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Epimorphisms of Generalized Polygons Part 1: Geometrical Characterizations

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1. Introduction and Preliminaries

A generalized polygon is a thick incidence geometry of rank 2 such that the girth of the incidence graph is twice the diameter of the incidence graph. These geometries are introduced by Tits [17] for group-theoretical purposes, but became an interesting research object in their own right. For an overview of the geometric study, see [19].

Obviously, the notion of morphism is an important one when dealing with geometries. For instance, monomorphisms are equivalent to embeddings of one geometry into the other (and on the group-theoretical level often give rise to maximal subgroups); isomorphisms clearly are needed to distinguish new geometries from old ones, but also to determine automorphism groups; epimorphisms can be used to construct quotient geometries or cover geometries (as the geometric counterpart of local fields). But in all these cases, the geometries considered in the literature are of the same kind, i.e., they have same gonality and diameter. In the present paper, we initiate the study of morphisms between generalized polygons of unequal gonality. We restrict ourselves to epimorphisms since we are motivated by some nice examples in this case. The study of monomorphisms and embeddings requires different techniques.

To see the problem, a good starting point is Pasini's theorem [13] that states that any epimorphism between two generalized n-gons, $n > 2$, is either an isomorphism or has infinite fibers. In particular, if an epimorphism is bijective if restricted to one point row, then it is a global isomorphism. This is no longer true for epimorphisms from a generalized m-gon to an n-gon, $m \neq n$, and a standard example is given is Section 3 below. It describes an epimorphism from the classical split Cayley hexagon over some field $\mathbb{F}$ to the ordinary Pappian projective plane over $\mathbb{F}$ with the property that line pencils and point rows are mapped bijectively onto line pencils and point rows, respectively.

The paper is organized as follows. In section 2, we propose a quite general classification system for epimorphisms of geometries, from the local point of view. In the next section, Section 3, we give some examples, and we characterize geometrically our stan-

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dard example mentioned above in Section 4. Finally, in Section 5, we give an alternative proof using ideas of the present paper to show part of a well know result of Bödi and Kramer [2].

The paper will have a sequel. In [7], we will treat free constructions (providing many more examples) and the finite case (mostly non-existence results).

Let's get down to precise definitions.

A **generalized** m**-gon** $\mathfrak{P}$ is a thick point-line geometry $(\mathcal{P}, \mathcal{L}, \mathcal{F})$ with an incidence graph of diameter m and girth $2m$. For a generalized m-gon $(\mathcal{P}_1, \mathcal{L}_1, \mathcal{F}_1)$ and a generalized n-gon $(\mathcal{P}_2, \mathcal{L}_2, \mathcal{F}_2)$ a **homomorphism** ϕ: $(\mathcal{P}_1, \mathcal{L}_1, \mathcal{F}_1) \to (\mathcal{P}_2, \mathcal{L}_2, \mathcal{F}_2)$ is a function that maps points to points, lines to lines, and preserves incidence, i.e. for any $(p, l) \in \mathcal{F}_1$ we have $(\phi(p), \phi(l)) \in (\phi \times \phi)(\mathcal{F}_1) \subseteq \mathcal{F}_2$. The **dual homomorphism** ϕ^{dual} is the mapping $(\mathcal{L}_1, \mathcal{P}_1, \mathcal{F}_1^{-1}) \to (\mathcal{L}_2, \mathcal{P}_2, \mathcal{F}_2^{-1})$ defined by $\phi^{\mathrm{dual}}|_{\mathcal{P}_1} := \phi|_{\mathcal{P}_1}$ and $\phi^{\mathrm{dual}}|_{\mathcal{L}_1} := \phi|_{\mathcal{L}_1}$ between the dual polygons.

Monomorphisms, isomorphisms and epimorphisms are always meant to be injective, bijective and surjective, respectively, on the point set, the line set and the flag set.

For a generalized m-gon $\mathfrak{P}$ and an integer $n \leq [\frac{m}{2}]$ a **distance-**n**-ovoid** O_n is a set of points of $\mathfrak{P}$ that are at distance $\geq 2n$ from each other with the property that given any vertex x of $\mathfrak{P}$ there is a point of O_n with distance $\leq n$ from x. Dually, a **distance-**n**-spread** is defined. Sometimes, if n is clear or not important, we will omit the distance and just write **ovoids** or **spreads**.

Given a partition $\mathcal{O}$ of the point set and a partition $\mathcal{S}$ of the line set of some point-line geometry, we define the **quotient geometry** as the geometry $(\mathcal{O}, \mathcal{S}, \mathcal{F})$ with the induced incidence, i.e. $(O, S) \in \mathcal{F}$ if and only if there is a point of O incident with a line of S; then we also write $O\mathtt{I}S$. If $O\mathtt{I}S$ such that there exist unique $p \in O$ and $l \in S$ with $p\mathtt{I}l$, then we sometimes write $O\mathtt{I}_!S$ and call O and S **uniquely incident**. Clearly, the canonical map from a geometry to a quotient is an epimorphism between these incidence structures.

A **partition of** $\mathfrak{P}$ **into ovoid-spread pairings** is a partition of the point set into ovoids and of the line set into spreads, such that given any incident ovoid-spread pair (O, S) (in the quotient geometry) any point of O is incident with some line of S and vice versa (in the original geometry).

2. Classification and Characterization

In this section we give a classification of epimorphisms between generalized polygons. The epimorphisms are classified by their properties restricted to the points, the lines, the flags, the point rows, and the line pencils. Then in 2.3, 2.4, and 2.5 we characterize the epimorphisms by describing the geometrical structure of the preimages of single vertices under these epimorphisms.

