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Some Constructions of Small Generalized Polygons

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We present several new constructions for small generalized polygons using small projective planes together with a conic or a unital, using other small polygons, and using certain graphs such as the Coxeter graph and the Pappus graph. We also give a new construction of the tilde geometry using the Petersen graph. © 2001 Academic Press

Key Words: generalized hexagon; generalized quadrangle; Coxeter graph; Pappus configuration; Petersen graph.

1. INTRODUCTION

A *generalized n-gon* Γ of order (s, t) is a rank 2 point-line geometry whose incidence graph has diameter n and girth $2n$, each vertex corresponding to a point has valency $t+1$ and each vertex corresponding to a line has valency $s+1$. These objects were introduced by Tits [5], who constructed the main examples. If $n=6$ or $n=8$, then all known examples arise from Chevalley groups as described by Tits (see, e.g., [6, or 7]). Although these examples are strongly group-related, there exist simple geometric constructions for large classes of them. In particular, all "classical" finite generalized quadrangles (*classical* means that the polygons arise naturally

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from Chevalley groups) “are” quadrics, Hermitian varieties, or the geometries of linear line complexes in projective 3-space. Similarly, the hexagons (of order (q, q)) related to Dickson’s group $\mathbf{G}_2(q)$ are constructed using non-singular quadrics in projective 6-space and linear line-complexes. In even characteristic, one deduces a construction in projective 5-space (in fact embedded in a symplectic geometry). For more details, we refer to the monograph [7].

Due to sporadic isomorphisms of small simple groups, some small polygons have alternative constructions. Some of these are rather geometric, others are rather combinatorial. The following example is probably the proto-type of this phenomenon. It is well-known that there is a unique quadrangle of order (2, 2). It may be constructed combinatorially as follows. The set of points is the set of pairs of a given 6-set. The lines are the triples of pairs partitioning the 6-set. Incidence is the natural one. Clearly, the symmetric group $\mathbf{S}_6$ (which is isomorphic to the symplectic group $\mathbf{PSp}_4(2)$) acts as an automorphism group on the quadrangle. A geometric construction runs as follows: the points are the points off a given hyperoval in $\mathbf{PG}(2, 4)$ and the lines are the secant lines (with respect to the hyperoval). Note that the stabilizer of the hyperoval inside $\mathbf{P\Gamma L}_3(4)$ is precisely the symmetric group $\mathbf{S}_6$. The following example illustrates the isomorphism $\mathbf{G}_2(2)' \cong \mathbf{PSU}_3(3)$. Let the point set of a geometry Γ be the set of points off a given Hermitian curve in $\mathbf{PG}(2, 9)$, and let the line set be the polar triangles with respect to the unital. Incidence is the natural one. Then Γ is the dual of the generalized hexagon $\mathsf{H}(2)$ of order (2, 2) embedded in $\mathbf{PG}(5, 2)$. Recently, the second author found another construction of $\mathsf{H}(2)$ (see [8]), based on a more complicated description due to the first author in [3]. We will give that construction below.

The aim of the present paper is to give more elementary constructions of small generalized polygons. In particular, we will construct the generalized hexagon $\mathsf{H}(2)$ in $\mathbf{PG}(2, 7)$, out of the Pappus configuration, and using the unique generalized quadrangle $\mathsf{Q}(4, 2)$ of order (2, 4). We will construct the generalized quadrangle $\mathsf{W}(2)$ of order (2, 2) out of a generalized digon of order (2, 2) (which is straightforward), and also in $\mathbf{PG}(2, 5)$ and $\mathbf{PG}(2, 9)$. We will construct $\mathsf{Q}(2, 4)$ in $\mathbf{PG}(2, 9)$, and also out of the Pappus configuration. Further, we will construct the famous triple cover of $\mathsf{W}(2)$ (the so-called *tilde geometry*) out of the Petersen graph.

