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Sharply 2-transitive groups of projectivities in generalized polygons

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Abstract

The group of projectivities of (a line of) a projective plane is always 3-transitive. It is well known that the projective planes with a sharply 3-transitive group of projectivities are classified: they are precisely the Pappian projective planes. It is also well known that the group of projectivities of a generalized polygon is 2-transitive. Here, we classify all generalized quadrangles, all finite generalized hexagons, and the parameter sets of all finite generalized octagons with a sharply 2-transitive group of projectivities. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction and statement of the main result

A *generalized polygon* Γ of order (s,t) is a rank 2 point-line geometry whose incidence graph has diameter n and girth $2n$, for some $n \in \mathbb{N}\backslash\{0,1\}$ (in which case the generalized polygon is also called a *generalized n-gon*), each vertex corresponding to a point has valency $t+1$ and each vertex corresponding to a line has valency $s+1$. If $s,t > 1$, then the geometry is usually called *thick*. Each non-thick generalized polygon can be obtained from a thick one, and so one usually only considers thick generalized polygons. These objects were introduced by Tits [12]. More information is gathered in my monograph [13], to which we refer for a general introduction and basic properties. Here, we recall some notation. For an element x of Γ, and a natural number i, we denote by $\Gamma_i(x)$, the set of elements of Γ at distance i from x in the incidence graph of Γ. The distance function in that incidence graph is denoted by δ. If two elements x and y are not at distance n, then there exists a unique element $\mathrm{proj}_y x$ incident with y and at distance $\delta(x,y)-1$ from x. We call that element the *projection of x onto*

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y. Also recall that the *dual* of Γ is obtained by interchanging the words 'point' and 'line'. The dual of a generalized n-gon is obviously again a generalized n-gon.

Let Γ be a generalized n-gon of order (s,t), and let x and y be two elements of Γ at distance n in the incidence graph (elements of Γ at distance n in the incidence graph of Γ are called *opposite*). Let $\Gamma_1(x)$ denote the set of elements of Γ incident with x, and similarly for $\Gamma_1(y)$. It is well known that the relation 'is not opposite' is a bijection from $\Gamma_1(x)$ to $\Gamma_1(y)$. This bijection is called a *perspectivity* and denoted by $[x,y]$. For a collection $\{x_0,x_1,\dots,x_\ell\}$ of points and lines, with x_{i-1} opposite x_i, $1\leqslant i\leqslant \ell$, we define the composition

$$[x_0,x_1,\dots,x_\ell]:=[x_0,x_1][x_1,x_2]\cdots[x_{\ell-1},x_\ell]$$

and call this bijection from $\Gamma_1(x_0)$ to $\Gamma_1(x_\ell)$ a *projectivity*. The set of all projectivities $\Gamma_1(L)\to\Gamma_1(L)$, for some line L of Γ, forms a group $\Pi(\Gamma)$, which is abstractly and as a permutation group, independent of L. It is called the *group of projectivities* of Γ. The 'Fundamental Theorem of Projective Plane Geometry' says that, for $n=3$ (a generalized 3-gon is nothing other than a projective plane), the (permutation) group of projectivities always acts 3-transitively, and it acts sharply 3-transitively if and only if the plane is Pappian (or equivalently, if and only if the projective plane arises from a three-dimensional vector space over a commutative field by taking the vector lines as points and the vector planes as lines, and inclusion as incidence). Now it is well known (for an explicit proof, see [8]) that in general, the group $\Pi(\Gamma)$ acts 2-transitively, and there are many examples of (finite and infinite) generalized 4-gons and generalized 8-gons with a group of projectivities which does not act 3-transitively (see e.g. [8] again, or Section 8.4 of the monograph [13]). In the present paper, we deal with the question $(*)$: 'what can be said about the generalized polygon Γ when $\Pi(\Gamma)$ acts sharply 2-transitively?'

Question $(*)$ has been suggested to me by Katrin Tent who, herself, classified in [11] all generalized quadrangles Γ with a sharply 2-transitive group of projectivities under the additional assumption that the one-point stabilizers of $\Pi(\Gamma)$ are abelian.

Note that for n even, the group $\Pi(\Gamma)$ has a subgroup (denoted by $\Pi^+(\Gamma)$) of index at most 2 consisting of all elements of $\Pi(\Gamma)$ associated to projectivities which are the composite of an even number of perspectivities (so-called *even projectivities*). Also, this group always acts 2-transitively, and hence, if $\Pi(\Gamma)$ acts sharply 2-transitively, then so does $\Pi^+(\Gamma)$. Consequently, the question: 'When exactly does $\Pi^+(\Gamma)$ act sharply 2-transitively?', is more general than the question $(*)$.

A few remarks should put this question in a better perspective.

(i) Characterizations of certain classes of projective and affine planes by properties of their groups of projectivities exist in abundance, see [10] for a survey. For generalized n-gons with $n>3$ (the case $n=2$ is trivial: the group of projectivities is in this case always the identity), only the results for n=4 of Brouns et al. [1] are available. Basically, the configurational properties induced by specific properties of the group of projectivities become too messy for $n>3$, and hence, they do not lead to anywhere. No classification result using groups of projectivities is known to me for generalized n-gons, with $n > 4$. The one we present here may not be very general (only a finite number of small polygons are characterized), but it can serve as a start for more results in this direction.