In the classification we distinguish between injectivity (i), surjectivity (s), bijectivity (b), and neither of those $(-)$. A homomorphism is 12345 with $1, 2, 3, 4, 5 \in \{i, s, b, -\}$, if it is 1, 2, 3, 4, 5 restricted to the points, lines, flags, point rows, and line pencils, respectively. Sometimes, we write $*$, if we do not know the property or if we do not want to specify.

We call a homomorphism **strictly** 12345, if it is 12345, but does not satisfy any additional condition (i.e., no surjection or injection is bijective).

The following theorem, in fact, holds for any epimorphism between any two point-line geometries, but we will only state it for generalized polygons.

THEOREM 2.1 *Any epimorphism between generalized polygons* $\mathfrak{P}_1$ *and* $\mathfrak{P}_2$ *is one of the following types:*

(i) *bbbbb, i.e. an isomorphism,*

(ii) $bsbib$,

(iii) $sbbbi$*, which is dual to* $bsbib$*,*

(iv) $ssbii$,

(v) $bssis$,

(vi) $sbssi$*, which is dual to* $bssis$*,*

(vii) $sss{*}{*}$.

In case (vii) there are the possibilities $sss--$, $sssi-$, $ssss-$, $sssb-$, $sss-i$, $sssi-$, $ssssi$, $sssbi$, $sss-s$, $sssis$, $sssss$, $sssbs$, $sss-b$, $sssib$, $ssssb$, $sssbb$.

Proof. The case "strictly $bbs{*}{*}$" cannot occur, as point and line bijectivity imply flag injectivity.

We show that $b{*}{*}{*}{*}$ implies $b{*}{*}is$, that ${*}b{*}{*}{*}$ implies ${*}b{*}si$ and that ${*}{*}b{*}{*}$ implies ${*}{*}bii$. Clearly, a point (line) bijective homomorphism has to be point row (line pencil) injective. Also a flag bijective homomorphism has to be point row and line pencil injective, since otherwise we would obtain flags having the same image. Moreover, suppose we have a point bijective epimorphism ϕ that is not line pencil surjective. Then for a point p of $\mathfrak{P}_1$ there exists a line $m \in [\phi(p)]\backslash\phi([p])$. Considering this line, we see a contradiction against point bijectivity, as a point distinct of p maps on $\phi(p)$, since the flag $(\phi(p), m)$ has to have a preimage. Dually, we prove the statement for line bijective epimorphisms.

Now we show that $b{*}{*}b{*}$ implies $bb{*}b{*}$, that ${*}b{*}{*}b$ implies $bb{*}{*}b$, that ${*}{*}bb{*}$ implies ${*}bbb{*}$ and that ${*}{*}b{*}b$ implies $b{*}b{*}b$. Let ϕ be a point bijective and point row bijective epimorphism. If ϕ is not line injective, there exist lines l_1, l_2 of $\mathfrak{P}_1$ with the same image under ϕ. By point row surjectivity, we find points on $\phi(l_1)$ which have two distance points in their preimage, a contradiction. Dually, for line bijective epimorphisms. The same proof holds for a flag bijective homomorphism.

Putting everything together still leaves to disprove existence of strictly $bssib$ and strictly $sbsbi$. Suppose a strictly $bssib$ homeomorphism exists. Then we have flags (p, l), (p', l') with the same image. By point bijectivity we get $p = p'$, but this gives a contradiction against line pencil bijectivity. Dually, we also exclude strictly $sbsbi$. ■

Epimorphisms between generalized polygons are either gonality preserving or gonality decreasing. More precisely, we have

COROLLARY 2.2 *Any homomorphism belonging to one of the cases (ii), (iii), (iv), (v) or (vi) is gonality decreasing. Moreover, any gonality preserving strict epimorphism is* $sss--$, $ssss-$, $sss-s$, *or* $sssss$ *with nonfinite preimage.*

Proof. All other epimorphisms are either point row or line pencil injective, hence injective by [2] (or by Theorem 5.1 below) if they are gonality preserving. Moreover, by [13] there are no gonality preserving proper epimorphisms with finite preimage. ■

The following theorems give geometrical characterizations of the interesting cases of epimorphisms between generalized polygons, i.e. of cases *(ii)* to *(vi)* and $sssbb$ of *(vii)* of 2.1:

THEOREM 2.3 *Let* $2 \leq n < m$. *The existence of a (strictly)* $sssbb$ *homomorphism from an* m*-gon* $\mathfrak{P}_1$ *onto an* n*-gon* $\mathfrak{P}_2$ *is equivalent to the existence of a partition of the point and line sets of* $\mathfrak{P}_1$ *into distance-*n*-ovoid spread pairings.*

Proof. Assume that the generalized m-gon $\mathfrak{P}$ allows a partition into distance-n-ovoid-spread pairings. First, we have to show that the quotient geometry is a generalized n-gon. Indeed, its diameter is at most n, as we are considering distance-n-ovoids and distance-n-spreads in $\mathfrak{P}_1$. To establish diameter n and girth $2n$ it suffices to rule out the existence of ordinary polygons of gonality $n' < n$. But if there was an ordinary n'-gon, we would either obtain an ordinary n'-gon in $\mathfrak{P}_1$ or two vertices x, y with $d(x, y) \leq 2n' < 2n$ and $\phi(x) = \phi(y)$, both of which is impossible. Thickness is inherited.

Now we have to prove that this epimorphism is of type $sssbb$. Indeed, it is neither point nor line nor flag injective (since we have ovoid-spread pairings), but injective on each point row and each line pencil. But again since we have ovoid-spread pairings the epimorphism is also surjective on each point row and each line pencil.