2. CONSTRUCTIONS USING SMALL PROJECTIVE PLANES TOGETHER WITH A CONIC OR A HERMITIAN UNITAL

In the following constructions I_{nat} is the *natural incidence relation*. Here a point p and a line l are *naturally incident* if, considered as set-theoretic objects, $p \in l$, $l \in p$, $p \subseteq l$, or $l \subseteq p$.

Construction 1. Let $\mathcal{O}$ be a conic in $\mathbf{PG}(2, 5)$. Let $\mathcal{P}$ be the set of exterior points of $\mathbf{PG}(2, 5)$ with respect to $\mathcal{O}$ and let $\mathcal{L}$ be the set of exterior lines together with the polar triangles consisting of exterior points with respect to $\mathcal{O}$. Then the triple $(\mathcal{P}, \mathcal{L}, \mathtt{I}_{nat})$ is a generalized quadrangle of order $(2, 2)$.

Construction 2. Let $\mathcal{O}$ be a conic in $\mathbf{PG}(2, 5)$. Let $\mathcal{P}$ be the set of interior points with respect to $\mathcal{O}$ together with the polar triangles consisting of secant lines of $\mathcal{O}$, and let $\mathcal{L}$ be the set of secant lines of $\mathcal{O}$. Then the triple $(\mathcal{P}, \mathcal{L}, \mathtt{I}_{nat})$ is a generalized quadrangle of order $(2, 2)$.

Construction 3. Let $\mathcal{O}$ be a conic in $\mathbf{PG}(2, 7)$. Let X be the set of polar triangles consisting of interior points, and let E be the set of non-disjoint pairs of elements of X. An element of E, that is, a pair of triples of points, can be identified with the unique interior point these two triples intersect in. Thus, E can be identified with the set of interior points with respect to $\mathcal{O}$, and (X, E) is the incidence graph of $\mathbf{PG}(2, 2)$. The derived group $G' \cong \mathbf{PSL}_2(7)$ of the automorphisms group $G \cong \mathbf{PGL}_2(7)$ of $\mathcal{O}$ in $\mathbf{PG}(2, 7)$ has two orbits on X. These correspond exactly to the sets of points and lines of $\mathbf{PG}(2, 2)$, respectively.

Construction 4. Let $\mathcal{O}$ be a conic in $\mathbf{PG}(2, 7)$. Let $\mathcal{P}$ be the set of points off $\mathcal{O}$ together with the polar triangles consisting of interior points with respect to $\mathcal{O}$, and let $\mathcal{L}$ be the set of interior points with respect to $\mathcal{O}$ together with the polar triangles containing exterior points with respect to $\mathcal{O}$. Then $(\mathcal{P}, \mathcal{L}, \mathtt{I}_{nat})$ is a generalized hexagon of order $(2, 2)$ isomorphic to $\mathsf{H}(2)$.

Construction 5. Let $\mathcal{O}$ be a conic in $\mathbf{PG}(2, 9)$. Let X be the set of polar triangles consisting of exterior points, and let E be the set of non-disjoint pairs of elements of X. As in Construction 3, the set E can be identified with the set of exterior points with respect to $\mathcal{O}$. Then (X, E) is the incidence graph of $\mathsf{W}(2)$. The derived group $G' \cong \mathbf{PSL}_2(9)$ of the linear automorphisms group $G \cong \mathbf{PGL}_2(9)$ of $\mathcal{O}$ in $\mathbf{PG}(2, 9)$ has two orbits on X. These correspond exactly to the sets of points and lines of $\mathsf{W}(2)$, respectively.

Construction 6. Let $\mathcal{U}$ be a Hermitian unital in $\mathbf{PG}(2, 9)$, and let L be a tangent line of $\mathcal{U}$. Let $\mathcal{P}$ be the set of polar triangles with respect to $\mathcal{U}$ with a vertex on L. Let $\mathcal{L}$ be the set of points of L not on $\mathcal{U}$, together with the polar triangles not having a vertex on L. If incidence $\mathtt{I}$ is inverse containment or being non-disjoint, then $(\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a generalized quadrangle of order $(2, 4)$.