(ii) If $s = 2$, then $\Pi(\Gamma) = \Pi^+(\Gamma)$ is automatically sharply 2-transitive (in fact, at the same time sharply 3-transitive). A classification of all generalized polygons Γ with $\Pi(\Gamma)$ or $\Pi^+(\Gamma)$ sharply 2-transitive would imply a classification of all generalized polygons of order $(2, t)$. The latter one is at the moment not a reasonable problem, since it would in particular settle the question whether t has necessarily to be finite for n even (and this is an open problem solved only for $n = 4$; see Appendix 5 of [13]). We will restrict ourselves here to the values $n = 3, 4, 6, 8$, which appear to be the most interesting ones by the existence of 'classical examples' related to simple groups.

(iii) If we consider for a moment only the finite case, then we see that a complete classification of polygons Γ with $\Pi(\Gamma)$ sharply 2-transitive requires, as above, the classification of generalized octagons of order $(2, 4)$. This is a long-standing problem that we will not try to solve in the present paper.

Our Main Result reads as follows.

Main Result. *Let Γ be a projective plane, a generalized quadrangle, a finite generalized hexagon, or a finite generalized octagon. Suppose that $\Pi^+(\Gamma)$ acts sharply 2-transitively. Then $\Pi(\Gamma) = \Pi^+(\Gamma)$ and one of the following holds*:

1. *Γ is the unique projective plane of order* $(2, 2)$,
2. *Γ is the unique generalized quadrangle of order* $(2, 2)$,
3. *Γ is the unique generalized quadrangle of order* $(2, 4)$,
4. *Γ is isomorphic to the generalized quadrangle* $\mathsf{Q}(4, 3)$ *of order* $(3, 3)$ *arising from a non-singular quadric in the four-dimensional projective space* $\mathbf{PG}(4, 3)$ *over the Galois field* $\mathbf{GF}(3)$ *of order* 3 (*see also* [9]),
5. *Γ is a generalized hexagon of order* $(2, 2)$ (*and there are exactly* 2 *such*; *each one the dual of the other*),
6. *Γ is the unique generalized hexagon of order* $(2, 8)$ *and*
7. *Γ is a generalized octagon of order* $(2, 4)$ *or* $(4, 2)$.

Concerning Cases 5 and 6, we remark that the finite generalized hexagons of order $(2, t)$ are classified by Cohen and Tits [3]. As for Case 7 of the Main result, we remark that for the known generalized octagons Γ of order $(2, 4)$ and $(4, 2)$ we actually have that $\Pi^+(\Gamma)$ acts sharply 2-transitively (this is proved in [8]).

Concerning our proof, we note that our argument for $n = 6, 8$ is typically a finite one, because we heavily use Lemma 2 of the next section. We could also use it for the case $n = 4$ to get rid of some small examples, but here there is a better geometric way, which also immediately gives us the examples without having to refer to the explicit calculation of the groups $\Pi^+(\Gamma)$ for some small finite generalized quadrangles Γ.

We subdivide our proof into the following parts. After two rather general lemmas (proving in particular that $\Pi^+(\Gamma) = \Pi(\Gamma)$ under the assumptions of the Main Result), we first deal with $n = 4$ (the case $n = 3$ follows from the 'Fundamental Theorem' stated above). Then we reduce the cases $n = 6, 8$ to a finite set of possible counterexamples. In the last part, we get rid of those.

2. Two useful lemmas

Lemma 1. *Let Γ be any generalized n-gon of order (s,t), $s,t > 1$ (and possibly infinite), $n > 3$. Suppose that $\Pi^+(\Gamma)$ acts sharply 2-transitively. Then $\Pi^+(\Gamma) = \Pi(\Gamma)$. Moreover, if s is finite, and n is not congruent to 2 modulo 3, then s is not congruent to 1 modulo 3.*

Proof. In this proof, we use the following observation, partly due to Norbert Knarr (private communication). Let L be any line of Γ. Pick any three points x, y, z incident with L. It is easy to see that there is an ordinary $(n+1)$-gon with sides $x_0{:=}L, x_2, x_4, \ldots, x_{2n}$, x_{2i} meeting x_{2i+2}, but not x_{2i+4} (subscripts to be taken modulo $2n+2$), such that x is incident with x_{2n}, y is incident with x_2 and z is the projection onto L of x_{n+1} (if n is odd) or of the intersection of x_n and x_{n+2} (if n is even). Let x_{2i+1} be the intersection of x_{2i} and x_{2i+2} (subscripts again modulo $2n+2$). Let $\theta : \Gamma_1(L) \to \Gamma_1(L)$ be the even projectivity defined by $\theta{:=}[x_0, x_n, x_{2n}, x_{3n}, \ldots, x_{(2n+2)n}]$ (subscripts modulo $2n+2$, and note that $x_{(2n+2)n} = x_0 = L$). It was observed by Norber Knarr that θ stabilizes $\{x, y, z\}$ and that θ^3 fixes x, y and z. In fact, it is not difficult to see that $\theta{:}x \mapsto y \mapsto z \mapsto x$ if $n \equiv 0 \bmod 3$, that $\theta : x \mapsto z \mapsto y \mapsto x$ if $n \equiv 1 \bmod 3$, and that θ fixes x, y, z if $n \equiv 2 \bmod 3$. If n is even, then $\theta' : \Gamma_1(L) \to \Gamma_1(L)$ defined by $\theta'{:=}[x_0, x_n, x_{2n}, \ldots, x_{(n+1)n}]$ does not possibly belong to $\Pi^+(\Gamma)$ (because it is composed of an odd number of perspectivities), and one checks that $\theta' : x \mapsto y \mapsto z \mapsto x$ if $n \equiv 1 \bmod 3$, that $\theta' : x \mapsto z \mapsto y \mapsto x$ if $n \equiv 0 \bmod 3$, and that θ' fixes x, y, z if $n \equiv 2 \bmod 3$. Note that $\theta'^2 = \theta$.