Conversely, suppose ϕ is a homomorphism of the given type. Vertices x, y of $\mathfrak{P}_1$ belonging to the same preimage under ϕ have to be at distance $\geq 2n$, since otherwise we find a chain from x to y that "collapses" under ϕ (i.e. the image does not contain any ordinary polygon) to avoid an ordinary polygon in the image with gonality $< n$. But if the chain collapses, then two confluent lines or two collinear points have the same image, contradicting the point row or the line pencil bijectivity. Now take two non-opposite vertices x, y in $\mathfrak{P}_2$ There exists a chain of length $< n$ from x to y which we can lift to $\mathfrak{P}_1$. There, by line pencil and point row bijectivity, we obtain a set of pairwise disjoint chains from vertices of $\phi^{-1}(x)$ to vertices of $\phi^{-1}(y)$ of the same length. This proves that for any vertex a of $\mathfrak{P}_1$ and any vertex x of $\mathfrak{P}_2$ there either exists a unique vertex $b \in \phi^{-1}(x)$ at a distance $< n$ from a or there exist vertices $b \in \phi^{-1}(x)$ at distance n from a. Hence the preimages are distance-n-ovoids and -spreads, cf. Lemma 7.2.2 of [19]. Lifting chains in the special case $x \mathtt{I} y$ gives pairings. ■

THEOREM 2.4 *Let* $2 \leq n < m$. *The existence of a (strictly)* $bsbib$, $ssbii$ *or* $sbbbi$ *homomorphism from an* m*-gon* $\mathfrak{P}_1$ *onto an* n*-gon* $\mathfrak{P}_2$ *is equivalent to the existence of a partition*

$\mathcal{O}$ of the point set of $\mathfrak{P}_1$ into sets O_α of points mutual at distance $\geq 2n$ and of a partition $\mathcal{S}$ of the line set of $\mathfrak{P}_1$ into sets S_β of lines mutually at distance $\geq 2n, \alpha, \beta \in I$ for some index set I, with the following properties:

(i) *If n odd,*

- *for any two points p, q there exist k unique sets $O_i, 1 \leq i \leq k \leq \frac{n-1}{2} + 1$, and $k - 1$ unique distance sets $S_j, 1 \leq j \leq k - 1 \leq \frac{n-1}{2}$, such that $p \in O_1 \mathtt{I}_! S_1 \mathtt{I}_! O_2 \mathtt{I}_! \cdots g \mathtt{I}_! S_{k-1} \mathtt{I}_! O_k \ni q$;*
- *for any two lines x, y there exist k unique distinct sets S_j, $1 \leq j \leq k \leq \frac{n-1}{2} + 1$, such that $x \in S_1 \mathtt{I}_! O_1 \mathtt{I}_! S_2 \mathtt{I}_! \cdots \mathtt{I}_! O_{k-1} \mathtt{I}_! S_k \in q$.*

(ii) *If n even, for any point p and any line x there exist k unique distinct sets O_i, $1 \leq i \leq k \leq \frac{n}{2}$, and k unique distinct sets S_j, $1 \leq j \leq k \leq \frac{n}{2}$, such that $p \in O_1 \mathtt{I}_! S_1 \mathtt{I}_! O_2 \mathtt{I}_! \cdots \mathtt{I}_! O_k \mathtt{I}_! S_k \ni x$*

More precisely, we have $bsbib$ if and only if $|O_\alpha| = 1$ for all α and $sbbbi$ if and only if $|S_\beta| = 1$ for all β and $ssbii$ otherwise.

Proof. Suppose we have a partition of the points and lines of $\mathfrak{P}_1$ with the given properties. The map $\mathfrak{P}_1 \to (\mathcal{O}, \mathcal{S}, \mathcal{F}')$ to the quotient geometry is an epimorphism. We will prove that the image is a generalized n-gon. For sure, the diameter is $\leq n$ (since $\frac{n}{2} + \frac{n}{2} = n = \frac{n-1}{2} + \frac{n-1}{2} + 1$), hence it remains to prove that there are no ordinary polygons of gonality $< n$. Thickness is inherited. Indeed, if we had an ordinary polygon of gonality $n' < n$ we would find vertices x, y of appropriate type (i.e. both points or both lines for n odd; one line, one point for n even) that have two n'-chains, respectively one $(n' - 1)$- and one $(n' + 1)$-chain connecting them, namely the two parts of the ordinary polygon, a contradiction since $n' < n$.

It remains to show that the obtained epimorphism ϕ between polygons is of one of the given types. But the homomorphism is clearly flag injective, as flags (x, y), (x', y'), $x \neq x'$, $y \neq y'$ cannot have the same image, since otherwise the preimages $\phi^{-1}(\phi(x))$ and $\phi^{-1}(\phi(y))$ would not be uniquely incident; other flags cannot have the same image because vertices belonging to the same set of the partition are mutually at distance $\geq 2n \geq 4$. Hence the homomorphism is even flag bijective, thus the classification 2.1 proves the claim, since the homomorphism obviously is not an isomorphism.

Conversely, suppose we have a homomorphism ϕ of one of the given types. We claim that the preimages form a partition of the points and lines of the given shape. We prove things only for n even, the other case being similar. Take any point p and any line x in $\mathfrak{P}_1$. In the image, $\phi(p)$ and $\phi(x)$ have a unique chain $\phi(p) = p_1 \mathtt{I} x_1 \mathtt{I} p_2 \mathtt{I} \cdots \mathtt{I} p_k \mathtt{I} x_k = \phi(x)$ of length at most $n - 1$ (i.e. $1 \leq k \leq \frac{n}{2}$) connecting them. We claim that the sets of preimages of all the vertices contained in this chain satisfy the above condition. Clearly, there is but one such chain and the inequalities for the indices are satisfied. Now we show the unique incidence. But by flag bijectivity we can lift any flag $(p_i x_j)$, $j \in \{i - 1, i\}$ uniquely, hence the claim. Vertices belonging to the same set of the partition have to be mutually at distance $\geq 2n$. For, let otherwise be x, y be two distinct vertices contained in the same set of the partition at distance $\leq 2n$. We find a chain from x to y that has to collapse under ϕ (i.e. the image does not contain any ordinary polygon), since otherwise we would obtain an ordinary polygon in the image with gonality $\leq n$. But collapsing of the chain is impossible, since two confluent lines or two collinear points cannot have the same image by flag bijectivity.