Finally, we list the two well-known constructions mentioned in the introduction; the second one is due to Tits [5].

Construction 7. Let $\mathscr{H}$ be a hyperoval in $\mathbf{PG}(2, 4)$. Let $\mathscr{P}$ be the set of points off $\mathscr{H}$, let $\mathscr{L}$ be the set of bisecant lines with respect to $\mathscr{H}$, and let $\mathtt{I}$ be the incidence relation of $\mathbf{PG}(2, 4)$. Then $(\mathscr{P}, \mathscr{L}, \mathtt{I})$ is a generalized quadrangle of order (2, 2).

Construction 8. Let $\mathscr{U}$ be a Hermitian unital in $\mathbf{PG}(2, 9)$. Let $\mathscr{P}$ be the set of points off $\mathscr{U}$, and let $\mathscr{L}$ be the set of polar triangles with respect to $\mathscr{U}$. Then $(\mathscr{P}, \mathscr{L}, \mathtt{I}_{nat})$ is a generalized hexagon of order (2, 2) isomorphic to the dual of $\mathsf{H}(2)$.

3. CONSTRUCTIONS USING SMALL GENERALIZED POLYGONS

Not in the spirit of the title of this section, we start off with a construction in a projective (or rather affine) space.

Construction 9. Let $\mathbf{PG}(3, 3)$ be the 3-dimensional projective space over the field $\mathbf{GF}(3)$ of four elements. Let σ be a symplectic polarity in $\mathbf{PG}(3, 3)$ and let p be an arbitrary but fixed point of $\mathbf{PG}(3, 3)$. Let $\{L, L', M, M'\}$ be the set of lines in p^σ incident with p. For a point x not in p^σ and a line $X \in \{L, L', M, M'\}$, we denote by xX the line in $\mathbf{PG}(3, 3)$ through x and the intersection of x^σ with X. Further, an affine object is an object not in p^σ. Let $\mathscr{P}$ be the set of affine points of $\mathbf{PG}(3, 3)$, together with all affine symplectic lines (a symplectic line is a line X with $X^\sigma = X$). Let $\mathscr{L}$ be the set of all affine lines of $\mathbf{PG}(3, 3)$ through p, together with, for each affine point x, the triples $\{x, xL, xL'\}$ and $\{x, xM, xM'\}$. If $\mathtt{I}$ is the natural incidence, or incidence in $\mathbf{PG}(3, 3)$, then $(\mathscr{P}, \mathscr{L}, \mathtt{I})$ is a generalized hexagon of order (2, 2) isomorphic to $\mathsf{H}(2)$.

A direct translation into the language of generalized quadrangles of the previous construction gives us the following construction.

Construction 10. Let $\mathsf{Q}(5, 2) = (\mathscr{P}', \mathscr{L}', \mathtt{I}')$ be a generalized quadrangle of order (2, 4). Let $\mathscr{S}$ be a normal spread of $\mathsf{Q}(5, 2)$, i.e., a partition of $\mathscr{P}'$ into (disjoint) lines such that, if $L, L' \in \mathscr{S}$, then the unique line of $\mathsf{Q}(5, 2)$ meeting every line that meets both L and L' is also contained in $\mathscr{S}$. Let $\{\mathscr{S}, \mathscr{U}, \mathscr{U}', \mathscr{V}, \mathscr{V}'\}$ be a partition of $\mathscr{L}$ into (disjoint) spreads of $\mathsf{Q}(5, 2)$ (note that only $\mathscr{S}$ is normal). Let $\mathscr{P}$ be the union of $\mathscr{P}'$ and $\mathscr{L}' \backslash \mathscr{S}$. Let $\mathscr{L}$ be the union of $\mathscr{S}$ and the set of triples $\{x, u, u'\}$ and $\{x, v, v'\}$, with $x \in \mathscr{P}'$, $(u, u', v, v') \in \mathscr{U} \times \mathscr{U}' \times \mathscr{V} \times \mathscr{V}'$ and $x \mathtt{I}' u, u', v, v'$. With the natural incidence relation $\mathtt{I}_{nat}$, the triple $(\mathscr{P}, \mathscr{L}, \mathtt{I}_{nat})$ is a generalized hexagon of order (2, 2) isomorphic to $\mathsf{H}(2)$.

