Some Remarks on Embeddings of the Flag Geometries of Projective Planes in Finite Projective Spaces

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Abstract

The flag geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ of a finite projective plane Π of order s is the generalized hexagon of order $(s, 1)$ obtained from Π by putting $\mathcal{P}$ equal to the set of all flags of Π, by putting $\mathcal{L}$ equal to the set of all points and lines of Π and where $\mathtt{I}$ is the natural incidence relation (inverse containment), i.e., Γ is the dual of the double of Π in the sense of [8]. Then we say that Γ is fully (and weakly) embedded in the finite projective space $\mathbf{PG}(d, q)$ if Γ is a subgeometry of the natural point-line geometry associated with $\mathbf{PG}(d, q)$, if $s = q$, if the set of points of Γ generates $\mathbf{PG}(d, q)$ (and if the set of points of Γ not opposite any given point of Γ does not generate $\mathbf{PG}(d, q)$). We have classified all such embeddings in [3, 4, 5, 6]. In the present paper, we weaken the hypotheses in some special cases, and we give an alternative formulation of the classification.

1 Definitions and statement of the main results

The problem we consider may be stated as follows. Let Π be a (finite) projective plane of order s. We define the *flag geometry* Γ of Π as follows. The points of Γ are the flags of Π (i.e., the incident point-line pairs of Π); the lines of Γ are the points and lines of Π. Incidence between points and lines of Γ is reverse containment. It follows that Γ is a (finite) generalized hexagon of order $(s, 1)$ (see (1.6) of [8]). The advantage of viewing Γ rather as a generalized hexagon than as a flag geometry of a projective plane is that one can apply results from the general theory of generalized hexagons. We will call Γ a *thin generalized hexagon* (since there are only 2 lines through every point of Γ).

Throughout, we assume that Γ is a thin generalized hexagon of order $(s,1)$ with corresponding projective plane $\pi(\Gamma)$. We introduce some further notation. Two elements of Γ are called *opposite* if they are at distance 6 from each other in the incidence graph of Γ. Two points of Γ are *collinear* if they are incident with a common line. For any point x of Γ, we denote by $x^{\perp\!\!\!\perp}$ the set of points of Γ not opposite x. Given a line L of Γ, we write $L^{\perp\!\!\!\perp}$ for the intersection of all sets $p^{\perp\!\!\!\perp}$ with p a point incident with L (in this notation we view L as the set of points incident with L). For an element x of Γ (point or line), we denote by $\Gamma_i(x)$ the set of elements of Γ at distance i from x in the incidence graph of Γ. In this notation, we have $x^{\perp\!\!\!\perp} = \Gamma_0(x) \cup \Gamma_2(x) \cup \Gamma_4(x)$ and $L^{\perp\!\!\!\perp} = \Gamma_1(L) \cup \Gamma_3(L)$, with x any point and L any line of Γ. Furthermore, an apartment of Γ is a thin subhexagon of order $(1,1)$. It corresponds with a triangle in $\pi(\Gamma)$.

Let $\mathbf{PG}(d,q)$ be the d-dimensional projective space over the Galois field $\mathbf{GF}(q)$. We say that Γ is *(weakly) embedded in* $\mathbf{PG}(d,q)$ if the point set of Γ is a subset of the point set of $\mathbf{PG}(d,q)$ which generates $\mathbf{PG}(d,q)$, if the line set of Γ is a subset of the line set of $\mathbf{PG}(d,q)$, if the incidence relation in $\mathbf{PG}(d,q)$ restricted to Γ is the incidence relation in Γ (and if for every point of Γ, the set $x^{\perp\!\!\!\perp}$ does not generate $\mathbf{PG}(d,q)$). If moreover $s = q$, then we say that the (weak) embedding is also *full*.