Now, if n is odd, then automatically $\Pi^+(\Gamma) = \Pi(\Gamma)$ (because a composition of an odd number of perspectivities always maps $\Gamma_1(\text{line})$ to $\Gamma_1(\text{point})$, and vice versa). Suppose now that n is even. Assume that $\Pi(\Gamma) \neq \Pi^+(\Gamma)$. Then θ'^3 of the previous paragraph fixes x, y, z and belongs to $\Pi(\Gamma) \backslash \Pi^+(\Gamma)$ (hence $\theta'^3 \neq \text{id}$). Let u be a point incident with L and not fixed by θ'^3. Noting that x, y, z were chosen arbitrarily, we can consider an element $\sigma : \Gamma_1(L) \to \Gamma_1(L)$ of $\Pi(\Gamma) \backslash \Pi^+(\Gamma)$ fixing x, y, u. Clearly, the composition $\sigma\theta'^3$ fixes x and y, but not u. But $\sigma\theta'^3 \in \Pi^+(\Gamma)$, a contradiction. Hence, θ'^3 is the identity and $\Pi^+(\Gamma) = \Pi(\Gamma)$.

Now suppose that $n \not\equiv 2 \bmod 3$, and let $s \equiv 1 \bmod 3$ be finite. Then the map θ above belongs to $\Pi^+(\Gamma)$ and is not trivial. Clearly, θ^3 is trivial, so θ defines a number of 3-cycles in $\Gamma_1(L)$. Since $s \equiv 1 \bmod 3$, there are at least two points on L fixed by θ, hence θ is trivial by the sharp 2-transitivity, a contradiction.

The lemma is proved. □

Remark 1. Considering θ'^3 of the previous proof again, we see that this fixes at least three points. If $\Pi(\Gamma)$ is a Zassenhaus group, i.e., if the pointwise stabilizer of three elements is automatically the identity, then θ'^3 is the identity, and hence $\Pi^+(\Gamma)=\Pi(\Gamma)$. This observation may be used to shorten the arguments in [8].

For the next lemma, we introduce some notation. Let Γ be a finite generalized n-gon, $n = 4, 6, 8$. Let p be any point of Γ, and fix two lines L and M through p. Now we consider the following subgeometry $\Gamma^{\{L,M\}}$ (respectively $\Gamma^{\{p\}}$) of Γ. The points of $\Gamma^{\{L,M\}}$ (respectively $\Gamma^{\{p\}}$) are the points of Γ opposite p; the lines of $\Gamma^{\{L,M\}}$ (respectively $\Gamma^{\{p\}}$) are the lines of Γ opposite both L and M (respectively at a distance $n-1$ from p); incidence is inherited from Γ.

Lemma 2. *With the above notation, the geometry $\Gamma^{\{L,M\}}$ is connected except possibly in the following cases*:

(a) *Γ is a quadrangle and $(s,t) \in \{(2,2),(2,4),(3,3),(4,2)\}$*,
(b) *Γ is a hexagon and $(s,t) \in \{(2,2),(2,8),(3,3),(4,4),(8,2)\}$*,
(c) *Γ is an octagon and $(s,t) \in \{(2,4),(3,6),(4,2),(6,3)\}$*.

Proof. The lemma will be proved by the method introduced by Brouwer [2], which he attributes to Willem Haemers. In fact, we can more or less copy Section 4 of Brouwer [2] (and we explicitly do so because we will need a slight modification later on). So, suppose that $\Gamma^{\{L,M\}}$ is disconnected. Let A be the adjacency matrix of the collinearity graph of $\Gamma^{\{p\}}$. Let U, V be two disjoint components whose union is $\Gamma^{\{L,M\}}$. Consider the corresponding partition of A and let B be the condensed form of average row sums of the blocks of A. Putting $r = (s-1)(t+1)$, which is the valency of the collinearity graph of $\Gamma^{\{p\}}$, $u = |U|$ and $v = |V|$, we find

$$B = \begin{pmatrix} r-\varepsilon & \varepsilon \\ \varepsilon u/v & r-\varepsilon u/v \end{pmatrix},$$

where ε is the average number of points in V collinear (in $\Gamma^{\{p\}}$) with a point of U. The eigenvalues of B are r and $r-\varepsilon-\varepsilon u/v$, and they must interlace the eigenvalues of A. So, as in [2], we must have

$$(s-1)(t+1) - \varepsilon(1+u/v) \leqslant s-1+\sqrt{ast},$$

with $a = n/2-2$. Similarly as in [2], the expression $\varepsilon(1+u/v)$ is maximized by having all lines of $\Gamma^{\{p\}}$ which do not belong to $\Gamma^{\{L,M\}}$ meet U in the same number of points, in which case $\varepsilon(1+u/v) = 2s$. Hence

$$(s-1)(t+1) - 2s \leqslant s-1+\sqrt{ast}.$$

For $n=4$, this reduces to $st \leqslant 2s+t$. We easily obtain $(s,t) \in \{(2,2),(2,4),(3,3),(4,2)\}$. For $n = 6$, this means that $st \leqslant 2s+t+\sqrt{st}$. Since st is a perfect square (see [4]) and since $s \leqslant t^3$ (see [6]), this implies that $(s,t) \in \{(2,2),(2,8),(3,3),(4,4),(8,2)\}$.