Finally, we mention two known constructions. The first is immediate, the second is contained in [8].

Construction 11. Let $\Gamma' = (\mathscr{P}', \mathscr{L}', \mathrm{I}')$ be a generalized digon of order (2, 2). Let $\mathscr{P}$ be the set of points, lines and flags of Γ' and let $\mathscr{L}$ be the set of triples $\{p, L, \{p, L\}\}$, where $p \in \mathscr{P}'$ and $L \in \mathscr{L}'$, together with the triples of flags $\{F, F', F''\}$ such that $F \cup F' \cup F'' = \mathscr{P}' \cup \mathscr{L}'$. Then $(\mathscr{P}, \mathscr{L}, \mathrm{I}_{nat})$ is a generalized quadrangle of order (2, 2).

Construction 12. Let $\Gamma' = (\mathscr{P}', \mathscr{L}', \mathrm{I}')$ be a projective plane (= generalized triangle) of order (2, 2). Let $\mathscr{P}$ be the set of points, lines, flags and antiflags of Γ', let $\mathscr{L}$ be the set of triples $\{p, L, \{p, L\}\}$, where $p \in \mathscr{P}'$, $L \in \mathscr{L}'$ and $p\mathrm{I}'L$, together with the triples $\{F, A, A'\}$, where $F = \{p, L\}$ is a flag (with $p \in \mathscr{P}'$ and $L \in \mathscr{L}'$) and A, A' are antiflags such that $F \cup A \cup A'$ is the set of points and lines of Γ' incident in Γ' with L and p, respectively. Then $(\mathscr{P}, \mathscr{L}, \mathrm{I}_{nat})$ is a generalized hexagon of order (2, 2) isomorphic to $\mathbf{H}(2)$.

4. CONSTRUCTIONS USING GRAPHS

In this section, we use four different graphs to construct small polygons. The graphs are the Petersen graph, the Pappus graph, the Coxeter graph, and the Heawood graph. We briefly recall how these graphs are constructed.

$*$ *The Petersen Graph.* Consider a general Desargues configuration. The vertices of the Petersen graph are the points of this configuration, and the edges are the pairs of points not on a line of the Desargues configuration. So, there are 10 vertices and 15 edges. Each edge of the Petersen graph has two *opposite* edges, i.e., two edges at maximal distance in the edge graph. These two edges are also mutually at maximal distance. The set of three edges thus obtained forms a *spread* in the following sense: each edge not of the spread is adjacent to a unique edge of the spread (this is the usual notion of a spread in a generalized hexagon; in [7] this is called a *distance*-3-*spread*). We have pictured the Petersen graph and a spread (thick edges) in Fig. 1.

The Petersen graph can also be constructed as follows. The vertices are the transpositions of the symmetric group $\mathbf{S}_5$. Two vertices are adjacent if the corresponding transpositions commute. This way, edges can be identified with the involutions of the unique alternating subgroup $\mathbf{A}_5$. Now a spread is just a set of non-trivial elements of any subgroup of $\mathbf{A}_5$ isomorphic to $\mathbf{A}_4$.