All weak full embeddings of thin generalized hexagons in finite projective spaces are classified in [3, 4, 5, 6]. They are just the examples presented in the next section. The motivation for classifying these objects is given by the fact that it is a crucial step for classifying all (weak) embeddings of line-regular generalized hexagons; line-regular just means that the hexagon is the dual of one of the classical examples naturally embedded in the triality quadric (these classical hexagons were discovered by J. Tits [7]). However, in order to obtain strong results, we need some lemmas on full embeddings of thin hexagons under slightly different hypotheses than in the papers [3, 4, 5, 6]. More exactly, we first note that all examples of weakly fully embedded finite thin hexagons live in d-dimensional space with $d = 6, 7, 8$ and that the hexagons themselves are the duals of the doubles of Desarguesian projective planes. In the present paper, we will show that every full embedding in $\mathbf{PG}(8,q)$ of a thin hexagon arising from a Desarguesian plane is automatically weak. Further, if q is a prime, then it follows from the classification that only for $d = 6, 7$, there exist weakly fully embedded thin hexagons in $\mathbf{PG}(d,q)$ (and $q = 3$ if $d = 6$). The fact that $d \neq 8$ can easily be shown directly under the assumption that the corresponding projective plane is Desarguesian, and we will do so below. We will also show that, if $d = 7$, then every full embedding in $\mathbf{PG}(7,q)$, q prime, is automatically weak (still assuming a Desarguesian plane corresponding to the embedded thin hexagon). Hence we will show the following result.

Theorem 1.1 *If the thin generalized hexagon Γ is fully embedded in $\mathbf{PG}(8,q)$, and if $\pi(\Gamma)$ is Desarguesian, then Γ is also weakly embedded and hence the embedding is known. If Γ is fully embedded in $\mathbf{PG}(7,q)$, with q prime (still assuming that $\pi(\Gamma)$ is Desarguesian), then again it is weakly embedded and the embedding is known.*

For the application to embeddings of line-regular hexagons, the case $d = 7$ plays a key role. It will be convenient to have to our disposal the following equivalent condition for a weak embedding.

Theorem 1.2 *If the thin generalized hexagon* Γ *is fully embedded in* $\mathbf{PG}(d,q)$*,* $d \geq 7$*, and if for every pair* $\{L,M\}$ *of opposite lines of* Γ *the set* $L^{\perp\!\!\perp} \cap M^{\perp\!\!\perp}$ *is contained in a plane, then* $d = 7$ *and the embedding is weak, and hence known.*

We will prove these theorems in Section 3. In Section 4, we present an alternative formulation of these results and of the results in [3, 4, 5, 6].

2 The examples

Let V be a 3-dimensional vector space over $\mathbf{GF}(q)$, and let V^* be the dual space. We choose dual bases. Then the vector lines of the tensor product $V \otimes V^*$ can be seen as the point-line pairs of the projective plane $\mathbf{PG}(2,q)$. Indeed, it is easily calculated that the pair $\{(x_0,x_1,x_2),[a_0,a_1,a_2]\}$ (we use parentheses for the coordinates of points and brackets for those of lines) corresponds to the vector line generated by the vector $(a_0x_0, a_0x_1, a_0x_2, a_1x_0, a_1x_1, a_1x_2, a_2x_0, a_2x_1, a_2x_2)$. Hence we have a mapping θ of the point-line pairs of $\mathbf{PG}(2,q)$ into the set of points of $\mathbf{PG}(8,q)$ (and the image of θ is the Segre variety $\mathcal{S}_{2;2}$, see Hirschfeld & Thas [1], §25.5). Let σ be any field automorphism of $\mathbf{GF}(q)$. We define a *twisted* version θ_σ of θ as follows. If p is a point of $\mathbf{PG}(2,q)$ and L a line of $\mathbf{PG}(2,q)$, then $\{p,L\}^{\theta_\sigma} = \{p^\sigma, L\}^\theta$, where p^σ is defined coordinatewise.