Similarly, for $n=8$, we have $st \leqslant 2s+t+\sqrt{2st}$. As $2st$ is a perfect square [4] and $s \leqslant t^2 \leqslant s^4$ [7], we obtain $(s,t) \in \{(2,4),(3,6),(4,2),(6,3)\}$.
The lemma is proved. □

3. Generalized quadrangles

In this section, we assume that Γ is a generalized quadrangle (4-gon) with $\Pi^+(\Gamma)$ sharply 2-transitive. All generalized quadrangles of order $(2,t)$ are classified, see for instance the monograph [13, 1.7.9]. Hence, we may assume that the order of Γ is (s,t) with $s>2$. We show that in this case $t \leqslant 3$. Let z be any point of Γ and let p,a,b be three mutually opposite points collinear with z, chosen in such a way that there exists a point x opposite p and collinear with both a,b (one easily checks that this is always possible). Let a' (respectively b') be the projection of p onto ax (respectively bx). Let L be any line through p distinct from pa', pb' and pz (if such a line L does not exist, then $t=2$ and we are done). Consider the even projectivity $\theta=[L,ax,pz,bx,L]$. It is clear that θ maps p onto itself, and that it also fixes the point $\mathrm{proj}_L\, x$. Hence θ also fixes $\mathrm{proj}_L\, a$, which is mapped onto $\mathrm{proj}_L\, b$. We conclude that $\mathrm{proj}_L\, a = \mathrm{proj}_L\, b$ and hence $|\Gamma_2(p)\cap\Gamma_2(a)\cap\Gamma_2(b)| = t-1$. Now let b^* be a point incident with bz but distinct from b, from z and from $\mathrm{proj}_{bz}\, a'$ (since $s>2$, we can find such a point b^*). Interchanging the roles of x and $\mathrm{proj}_{ax}\, b^*$, and of b and b^*, we see that $|\Gamma_2(p)\cap\Gamma_2(a)\cap\Gamma_2(b^*)| = t-1$. But no element of $\Gamma_2(p)\cap\Gamma_2(a)\cap\Gamma_2(b)$ is collinear with b^*, except for z. Moreover, also a' does not belong to $\Gamma_2(b^*)$. Hence $\Gamma_2(p)\cap\Gamma_2(a)\cap\Gamma_2(b^*)$ contains at most 2 elements (namely z and possibly a point incident with pb'). This implies $t-1 \leqslant 2$.

So we have shown that $t \leqslant 3$. But now Γ is finite and is known (see 1.7 of the monograph [13], cp. 6.1 and 6.2 of [9]). The result now follows from the explicit determination of $\Pi^+(\Gamma)$, with Γ a quadrangle of order $(s,2)$ or $(s,3)$. This is done in [8] for the orders $(4,2)$, $(3,3)$ and $(9,3)$, and in [5] for the quadrangle of order $(5,3)$.

Alternatively, we may argue as follows. Let L and M be two opposite lines of Γ. Let L' and M' be two opposite lines each meeting both L and M. Finally, let N be opposite both L and M, and meeting both L' and M'. Since $\Pi^+(\Gamma)=\Pi(\Gamma)$ by Lemma 1, the projectivity $[L,M,N,L]$ is trivial, and this readily implies that, in the terminology of Payne and Thas [9], the pair $\{L,M\}$ is *regular*, and hence that each line of Γ is *regular*. Hence, by 2.2.2(i) of [9], we have $t \geqslant s$. Hence, only the quadrangles of order $(2,2)$ and $(3,3)$ must be considered (this argument also works for s infinite!). Moreover, for order $(3,3)$, all lines are regular, and hence we have the generalized quadrangle $\mathsf{Q}(4,3)$ arising from a non-degenerate quadric in the four-dimensional projective space $\mathbf{PG}(4,3)$ over the Galois field $\mathbf{GF}(3)$ of order 3. Now Knarr [8] tells us that $\Pi^+(\mathsf{Q}(4,3)) \cong \mathbf{PSL}_2(4)$ and so Case 4 of the Main Result follows.

Remark 2. Completely similar as in the beginning of this section, one shows the following more general fact. If Γ is a generalized n-gon, $n \geqslant 4$ even, of order (s,t), with $\Pi^+(\Gamma)$ sharply 2-transitive, p is some point of Γ, and x,y,z are points opposite p with

x and y collinear with z, but x not collinear with y, then $|\Gamma_2(p)\cap\Gamma_{n-2}(x)\cap\Gamma_{n-2}(y)| \in \{0, t-1\}$.

4. Finite generalized hexagons and octagons

In this section, we suppose that Γ is a finite generalized hexagon or octagon of order (s,t), and that $\Pi^+(\Gamma)$ acts sharply 2-transitively. Let n be the diameter of the incidence graph of Γ (so $n = 6$ or 8).

