 0, 0)$ and $(0, 0, 0, 1)$ is preserved. If θ fixes $(0, 1, 0, 0)$, then $\mathscr{A} = \mathscr{A}'$ and so $\sigma = \sigma'$. If θ interchanges $(0, 1, 0, 0)$ and $(0, 0, 1, 0)$, then one easily deduces $\sigma' = \sigma^{-1}$.

Conversely, if either $\sigma = \sigma'$ or $\sigma' = \sigma^{-1}$, then it is clear that $\mathscr{A}$ and $\mathscr{A}'$ are isomorphic. ∎

COROLLARY 3. *Let σ and σ' be two automorphisms of* **GF**(q). *Let Γ be the dual of the double of* **PG**$(2, q)$. *Then a semi-classical embedding of Γ in* **PG**$(8, q)$ *with respect to σ is isomorphic, with respect to* $\mathbf{PGL}_9(q)$, *to a semi-classical embedding of Γ in* **PG**$(8, q)$ *with respect to σ' if and only if* $\sigma' \in \{\sigma, \sigma^{-1}\}$.

Proof. In view of the previous corollary, we only have to show that two embeddings of Γ with respect to σ and σ^{-1}, respectively, are isomorphic for $\mathbf{PGL}_9(q)$. But this is obvious by considering the map sending the point in the first representation corresponding with the flag $\{(x, y, z), [a, b, c]\}$, with $x, y, z, a, b, c \in \mathbf{GF}(q)$ and $ax + by + cz = 0$, of **PG**$(2, q)$ to the point in the second representation corresponding with the flag $\{(a^{\sigma}, b^{\sigma}, c^{\sigma}), [x^{\sigma}, y^{\sigma}, z^{\sigma}]\}$. It is easy to check that this induces a linear isomorphism from the above defined embedding of Γ with respect to σ to the one with respect to σ^{-1}. ∎

Remark 4. Let Γ be the thin generalized hexagon of order $(q, 1)$ arising from the Desarguesian plane **PG**$(2, q)$, and consider the semi-classical embedding of it in **PG**$(8, q)$, with respect to the field automorphism σ, as described above. Let L and M be two opposite lines of Γ (respectively corresponding with a point and a line of **PG**$(2, q)$). Let x_1, x_2, x_3, x_4 be four points of Γ on L and let y_i, $i = 1, 2, 3, 4$, be the unique point of M not opposite x_i. Then we claim that $(x_1, x_2; x_3, x_4)^{\sigma} = (y_1, y_2; y_3, y_4)$.

Indeed, without loss of generality, we may assume that L corresponds with the point $(1, 0, 0)$ of $\mathbf{PG}(2, q)$, and M with the line $[1, 0, 0]$. We may then identify the points x_i and y_i with the following flags of $\mathbf{PG}(2, q)$,

$$x_1 \rightsquigarrow \{(1, 0, 0), [0, 1, 0]\}, \qquad y_1 \rightsquigarrow \{(0, 0, 1), [1, 0, 0]\}$$

$$x_2 \rightsquigarrow \{(1, 0, 0), [0, 0, 1]\}, \qquad y_2 \rightsquigarrow \{(0, 1, 0), [1, 0, 0]\},$$

$$x_3 \rightsquigarrow \{(1, 0, 0), [0, 1, 1]\}, \qquad y_3 \rightsquigarrow \{(0, 1, -1), [1, 0, 0]\},$$

$$x_4 \rightsquigarrow \{(1, 0, 0), [0, k, 1]\}, \qquad y_4 \rightsquigarrow \{(0, 1, -k), [1, 0, 0]\},$$

for some $k \in \mathbf{GF}(q)$. It is now an elementary exercise to calculate the coordinates of the points x_i and y_i, $i = 1, 2, 3, 4$, and to deduce the above given relation between the cross-ratios. Note also that everything in this section can be generalized to the infinite case without notable change.

3. PRELIMINARY RESULTS

3.1. *Some Known Results*

Standing Hypotheses. *From now on we suppose that* $\Gamma = (\mathscr{P}, \mathscr{L}, \mathbf{I})$ *is a generalized hexagon of order* $(q, 1)$ *weakly embedded in* $\mathbf{PG}(d, q)$, *and we denote by* $\pi(\Gamma)$ *the projective plane for which the dual of the double is isomorphic to* Γ.

We now recall some facts and definitions from [5–7].

Let $x \in \mathscr{P}$. The set $x^{\perp\!\!\!\perp}$ does not generate $\mathbf{PG}(d, q)$; hence it generates some (proper) subspace of $\mathbf{PG}(d, q)$ which we will denote by ζ_x. For any line L of Γ, we denote by ξ_L the subspace of $\mathbf{PG}(d, q)$ generated by $\Gamma_3(L)$.

Lemma 5. *For every* $x \in \mathscr{P}$, *the space* $\zeta_x = \langle x^{\perp\!\!\!\perp} \rangle$ *is a hyperplane which does not contain any point of* $\Gamma_6(x)$. *In particular,* $\zeta_x \neq \zeta_y$ *for* $x, y \in \mathscr{P}$ *with* $x \neq y$. *Also, there is a unique* $(d-2)$*-space* $\tilde{\xi}_L$ *contained in all* ζ_x, $L \in \mathscr{L}$ *and* $x\mathbf{I}L$.

Lemma 6. *For every line* $L \in \mathscr{L}$, *the space* $\xi_L = \langle L^{\perp\!\!\!\perp} \rangle$ *has dimension either* $d-3$ *or* $d-2$, *and it contains no point of* $\Gamma_5(L)$.

LEMMA 7. *Every apartment Σ of Γ generates a 5-dimensional subspace of* **PG**(d, q).

LEMMA 8. *Let U be any subspace of* **PG**(d, q) *containing an apartment Σ of Γ. Then the points x of Γ in U for which $\Gamma_1(x) \subseteq U$ together with the lines of Γ in U form a (weak) subhexagon Γ' of Γ. Let L, M be two concurrent lines of Σ and let x, y be two points not contained in Σ but incident with respectively L and M. If U contains $\Gamma_1(x)$ and $\Gamma_1(y)$, then Γ' has some order $(s, 1)$, $1 < s \leqslant q$.*

LEMMA 9. *Let Γ be weakly and fully embedded in* **PG**(d, q). *Then $6 \leqslant d \leqslant 8$.*

LEMMA 10. *The projective plane $\pi(\Gamma)$ is isomorphic to* **PG**$(2, q)$.

LEMMA 11. *Let L and M be two arbitrary opposite lines of Γ. Let $L_0, L_1, ..., L_k$ be $k+1$ distinct elements of $\Gamma_2(L)$, $1 \leqslant k \leqslant q$, and put $\Gamma_2(M) \cap \Gamma_2(L_i) = \{M_i\}$, $0 \leqslant i \leqslant k$. Then the dimension of the subspace U of* **PG**(d, q) *generated by $L_0, L_1, ..., L_k$ is equal to the dimension of the subspace V generated by $M_0, M_1, ..., M_k$.*

LEMMA 12. *Let L_0, L_1, L_2 be three distinct lines of Γ concurrent with some line $L \in \mathscr{L}$. Then $U := \langle L_0, L_1, L_2 \rangle$ has dimension* 4.

LEMMA 13. *Let L be any line of Γ, and let x_0, x_1, x_2, x_3 be four distinct points on L. Without loss of generality, we may assume that L corresponds with a line L' of $\pi(\Gamma)$. Let x_i, $0 \leqslant i \leqslant 3$, correspond in $\pi(\Gamma)$ with the flag $\{x_i', L'\}$. Let θ be any self-projectivity of L' in $\pi(\Gamma)$, that is, θ is induced by perspectivities of $\pi(\Gamma)$, and suppose that the point y_i of Γ corresponds with the flag $\{x_i'^{\theta}, L'\}$ of $\pi(\Gamma)$. Then the cross ratios $(x_0, x_1; x_2, x_3)$ and $(y_0, y_1; y_2, y_3)$ (considered as cross-ratios of points in* **PG**(d, q)) *are equal.*

The last lemma follows directly from Lemma 5 and the proof of Proposition 6 in [5].