Restricting θ_σ to the incident point-line pairs of $\mathbf{PG}(2,q)$, we obtain a weak full embedding of the flag geometry Γ of $\mathbf{PG}(2,q)$ in $\mathbf{PG}(8,q)$ (if $\sigma \neq 1$) or in $\mathbf{PG}(7,q)$ (if $\sigma = 1$; in this case the images of all flags of $\mathbf{PG}(2,q)$ are contained in the hyperplane $\mathbf{PG}(7,q)$ with equation $X_{00} + X_{11} + X_{22} = 0$, where X_{ij} refers to the coordinate corresponding to a_ix_j in the above expression); see [6]. If $\sigma = 1$ and q is a power of 3, then one can project the embedded geometry from the point with coordinates $x_{00} = x_{11} = x_{22} = 1$, $x_{ij} = 0$, $i \neq j$, onto any hyperplane $\mathbf{PG}(6,q)$ of $\mathbf{PG}(7,q)$ to obtain a weak full embedding of Γ in $\mathbf{PG}(6,q)$ (see [3]).

3 Proofs of the main results

Throughout, we assume that Γ is a thin generalized hexagon, fully embedded in $\mathbf{PG}(d,q)$, $d = 6,7,8$.

We will need the following basic lemma.

Lemma 3.1 *Let* H *be any hyperplane of* $\mathbf{PG}(d,q)$*. If* H *contains an apartment of* Γ*, then the set of points* x *of* Γ *such that both elements of* $\Gamma_1(x)$ *are contained in* H *is the point set of a thin subhexagon* Γ' *(and hence* $\pi(\Gamma')$ *is a (generalized) subplane of* $\pi(\Gamma)$*).*

PROOF. Since Γ is thin, the embedding is *flat* in the sense of [2]. The result follows from Lemma 3 of [2]. □

With the notation of the lemma, we say that Γ' is induced by H.

Now assume that $\pi(\Gamma)$ is Desarguesian, that $d = 7$, and that q is prime. Let x be any point of Γ. Suppose $\Gamma_1(x) = \{L_1, L_2\}$. Choose points y_1 and y_2 on L_1 and L_2 respectively, $y_1 \neq x \neq y_2$, and put $\{L_i, M_i\} = \Gamma_1(y_i)$, $i = 1, 2$. Further, consider an arbitrary point z on L_1, $x \neq z \neq y_1$. Suppose $\{L_1, N\} = \Gamma_1(z)$. Put $\{x, x_1, x_2, \ldots, x_q\} = M_1^{\perp\perp} \cap M_2^{\perp\perp}$. The subspace U generated by L_1, L_2, M_1, M_2, N is at most 5-dimensional. The subspace V generated by U and x_1, x_2 contains an apartment of Γ, three lines of Γ meeting L_1 and three lines of Γ meeting M_1. It follows from 1.6.2 of [8] that the subhexagon Γ' induced by V is the dual of the double of a (thick) projective subplane $\pi(\Gamma')$, which must coincide with $\pi(\Gamma)$ since Desarguesian projective planes over prime fields do not admit proper subplanes. Hence V is 7-dimensional and so U is at least 5-dimensional. It follows that U is 5-dimensional, and that $U_1 = \langle U, x_1\rangle$ is 6-dimensional. Similarly, $U_i = \langle U, x_i\rangle$ is 6-dimensional, $i \in \{1, 2, \ldots, q\}$. If $U_i = U_j$, for $i \neq j$, then putting, without loss of generality, $i = 1$, $j = 2$, the above argument shows that $U_i = \mathbf{PG}(7, q)$, a contradiction. Hence $U_i \neq U_j$ for $i \neq j$. Consequently $\{U_i \,|\, i = 1, 2, \ldots, q\}$ is a set of q hyperplanes through U. Let H be the unique hyperplane through U distinct from U_i, for all $i \in \{1, 2, \ldots, q\}$. We remark that U_i does not contain any element of $\Gamma_2(L_2) \setminus \{L_1, M_2\}$, since otherwise the subhexagon induced by U_i must again coincide with Γ itself by a similar argument as before.