Let p be any point of Γ, and fix two lines L and M through p. Let x be some fixed point opposite p. Let y be a point in the same connected component of $\Gamma^{\{L,M\}}$ as x. Suppose that $\mathrm{proj}_L\, x = \mathrm{proj}_L\, y$. If x and y are collinear, then the line xy does not belong to $\Gamma^{\{L,M\}}$, and hence x and y are never collinear in $\Gamma^{\{L,M\}}$. If x and y are at distance 4 (measured in the incidence graph of $\Gamma^{\{L,M\}}$), and if $\{z\} = \Gamma_2(x) \cap \Gamma_2(y)$, then by considering the projectivity $[M, xz, L, yz, M]$, we see that $\mathrm{proj}_M\, x = \mathrm{proj}_M\, y$. Suppose now that x and y are at distance $d > 4$ (again measured in the incidence graph of $\Gamma^{\{L,M\}}$). Let γ be a minimal path from x to y in $\Gamma^{\{L,M\}}$. Let y' be the projection of the point $\mathrm{proj}_L\, x$ onto the second line of γ. By the previous argument we have $\mathrm{proj}_M\, y' = \mathrm{proj}_M\, x$. An induction argument on the length of γ now implies that $\mathrm{proj}_M\, y = \mathrm{proj}_M\, y'$. Hence $\mathrm{proj}_M\, x = \mathrm{proj}_M\, y$. It is of course clear that there exists a point a opposite p with $\mathrm{proj}_L\, a = \mathrm{proj}_L\, x$ and $\mathrm{proj}_M\, a \neq \mathrm{proj}_M\, x$. This shows that the geometry $\Gamma^{\{L,M\}}$ cannot be connected (and must have at least s components since there are s choices for $\mathrm{proj}_M\, a$).

Now we apply Lemma 2. The cases $s = 2$ and $t = 2$ give rise to Cases 5, 6 and 7 of our Main Result (because the unique generalized hexagon of order $(8,2)$ has a 3-transitive group of projectivities; see [8]). Also, the case $(n,s,t) = (6,4,4)$ has been taken care of by Lemma 1.

Hence, we are left to show that for no generalized hexagon Γ of order $(3,3)$, and for no generalized octagon of order $(3,6)$ or $(6,3)$, the permutation group $\Pi^+(\Gamma)$ acts sharply 2-transitively. In the next section, we will use the geometry of traces to rule these cases out.

5. The remaining small cases

5.1. The case $(n,s,t) = (8,3,6)$

Let Γ be a generalized octagon of order $(3,6)$ with $\Pi^+(\Gamma)$ sharply 2-transitive. Let p be any point of Γ, and let x_0 be a point of Γ opposite p. If L is some line through p, then we label the point $\mathrm{proj}_L\, x_0$ by $(L, 0 \bmod 3)$. We now choose an arbitrary order $(L_1, L_2, L_3, L_4, L_5, L_6, L_7)$ of the lines through p, and we label the two points on L_1 distinct from p and from $\mathrm{proj}_L\, x_0$ arbitrarily by $(L_1, 1 \bmod 3)$ and $(L_1, 2 \bmod 3)$. For convenience, we usually omit 'mod 3' when it is clear it should be there. Let θ_i,

$2 \leqslant i \leqslant 7$ be any even projectivity from L_1 to L_i which maps p to p and $(L_1,0)$ to $(L_i,0)$ (θ_i exists by the 2-transitivity of $\Pi^+(\Gamma)$). Then we label the image of (L_1,ℓ), $\ell \in \{1,2\}$, by (L_i,ℓ). This labeling is independent of the choice of θ_i by the sharp 2-transitivity of $\Pi^+(\Gamma)$. Now with every point x opposite p, we can associate a unique 7-tuple $7(x){:=}(i_1,i_2,\ldots,i_7) \in \{0,1,2\}^7$ defined by $\mathrm{proj}_{L_j}x = (L_j,i_j)$, $1 \leqslant j \leqslant 7$. Now let y be any point opposite p collinear with x. Without loss of generality we may assume that the line xy is not opposite L_1. Hence $7(y)$ is of the form $(i_1,j_2,j_3,\ldots,j_7)$. Consider the even projectivity $\sigma_\ell{:=}[L_2,xy,L_\ell]$, $3 \leqslant \ell \leqslant 7$. Clearly it maps (L_2,i_2) to (L_ℓ,i_ℓ). We now claim that it maps (L_2,j_2) to $(L_\ell,i_\ell + j_2 - i_2)$. First, remark that every even projectivity from L_ℓ to L_2 which maps p to p and $(L_\ell,0)$ to $(L_2,0)$ maps $(L_\ell,1)$ to $(L_2,1)$. Now let σ be any projectivity from L_2 to L_ℓ mapping p to p and $(L_2,0)$ to $(L_\ell,1)$. Suppose σ maps $(L_2,1)$ to $(L_\ell,0)$. Then we may compose σ with an even projectivity σ' from L_ℓ to L_2, where σ' fixes p and maps (L_ℓ,k) to (L_2,k), $k=0,1,2$, and we obtain an even projectivity $\sigma\sigma'$ from L_2 onto itself fixing p and $(L_2,2)$ and swapping $(L_2,0)$ with $(L_2,1)$. This contradicts the sharp 2-transitivity of $\Pi^+(\Gamma)$. Hence σ maps $(L_2,1)$ to $(L_\ell,2)$ and $(L_2,2)$ to $(L_\ell,0)$. Similarly, every even projectivity from L_2 to L_ℓ mapping p to p and $(L_2,0)$ to $(L_\ell,2)$, maps $(L_2,1)$ to $(L_\ell,0)$ and $(L_2,2)$ to $(L_\ell,1)$. Consequently, we have shown that the even projectivities from L_2 to L_ℓ fixing p are of the form $(L_2,k) \mapsto (L_\ell,k+\varepsilon)$, with $\varepsilon \in \{0,1,2\}$ (modulo 3). Our claim now follows easily. Putting $\varepsilon = j_2 - i_2$, we now have that $7(y) = (i_1,i_2+\varepsilon,i_3+\varepsilon,\ldots,i_7+\varepsilon)$. Since ε appears 6 times, we deduce that the sum of all entries of $7(y)$ is congruent modulo 3 to the sum of all entries of $7(x)$. We can draw two conclusion out of this.