An immediate consequence of Lemma 11 is the following

COROLLARY 14. *Let L and M be two arbitrary lines of Γ. Let L_0, L_1, L_2 be three distinct elements of $\Gamma_2(L)$, and let M_0, M_1, M_2 be three distinct elements of $\Gamma_2(M)$. Then the number of elements of $\Gamma_2(L)$ contained in the space $\langle L_0, L_1, L_2 \rangle$ is equal to the number of elements of $\Gamma_2(M)$ contained in the space $\langle M_0, M_1, M_2 \rangle$.*

Finally, we have:

LEMMA 15. *Let L and M be two arbitrary opposite lines of Γ and suppose that $\langle L^{\perp\!\!\perp} \rangle$ has dimension* 5. *If there exist four distinct lines L_1, L_2, L_3,*

$L_4 \in \Gamma_2(L)$ such that $\langle L_1, L_2, L_3, L_4 \rangle$ is a 4-dimensional subspace, then the set $\Gamma_3(L) \cap \Gamma_3(M)$ is contained in a 3-dimensional space and is projectively equivalent (with respect to $\mathbf{PGL}_4(q)$) with the following set of points of $\mathbf{PG}(3, q)$,

$$\{(1, x, x^{\sigma}, x^{\sigma+1}) \mid x \in \mathbf{GF}(q)\} \cup \{(0, 0, 0, 1)\},$$

where σ is some field automorphism with at least 3 fixed elements. Hence it is a σ-curve.

3.2. *Case Distinction*

Suppose that $\langle L^{\perp\!\!\!\perp} \rangle$ is a subspace of dimension $\rho_L \geqslant 5$, for all lines L of Γ (the case $\rho_L \leqslant 4$ has already been taken care of by the Theorem in the introduction and Lemmas 6 and 12). By Lemma 11, ρ_L is independent of L, and we write $\rho_L = \rho$. Clearly $\rho \leqslant d-2$, hence $d=7, 8$. If $d=7$, then $\rho=5$ and we distinguish the following cases (where NE stands for non-existence):

NE(7.1) $d=7$, $\rho=5$, and for every line L of Γ, there exists a set $\{L_1, L_2, L_3, L_4\} \subseteq \Gamma_2(L)$ of cardinality 4 such that the subspace generated by L_1, L_2, L_3, L_4 has dimension 4.

NE(7.2) $d=7$, $\rho=5$, and for every line L of Γ, and every set $\{L_1, L_2, L_3, L_4\} \subseteq \Gamma_2(L)$ of cardinality 4, the subspace generated by L_1, L_2, L_3, L_4 has dimension 5.

If $d=8$, then we have $\rho=5$ or $\rho=6$. Here, we distinguish the following cases (where EX stands for existence of examples):

NE(8.1) $d=8$ and $\rho=6$.

EX(8.2) $d=8$ and $\rho=5$.

In [7] it was proved that Case NE(7.1) cannot occur. Some preliminary results to Case EX(8.2) were also proved (as, for instance, Lemma 15 above). In the present paper, we take care of Cases NE(7.2) (Section 4), NE(8.1) (Section 5), and EX(8.2) (Section 6).

4. THE CASE $d=7$

Let L and M be two opposite lines of Γ. It is clear that $\xi_L \neq \xi_M$. Hence the space $\eta_{L,M} = \xi_L \cap \xi_M$ has dimension either 4 or 3. Suppose that the dimension of $\eta_{L,M}$ is equal to 4. Then there is a point x of L which belongs to $\eta_{L,M}$, and hence to ξ_M. This contradicts Lemma 6. Hence the dimension of $\eta_{L,M}$ is 3. Now let $\mathscr{A}_{L,M}$ be the set of points of Γ in $\eta_{L,M}$, then $\mathscr{A}_{L,M} = \Gamma_3(L) \cap \Gamma_3(M)$. From our assumption (see NE(7.2)) it readily follows that no four points of $\mathscr{A}_{L,M}$ are contained in a plane of $\eta_{L,M}$. Hence $\mathscr{A}_{L,M}$ is a $(q+1)$-arc. These objects are well studied, and we summarize some elementary properties which we will need below (see [1]). Let $\mathscr{A}$ be a $(q+1)$-arc in $\mathbf{PG}(3, q)$.

(ARC1) Let $p \in \mathscr{A}$ be arbitrary, and let Π be a plane of $\mathbf{PG}(3, q)$ not containing p. Then the projection of $\mathscr{A} \backslash \{p\}$ from p onto Π is a q-arc, which can be completed in a unique way to a $(q+1)$-arc by adding one point p' if q is odd, and which can be completed in a unique way to a $(q+2)$-arc by adding two points p', p'' if q is even. The line pp' (q odd) and the pair of lines $\{pp', pp''\}$ (q even) are independent of Π and are called *special unisecants or tangents to* $\mathscr{A}$ *at* p.

(ARC2) For each point $p \in \mathscr{A}$, there is a unique plane π_p containing at least one tangent to $\mathscr{A}$ at p and intersecting $\mathscr{A}$ in $\{p\}$. This plane is called the *osculating plane* of $\mathscr{A}$ at p. For q even, the osculating plane at p contains the two tangents at p.

(ARC3) Let q be even. Then all the tangents to $\mathscr{A}$ form the set of lines of a hyperbolic quadric. If p and p' are two points of $\mathscr{A}$, and if L is the intersection of the respective osculating planes at p and p', then there are two points s, s' on L such that $ps, ps', p's, p's'$ are tangent to $\mathscr{A}$; intersecting a plane through p, p' and $x \in \mathscr{A} \backslash \{p, p'\}$ with L is a bijection from the set of $q-1$ planes through p, p' not containing any tangent to $\mathscr{A}$ at p (or equivalently p') to the set of points of L different from s and s'.

(ARC4) Let T be a line through some point $p \in \mathscr{A}$. If no plane through T contains at least 3 points of $\mathscr{A}$, then T is tangent to $\mathscr{A}$ at p. Also, no point of $\mathbf{PG}(3, q)$ is contained in every tangent of $\mathscr{A}$.

(ARC5) If q is even, then there exists an automorphism σ of $\mathbf{GF}(q)$, generating the automorphism group of $\mathbf{GF}(q)$, such that, with respect to a suitably chosen coordinate system, $\mathscr{A}$ consists of the points $(0, 0, 0, 1)$ and $(1, x, x^\sigma, x^{\sigma+1})$, with $x \in \mathbf{GF}(q)$.

Now we put $\Gamma_2(L) = \{L_0, L_1, ..., L_q\}$ and $\Gamma_2(L_i) \cap \Gamma_2(M) = \{M_i\}$, $0 \leqslant i \leqslant q$. It is clear that $\langle \xi_L, \xi_{L_0} \rangle = \zeta_{x_0}$, where x_0 is the intersection of L and L_0. Consequently the space $U := \xi_L \cap \xi_{L_0}$ is 4-dimensional. Now the 3-dimensional space $\eta_{L,M}$ is not contained in U (since it contains $L_1 \cap \eta_{L,M}$ which is not contained in U). Furthermore, both spaces U and $\eta_{L,M}$ are contained in ξ_L, which is 5-dimensional. Hence $U' = \eta_{L,M} \cap U = \eta_{L,M} \cap \xi_{L_0}$ is a plane. Let y_i be the intersection of L_i and M_i, $0 \leqslant i \leqslant q$. It is obvious that y_0 is contained in U', but y_i, $1 \leqslant i \leqslant q$, is not. We now claim that U' contains a tangent to $\mathscr{A}_{L,M}$ at y_0. Therefore, we consider a point s on L_0, with $x_0 \neq s \neq y_0$, a line $R \neq L_0$ of Γ through s, and a point $r \neq s$ on R. It is easy to see (translating the situation to the projective plane $\pi(\Gamma)$) that $|\Gamma_4(r) \cap \mathscr{A}_{L,M}| = 2$. So we may put $\Gamma_4(r) \cap \mathscr{A}_{L,M} = \{y_0, y_{\ell(r)}\}$, for some $\ell(r) \in \{1, 2, ..., q\}$. Obviously $U_r = \zeta_r \cap \eta_{L,M}$ is a plane of $\eta_{L,M}$ containing the intersection $\xi_R \cap \eta_{L,M}$, which is a line T in $\eta_{L,M}$ (as $y_{\ell(r)}$ is not in ξ_R). The planes U_r and $\zeta_s \cap \eta_{L,M} = \xi_{L_0} \cap \eta_{L,M} = U \cap \eta_{L,M} = U'$ contain y_0, q of them meet $\mathscr{A}_{L,M}$ in at most two points, and exactly one (namely, U') meets $\mathscr{A}_{L,M}$ in exactly one point. By property (ARC4), T is a tangent and (ARC2) implies that U' is the osculating plane of $\mathscr{A}_{L,M}$ at y_0, and as such independent of the line of Γ through y_0, i.e., $U' = \xi_{M_0} \cap \eta_{L,M}$. Now it is also clear that $U' = \zeta_{s'} \cap \eta_{L,M}$ for any $s' \in \Gamma_1(L_0) \cap \Gamma_1(M_0)$ and $s' \notin \Gamma_1(L) \cup \Gamma_1(M)$.