It follows that H contains all elements of $\Gamma_2(L_2)$. Considering a line $N' \in \Gamma_2(L_2) \setminus \{L_1, M_2\}$, we similarly deduce that there is a hyperplane H' containing all elements of $\Gamma_2(L_1) \cup \{N', M_2\}$. The intersection $H \cap H'$ together with the point x_1 generates a subspace which induces a subhexagon Γ', and again Γ' must coincide with Γ, as before. This implies that $H = H'$ and the embedding is weak. The first part of Theorem 1.1 is proved.

Now we assume that $\pi(\Gamma)$ is Desarguesian and that $d = 8$. We aim at proving that the embedding is weak. Let $x, L_1, L_2, y_1, y_2, M_1, M_2, z, N$ and x_i, $i \in \{1, 2, \ldots, x_q\}$, be as previously. Further, let z' be a point on L_1 such that, if we put $\{L_1, N'\} = \Gamma_1(z')$, then the cross ratio of the points of $\pi(\Gamma)$ corresponding with the quadruple (M_1, L_2, N, N') of lines in Γ is a primitive element of $\mathbf{GF}(q)$ (i.e., an element generating the multiplicative group of $\mathbf{GF}(q)$). Consequently the projective plane $\pi(\Gamma)$ is generated by the flags corresponding with the points x, x_i, x_j, z, z' (with $i, j \in \{1, 2, \ldots, q\}$, $i \neq j$) of Γ, respectively by the flags corresponding with the points x, x_i, z, z', u, where u is any point on L_2, $x \neq u \neq y_2$, and where i is arbitrary in $\{1, 2, \ldots, q\}$. This implies immediately (by Lemma 3.1) that $\mathbf{PG}(8, q)$ is generated by $x, M_1, M_2, x_i, x_j, N, N'$ (for any $i, j \in \{1, 2, \ldots, q\}$, $i \neq j$) or by $x, M_1, M_2, x_i, N, N', R$, where $\{L_2, R\} = \Gamma_1(u)$, u as above. As before we infer that $\{H_i = \langle x, M_1, M_2, x_i, N, N'\rangle \,|\, i \in \{1, 2, \ldots, q\}\}$ is a set of q (mutually distinct) hyperplanes, each of them containing the 6-dimensional space $U = \langle x, M_1, M_2, N, N'\rangle$. Further, H_i does not contain any element of $\Gamma_2(L_2) \setminus \{L_1, M_2\}$. Hence the unique "missing" hyperplane H through U contains all elements of $\Gamma_2(L_2)$. Similarly, there is a hyperplane H' containing M_2, R, R', with $R, R' \in \Gamma_2(L_2) \setminus \{L_1, M_2\}$, and all elements of $\Gamma_2(L_1)$. As in the previous case, one deduces $H = H'$ and, since x was arbitrary, the embedding is weak.

Theorem 1.1 is proved.

Finally, we assume that $d \geq 7$ and that for every pair $\{L, M\}$ of opposite lines of Γ the set $L^{\perp\perp} \cap M^{\perp\perp}$ is contained in a plane of $\mathbf{PG}(d, q)$.

Let x be any point of Γ and suppose again $\Gamma_1(x) = \{L_1, L_2\}$. Let M be any line of Γ opposite

L_1. The space U_1 generated by all elements of $\Gamma_2(L_1)$ coincides with the space generated by L_1 and all elements of $L_1^{\perp\!\!\perp} \cap M^{\perp\!\!\perp}$ (since every element of $\Gamma_2(L_1)$ contains a point of L_1 and a point of $L_1^{\perp\!\!\perp} \cap M^{\perp\!\!\perp}$). Hence the dimension of U_1 is at most 4. Similarly, the dimension of U_2, the subspace generated by all elements of $\Gamma(L_2)$, is at most 4. Since the intersection $U_1 \cap U_2$ contains L_1 and L_2, and hence has dimension at least 2, we see that the subspace H generated by $\Gamma_4(x)$ has dimension at most 6.