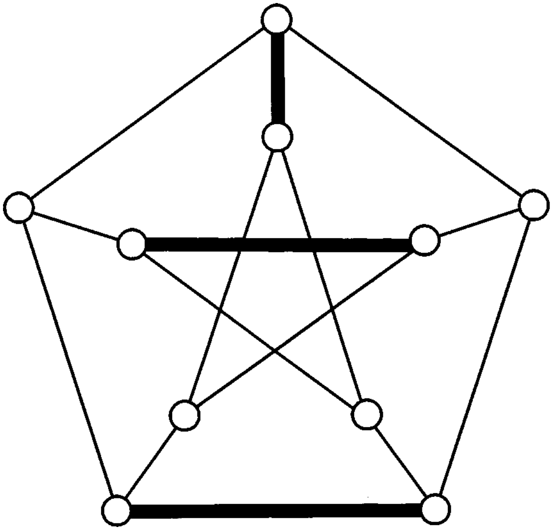

FIGURE 1

∗ *The Pappus Graph.* Consider a Pappus configuration (this is a biaffine plane of order 3). The vertices of the Pappus graph are the points and lines of this configuration and adjacency is incidence. So, there are 18 vertices and 27 edges. Here, too, each edge has two *opposites* in the above sense (in the edge graph, opposite edges are vertices at distance 4 from each other). And again, these two opposite edges are mutually opposite. Hence we have a set of three edges which can be called a *spread* in the following sense: each vertex not on one of these three edges is adjacent to a (unique) vertex of a unique edge of the spread (this is the usual definition of a spread in a generalized octagon; in [7] this is called a *distance*-4-*spread*). We have pictured the Pappus graph together with a spread (thick edges) in Fig. 2.

Figure 3 is the *edge graph* of the Pappus graph: it arises from the Pappus graph (Fig. 2) by taking as vertices the midpoints of all edges and joining two such vertices if the corresponding edges are adjacent (share a vertex) in the Pappus graph. The labeling refers to the proof of Lemma 1 below. Figure 3 also shows three spreads (the "square, circle, and triangle spreads"), which, together with their images under rotation through 120 and 240 degrees around the centre of the diagram, are a partition of the edges into 9 spreads.

Also, each edge e of the Pappus graph is at distance 3 in the edge graph from exactly twelve other edges. Eight amongst them share a vertex with some edge opposite e; the other four are contained in a unique ordinary hexagon which also includes e, and will be called *half opposite* e. For instance, in Figs. 2 and 3, half opposite $a1$ are $b8$, $c9$, $d7$, and $e6$.