We now have to distinguish between q odd and q even.

4.1. *The Case q Odd*

In this case, the tangent T is unique (see (ARC1)). Also, any $(q+1)$-arc in $\mathbf{PG}(3, q)$ is a twisted cubic (see [1]) and the $(q+1)$-arc in $\mathbf{PG}(2, q)$ mentioned in (ARC1) is a conic.

Since T is unique, it is independent of R. Also, we can write $T = \xi_R \cap \eta_{L,M} = \xi_R \cap \xi_M \cap \xi_L = \eta_{R,M} \cap \xi_L$. Consequently, T is also tangent to $\mathscr{A}_{R,M}$ at y_0. Hence, if we consider any two lines X and Y of $\Gamma_6(M)$, then the $(q+1)$-arcs $\mathscr{A}_{X,M}$ and $\mathscr{A}_{Y,M}$ have the same tangents at their intersection. We now choose a subspace $\mathbf{PG}(3, q)$ of ξ_M skew to M_0 and we project $\xi_M \backslash \Gamma_1(M_0)$ from M_0 onto $\mathbf{PG}(3, q)$. Let $\mathscr{C}_L$ be the projection of $\mathscr{A}_{L,M} \backslash \{y_0\}$ from M_0 onto $\mathbf{PG}(3, q)$ together with the point $\langle M_0, T \rangle \cap \mathbf{PG}(3, q) =: y'$. Then $\mathscr{C}_L$ is a conic. The projections M'_i of M_i, $1 \leqslant i \leqslant q$, are q generators of the quadratic cone $\mathscr{Q}$ containing $\mathscr{C}_L$ and with vertex M', where M' is the intersection of $\langle M_0, M \rangle$ and $\mathbf{PG}(3, q)$. Let π_0 be the plane $\xi_{M_0} \cap \mathbf{PG}(3, q)$. By the arguments above, ξ_{M_0} contains the osculating plane, and hence the tangent line, of each $\mathscr{A}_{X,M}$ at the common point of $\mathscr{A}_{X,M}$ and M_0, with X opposite M. Hence π_0 is a plane meeting M'_i exactly in the point M', $1 \leqslant i \leqslant q$, and it contains the projection of all tangents. It follows that π_0 is the tangent plane of the cone $\mathscr{Q}$ at $M'y'$. Let G be the generator of the cone containing y'. Let X be an arbitrary line of Γ opposite M. Then the projection $\mathscr{C}'$ of $\mathscr{A}_{X,M} \backslash \{x\}$ (where $\{x\} = \Gamma_3(X) \cap \Gamma_1(M_0)$) from M_0 onto $\mathbf{PG}(3, q)$ lies on the cone $\mathscr{Q}$. If x' is the projection of the tangent (minus x) of $\mathscr{A}_{X,M}$ at x, then obviously $\mathscr{C}' \cup \{x'\}$ is a conic; also, x' is on $\pi_0 \cap \pi$, where π is the plane of $\mathscr{C}'$. Further, $\mathscr{C}' \cup \{x''\}$, with $\{x''\} = G \cap \pi$, is a conic. As $x', x'' \in \pi \cap \pi_0$ and both extend $\mathscr{C}'$ to a conic, we necessarily have $x' = x''$. Hence $x' \in G$.

Let $y_1 \in M_0 \backslash \{y_0\}$, and let T_1 be the tangent at y_1 of the twisted cubic $\mathscr{A}_{X,M}$, with X opposite M and $y_1 \in \Gamma_3(X)$. Assume, by way of contradiction, that $T \cap T_1 = \{t\}$. Then $t \in \zeta_{s'}$ with $s' \in \Gamma_i(y_0) \cup \Gamma_j(y_1)$, with $i, j = 0, 2, 4$. It easily follows that t belongs to $\zeta_{s'}$ for each point s' of Γ. In particular, t belongs to any tangent of the twisted cubic $\mathscr{A}_{L,M}$, clearly a contradiction by (ARC4).

Let $\mathscr{C}_R$ be the unique conic on $\mathscr{Q}$ corresponding with $\mathscr{A}_{R,M}$. Then, since $\mathscr{C}_L \cap \mathscr{C}_R = \{y'\}$, the conics $\mathscr{C}_L$ and $\mathscr{C}_R$ have the same tangent line at y'. Now let p be an internal point of $\mathscr{C}_L$. Consider an arbitrary line P in $\mathbf{PG}(3, q)$ through p not containing the vertex M' of $\mathscr{Q}$, not contained in the plane of $\mathscr{C}_L$, and such that P meets $\mathscr{Q}$ in two points p'_1, p'_2. Then p'_1, p'_2 are the projections from M_0 onto $\mathbf{PG}(3, q)$ of two points p_1, p_2 (non-collinear in Γ), respectively, belonging to $\Gamma_3(M)$. Now, there is a unique line X opposite M with $p_1, p_2 \in \mathscr{A}_{X,M}$. Let $\mathscr{C}_X$ be the unique conic on $\mathscr{Q}$ corresponding to $\mathscr{A}_{X,M}$. The sets $\mathscr{A}_{L,M}$ and $\mathscr{A}_{X,M}$ have just one point in common. This common point is not on M_0, as otherwise $\mathscr{C}_X$ contains y', and $\mathscr{C}_X$ and $\mathscr{C}_L$ have a common tangent line at y', contradicting the fact that p belongs to the planes of $\mathscr{C}_L$ and $\mathscr{C}_X$. So $\mathscr{A}_{L,M} \cap \mathscr{A}_{X,M}$ does not belong to M_0. Hence $\mathscr{C}_X$ and $\mathscr{C}_L$ meet in a unique point not on G (by a previous paragraph $\mathscr{C}_X$ and $\mathscr{C}_L$ do not have a common point on G). At that point the tangent lines of $\mathscr{C}_X$ and $\mathscr{C}_L$ coincide. Hence p is contained in that common tangent line and therefore cannot be an internal point of $\mathscr{C}_L$. This is a contradiction.

4.2. *The Case q Even*

The way to handle this case is by looking at subhexagons arising from subplanes of order 2 of $\pi(\Gamma)$. We first prove a lemma about these structures.

LEMMA 16. *Let Γ' be a thin generalized hexagon of order* $(2, 1)$, *weakly embedded in* $\mathbf{PG}(n, \mathbb{K})$, *for some* (*not necessarily finite*) *field* $\mathbb{K}$. *Then* $n=7$ *and, if* $\mathbb{F}$ *is the prime subfield of* $\mathbb{K}$, *then* Γ' *is contained in a subspace* $\mathbf{PG}(7, \mathbb{F})$ *of* $\mathbf{PG}(7, \mathbb{K})$. *In particular, if the characteristic of* $\mathbb{K}$ *is equal to* 2, *then* Γ' *is fully and weakly embedded in* $\mathbf{PG}(7, \mathbb{F})$. *Also, the embedding is unique up to a linear projectivity of* $\mathbf{PG}(n, \mathbb{K})$.