Now let y be a point of Γ opposite x. The space $U = \langle H, y \rangle$ induces a subhexagon Γ' of Γ, which must coincide with Γ (since $\pi(\Gamma')$ contains a full point row and a full line pencil of $\pi(\Gamma)$). Hence H is a hyperplane, has dimension 6, $d = 7$, and the embedding is weak.

Theorem 1.2 is proved.

4 An alternative formulation

Our main results and the results of [3, 4, 5, 6] can be stated in a combinatorial way, without mentioning embeddings of geometries. We first restate the final result in [6]. Therefore, we introduce the following notation: if two lines L and M of some projective space meet, then we write $L \sim M$.

Theorem 4.1 *Let $\mathcal{S}$ and $\mathcal{S}'$ be two sets of q^2+q+1 mutually skew lines in $\mathbf{PG}(d,q)$ which together generate $\mathbf{PG}(d,q)$, $d \geq 3$. Suppose that $\pi = (\mathcal{S}, \mathcal{S}', \sim)$ is a projective plane, and suppose that for every pair $(L, L') \in \mathcal{S} \times \mathcal{S}'$ with $L \sim L'$, the set of lines $\{X \in \mathcal{S} \cup \mathcal{S}' \mid X \sim L \text{ or } X \sim L'\}$ does not span $\mathbf{PG}(d,q)$. Then, if we denote by $\mathcal{P}$ the set of points incident with some member of $\mathcal{S} \cup \mathcal{S}'$, and if we denote the incidence relation in $\mathbf{PG}(d,q)$ by $\mathtt{I}$, the geometry $\Gamma = (\mathcal{P}, \mathcal{S} \cup \mathcal{S}', \mathtt{I})$ is a fully and weakly embedded thin hexagon and hence known.*

The proof of this theorem is a straightforward translation, if one remarks that the condition that the elements of both $\mathcal{S}$ and $\mathcal{S}'$ are mutually non-intersecting implies that $|\mathcal{P}| = (q+1)(q^2+q+1)$ and that any point of $\mathcal{P}$ is on two lines of $\mathcal{S} \cup \mathcal{S}'$. That condition cannot be dispensed with as the following example shows.

In $\mathbf{PG}(5,q)$ we choose two non-intersecting planes Π and Π' and an isomorphism $\Theta : \Pi \to \Pi'$. Define $\mathcal{S}$ to be the set of lines of Π, and $\mathcal{S}'$ to be the set of lines $\{xx^\Theta \mid x \text{ is a point of } \Pi\}$. One can check that $\mathcal{S}, \mathcal{S}'$ satisfy the conditions of Theorem 4.1, except that the elements of $\mathcal{S}$ are not mutually skew. But the conclusion of Theorem 4.1 is false for this example.

To finish, we restate Theorem 1.1 of the present paper, and leave the restatement of Theorem 1.2 for the interested reader.

Theorem 4.2 *Let $\mathcal{S}$ and $\mathcal{S}'$ be two sets of q^2+q+1 mutually skew lines in $\mathbf{PG}(d,q)$ which together generate $\mathbf{PG}(d,q)$, $d \geq 3$. Suppose that $\pi = (\mathcal{S}, \mathcal{S}', \sim)$ is a Desarguesian projective plane. If either $d = 8$ or q is prime and $d = 7$, then the conditions of* Theorem 4.1 *are satisfied and consequently, with the same notation as above, the geometry $\Gamma = (\mathcal{P}, \mathcal{S} \cup \mathcal{S}', \mathtt{I})$ is a fully and weakly embedded thin hexagon and hence known.*

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