Finally, the last statement is obvious. ■

THEOREM 2.5 *Let* $2 \leq n < m$. *The existence of a (strictly)* $bssis$ *homomorphism from an* m*-gon* $\mathfrak{P}_1$ *onto an* n*-gon* $\mathfrak{P}_2$ *is equivalent to the existence of a partition* $\mathcal{S}$ *of the line set of* $\mathfrak{P}_1$ *into sets* $\mathcal{S}_\alpha, \alpha \in I$ *for some index set* I*, with the following properties:*

(i) *If n odd,*

- *for any two points* p, q *there exist k unique points* p_i, $1 \leq i \leq k \leq \frac{n-1}{2} + 1$, *and* $k-1$ *unique distinct sets* S_j, $1 \leq j \leq k-1 \leq \frac{n-1}{2}$, *such that* $x \in S_1 \mathtt{I} p_2 \mathtt{I} \cdots \mathtt{I} S_{k-1} \mathtt{I} p_k = q$;
- *for any two lines* x, y *there exist k unique distinct sets* S_j, $1 \leq j \leq k \leq \frac{n-1}{2} + 1$, *and* $k-1$ *unique distinct points* p_i, $1 \leq i \leq k-1 \leq \frac{n-1}{2}$, *such that* $p = p_1 \mathtt{I} S_1 \mathtt{I} p_1 \mathtt{I} S_2 \mathtt{I} \cdots \mathtt{I} p_{k-1} \mathtt{I} S_k \ni y$;
- *there exists a set* S_α *containing two confluent lines.*

(ii) *If n even,*

- *for any point p and any line x there exist k unique distinct points* p_i, $1 \leq i \leq k \leq \frac{n}{2}$, *and k unique distinct sets* S_j, $1 \leq j \leq k \leq \frac{n}{2}$, *such that* $p = p_1 \mathtt{I} S_1 \mathtt{I} p_2 \mathtt{I} \cdots \mathtt{I} p_k \mathtt{I} S_k \ni x$;
- *there exists a set* S_α *containing two confluent lines.*

The dual statement holds for $sbssi$ *homomorphisms.*

Proof. Given a partition of the lines with the above properties, the canonical map from $\mathfrak{P}_1$ onto its quotient geometry is an epimorphism. The proof that the quotient geometry is a generalized *n*-gon is similar to that of Theorem 2.4 (note that the uniqueness of the p_i is responsible for the nonexistence of ordinary polygons with too small a gonality). By the classification 2.1 the type of the epimorphism is $bssib$, since it is point bijective and not flag injective.

Conversely, we only have to prove the existence of a set S_α containing two confluent lines, the remainder being as in the proof of Theorem 2.4. But this is an immediate consequence of point bijectivity and flag non-injectivity.

The statement about duality is obvious. ■

COROLLARY 2.6 *Strictly* $bsbib$, $ssbii$, $sbbbi$, $sssbb$ *homomorphisms from an* m*-gon to and* n*-gon only exist for* $n \leq \left[\frac{m}{2}\right]$. ■

Remark 2.7.

(i) There are examples for all classes of epimorphisms between generalized polygons as described in Theorem 2.1. In section 3 we give some concrete examples. Moreover, see [7] for some general existence and non-existence results.

(ii) The *sssbb* epimorphisms are covers in geometrical sense. An *sssbb* epimorphism from an m-gon to an n-gon maps the set of vertices at distance $\leq n-1$ from any vertex x bijectively onto the set of vertices at distance $\leq n-1$ from the image of x. We call them **local isomorphisms**.

(iii) Note that in case of *bsbib*/*sbbbi* onto a digon (i.e. $n = 2$) the sets S_j and O_i are distance-2-spreads and distance-2-ovoids, respectively. Indeed, for *sbbbi*, if we fix some i each line is incident with a unique vertex of O_i.

(iv) Corollary 2.6 does not hold for *bssis*. It is quite easy to construct a *bssis* homomorphism from a projective plane to a digon. Indeed, it suffices to find a partition of the line set of the plane in at least three dual blocking sets (a dual blocking set is just a set of lines covering all points of the plane). The digon then is the quotient geometry.

(v) Since any homomorphism between two generalized polygons factors into an epimorphism and an embedding, it would be natural to look at embeddings of polygons. Of course, with free constructions (cf. [18]) it is possible to embed polygons of different gonality in each other. The easiest embedding is $(\mathcal{P}, \mathcal{L}, F) \to (\mathcal{P}, \mathcal{L}, \mathcal{P} \times \mathcal{L})$.

3. Concrete Examples

In this section we present some "nice" examples. With "nice" examples we understand *finite, continuous* or using *classical objects* such as classical ovoids and spreads. All the other examples we are aware of involve some free construction and will be given in [7].

Example 3.1. This first example was the origin of our research which resulted in this article. It was first mentioned in [6].

Consider Tits's description of the split Cayley hexagon (see e.g. [17]). The points of $\mathcal{H}(\mathbb{F})$ are the points of the quadric $\mathcal{Q}(6, \mathbb{F})$ in $PG(6, \mathbb{F})$ given by $X_0X_4 + X_1X_5 + X_2X_6 = X_3^2$, where $\mathbb{F}$ is a (commutative) field. From the Grassmann coordinates of the lines, it is easily seen that each point star has (algebraic) dimension 3, i.e., the point stars form point sets of (Pappian) projective planes.