Proof. Let e_i be the point of $\mathbf{PG}(n, \mathbb{K})$ with coordinates $(0, 0, ..., 0, 1, 0, ..., 0)$, where there are i zeros preceding the coordinate 1, $0 \leqslant i \leqslant n$. Now the points of any apartment Σ of Γ' span a 5-dimensional space. For if a point p of Σ is contained in the space generated by the other points of Σ, then p is contained in $U := \langle p'^{\perp\!\!\perp} \rangle$, where p' is in Σ opposite p. It is now easily seen that Γ' must be contained in U, a contradiction. Hence $n \geqslant 5$. Similarly, one shows that, for every point p, the space $\langle p^{\perp\!\!\perp} \rangle$ is a hyperplane which does not contain points of Γ' opposite p, and that the six hyperplanes thus obtained from the points of any apartment are linearly independent. Also, it is easily seen that the subspace of $\mathbf{PG}(n, \mathbb{K})$ generated by an apartment and two well chosen points must contain all the points of Γ'. Hence $n \leqslant 7$.

So we may choose any apartment of Γ' and identify its points with e_0, $e_1, ..., e_5$, with $e_i \in e_{i+1}^{\perp}$, $i = 0, 1, ..., 4$, and $e_0 \in e_5^{\perp}$. For any point p of Γ' we denote the subspace $\langle p^{\perp\!\!\!\perp} \rangle$ by ζ_p. It follows from an argument above that ζ_p does not contain any point of Γ' opposite p. So all ζ_p's are distinct, if p runs through the set of points of Γ'.

Suppose first $n=5$. Let the point p of Γ' be defined by $\Gamma'_3(e_1e_2) \cap \Gamma'_3(e_4e_5) = \{e_0, e_3, p\}$. Then $p \in \zeta_{e_1} \cap \zeta_{e_2} \cap \zeta_{e_4} \cap \zeta_{e_5}$, and the latter is a 1-dimensional space. Hence p is on the line joining e_0 and e_3 in $\mathbf{PG}(5, \mathbb{K})$. Similarly, if $\{p_{126}\} = \Gamma'_3(e_1e_2) \cap \Gamma'_2(p)$ and $\{p_{23}\} = \Gamma'_2(e_2) \cap \Gamma'_2(e_3)$, then $\{e_0, p_{23}, p_{126}\} = \Gamma'_3(e_1e_2) \cap \Gamma'_3(L)$, with $\{L\} = \Gamma'_3(e_0) \cap \Gamma'_3(e_5)$, and so e_0, p_{23}, p_{126} are collinear in $\mathbf{PG}(5, \mathbb{K})$. This implies that e_0, e_1, e_2 and e_3 are contained in a plane, a contradiction.

Now let $n=6, 7$. Without loss of generality, we may assume that p, with $\Gamma'_3(e_1e_2) \cap \Gamma'_3(e_4e_5) = \{e_0, e_3, p\}$, is not contained in the subspace of $\mathbf{PG}(n, \mathbb{K})$ generated by the points of Σ (with notation as above). Hence we may put $p = e_6$. Let p' (respectively p'') be the unique point of Γ' not contained in Σ and at distance 3 from both the lines e_0e_5 and e_2e_3 (respectively e_0e_1 and e_3e_4). If $n=7$, we may assume that e_0, $e_1, \ldots, e_6$, p' generate $\mathbf{PG}(n, \mathbb{K})$. Let p_{ij} be the point of Γ' on the line e_ie_j, if the latter corresponds to a line of Γ', $i<j$. Then we may choose coordinates in such a way that $p_{ij} = e_i + e_j$, for all suitable pairs $\{i, j\}$. Also, we may put $p_{126} = e_1 + e_2 + e_6$ (where p_{126} is defined as above). Let p_{057} (respectively p_{348}) be the unique point of Γ' on the line $p'p_{05}$ (respectively $p''p_{34}$) distinct from p' and from p_{05} (respectively p'' and p_{34}). Then p_{126}, p_{057} and p_{348} are collinear in Γ'. Hence, if we identify a point with its coordinates, we may put

$$
\begin{aligned}
p_{057} &= e_0 + e_5 + xp', \\
p_{348} &= (e_1 + e_2 + e_6) + c(e_0 + e_5 + xp'), \\
p'' &= (e_1 + e_2 + e_6) + c(e_0 + e_5 + xp') + d(e_3 + e_4),
\end{aligned}
$$

for some non-zero elements $c, d, x \in \mathbb{K}$. Similarly we define the points p_{237}, p_{456}, p_{018} of Γ' respectively on the lines $p'p_{23}$, e_6p_{45}, $p''p_{01}$, and such that they form a line of Γ'. We may now put

$$
\begin{aligned}
p_{237} &= e_2 + e_3 + ap', \\
p_{456} &= e_4 + e_5 + be_6, \\
p_{018} &= (e_2 + e_3 + ap') + f(e_4 + e_5 + be_6), \\
p'' &= (e_2 + e_3 + ap') + f(e_4 + e_5 + be_6) + g(e_0 + e_1),
\end{aligned}
$$

for some non-zero elements $a, b, f, g \in \mathbb{K}$.

We now suppose that $n=7$. Then p' can be chosen such that $p' = e_7$ and $x=1$. Comparing the two expressions for p'' obtained above, we easily deduce $a=b=c=d=f=g=1$. Hence Γ' is contained in the subspace $\mathbf{PG}(7, \mathbb{F})$ of $\mathbf{PG}(7, \mathbb{K})$, where $\mathbb{F}$ is the prime field of $\mathbb{K}$. The embedding is automatically weak since ζ_p, p any point of Γ', is generated by 7 points, as can be easily checked.

Now let $n=6$. Our first aim is to write p' as a linear combination of e_0, $e_1, ..., e_6$. Therefore, we note that ζ_{e_2} contains (and hence is generated by) $e_0, e_1, e_2, e_3, e_4, e_6$. Since $p' \in \zeta_{e_2}$, we deduce that e_5 is not involved in p'. Similarly, e_2 is not involved in p'. Moreover, $\zeta_{p'}$ contains (and hence is generated by) $e_1, e_2, e_3, e_5, p_{126}$ and p_{456}. So we may put

$$p' = r(e_4 + be_6) + ue_3 + ve_0 + w(e_1 + e_6),$$

for some non-zero r, $w \in \mathbb{K}$ (non-zero indeed because otherwise p' is contained in ζ_{e_i} for $i \in \{1, 4\}$, a contradiction), and u, $v \in \mathbb{K}$, with $(u, v) \neq (0, 0)$ (for otherwise $p' \in \zeta_{e_6}$). So we may put $r=1$.

Now we note that $\zeta_{e_0} \cap \cdots \cap \zeta_{e_5}$ is a unique point p^*. We also know that $\zeta_{e_1} \cap \zeta_{e_2} \cap \zeta_{e_4} \cap \zeta_{e_5}$ is 2-dimensional and that it contains e_0, e_3, e_6. Hence p^* belongs to $\pi_{036} := \langle e_0, e_3, e_6 \rangle$. Similarly p^* belongs to $\pi_{147} := \langle e_1, e_4, p' \rangle$ (and this is a plane, for if it were a line, then p' would belong to ζ_{e_6}, a contradiction) and to $\pi_{258} := \langle e_2, e_5, p'' \rangle$. It is clear that the planes π_{036} and π_{147} intersect in the point $(b+w)e_6 + ue_3 + ve_0 = p^*$. Since p^* must also belong to π_{258}, we conclude that we can write

$$p'' = (b+w)e_6 + ue_3 + ve_0 + se_2 + te_5,$$

for some non-zero s, $t \in \mathbb{K}$ (if $s=0$, then $p'' \in \zeta_{e_5}$, a contradiction; if $t=0$, then $p'' \in \zeta_{e_2}$, a contradiction). Plugging in the expression of p' in the two expressions above for p'', we obtain

$$p'' = (c + cxv)e_0 + (1 + cxw)e_1 + e_2 + (d + cxu)e_3$$
$$+ (d + cx)e_4 + ce_5 + (1 + cxw + cxb)e_6$$

and

$$p'' = (g + av)e_0 + (g + aw)e_1 + e_2 + (1 + au)e_3$$
$$+ (f + a)e_4 + fe_5 + (fb + aw + ab)e_6.$$