First. With the usual subtraction, we have that $7(x) - 7(y)$ contains a unique zero entry and either six 1's or six 2's when x and y are distinct collinear points opposite p. The zero entry is at position i if and only if xy is not opposite L_i, $i \in \{1,2,\ldots,7\}$.

Second. Since we can reach every point opposite p by a sequence of collinear points (because $\Gamma^{\{p\}}$ is connected, see [2]), we have exactly 3^6 7-tuples which are actually equal to $7(z)$, for some point z of Γ opposite p. Since there are $3^4 \cdot 6^3$ points in Γ opposite p, this means that on the average, every admissible 7-tuple appears as $7(x)$ for 24 points x (an *admissible* 7-tuple is one which is equal to $7(u)$, for some point u opposite p).

Now we consider any admissible 7-tuple, and without loss of generality we may take $7(x_0) = (0,0,\ldots,0)$. Let x_1 be any point opposite p collinear with x_0 and such that the line x_0x_1 is not opposite L_1 (there are 2 choices for x_1). Without loss of generality we may assume that $7(x_1) = (0,1,1,\ldots,1)$. Now we consider any point x_2 opposite p, collinear with x_1 and not on the line x_0x_1 (fixing x_1, there are 12 choices for x_2; hence in total we have 24 choices). Without loss of generality, we may assume that x_1x_2 is not opposite L_7. Then, since $7(x_1) - 7(x_2)$ contains either six 1's or six 2's (and the zero entry appears at the last position because $\mathrm{proj}_{L_7}x_1 = \mathrm{proj}_{L_7}x_2$) we have two possibilities.

1. $7(x_2) = (1,2,2,2,2,2,1)$. In this case there is a unique point x_3 collinear with x_2, opposite p, such that x_2x_3 is not opposite L_1, and with $7(x_3) = (1,1,1,1,1,1,0)$. It

is now easily seen that a point x_4 opposite p and collinear with x_3 exists such that $7(x_4)=7(x_0)$.

2. $7(x_2)=(2,0,0,0,0,0,1)$. In this case we can take for x_3 the unique point opposite p, collinear with x_2, such that x_2x_3 is not opposite L_1, and with $7(x_3)=(2,2,2,2,2,2,0)$. Also in this case, there is now a point x_4 collinear with x_3 opposite p with $7(x_4)=7(x_0)$.

Hence, each of the 24 choices for x_2 gives rise to a point x_4 at distance 7 from x_0x_1 with $7(x_4)=7(x_0)$. If two such points coincide, then there unique paths to x_0x_1 must coincide, a contradiction (they are all different by construction). Hence, we have a set of 25 points (all points x_4 and in addition the point x_0) giving rise to the same prechosen 7-tuple. Hence, the average of points x with $7(x)$ prechosen must be at least 25, a contradiction to our previous paragraph.

Hence Γ cannot exist.

5.2. The case $(n,s,t)=(8,6,3)$

Let Γ be a generalized octagon of order $(6,3)$ with $\Pi^+(\Gamma)$ sharply 2-transitive. Let p be any point of Γ, and let x_0 be a point of Γ opposite p. As in the previous case, we can associate a 4-tuple $(0,0,0,0)$ to x_0 by taking an order (L_1,L_2,L_3,L_4) of the lines through p, and by labeling the point $\mathrm{proj}_{L_i}x_0$ as $(L_i,0 \bmod 6)$, $1\leqslant i\leqslant 4$ (and we will omit 'mod 6' again in the sequel). We now choose a point on L_1 distinct from p and from $(L_1,0)$ and label it $(L_1,1)$. There is a unique element θ of $\Pi^+(\Gamma)$ mapping L_1 to itself, fixing p and mapping $(L_1,0)$ to $(L_1,1)$. We define $(L_1,j)^\theta=(L_1,j+1)$ inductively, for all j (modulo 6). As before, this induces a unique labeling on the lines L_i, $i=2,3,4$, and we can associate a 4-tuple $4(x)$ with every point x opposite p, in exactly the same way as before. One also shows similarly that the sum of the labels is congruent 3 modulo 6, and that for collinear points x and y, the 4-tuples $4(x)$ and $4(y)$ have the same entry at a certain position, and the entries in the other positions have a constant difference.

