Distance transitive generalized quadrangles of prime order

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Dedicated to Reiner Salzmann on the occasion of his 70th birthday

Abstract

Without using the classification of finite simple groups, we classify the finite generalized quadrangles of prime order admitting a group acting distance transitively on the collinearity graph. Our method uses combinatorial geometry and permutation groups.

1 Introduction

A *generalized quadrangle* Γ of order (s,t) is a rank 2 point-line geometry whose incidence graph has diameter 4 and girth 8, each vertex corresponding to a point has valency $t+1$ and each vertex corresponding to a line has valency $s+1$. It is easy to see, and in fact well known (see e.g. PAYNE & THAS [4]), that the collinearity graph of Γ is a distance regular graph with parameters $k = s(t+1)$, $\lambda = s-1$ and $\mu = t+1$. Using the classification of finite simple groups, BUEKENHOUT & VAN MALDEGHEM [1] classified all finite generalized quadrangles Γ for which the collinearity graph is distance transitive, i.e., the automorphism group of the collinearity graph of Γ (which coincides with the automorphism group of Γ) has rank 3 on the graph. This seems to be a very difficult problem to solve without the classification of finite simple groups. In this paper, we do this for finite generalized quadrangles of order (p,p), with p a prime number.

Note that KANTOR (unpublished) also has proved this result (independently and, in fact, quite earlier than I did), but he never published his proof. It is, however,

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worthwhile to have a published proof. Group-theoretic arguments and geometric arguments interfere with each other in a rather beautiful way. Moreover, arguments as in the present paper are needed to handle related questions, even if one is allowed to use the classification of finite simple groups.

Generalized quadrangles were introduced by Tits [6]. The standard reference is Payne & Thas [4], and I will freely use results from that book.

I have announced a proof of the Main Result in Van Maldeghem [7], Section 4.8.

2 Statement and proof of the Main Result

In this paper we prove the following theorem.

Main Result. *A finite generalized quadrangle Γ of order (p,p), p prime, whose collinearity graph admits a rank 3 automorphism group is isomorphic to the symplectic quadrangle $\mathsf{W}(p)$ or to the orthogonal quadrangle $\mathsf{Q}(4,p)$.*

The quadrangle $\mathsf{W}(p)$ is the geometry of points and isotropic lines of a symplectic polarity in the projective space $\mathbf{PG}(3,p)$ of dimension 3 over the Galois field $\mathbf{GF}(p)$; the quadrangle $\mathsf{Q}(4,p)$ is the geometry of points and lines of a non-degenerate quadric in the projective space $\mathbf{PG}(4,p)$. In our proof, we will use the notation $\Gamma_i(a)$ for the set of elements of Γ at distance i (measured in the incidence graph) from some given element a.

Proof of the Main Result. Let G be the full automorphism group of Γ. Let x be any point of Γ. Then the stabilizer G_x acts transitively on the $p(p+1)$ points collinear with x, and on the p^3 points of Γ not collinear with p. Hence the order of G_x is divisible by $p^3(p+1)$. Since G_x acts transitively on $\Gamma_2(x)$, it also acts transitively on the set of $p+1$ lines through x. Hence, if L is any such line, the order of the stabilizer $G_{x,L}$ is divisible by p^3. Let P be a subgroup of $G_{x,L}$ of order p^3.

First suppose that P acts trivially on the set $\Gamma_1(x)$ of lines through x. Then the subgroup of P fixing two given non-collinear points y and z both collinear with x, has order p, fixes all lines through x and all points on the lines xy and xz, and acts transitively on the p points collinear with both y and z but different from x. In the terminology of Thas, Payne & Van Maldeghem [5], this means that Γ is a half Moufang quadrangle. The Main Result now follows from the paper we just have cited.