Comparing coefficients, we obtain after a tedious computation $u=v=2$ (hence the characteristic of $\mathbb{K}$ is not equal to 2), $s=t=-2$ and $b=c=d=f=g=w=1=-a=-x$. Hence $\zeta_{p_{057}}$ contains the points e_2+e_3, e_0, e_5, $p' = 2e_0 + e_1 + 2e_3 + e_4 + 2e_6$, e_6, $e_1 + e_2$ and $e_3 + e_4$. These points, however, generate the space $\langle e_0, e_1 + e_2, e_3 + e_2, e_4 + e_3, e_5, e_6, 2e_3 \rangle = \mathbf{PG}(6, \mathbb{K})$ (remember that the characteristic of $\mathbb{K}$ is different from 2), a contradiction.

The lemma is proved. ∎

Remark 17. If $n=7$ in the above lemma, then it is easy to check by a direct computation (given the coordinates of the points and lines of Γ' in the proof) that, with the notation of the above proof, the subspaces $\langle \Gamma'_3(e_0e_1) \cap \Gamma'_3(e_3e_4)\rangle$ and $\langle \Gamma'_3(e_1e_2) \cap \Gamma'_3(e_4e_5)\rangle$ are disjoint.

Remark 18. It is easy to check that the automorphism group induced by $\mathbf{PGL}_8(\mathbb{K})$ on any embedded thin hexagon of order (2, 1) in $\mathbf{PG}(7, \mathbb{K})$, for any field $\mathbb{K}$, is equal to $\mathbf{PGL}_3(2)$. Since we will not need this fact, we will not explicitly prove it. It can be done as an elementary exercise by the interested reader.

We can now continue with our proof of the Main Result for $d=7$ and q even. Notice that $q>2$. We use the same notation as in Subsection 4.1. Let x_i and z_i be the intersection of L and L_i, and of M and M_i, respectively, $0 \leqslant i \leqslant q$. Then the space $\zeta_{x_0} \cap \zeta_{y_0} \cap \zeta_{z_0} \cap \zeta_{x_q} \cap \zeta_{y_q} \cap \zeta_{z_q}$ is a line N, and by previous considerations it is also equal to the intersection of the osculating planes of $\mathscr{A}_{L,M}$ at y_0 and at y_q.

Now the projective plane $\pi(\Gamma)$ is Desarguesian, hence if s is an arbitrary point of Γ on L_0, $x_0 \neq s \neq y_0$, then every point x_i, $1 \leqslant i \leqslant q-1$, is together with s, x_0, y_0, z_0, x_q, y_q, z_q contained in a subhexagon $\Gamma_{s,i}$ of order (2, 1), which is weakly embedded in a subspace of $\mathbf{PG}(7, q)$. By Lemma 16 above, $\Gamma_{s,i}$ is fully and weakly embedded in a subspace $\mathbf{PG}(7, 2)$ of $\mathbf{PG}(7, q)$. For p a point of $\Gamma_{s,i}$, it is clear that the space generated by the points not opposite p in $\Gamma_{s,i}$ coincides with ζ_p. Hence N is the intersection of all hyperplanes $\mathscr{H}_p$ generated by the points of $\Gamma_{s,i}$ not opposite (in $\Gamma_{s,i}$) the point p, with $p \in \{x_0, y_0, z_0, x_q, y_q, z_q\}$.

Now let s' be the unique point of $\Gamma_{s,i}$ collinear with s and at distance 3 from M_q. Then we can choose s such that the intersection x of the plane $\langle z_0, s', x_q\rangle$ with N does not lie on any tangent to $\mathscr{A}_{L,M}$ at y_0 (because there are $q-1$ choices for s giving rise to $q-1$ choices for x by property (ARC3) above, because there are only 2 "forbidden" points on N, and as $q \geqslant 4$). Hence we can now choose i such that x belongs to the plane $\langle y_0, y_i, y_q\rangle$. But then the 6-dimensional space generated by x_0, y_0, z_0, x_q, y_q, z_q and x contains also s' and y_i and hence $\Gamma_{s,i}$, contradicting Lemma 16. This completes the case $d=7$.

5. THE CASE $d=8$ AND $\rho=6$

From now on we put $d=8$ and in the present section we assume that the dimension of ξ_L is equal to 6, for all lines L of Γ.

We will use the following notation: a sequence $(X_0, X_1, ..., X_q)$ of distinct subspaces of $\mathbf{PG}(d, q)$ of a certain dimension m, where all X_i contain a fixed subspace of dimension $m-1$, and where all X_i are contained in a fixed subspace of dimension $m+1$ of $\mathbf{PG}(d, q)$, is *projective with* another such sequence $(Y_0, Y_1, ..., Y_q)$ (where the dimension m' of Y_i is not necessarily equal to m) if the cross-ratios $(X_i, X_j; X_k, X_\ell)$ and $(Y_i, Y_j; Y_k, Y_\ell)$ are the same for all $\{i, j, k, \ell\} \subseteq \{0, 1, ..., q\}$, with $|\{i, j, k, \ell\}| = 4$.

Choose $L \in \mathscr{L}$ and let $M \in \mathscr{L}$ be opposite L. Put $\eta_{L,M} := \xi_L \cap \xi_M$. Lemma 6 implies that $\langle \xi_L, \xi_M \rangle = \mathbf{PG}(8, q)$, hence $\eta_{L,M}$ is 4-dimensional. Let L_0, L_1 be two elements of $\Gamma_2(L)$ and put $\{M_i\} = \Gamma_2(L_i) \cap \Gamma_2(M)$, $i = 0, 1$. Let $\{N_1, N_2, ..., N_q = M\} = \Gamma_2(M_0) \backslash \{L_0\}$. Furthermore, put $U := \eta_{L,M} \cap \zeta_u$, where u is the intersection point of L_0 and M_0. Then U has dimension 3. Also, we let x_i, $i \in \{1, 2, ..., q\}$, be the intersection point of M_0 and N_i. Since all ζ_{x_i}, $i = 1, 2, ..., q$, and ζ_u share the same 6-dimensional space ξ_{M_0}, we deduce that U contains $\pi_i := \eta_{L,N_i} \cap \eta_{L,M}$ for all $i \in \{1, 2, ..., q-1\}$. It is clear that π_i is either 2-dimensional or 3-dimensional. Suppose, for some $i < q$, that π_i is 3-dimensional. Then $V := \langle \eta_{L,M}, \eta_{L,N_i} \rangle$ is 5-dimensional. But clearly the intersection of L_1 with V contains at least two points (one of $\eta_{L,M}$ and one of η_{L,N_i}). Interchanging the roles of L_1 and any element of $\Gamma_2(L) \backslash \{L_0\}$, we see that V contains L and hence ξ_L, a contradiction. Hence π_i is a plane for all $i \in \{1, 2, ..., q-1\}$. Now we fix an arbitrary $i \in \{1, 2, ..., q-1\}$. We consider a 3-dimensional space W_i inside ξ_L skew to π_i, and we project all elements of $\Gamma_1(L)$, $\Gamma_2(L) \backslash \{L_0\}$, $(\Gamma_3(L) \cap \Gamma_3(M)) \backslash \{u\}$ and $(\Gamma_3(L) \cap \Gamma_3(N_i)) \backslash \{u\}$ from π_i onto W_i. The spaces $\eta_{L,M}$, η_{L,N_i} and L are mapped onto three different pairwise skew lines, say, M', N_i', L'. Hence the projection of the elements of $\Gamma_2(L) \backslash \{L_0\}$ forms a set of q lines of a regulus $\mathscr{R}_i$. The projection of $\langle U, L_0 \rangle \backslash \{\pi_0\}$ clearly must complete this regulus (the projection of $U \backslash \{\pi_i\}$ is a point on M' distinct from the projection of any element of $(\Gamma_3(L) \cap \Gamma_3(M)) \backslash \{u\}$). As the hyperplanes ξ_n, with $n \in N_i$, are the $q+1$ hyperplanes containing ξ_{N_i}, we have also that $\langle \pi_i, u_1 \rangle \neq \langle \pi_i, u_2 \rangle$ for distinct points u_1, u_2 on $(\Gamma_3(L) \cap \Gamma_3(M)) \backslash \{u\}$.