It is now a little elementary exercise to show that, if (a,b,c,d) is an admissible 4-tuple (as before, this means that there exists a point x opposite p with $4(x)=(a,b,c,d)$), then

$$(c-a,d-b)\in\{(0,0),(2,4),(4,2),(3,3),(1,5),(5,1),(0,3),(2,1),(4,5),(3,0),(1,2),(5,4)\}=:\mathscr{A}.$$

For $(i,j)\in\mathscr{A}$, we put $\mathscr{S}(i,j)=\{x\in\Gamma_8(p)\,|\,4(x)=(a,b,a+i,b+j),\text{ for some } a,b\}$. Suppose now two points x and y are collinear in $\Gamma^{\{L_3,L_4\}}$. Then xy is opposite both L_3 and L_4, hence we may assume it is not opposite L_1. So, $4(x)=4(y)+(0,\varepsilon,\varepsilon,\varepsilon)$, and we see that x and y belong to the same set $\mathscr{S}(i,j)$ for some suitable (i,j). This means that each $\mathscr{S}(i,j)$ is the union of connected components of $\Gamma^{\{L_3,L_4\}}$, and hence there are at least 12 connected components. Now we set $\mathscr{S}_1=\mathscr{S}(0,0)\cup\mathscr{S}(2,4)\cup\mathscr{S}(4,2)$, $\mathscr{S}_2=\mathscr{S}(3,3)\cup\mathscr{S}(1,5)\cup\mathscr{S}(5,1)$, $\mathscr{S}_3=\mathscr{S}(0,3)\cup\mathscr{S}(2,1)\cup\mathscr{S}(4,5)$ and $\mathscr{S}_4=\mathscr{S}(3,0)\cup$

$\mathscr{S}(1,2)\cup\mathscr{S}(5,4)$. It is easy to check that an arbitrary member of $\mathscr{S}_1$ (respectively $\mathscr{S}_2$, $\mathscr{S}_3$, $\mathscr{S}_4$) is collinear (in $\Gamma^{\{p\}}$) with exactly 14 members of $\mathscr{S}_1$ (respectively $\mathscr{S}_2$, $\mathscr{S}_3$, $\mathscr{S}_4$), with no members of $\mathscr{S}_2$ (respectively $\mathscr{S}_1$, $\mathscr{S}_4$, $\mathscr{S}_3$), with exactly three members of both $\mathscr{S}_3$ and $\mathscr{S}_4$ (respectively $\mathscr{S}_3$ and $\mathscr{S}_4$, $\mathscr{S}_1$ and $\mathscr{S}_2$, $\mathscr{S}_1$, $\mathscr{S}_2$). Indeed, let us check this for instance for a point x with $4(x)=(0,0,0,0)\in\mathscr{S}_1$. The neighbors of x have corresponding 4-tuple (and '$\rightsquigarrow$' means 'gives rise to members of')

$$\begin{aligned}
(0,\ell,\ell,\ell),(\ell,0,\ell,\ell) &\rightsquigarrow \mathscr{S}(0,0)\subseteq\mathscr{S}_1, \quad \ell\in\{1,2,3,4,5\},\\
(2,2,0,2),(4,4,4,0) &\rightsquigarrow \mathscr{S}(2,4)\subseteq\mathscr{S}_1,\\
(2,2,2,0),(4,4,0,4) &\rightsquigarrow \mathscr{S}(4,2)\subseteq\mathscr{S}_1,\\
(1,1,0,1),(3,3,0,3),(5,5,0,5) &\rightsquigarrow \mathscr{S}_3,\\
(1,1,1,0),(3,3,3,0),(5,5,5,0) &\rightsquigarrow \mathscr{S}_4.
\end{aligned}$$

The condensed form of the adjacency matrix with corresponding partition is thus

$$\begin{pmatrix} 14 & 0 & 3 & 3\\ 0 & 14 & 3 & 3\\ 3 & 3 & 14 & 0\\ 3 & 3 & 0 & 14 \end{pmatrix}$$

and this has eigenvalues 20 (multiplicity 1), 14 (multiplicity 2) and 8 (multiplicity 1). As before, by interlacing, we must have $14\leqslant s-1+\sqrt{2st}=11$, a contradiction.

5.3. The case $(n,s,t)=(6,3,3)$

Let Γ be a generalized hexagon of order $(3,3)$ such that $\Pi^+(\Gamma)$ is sharply 2-transitive. Let p be a point of Γ. Exactly in the same way as in the two previous subsections, we can associate a 4-tuple $4(x)$ with every point x opposite p, and such a 4-tuple (i_1,i_2,i_3,i_4) consists of 4 integers i_ℓ modulo 3 which sum up to 0 modulo 3. Adjacent to x in $\Gamma^{\{p\}}$ are 8 points with corresponding 4-tuples $(i_1,i_2+\varepsilon,i_3+\varepsilon,i_4+\varepsilon)$, $(i_1+\varepsilon,i_2,i_3+\varepsilon,i_4+\varepsilon),\ldots,(\ldots,i_3+\varepsilon,i_4)$. We observe that no two of these 8 quadruples share in exactly one position an element. Hence, since $\Gamma^{\{p\}}$ is connected (see [2]), we have 27 admissible quadruples, and if we consider the graph G with vertex set the admissible quadruples, and we call two quadruples adjacent if they share in exactly one position an element, then we obtain a (strongly regular) graph without triangles. It can also be easily seen that there are no two quadruples differing in exactly one position.

Since there are 27 admissible quadruples, and 3^5 points opposite p, there must be at least one admissible quadruple equal to $4(x)$, for at least 9 points x opposite p. Now suppose, without loss of generality, that $(0,0,0,0)$ is such a quadruple, and let $4(x)=4(y)=(0,0,0,0)$ for two distinct points x and y. We now determine the mutual position of x and y by ruling out some possibilities.