Now we project the points of $\mathcal{H}$ into the subspace given by $X_0 - X_4 = 0$, $X_1 - X_5 = 0$, $X_2 - X_6 = 0$, and $X_3 = 0$: map a point $(x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6)$ on $(x_0 + x_4 : x_1 + x_5 : x_2 + x_6)$. Obviously, this defines a projection π of the points, but it does not necessarily map points to points.

However, if -1 is not the sum of three squares in the field $\mathbb{F}$, this projection defines a homomorphism from the split Cayley hexagon into the projective plane. Indeed, a point is mapped on a point, as $x_0 + x_4 = x_1 + x_5 = x_2 + x_6 = 0$ implies $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$ which in the considered fields is equivalent to $x_0 = x_1 = x_2 = x_3 = 0$. Lines are mapped on lines, as they are linearly spanned by two collinear points and no two collinear points have the same image. (Otherwise, let p and q be two collinear points with $\pi(p) = \pi(q)$. Then $p - q$ is a point of $\mathcal{H}(\mathbb{F})$ which is mapped to 0.) Obviously, incidence is preserved.

Moreover, the restriction of π to some point star is injective by the same reasoning as in the above paragraph, since for any two points p, q of one point star also $p - q$ is a point of the same star. By dimension, this means that π maps any point star bijectively on the point set of a projective plane $PG(2,\mathbb{F})$. This then implies bijectivity on point rows and line pencils. For π being a local isomorphism it remains to show that π is flag surjective. We will make use of the point surjectivity of π: fix any point p in $PG(2,\mathbb{F})$, take a point of the preimage and consider the point star of this preimage. The algebraic dimension of this point star is 3, and bijectivity of π restricted to point stars implies that any line through p is an image under π. But this gives flag surjectivity. Hence π is a local isomorphism. We call this the **standard local isomorphism from $\mathcal{H}(\mathbb{F})$ to $PG(2,\mathbb{F})$**.

The original construction of π, however, was made by means of algebra. We will now give a sketch: The split Cayley algebra of octaves $\mathbb{O}$ is defined as the set of matrices

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a, d \in \mathbb{F}$ and $b, c \in \mathbb{F}^3$ with $\mathbb{F}$ being a field. Let $\mathbb{O}$ be equipped with the usual eight dimensional vector space structure over $\mathbb{F}$ and the following multiplication (where $\times$ denotes the standard outer product and $\cdot$ denotes the standard inner product)

$$\begin{aligned} xy &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \\ &= \begin{pmatrix} aa' - b\cdot c' & ab' + d'b + c\times c' \\ a'c + dc' + b\times b' & dd' - c\cdot b' \end{pmatrix} \end{aligned}$$

which makes it an algebra over $\mathbb{F}$.

$\mathbb{H} := \{x \in \mathbb{O} \mid a = d, b = c\}$ is a subalgebra of $\mathbb{O}$ isomorphic to Hamilton's (split) quaternions. Moreover, if the characteristic of $\mathbb{F}$ is not 2, then for $v := \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ we have $\mathbb{O} = \mathbb{H} \oplus v\mathbb{H}$.

The incidence geometry $\mathcal{H}(\mathbb{F}) = (\mathcal{P}, \mathcal{L}, \subseteq)$ with the point set $\mathcal{P} = \{x\mathbb{F} | x \in \mathbb{O}, x_2 = 0\}$ and the set of lines $\mathcal{L} = \{x\mathbb{F}+y\mathbb{F} | x\mathbb{F}, y\mathbb{F} \in \mathcal{P}, xy = 0\}$, i.e. the point-line geometry consisting of the one dimensional and two dimensional subspaces of $\mathbb{O}$ with trivial multiplication, is the split Cayley hexagon.

Now if -1 is not a sum of three squares in $\mathbb{F}$, the projection

$$\pi\colon \mathcal{H}(\mathbb{F}) \to PG(2,\mathbb{F}) : x\mathbb{F} \mapsto b_1\mathbb{F}$$

with $x = \left(\begin{smallmatrix} 0 & b_1 \\ b_1 & 0 \end{smallmatrix}\right) + v\left(\begin{smallmatrix} a_2 & b_2 \\ b_2 & a_2 \end{smallmatrix}\right)$ mapping a point of the split Cayley hexagon into the projective plane over $\mathbb{F}$ defines a local isomorphism.

The construction of the split Cayley hexagon as given above can be found in [3], the algebraic proof for π being a local isomorphism in [6].

Example 3.2. The following examples arise from work of Linus Kramer who in [10] gives compact quadrangles with a partition of the point space into compact ovoids and a partition of the line space into compact spreads. Our job is just to apply Theorem 2.3. We will give a short summary of the results.

Let $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and let V be an $\mathbb{F}$-vector space of dimension $n+1$. Moreover, let f be a σ-hermitian form of Witt index 2 on V where $\sigma\colon x \mapsto \overline{x}$ is the standard involution of $\mathbb{F}$. The geometry of totally isotropic subspaces defines the classical standard hermitian quadrangles.

For any $0 \neq x = (x_0, x_1, 0) \in V$ (relative to a basis $\{e_0, \dots, e_n\}$ with f being positive definite on the $\mathbb{F}$-linear span of $\{e_2, \dots, e_n\}$) we have a hyperplane $h_x = \{v \in V \mid f(v, x) = 0\}$ and the set $\mathcal{O}_{H_x} = \{p \in \mathcal{P} \mid f(p, x) = 0\}$ is an ovoid, the Thas ovoid. Indeed, every line of the quadrangle meets the hyperplane H_x (in a point of the quadrangle) and the hyperplane does not contain a line of the quadrangle, as $f|_{H_x \times H_x}$ has Witt index 1. Moreover, any two distinct Thas ovoids have trivial intersection, as the two corresponding hyperplanes do not have any nonzero totally isotropic subspaces in common. Finally, since for any point p of the quadrangle there is an $0 \neq x = (x_0, x_1, 0)$ with $f(p, x) = 0$, we can cover the whole point set by Thas ovoids, and we have a partition.