Now suppose that P acts non-trivially on $\Gamma_1(x)$. Since p is prime, P acts transitively on $\Gamma_1(x) \setminus \{L\}$. Hence G_x acts doubly transitively on $\Gamma_1(x)$, and consequently G acts transitively on all pairs of concurrent lines. Let $M \in \Gamma_1(x) \setminus \{L\}$ and let

$y \in \Gamma_1(L) \setminus \{x\}$. Then the stabilizer in P of y and M has order p or p^2. Hence we may suppose that there is a group Q of order p fixing x, y, L, M. If Q fixes every line through y, then it must act transitively on $\Gamma_1(M) \setminus \{x\}$. Consequently, using the transitivity of G on pairs of collinear points, we conclude that Γ is a half Moufang quadrangle and the Main Result follows from THAS, PAYNE & VAN MALDEGHEM [5]. If Q fixes every point of $\Gamma_1(M)$, then we again conclude that Γ is half Moufang (using the transitivity of G on pairs of concurrent lines established above), and the result follows again from THAS, PAYNE & VAN MALDEGHEM [5]. Hence we may assume that Q acts transitively on both $\Gamma_1(M) \setminus \{x\}$ and $\Gamma_1(y) \setminus \{L\}$. Note that Q fixes all points on L and all lines through x. It is also an easy consequence of the foregoing that G acts transitively on the set of triples of points (a, b, c) with $a, c \in \Gamma_2(b)$ and $a \in \Gamma_4(c)$. Moreover, if the stabilizer in G of x, y, L, M contains a group of order p^2, then as above we may conclude that Γ is half Moufang and hence isomorphic to either $\mathsf{W}(p)$ or $\mathsf{Q}(4, p)$.

Hence we may assume that P is a Sylow p-subgroup of $G_{x,L}$, and hence also of G. Also, the group S fixing $\Gamma_1(L) \cup \Gamma_1(y)$ pointwise has order p and acts regularly on $\Gamma_1(x) \setminus \{L\}$. And clearly, a Sylow p-subgroup H of $G_{x,L,M}$ has order p^2 and is a normal elementary Abelian subgroup of G_x (because it is the pointwise stabilizer of $\Gamma_1(x)$ in G_x). Now P acts by conjugation on H and, identifying H with a two-dimensional vector space over the field $\mathbf{GF}(p)$, we see that P contains transvections (elements of order p fixing only one point at infinity). Interchanging the roles of L and M, we see that we induce $\mathbf{PSL}_2(p)$ (or possibly $\mathbf{PGL}_2(p)$) on the line at infinity of H (this line at infinity can obviously be identified with the line pencil in x). Let $z \in \Gamma_4(x) \cup \Gamma_2(y)$. It follows that $|G_{x,y,z}|(p+1)p^3 = |G_x| = Ap^2$, where A is either equal to $|\mathbf{PSL}_2(p)|$ or to $|\mathbf{PGL}_2(p)|$. Hence $|G_{x,y,z}| = p - 1$ or $\frac{1}{2}(p-1)$. By the transitivity of G on triples of points (a, b, c) mentioned above, the group $G_{x,z}$ acts transitively on $\Gamma_2(x) \cap \Gamma_2(z)$ and so $|G_{x,z}| = (p+1)(p-1)$ or $\frac{1}{2}(p+1)(p-1)$. But also $G_{x,z}$ acts by conjugation on H and hence on its line at infinity. This action is faithful since, if some element $\theta \in G_{x,y,z}$ fixes all lines through x, then θ centralizes S, and so it also fixes all points on the line yz, implying that θ is the identity. So in any case $G_{x,z}$ is (or induces) a subgroup of $\mathbf{PSL}_2(p)$ of order $\frac{1}{2}(p+1)(p-1)$. No subgroup of $\mathbf{PSL}_2(p)$ is this big (by Dickson's classification, see e.g. HUPPERT [2], Satz 8.27), except three examples for small p (assuming $p > 3$). Namely, for $p = 5$, we have $\mathbf{A}_4$; for $p = 7$, we have $\mathbf{S}_4$; and for $p = 11$, we have $\mathbf{A}_5$. But these subgroups also act on H as a vector space, hence they are subgroups of $\mathbf{SL}_2(p).\mathbf{Sc}_2(p)$, where $\mathbf{Sc}_2(p)$ is the group of scalar 2×2 matrices over $\mathbf{GF}(p)$. But it follows from an easy computation that this cannot be the case (for instance for $p = 5$, the four group of $\mathbf{A}_4$ in $\mathbf{PSL}_2(5)$ does not have a "section" in $\mathbf{SL}_2(5).\mathbf{Sc}_2(5)$).

This contradiction shows that P cannot be a Sylow p-subgroup and the proof of the Main Result is complete. $\square$

Acknowledgment. I would like to thank Tom De Medts for requiring a correction in the last part of the proof.

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