Hence, if we denote the set of points incident with L by $\{y_0, y_1, ..., y_q\}$, with $y_0 \mathbf{I} L_0$, and if we denote the unique element of $\Gamma_3(L) \cap \Gamma_3(M)$ collinear in Γ with y_j, $j \in \{1, 2, ..., q\}$, by z_j, then the sequence $(y_0, y_1, ..., y_q)$ is projective with the sequence $(U, \langle \pi_i, z_1 \rangle, ..., \langle \pi_i, z_q \rangle)$. Now fix $j \in \{1, 2, ..., q-1\}$, $j \neq i$. Then the sequence $(U, \langle \pi_i, z_1 \rangle, ..., \langle \pi_i, z_q \rangle)$ is projective with $(U, \langle \pi_j, z_1 \rangle, ..., \langle \pi_j, z_q \rangle)$. Suppose first that $\pi_i \neq \pi_j$. Let K be the intersection line of π_i and π_j. We choose a plane π' in $\eta_{L,M}$ skew to K and project the above sequences from K onto π'. Denote by U' the projection of $U \backslash K$, by p_i' (respectively p_j') the projection of $\pi_i \backslash K$ (respectively $\pi_j \backslash K$), and by z_k' the projection of z_k, $k = 1, 2, ..., q$. Then we obtain that the sequence $(U' = p_i' p_j', p_i' z_1', p_i' z_2', ..., p_i' z_q')$ is projective with $(U' = p_j' p_i', p_j' z_1', p_j' z_2', ..., p_j' z_q')$. Hence the points $p_i', p_j', z_1', z_2', ..., z_q'$ lie on a conic, which must be degenerate.

So the points $z'_1, z'_2, ..., z'_q$ are incident with a common line K'. Hence ξ_L is equal to the 5-dimensional space $\langle L, K, K' \rangle$, a contradiction.

So we have $\pi_i = \pi_j =: \pi$, for every $i, j \in \{1, 2, ..., q-1\}$. Let H be a line of Γ concurrent with M_1, but distinct from L_1 and from M. Let h be a point on H not incident with M_1. Then $\pi \cap \zeta_{z_1} \cap \zeta_h$ contains a point w. It is clear that w is contained in ζ_x, for all points x incident with one of the lines L, L_0, L_1, M_0, M_1, N_i, M, H, with $i \in \{1, 2, ..., q\}$. Clearly, for every point v on a line X meeting two of these lines, the space ξ_v also contains w. Hence w is contained in ζ_x for every point x which corresponds to a flag of the subplane of $\pi(\Gamma)$ generated by the elements corresponding with L, L_0, L_1, M_0, M_1, N_i, M, H, $i \in \{1, 2, ..., q\}$. But this subplane obviously is $\pi(\Gamma)$ itself. Consequently w is contained in ζ_x, for all points x of Γ. We now project Γ from w onto some hyperplane $\mathbf{PG}(7, q)$ of $\mathbf{PG}(8, q)$ not containing w. We claim that we obtain a full weak embedding in $\mathbf{PG}(7, q)$ of a generalized hexagon isomorphic to Γ, with $\xi_{L'}$ 5-dimensional for every line L' of the embedding. Indeed, it suffices to show that there is no line in $\mathbf{PG}(8, q)$ incident with w and two points t_1, t_2 of Γ. It is easy to see that there would be a point $t \in \Gamma_6(t_1) \cap \Gamma_4(t_2)$. Then $w, t_2 \in \zeta_t$ and hence $t_1 \in \zeta_t$, a contradiction. Our claim follows. But now we contradict the classification of all such embeddings in $\mathbf{PG}(7, q)$; see the theorem in the Introduction and the previous section.

6. THE CASE $d = 8$ AND $\rho = 5$

Let L, M be opposite lines of Γ. As before, our assumption implies that $\eta_{L, M}$ is 3-dimensional. Let $\mathcal{A}_{L, M}$ be the set of points of $\eta_{L, M}$ at distance 3 in Γ from both L and M.

LEMMA 19. *The set $\mathcal{A}_{L, M}$ is a σ-curve in $\eta_{L, M}$, for some automorphism σ of $\mathbf{GF}(q)$.*

Proof. If q is even, then either $\mathcal{A}_{L, M}$ is a $(q+1)$-arc, or there exists a set of 4 different lines of Γ concurrent with L and contained in a 4-space. In the former case, $\mathcal{A}_{L, M}$ is a σ-curve by (ARC5). In the latter case, the result follows from Lemma 15.

If q is odd, then the result again follows from Lemma 15 if there exists a set of 4 different lines of Γ concurrent with L and contained in a 4-space. Suppose now that $\mathcal{A}_{L, M}$ is a $(q+1)$-arc.

Let L_0, L_1, $L_2 \in \Gamma_2(L)$ and put $\{M_i\} := \Gamma_2(L_i) \cap \Gamma_2(M)$, $i = 0, 1, 2$. Also, let N be any element of $\Gamma_2(L_0)$ distinct from both L and M_0. Assume, by way of contradiction, that the subspace $U_N := \langle L, L_0, M_0, M, M_1, L_1,$

$L_2, N\rangle$ is k-dimensional, with $k<7$. Let $N' \in \Gamma_2(L_0) \backslash \{L, M_0, N\}$. The subspace $\langle U_N, N'\rangle$ induces a subhexagon which corresponds with some subplane π' of $\pi(\Gamma)$. Then N' can be chosen in such a way that $\pi' = \pi(\Gamma)$. So the subhexagon coincides with Γ, hence Γ is in $\langle U_N, N'\rangle$, which is at most 7-dimensional, a contradiction. Consequently U_N is 7-dimensional. Now U_N induces a subhexagon Γ^* of order $(q^*, 1)$, with $q^* > 2$ (indeed, since a subhexagon of order $(q^*, 1)$ corresponds with a subplane of $\pi(\Gamma)$ of order q^*, we have that q^* divides q, hence $q^* > 2$ as q is odd). Let N' be an element of $\Gamma_2(L_0)$ not contained in Γ^*. We define $U_{N'}$ similarly as U_N above and we obtain a subhexagon Γ^{**} of Γ induced by $U_{N'}$. The intersection of U_N and $U_{N'}$ is a 6-dimensional space U and it contains L, L_0, L_1, L_2, M, M_0, M_1, M_2. If U contains $\xi_L \cap \xi_M$, then there must be a point u of $\xi_L \cap \xi_M$ inside $\langle L, M\rangle$ (by comparing dimensions). Hence the plane $\langle u, M\rangle$ (this is indeed a plane as follows from Lemma 6) meets the line L, and so there is a point of L in ξ_M, contradicting Lemma 6. Hence $U \cap \xi_L \cap \xi_M$ is the plane π containing $L_i \cap M_i$, $i = 0, 1, 2$, and so, as π contains just three points of $\mathscr{A}_{L,M}$, $\Gamma^* \cap \Gamma^{**}$ is the configuration formed by L, L_0, L_1, L_2, M, M_0, M_1, M_2. But as $\pi(\Gamma)$ is Desarguesian over a field of odd characteristic, this is a contradiction.