So, especially the classical quadrangles $\mathcal{Q}_5(\mathbb{R})$, $Q_9(\mathbb{R})$, $H_3(\mathbb{C})$ and $H_5(\mathbb{C})$ admit partitions of the point set into ovoids. By $Q_5(\mathbb{R})^{\text{dual}} \cong H_3(\mathbb{C})$, we also have a partition of the line set of these two quadrangles into spreads, hence (by distance 2) a partition into ovoid-spread pairings. For Q_9 and $H_5(\mathbb{C})$ we will use the isomorphisms $Q_9(\mathbb{R})^{\text{dual}} \cong FKM(6, 8)$ and $H_5(\mathbb{C})^{\text{dual}} \cong FKM(5, 8)$ where FKM denote the Clifford quadrangles, see e.g. [5] or [11]. In [11] it is also shown that the point space of the Clifford quadrangles can be partitioned into ovoids.

This gives the following: the generalized quadrangles $Q_5(\mathbb{R})$, $Q_9(\mathbb{R})$, $H_3(\mathbb{C})$, $H_5(\mathbb{C})$, $FKM(5, 8)$, and $FKM(6, 8)$ admit a partition into ovoid-spread pairings, hence there exist local isomorphisms from these quadrangles into generalized digons.

Example 3.3. Another example was pointed out to us by Jef Thas. Consider the generalized quadrangle $T_2^*(O)$. To construct this, cf. [1] or [8], let O be a complete oval of $PG(2, q)$, $q = 2^h$, and embed $PG(2, q)$ as a plane P in $PG(3, q)$. The incidence geometry $T_2^*(O)$ consisting of the points of $PG(3, q)$ not in P and the lines of $PG(3, q)$ not in P meeting O is a generalized quadrangle with parameters $s = q - 1$, $t = q + 1$. The set of all lines of $T_2^*(O)$ meeting in a unique point of O form a spread (no two lines intersect and each point lies on one such line) and all such spreads partition the line set. Fixing a line in P not meeting O the set of all (hyper)planes H distinct from P gives rise to a partition of the point set into ovoids. Clearly, one such plane establishes an ovoid (no two points are collinear and each line contains such point) and the planes do not have points in common.

Example 3.4. We finish this section with a nice example of a class of $sss--$ epimorphisms from the symplectic quadrangle to the projective plane. It was suggested by Theo Grundhöfer. Take a (commutative) field $\mathbb{F}$ and let $f\colon \mathbb{F}^4 \to \mathbb{F}^3$ be a linear projection with one dimensional kernel. This gives rise to a homomorphism from the symplectic quadrangle $W(\mathbb{F})$ (the geometry consisting of the totally isotropic one and two dimensional subspaces of $\mathbb{F}^4$ of the bilinear form $x_0y_1 - x_1y_0 + x_2y_3 - x_3y_2$, cf. [14]) to the projective plane where we can freely choose the image of the point corresponding to the kernel of f. For the right choice of this image we even get an epimorphism.

4. A Geometric Characterization of the Standard Local Isomorphism from $\mathcal{H}(\mathbb{F})$ to $PG(2,\mathbb{F})$

Let π be the standard local isomorphism from $\mathcal{H}(\mathbb{F})$ to $PG(2,\mathbb{F})$. Let l be any line of $PG(2,\mathbb{F})$ and consider the inverse image H under π of l. Then this is a distance-3-spread of $\mathcal{H}(\mathbb{F})$ all elements of which lie in some 5-dimensional subspace U of $PG(6,\mathbb{F})$.

If l' is another line of $PG(2,\mathbb{F})$, and if H' is the corresponding distance-3-spread in the hyperplane U', then the inverse image of the point $L \cap L'$ is the distance-3-ovoid of $\mathcal{H}(\mathbb{F})$ obtained by intersecting the quadric $Q(6,\mathbb{F})$ with the spaces U and U'. If $U \cap Q(6,\mathbb{F})$ contained planes of $Q(6,\mathbb{F})$, then $U \cap U'$ would contain lines of $Q(6,\mathbb{F})$ and hence points of $\mathcal{H}(\mathbb{F})$ at distance ≤ 4, contradicting the fact that $U \cap U'$ meets $Q(6,\mathbb{F})$ in a distance-3-ovoid of $\mathcal{H}(\mathbb{F})$.

The construction of H with the subspace U parallels the construction by Thas [16] of Hermitian spreads in the finite case. We now show that this is a general fact (and note that the proof in [16] relies heavily on finiteness; our proof provides an alternative argument in the finite case).

PROPOSITION 4.1 *Let U be a hyperplane of $PG(6,\mathbb{F})$ meeting the quadric $Q(6,\mathbb{F})$ in a quadric $Q(5,\mathbb{F})$ which does not contain planes. Then the set H of lines of U contained in $\mathcal{H}(\mathbb{F})$ forms a distance-2-spread of the generalized quadrangle $Q(5,\mathbb{F})$, and a distance-3-spread of the generalized hexagon $\mathcal{H}(\mathbb{F})$.*

Proof. Let p be any point of $Q(5,\mathbb{F})$. We first show that there is a line of H containing p. Consider the plane P of $Q(6,\mathbb{F})$ containing all lines of $\mathcal{H}(\mathbb{F})$ through p (this exists, see 2.4.16 of [19]). Since $Q(5,\mathbb{F})$ does not contain planes, it must meet P in a unique line $m \in H$.