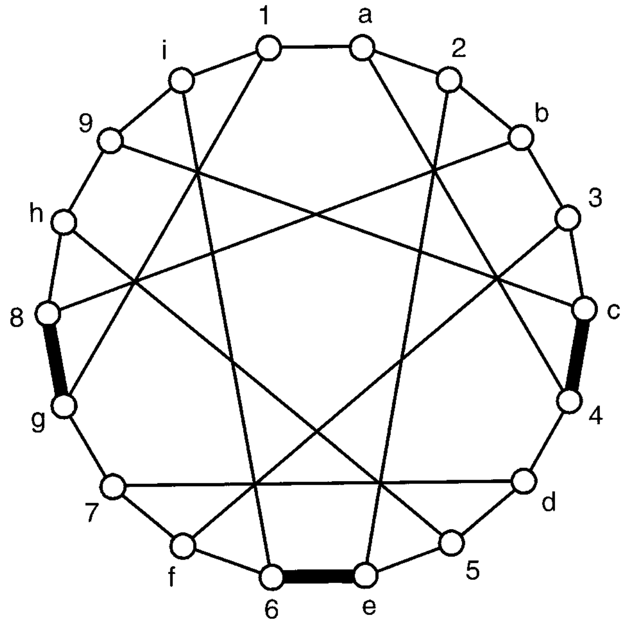

FIGURE 2

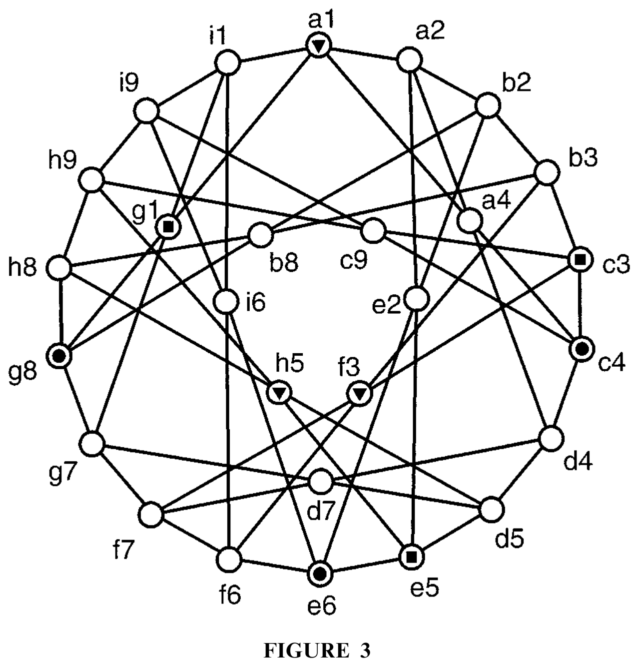

FIGURE 3

We will show below (see Lemma 2) that, up to left and right compositions with isomorphisms of the Pappus graph, and up to taking inverses, there is a unique permutation of the edges of the Pappus graph preserving opposition (or, equivalently, preserving all spreads), and mapping half opposite pairs of edges to adjacent pairs of edges. We call such a permutation a *hexagon permutation.*

There is one other remarkable property of the Pappus graph. For every pair of half opposite edges, there exists a unique edge half opposite both. Such a set of three mutually half opposite edges will be called a *half spread.* For instance, the edges $a1$, $d7$, $e6$ (see Figs. 2 and 3) form a half spread. To complete terminology, we will call a set of three mutually adjacent edges a *clique of edges.*

$*$ *The Coxeter Graph.* Consider $\mathbf{PG}(2,2)$, the projective plane of order 2. The vertices of the Coxeter graph are the antiflags of $\mathbf{PG}(2,2)$. Two antiflags $\{p, L\}, \{p', L'\}$ (with p, p' points and L, L' lines of $\mathbf{PG}(2,2)$) form an edge precisely when $p \neq p'$, $L \neq L'$ and the intersection point of L and L' is incident with the line joining p and p'. We have pictured the Coxeter graph in Fig. 4.

For every two vertices at distance 4 (which is the maximal distance), there are exactly two vertices at distance 4 from both of these, and these two vertices are also mutually at maximal distance. Hence we obtain a set of four vertices mutually at maximal distance. We call any such set a