The lemma is proved. ∎

Now let L and M again be two opposite lines of Γ, put $\Gamma_2(L) = \{L_0, L_1, ..., L_q\}$ and let $\{M_i\} = \Gamma_2(M) \cap \Gamma_2(L_i)$, $i = 0, 1, ..., q$. Also, let N be a line of Γ concurrent with L_0, but distinct from L and from M_0. We put $\mathscr{A} := \mathscr{A}_{L,M}$ and $\eta := \eta_{L,M}$. By Lemma 19, we know that $\mathscr{A}$ is a σ-curve in the 3-dimensional space η. Put $\mathscr{A} =: \{x_0, x_1, ..., x_q\}$, with $L_i \mathbf{I} x_i \mathbf{I} M_i$, $i \in \{0, 1, ..., q\}$, and put $\Gamma_2(M_0) = \{N_0 = L_0, N_1 = M, N_2, ..., N_q\}$. First, let $\mathscr{A}$ be a $(q+1)$-arc. Choose $i \in \{2, 3, ..., q\}$. The $q+1$ hyperplanes ζ_n, with n incident with N_i, intersect η in $q+1$ planes which share a common line T_i. If n is not incident with M_0, then such a plane contains x_0 and one other point of $\mathscr{A}$; if n is incident with M_0, then $\zeta_n \cap \mathscr{A} = \{x_0\}$. Hence T_i is a tangent of $\mathscr{A}$ at x_0 and ζ_n, with n on M_0, is the osculating plane π_0 of $\mathscr{A}$ at x_0. We also have $\pi_0 = \eta \cap \tilde{\xi}_{M_0}$ and $T_i = \eta \cap \tilde{\xi}_{N_i}$. If $\mathscr{A}$ is not a $(q+1)$-arc, then the same conclusions follow from the proof of Lemma 13 in [7]. Suppose now that there exist indices $i, j \in \{2, 3, ..., q\}$ such that $T_i \neq T_j$. Without loss of generality, we may assume that for a certain tangent T of $\mathscr{A}$ at x_0, we have $T = T_i$, for all $i \in \{2, 3, ..., (q+2)/2\}$, and there exists at least one line $N' \in \Gamma_2(L_0)$ with $T = \eta \cap \tilde{\xi}_{N'}$. Denote by x the intersection point of T and ζ_{x_1}. Clearly, the set of lines K of Γ with $x \in \tilde{\xi}_K$ corresponds to a closed subconfiguration in $\pi(\Gamma)$. But this configuration contains a triangle (corresponding to L, L_0, M_0, M, M_1, L_1), up to duality at least $(q+4)/2$ points on one side of the triangle (the points corresponding to the lines L_0, M, N_i, $2 \leqslant i \leqslant (q+2)/2$), and one line, but not a side, through a vertex incident with that side (namely, the line corresponding with N'). We conclude that the closed subconfiguration must be a subplane, and that it must coincide with $\pi(\Gamma)$. Hence x belongs to the hyperplane ζ_z, for every $z \in \mathcal{P}$. Hence, similarly as at the end of Section 5, we can project the embedded hexagon Γ from x onto a hyperplane $\mathbf{PG}(7, q)$ (not containing x) of $\mathbf{PG}(8, q)$ to obtain a full weak embedding in $\mathbf{PG}(7, q)$ of a generalized hexagon isomorphic to Γ. Since the point x belongs to η, we see that this embedding in $\mathbf{PG}(7, q)$ satisfies the assumption of the last part of the theorem in the introduction (with the projection L' of L playing the role of L). Hence the projection of $\mathscr{A}$ from x onto a hyperplane of η not containing x is a conic. Take, with respect to a suitable coordinatization, for $\mathscr{A}$ the set of points $\{(1, r, r^\sigma, r^{1+\sigma}) \mid r \in \mathbf{GF}(q)\} \cup \{(0, 0, 0, 1)\}$, with σ an automorphism of $\mathbf{GF}(q)$; take $x_0 = (1, 0, 0, 0)$ and $x_1 = (0, 0, 0, 1)$, which is allowed by Lemma 1. Then x is either $(0, 1, 0, 0)$ or $(0, 0, 1, 0)$. No choice leads to a conic, a contradiction.

Hence $T_i = T_j =: T$ for all $i, j \in \{2, 3..., q\}$. Let T' be the tangent of $\mathscr{A}$ at x_0 which is the intersection of η with $\tilde{\xi}_{N'}$, for any $N' \in \Gamma_2(L_0) \backslash \{L, M_0\}$. The same argument as in the previous paragraph shows that $T \neq T'$.

Let $\{y_i\} := \Gamma_1(L) \cap \Gamma_1(L_i)$ and $\{z_i\} = \Gamma_1(M) \cap \Gamma_1(M_i)$, $i \in \{0, 1, ..., q\}$. Put $\eta' = \eta_{L_1, M_0}$. It is clear that the subspace $U := \langle \eta, \eta' \rangle$ is contained in the 6-dimensional space $\zeta_{y_1} \cap \zeta_{z_0}$. But clearly $\langle \eta, \eta', y_1, z_0 \rangle$ is the whole space $\mathbf{PG}(8, q)$, hence U is 6-dimensional and $\eta \cap \eta'$ is a point u.

Now consider the space $W := \langle \xi_{N'}, \eta \rangle$. Since it clearly contains M, it contains ξ_M, and hence $W = \langle \xi_{N'}, \xi_M \rangle$. But then $W = \langle M, N', \eta_{M, N'} \rangle$, so W has dimension 7. Consequently $\xi_{N'} \cap \eta$ is a line, which is contained in $\tilde{\xi}_{N'} \cap \eta = T'$. Hence $\xi_{N'} \cap \eta = T'$. Now since $q > 2$, we deduce that T' is contained in the space $\xi_{N'} \cap \xi_{N''}$, with $N'' \in \Gamma_2(L_0) \backslash \{L, M_0, N'\}$. Now note that $\langle \xi_{N'}, \xi_{N''} \rangle$ induces a subhexagon of order $(q, 1)$ of Γ, hence $\langle \xi_{N'}, \xi_{N''} \rangle = \mathbf{PG}(8, q)$ and so $\xi_{N'} \cap \xi_{N''}$ is a plane κ_{L_0}.

Of course, in a similar way, there must be a tangent T^* of $\mathscr{A}_{L_1, M_0}$ at y_0 contained in κ_{L_0}. Hence u is the intersection of T' and T^*. Also similarly, we deduce that the intersection point u' of η and η_{L_0, M_1} is incident with T. If u'' is the intersection point of η' and η_{L_0, M_1}, then u, u' and u'' form a triangle in the plane $\zeta_{x_0} \cap \zeta_{x_1} \cap \zeta_{y_0} \cap \zeta_{y_1} \cap \zeta_{z_0} \cap \zeta_{z_1}$ (indeed, $\tilde{\xi}_L \cap \tilde{\xi}_M = \tilde{\eta}_{L, M}$ is 4-dimensional and $\tilde{\eta}_{L, M} \cap \zeta_{x_0} \neq \tilde{\eta}_{L, M} \cap \zeta_{x_1}$, as equality would imply $\eta \subseteq \zeta_{x_0} \cap \zeta_{x_1}$, a contradiction; if u, u', u'' were collinear, then they all would belong to $\eta \cap \eta' \cap \eta_{L_0, M_0}$, a contradiction as $\eta \cap \eta'$ is a point). Moreover, we see that $\xi_L = \langle L, L_0, L_1, u, u' \rangle$, $\xi_{L_1} = \langle L_1, L, M_1, u, u'' \rangle$, $\xi_{M_1} = \langle M_1, L_1, M, u', u'' \rangle$, $\xi_M = \langle M, M_0, M_1, u, u' \rangle$, $\xi_{M_0} = \langle M_0, M, L_0, u, u'' \rangle$ and $\xi_{L_0} = \langle L_0, L, M_0, u', u'' \rangle$. Remark also that u'' is not in η, and so $\tilde{\xi}_L \cap \tilde{\xi}_M = \langle \eta, u'' \rangle$, that is, $\mathbf{PG}(8, q) = \langle x_0, x_1, y_0, y_1, z_0, z_1, u, u', u'' \rangle$.

We now show that Γ is completely determined by the σ-curve.