So we have already shown that H is a distance-2-spread of the generalized quadrangle $Q(5,\mathbb{F})$. In order to show that it is a distance-3-spread of $\mathcal{H}(\mathbb{F})$, we have, according to 7.2.2 of [19], to prove that every line n of $\mathcal{H}(\mathbb{F})$ is at distance ≤ 2 from a unique element of H. If n lies in U, then this follows easily from the previous paragraph. If n is not contained in U, then it has a unique point x in common with U; the unique line m through x of H is at distance 2 from n and is unique with that property (again by the previous paragraph which says that H is a distance-2-spread of $Q(5,\mathbb{F})$). ∎

Any spread of $\mathcal{H}(\mathbb{F})$ obtained as in the previous proposition will be called a **Hermitian spread**.

Similarly, we have:

PROPOSITION 4.2 *Let U be a four dimensional subspace $PG(6,\mathbb{F})$ meeting the quadric $Q(6,\mathbb{F})$ in a quadric $Q(4,\mathbb{F})$ which does not contain lines. Then the set O of points of $Q(5,\mathbb{E})$ forms a distance-3-ovoid of the generalized hexagon $\mathcal{H}(\mathbb{F})$.*

Proof. Clearly no two points of O can be collinear. Now let p be a point of $\mathcal{H}(\mathbb{F})$ not in O. The set of points collinear with p in $\mathcal{H}(\mathbb{F})$ is the point set of a projective plane $PG(2,\mathbb{F})$ of $PG(6,\mathbb{F})$ lying on $Q(6,\mathbb{F})$. This plane meets U in a unique point (otherwise $Q(4,\mathbb{F})$ would contain at least one line), which belongs to O by definition. The result now follows from 7.2.2 of [19]. ∎

We call the ovoid $\mathcal{O}$ of the previous proposition a **classical ovoid**.

We can now prove a characterization of the standard local isomorphism.

THEOREM 4.3 *Let $\mathfrak{P}$ be a projective plane with point set $\mathcal{P}$ and line set L. Let ρ be a local isomorphism from $\mathcal{H}(\mathbb{F})$ onto $\mathfrak{P}$. If all line fibers are Hermitian spreads, then $\mathfrak{P}$ is the Pappian plane over $\mathbb{F}$ and $\rho = \pi$ is the standard local isomorphism.*

Proof. Let S be any line fiber of ρ. By assumption all elements of S are contained in a fixed hyperplane H of $PG(6, \mathbb{F})$. Let S' be a second line fiber, contained in the hyperplane $H' \neq H$. Choose arbitrarily a line $l \in S$. Since S' is a distance-3-spread, there is a unique element $l' \in S'$ concurrent with l. let p be the intersection point of l and l'. The point fiber containing p is contained in the 4-dimensional space $H \cap H'$. Hence it is easily seen that every point fiber is a classical ovoid. Now let S'' be a third line fiber contained in the hyperplane H'', $H \neq H'' \neq H'$. Let $PG(2, \mathbb{F})$ be the projective plane containing all lines of $\mathcal{H}(\mathbb{F})$ through p. It meets H'' in a line l'' of the quadric $Q(6, \mathbb{F})$. Hence we can identify the line fibers with the lines of $PG(2, \mathbb{F})$. Consider three line fibers corresponding to concurrent lines in $\mathfrak{P}$. The fiber corresponding to the intersection point contains a point p' collinear with p, hence $p' \in PG(2, \mathbb{F})$. The three fibers now all contain a line through p', hence it is clear the three lines of $PG(2, \mathbb{F})$ corresponding to the three fibers as above are collinear. This shows that $\mathfrak{P}$ is isomorphic to $PG(2, \mathbb{F})$.

Now we claim that the four dimensional spaces of all point fibers and the hyperplanes corresponding to all line fibers share a common fixed three dimensional space U. Indeed, it is clear that $H \cap H' \cap H'' =: U$ is three dimensional (H'' does not contain $H \cap H'$ because p does not lie in H''). Let m be any line of $\mathcal{H}(\mathbb{F})$ through p. Every element of the line fiber containing m is incident with an element of the point fiber containing p, and vice versa. Hence the hyperplane H_m containing all elements of the line fiber containing m contains $H \cap H'$. Similarly, if m' is a line of $PG(2, \mathbb{F})$ through the intersection point of l and l'', then the hyperplane containing all elements of the line fiber containing m' contains $H \cap H''$. It follows that the four dimensional space corresponding to the point fiber of the point $m \cap m'$ contains the space $(H \cap H') \cap (H \cap H'') = U$. Varying m and m', we see that the four dimensional space corresponding to the point fiber of any point of $PG(2, \mathbb{F})$ not on $l \cup l' \cup l''$ contains U. Also, the four dimensional space corresponding to the point fiber of the point $m \cap l''$ contains the space $H \cap H' \cap H'' = U$. So the four dimensional space of the point fiber of every point of l'' contains U; similarly for the point of l' and l''. Hence the space of every point fiber contains U. Since these spaces are intersections of hyperplanes spanned by the line fibers, the claim follows.

But now ρ can be described as follows. Consider a line n of $\mathcal{H}(\mathbb{F})$; consider intersection of the hyperplane spanned by the line fiber of n with $PG(2, \mathbb{F})$: this line is the image of n under ρ. But this coincides with the projection $\pi(n)$ of n from U onto $PG(2\mathbb{F})$.

The theorem is proved. ∎

5. The Bödi-Kramer Theorem

This section serves as an addendum to the present article. It gives an entirely geometric proof of Theorem 2.8 of Bödi-Kramer [2] (as opposed to the original proof which uses a weak form of coordinates). However, we use complementary assumptions. Indeed, in [2], the assumptions are that ϕ is injective on at least one point row, and that the image is a generalized n-gon; one then immediately deduces in a geometric way that ϕ is injective on all point rows. So we assume the latter and weaken instead the second condition.