Suppose that $|\Gamma_3(p)\cap\Gamma_3(x)\cap\Gamma_3(y)|=0$. Let M be any line through y and put $N=\mathrm{proj}_x M$. The point $\mathrm{proj}_N y$ is opposite p since otherwise it would coincide with $\mathrm{proj}_N p$, and the latter is opposite p (because, if $U=\mathrm{proj}_p N$ and $u=\mathrm{proj}_U N$, we have by assumption that $\mathrm{proj}_u\, y \neq \mathrm{proj}_u\, x$). Similarly, the point $\mathrm{proj}_M\, x$ is opposite p. By Remark 2, the sets $\Gamma_2(p) \cap \Gamma_2(x)$ and $\Gamma_2(p) \cap \Gamma_2(\mathrm{proj}_M\, x)$ have exactly two elements in common. But since y and $\mathrm{proj}_M\, x$ are collinear, the sets $\Gamma_2(p)\cap\Gamma_2(y)$ and $\Gamma_2(p) \cap \Gamma_2(\mathrm{proj}_M\, x)$ have exactly one element in common, a contradiction (because $\Gamma_2(p) \cap \Gamma_2(x) = \Gamma_2(p) \cap \Gamma_2(y)$ by assumption).

Suppose now $\Gamma_3(p)\cap\Gamma_3(x)\cap\Gamma_3(y)=\{L\}$. Suppose, moreover, that $\mathrm{proj}_L\, x \neq \mathrm{proj}_L\, y$. Then $\delta(x,y) = 6$ and considering a line $M \neq \mathrm{proj}_y\, L$ through y, we can copy the argument in the previous paragraph to reach a contradiction.

Similarly, we can rule out the case $\Gamma_3(p)\cap\Gamma_3(x)\cap\Gamma_3(y)=\{L,L'\}$, $L \neq L'$ ($\mathrm{proj}_L\, x \neq \mathrm{proj}_L\, y$ is automatic since $x \neq y$). Note that an analogous argument shows that $|\Gamma_3(p)\cap \Gamma_3(x) \cap \Gamma_3(y)| \neq 3$.

Suppose now $|\Gamma_3(p)\cap\Gamma_3(x)\cap\Gamma_3(y)|=4$. There is at most one further point z opposite p with $|\Gamma_3(p)\cap\Gamma_3(x)\cap\Gamma_3(z)|=4$. Since there are at least nine points u with $4(u)=4(x)$, there is at least one point w opposite p with $4(w)=4(x)$ and $\Gamma_3(p)\cap\Gamma_3(x)\cap\Gamma_3(w)=\{L\}$, for some line L, and $\mathrm{proj}_L\, x = \mathrm{proj}_L\, w$. But then $\Gamma_3(p) \cap \Gamma_3(y) \cap \Gamma_3(w) = \{L\}$ with $\mathrm{proj}_L\, y \neq \mathrm{proj}_L\, w$. So $4(w) \neq 4(y)$, a contradiction.

Hence we have shown that $\Gamma_3(p) \cap \Gamma_3(x) \cap \Gamma_3(y)$ consists of a unique line L with $\mathrm{proj}_L\, x{=}\mathrm{proj}_L\, y$. It is clear that each such line L gives rise to at most two points y, $y \neq x$, with $4(y){=}4(x)$, because on each line K through $\mathrm{proj}_L\, x$, $K \neq L$, K not through x, the point y must be equal to the projection of every element of $(\Gamma_2(p)\cap\Gamma_4(x))\setminus\{\mathrm{proj}_L\, p\}$. Since there are four lines in $\Gamma_3(p) \cap \Gamma_3(x)$, there are at most nine elements y with $4(y) = 4(x)$. Our assumption now implies that there are exactly nine such elements. We can do the same with a second admissible quadruple, and continuing this way, we finally have that every admissible quadruple arises from exactly nine points opposite p. We can show that such a set of nine points is contained in a subhexagon of order $(1,3)$, but we will not need this fact.

Now put $\Gamma_3(p)\cap\Gamma_3(x)=\{L_0,L_1,L_2,L_3\}$. Let u be a point on L_0 distinct from $\mathrm{proj}_{L_0}\, x$. Let $(u, uw_i, w_i, w_iu_i, L_i)$, $i = 1,2$ be path from u to L_i. Then $w_1 \neq w_2$ (otherwise $4(w_1)$ and $4(x)$ differ in at most one position, a contradiction). Since $\Pi(\Gamma) = \Pi^+(\Gamma)$, the projectivity $[L_2, L_0, L_1, L_2]$ is the identity. Hence $\delta(u_1,u_2) = 4$, and there is a path $(u_1, u_1u_{12}, u_{12}, u_{12}u_2, u_2)$ from U_1 to U_2. By an argument in the previous paragraph, we know that on the line uw_2, there is a unique point v with $4(v) = 4(w_1)$. Hence $4(w_1)$ and $4(w_2)$ differ in exactly three positions (because if they were equal, then they would have to be equal to $4(x)$, a contradiction). Similarly, $4(w_1)$ (respectively $4(w_2)$) and $4(w_{12})$ differ in exactly three positions. But this induces a triangle in the graph G (see above), a contradiction.

This completes the proof of our Main Result. □

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