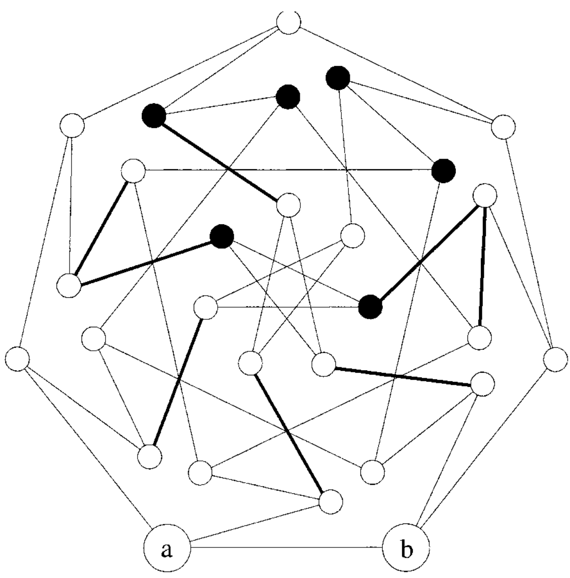

FIGURE 4

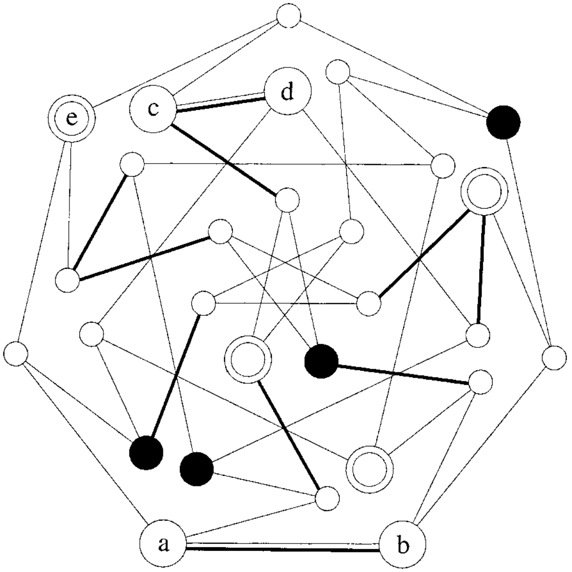

FIGURE 5

variety of the Coxeter graph. Note that a variety corresponds to a unique point or line of $\mathbf{PG}(2, 2)$ (because it is easily checked that vertices at maximal distance correspond to antiflags which share a point or a line; a variety then corresponds to a set of four antiflags sharing the same point or line). The black vertices and the doubly circled vertices in Fig. 5 are two varieties.

Consider an arbitrary vertex a of the Coxeter graph. There are six vertices at maximal distance from a (these are the solid black circles in Fig. 4). Now let $\{a, b\}$ be an edge of the Coxeter graph. There are exactly two vertices c, d at distance 4 from both a and b and $\{c, d\}$ happens to be an edge (see Fig. 5). We call $\{a, b\}$ and $\{c, d\}$ *opposite* edges. We have also the following property. There are exactly two vertices at distance 2 from both a and c (respectively a and d, or b and c, or b and d), for example e is such a vertex. With the eight vertices thus obtained one can form two uniquely determined varieties (see, again, Fig. 5). We call these two varieties *in the middle* of $\{a, b\}$ and $\{c, d\}$. Note that these two varieties correspond to a unique incident point-line pair of $\mathbf{PG}(2, 2)$ (hence a flag).

$*$ *The Heawood Graph.* This is the incidence graph of $\mathbf{PG}(2, 2)$. Here, too, we call two vertices at maximal distance *opposite*. Figure 6 pictures the Heawood graph and two opposite vertices (the solid black circles).

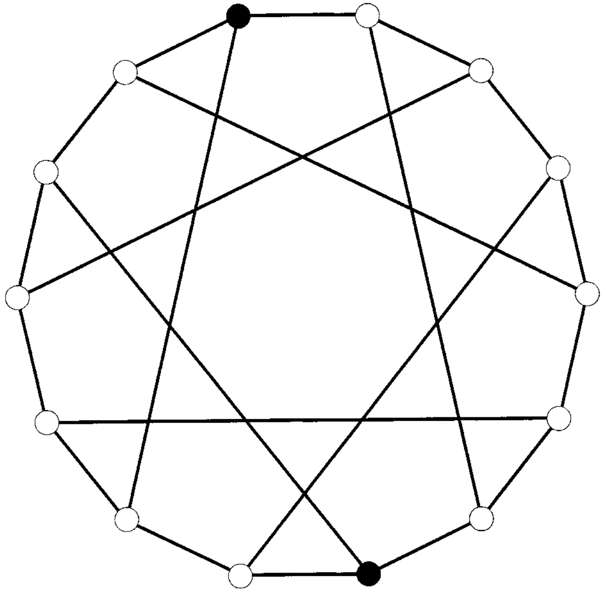

FIGURE 6

We start with some constructions of small generalized quadrangles.

Construction 13. Let $G=(X, E)$ be the Petersen graph. Let $\mathscr{P}=E$ and let $\mathscr{L}$ be the set of vertices union the set of spreads of G. Then $(\mathscr{P}, \mathscr{L}, \mathrm{I}_{nat})$ is a generalized quadrangle of order (2, 2).