THEOREM 5.1 *A homomorphism between generalized n-gons that is point row injective and the point-line image under which contains two opposite elements is injective.*

Proof. Let ϕ be this homomorphism. If two objects have the same image under ϕ, then considering a minimal chain connecting these objects, we see that there must be a subchain $x \mathtt{I} y \mathtt{I} z$ with $\phi(x) = \phi(z)$. If y is a line, this is impossible by assumption. Hence it suffices to show that ϕ is line pencil injective.

First we show that, whenever two points $\phi(p)$ and $\phi(q)$ are opposite in the image of ϕ, then ϕ is injective on the line pencil through p. Indeed, suppose the lines l and l' through p are mapped under ϕ onto the same line. Let m be the line incident with q and nearest to l. If r is an arbitrary point on m, then the point on l' nearest to r must clearly be mapped onto the point on $\phi(l') = \phi(l)$ nearest to $\phi(m)$, contradicting point row injectivity.

Let us call an ordinary n-gon in the domain of ϕ *stable* if it is mapped onto an ordinary n-gon under ϕ. We show that there exists at least one stable n-gon. Indeed, this follows from the previous paragraph if there are two opposite points in the image. Now let $\phi(l)$ and $\phi(x)$ be two arbitrary opposite elements in the image of ϕ with l a line. Then l and x are also opposite (ϕ diminishes distances). Since ϕ is injective on point rows, every n-gon through l and x is stable. Notice that ϕ is automatically injective on $[x]$, i.e., the line pencil, respectively point row of x. Hence, in view of this and the previous paragraph, it suffices to show that every point is contained in a stable n-gon.

Therefore, let p be a point collinear with a point p' contained in a stable n-gon $\mathcal{A}$. If p is incident with a line l of $\mathcal{A}$, then we obtain a stable n-gon containing p by considering the unique minimal path connecting p with the line (point, respectively) of $\mathcal{A}$ opposite l. Similarly, every line m through p' is contained in a stable n-gon (by also noticing that $\phi(m)$ is distinct from at least one of the images under ϕ of the lines of $\mathcal{A}$ through p'). Putting $pp' = m$, we see that p is incident with a line of a stable n-gon, and hence is contained itself in a stable n-gon by our previous argument. Continuing like that, we see that, by connectedness, all points are contained in a stable n-gon.

The theorem is proved. ∎

Remark 5.2. It is clear that the condition of point row injectivity in the previous theorem cannot be dispensed with. But also the second condition is necessary, even under the stronger condition of point row bijectivity, as one may project a quadric of Witt index 2 in a k-dimensional projective space from a $(k-2)$-dimensional subspace not intersecting the quadric onto a line of the quadric. More sophisticated examples arise from free constructions.

References

1. R. W. Ahrens and G. Szekeres, On a combinatorial generalization of 27 lines associated with a cubic surface, *J. Austral. Math. Soc.*, Vol. 10 (1969) pp. 485–492.

2. R. Bödi and L. Kramer, On homomorphisms between generalized polygons, *Geom. Dedicata*, Vol. 58 (1995) pp. 1–14.
3. F. Buekenhout and A. M. Cohen, *Diagram Geometry*. In preparation.
4. M. De Soete and J. A. Thas, A characterization theorem for the generalized quadrangle $T_2^*(O)$ of order $(s, s+2)$, *Ars Combin.*, Vol. 17 (1984) pp. 225–242.
5. D. Ferus, H. Karcher and M. F. Münzer, Cliffordalgebren und neue isoparametrische Hyperflächen, *Math. Z.*, Vol. 177 (1981) pp. 479–502.
6. R. Gramlich, *Homomorphisms Of Generalized Hexagons*, Diplomarbeit (Master Thesis), Würzburg (1998).
7. R. Gramlich and H. Van Maldeghem, Epimorphisms of generalized polygons, Part 2: Some existence and non-existence results. In preparation.
8. M. Hall, Jr., Affine quadrilaterals, *Studies in Pure Mathematics* (L. Mirsky, ed)., Academic Pres, London-New York (1971) pp. 113–116.
9. L. Kramer and H. Van Maldeghem, Compact ovoids in quadrangles I: Geometric Constructions, *Geom. Dedicata*, Vol. 78 (1999) pp. 279–300.
10. L. Kramer, Compact ovoids in quadrangles II: the classical quadrangles, *Geom. Dedicata*, Vol. 79 (2000) pp. 179–188.
11. L. Kramer, Compact ovoids in quadrangles III: Clifford algebras and isoparametric hypersurfaces, *Geom. Dedicata*, Vol. 79 (2000) pp. 321–339.
12. L. Kramer, *Compact Polygons and Isoparametric Hypersurfaces*. In preparation.
13. A. Pasini, The nonexistence of proper epimorphisms of finite thick generalized polygons, *Geom. Dedicata*, Vol. 15 (1984) pp. 389–397.
14. S. E. Payne and J. A. Thas, *Finite Generalized Quadrangles*, Pitman, Boston (1984).
15. G. L. Schellekens, On a hexagonic structure I,II, *Indag. Math.*, Vol. 24 (1962) pp. 201–234.
16. J. A. Thas, Polar spaces, generalized hexagons and perfect codes, *J. Combin. Theory Ser. A*, Vol. 29 (1980) pp. 87–93.
17. J. Tits, Sur la trialité et certains groupes que s'en déduisent, *Inst. Hautes Études Sci. Publ. Math.*, Vol. 43 (1959) pp. 13–60.
18. J. Tits, Endliche Spiegelungsgruppen, die als Weylgruppen auftreten, *Invent. Math.*, Vol. 43 (1977) pp. 283–295.
19. H. Van Maldeghem, *Generalized Polygons*, Birkhäuser, Basel (1998).