In order to do this explicitly, we choose coordinates in $\mathbf{PG}(8, q)$ as

$$y_0 = (1, 0, 0; 0, 0, 0; 0, 0, 0), \qquad x_0 = (0, 0, 0; 0, 0, 1; 0, 0, 0)$$
$$y_1 = (0, 1, 0; 0, 0, 0; 0, 0, 0), \qquad u = (0, 0, 0; 0, 0, 0; 1, 0, 0)$$
$$x_1 = (0, 0, 1; 0, 0, 0; 0, 0, 0), \qquad u' = (0, 0, 0; 0, 0, 0; 0, 1, 0)$$
$$z_1 = (0, 0, 0; 1, 0, 0; 0, 0, 0), \qquad u'' = (0, 0, 0; 0, 0, 0; 0, 0, 1).$$
$$z_0 = (0, 0, 0; 0, 1, 0; 0, 0, 0),$$

We may choose coordinates such that $\mathscr{A}_{L_1, M_0}$ is then equal to the set

$$\{(x^{1+\sigma}, 0, 0; 1, 0, 0; x, 0, x^\sigma) \mid x \in \mathbf{GF}(q)\} \cup \{y_0\},$$

with σ the automorphism of $\mathbf{GF}(q)$ of the σ-curve under consideration. Moreover, we can choose coordinates such that the point $v_1 := (1, 0, 0; 1, 0, 0; 1, 0, 1)$ is in Γ collinear with $w_2 := (0, 0, 0; 0, 1, 1; 0, 0, 0)$. Now let $w_i = (0, 0, 0; 0, 1, a_i; 0, 0, 0) \in \Gamma_1(M_0)$, $i \in \{1, 2, ..., q\}$, $a_i \in \mathbf{GF}(q)$, $(a_1, a_2) = (0, 1)$. Fix $i \in \{3, 4, ..., q\}$. Then the cross-ratio $(x_0, z_0; w_2, w_i)$ is equal to a_i. Let $K \in \Gamma_2(M_1) \backslash \{L_1, M\}$, and denote by x'_0, z'_0, w'_2, w'_i the unique point of $\Gamma_2(K)$ at distance 4 in Γ from x_0, z_0, w_2, w_i, respectively. Clearly, we have $(x_0, z_0; w_2, w_i) = (\zeta_{x'_0}, \zeta_{z'_0}; \zeta_{w'_2}, \zeta_{w'_i})$. Now let π_{x_0}, π_{z_0}, π_{w_2}, π_{w_i} be the intersection of η' with $\zeta_{x'_0}$, $\zeta_{z'_0}$, $\zeta_{w'_2}$, $\zeta_{w'_i}$, respectively. Then $(\zeta_{x'_0}, \zeta_{z'_0}; \zeta_{w'_2}, \zeta_{w'_i}) = (\pi_{x_0}, \pi_{z_0}; \pi_{w_2}, \pi_{w_i})$. Now π_{z_0} is clearly the osculating plane of $\mathscr{A}_{L_1, M_0}$ at z_1. By an above argument, the planes π_{x_0}, π_{w_2} and π_{w_i} contain the tangent uz_1 to $\mathscr{A}_{L_1, M_0}$ at z_1. Moreover, the plane π_{x_0} contains $y_0 \in \mathscr{A}_{L_1, M_0}$ and the plane π_{w_2} contains v_1. Also, the plane π_{w_i} contains a unique point v_i of $\mathscr{A}_{L_1, M_0}$ (namely, the unique point of $\mathscr{A}_{L_1, M_0}$ collinear in Γ with w_i). One easily calculates that $(\pi_{x_0}, \pi_{z_0}; \pi_{w_2}, \pi_{w_i}) = a_i$ if and only if $v_i = (a_i^{\sigma+1}, 0, 0; 1, 0, 0; a_i, 0, a_i^\sigma)$. Hence we may choose subscripts in such a way that the line N_i of Γ contains the points $(0, 0, 0; 0, 1, a_i; 0, 0, 0)$ and $(a_i^{\sigma+1}, 0, 0; 1, 0, 0; a_i, 0, a_i^\sigma)$, $a_i \in \mathbf{GF}(q)$ (for $\mathscr{A}$ not a $(q+1)$-arc, this also follows from Remark 17 of [7]). We put $N_\infty = L_0$ (remark that $N_1 = M$).

Now consider again the line K of the previous paragraph. For every line $N_i \in \Gamma_2(M_0)$, there exists a unique line K_i of $\Gamma_2(K) \cap \Gamma_2(N_i)$. The line K_i meets the space ξ_L is a unique point u_i, which is also the intersection of the space $\langle N_i, K \rangle$ with ξ_L (the space $\langle N_i, K, \xi_L \rangle$ induces Γ, hence is 8-dimensional; therefore the 3-space $\langle N_i, K \rangle$ and the 5-space ξ_L meet in a point). Now $u_i \in \Gamma_3(L) \cap \Gamma_3(K)$. Hence the set $\{u_i \mid i \in \{\infty, 1, ..., q\}\}$ is contained in a 3-dimensional subspace $\eta_{L,K}$ of $\mathbf{PG}(8, q)$.

By choosing the unit point $(1, 1, 1; 1, 1, 1; 1, 1, 1)$ in a suitable way, we may assume that K contains the points $(0, 0, 1; 1, 0, 0; 0, 0, 0)$ and $(0, 1, 0; 0, 1, 0; 0, 1, 1)$, because the map $(X_0, X_1, ..., X_8) \mapsto (X_0, AX_1, BX_2; X_3, CX_4, CX_5; X_6, DX_7, X_8)$, with $A, B, C, D \in \mathbf{GF}(q)^\times$ arbitrary, preserves the coordinates of points introduced so far. We can now explicitly compute the coordinates of the point u_i and we obtain

$$u_i = (-a_i^{\sigma+1}, a_i^{\sigma}, 1; 0, 0, -a_i^{1+\sigma}; -a_i, a_i^{\sigma}, 0),$$

$a_i \in \mathbf{GF}(q)$, $i = 1, 2, ..., q$, and $u_\infty = (1, 0, 0; 0, 0, 1; 0, 0, 0)$. In order to find the unique point y_j on $\Gamma_1(L)$ collinear in Γ with u_i, $i = 1, 2, ..., q$, we use the property that $u_i y_j$ meets the space $\langle x_0, x_1, u, u' \rangle$ in a point. We obtain $y_j = (-a_i, 1, 0; 0, 0, 0; 0, 0, 0)$ and we may reindex the set $\Gamma_1(L)$ so that this point is called y_i.

There is a unique line $L_{i,j}$ of Γ meeting both $u_i y_i$ and N_j, $1 < i \leqslant q$ and $N_j \notin \{L_0, M\}$. There is also a unique point $p_{i,j}$ on $L_{i,j}$ at distance 3 from M_1. As above, we can explicitly calculate the coordinates of $p_{i,j}$. After an elementary computation, we obtain

$$p_{i,j} = (0, a_j^{\sigma+1}, a_j; a_i, a_i^{\sigma+1}, 0; 0, a_i^{\sigma} a_j, a_i a_j^{\sigma}).$$

It is then easy to calculate the coordinates of the point $q_{i,j}$ on N_j collinear with $p_{i,j}$. We obtain

$$q_{i,j} = (a_j^{\sigma+1}, 0, 0; 1, a_i^{\sigma}, a_i^{\sigma} a_j; a_j, 0, a_j^{\sigma}).$$

Hence we know the coordinates of $L_{i,j} = p_{i,j} q_{i,j}$.

Now similarly as above, the space $\langle L_{2,2}, L_{i,i} \rangle$, $i \neq 2$, meets the space ξ_{L_1} in a unique point r_i of Γ, and this determines the lines of $\Gamma_2(L_1)$, just in the same way as above. Similarly, we can determine the elements of $\Gamma_2(L_0)$ and of $\Gamma_2(M)$. This means that we have determined all lines at distance at most 4 from L. In a completely similar way, all lines at distance 4 from M can be calculated, hence all elements of Γ are determined and the embedding is uniquely determined.

This concludes the proof of our Main Result.

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