Construction 14. Let $G=(X, E)$ be the Pappus graph. Let $\mathscr{P}=E$ and let $\mathscr{L}$ be the set of spreads, half spreads and cliques. Then $(\mathscr{P}, \mathscr{L}, \mathrm{I}_{nat})$ is a generalized quadrangle of order (2, 4).

We now continue with some constructions of $\mathbf{H}(2)$.

Construction 15. Let $G=(X, E)$ be the Coxeter graph. Let $\mathscr{P}$ be the set of vertices, varieties and pairs of opposite edges of G. Let $\mathscr{L}$ be the set of triples $\{a, b, \{e, e'\}\}$, where $a, b \in X$, $e=\{a, b\} \in E$ and $e' \in E$ opposite e, together with the triples $\{V, V', \{e, e'\}\}$, where $e \in E$ is opposite $e' \in E$ and V, V' are the two varieties in the middle of e and e'. Then the triple $(\mathscr{P}, \mathscr{L}, \mathrm{I}_{nat})$ is a generalized hexagon isomorphic to $\mathbf{H}(2)$.

The next construction is a direct translation of Construction 12.

Construction 16. Let $G=(X, E)$ be the Heawood graph. Let $\mathscr{P}$ be the set of vertices, edges and pairs of opposite vertices. Let $\mathscr{L}$ be the union of E and the set of triples $\{e, \{a, b\}, \{a', b'\}\}$, where $e \in E$, $\{a, a', b, b'\}$ is the set of vertices adjacent to a vertex of e and a and a' are opposite b and b', respectively. Then the triple $(\mathscr{P}, \mathscr{L}, \mathrm{I}_{nat})$ is a generalized hexagon of order $(2, 2)$ isomorphic to $\mathbf{H}(2)$.

Construction 17. Let $G=(X, E)$ and $G'=(X', E')$ be two copies of the Pappus graph. We may abstractly identify E with E' in such a way that triples of opposite edges are identified with triples of opposite edges and triples of adjacent edges are identified with triples of half opposite edges. We denote the corresponding quotient set by E''. Now put $\mathscr{P}=X\cup X'\cup E''$. Let $\mathscr{L}$ be the union of the set E'' and the set of triples $\{e, e', e''\}$, where $e, e', e''\in E''$ correspond to a spread in G (and hence also in G'). Then the triple $(\mathscr{P}, \mathscr{L}, \mathrm{I}_{nat})$ is a generalized hexagon of order $(2, 2)$ isomorphic to $\mathbf{H}(2)$.

5. CONSTRUCTION OF THE TILDE GEOMETRY USING THE PETERSEN GRAPH

First we mention a property of the Petersen graph $G=(X, E)$. Let v be any vertex and let e_1, e_1', e_1'' be the edges containing v. Let $\{e_1, e_2, e_3\}$, $\{e_1', e_2', e_3'\}$, and $\{e_1'', e_2'', e_3''\}$ be the three spreads containing these edges. Then the graph induced by the edges $e_2, e_3, e_2', e_3', e_2'', e_3''$ is an ordinary hexagon covering the vertices at distance 2 from v in G. We can choose e_2, e_2', e_2'' mutually not adjacent. This defines a cyclic order on these three spreads. By connectivity, we can put a cyclic order on any spread. The question arises whether this is well-defined. Therefore, we look at this ordering in a different way.

Let $\mathscr{S}_1$ be any 5-subset of E inducing a partition of X. We fix $\mathscr{S}_1$ for the rest of this section (the triply bonded edges in Fig. 7). Then it is easy to see that $E\backslash\mathscr{S}_1$ is the union of the sets of edges of two ordinary pentagons. We denote the sets of edges of these pentagons by $\mathscr{S}_2$ (in Fig. 7 the simply laced pentagon) and $\mathscr{S}_3$ (in Fig. 7 the doubly bonded edges). Every spread of G has an element in each of $\mathscr{S}_1, \mathscr{S}_2, \mathscr{S}_3$, and hence, if we choose an arbitrary but fixed cyclic ordering of the set $\{\mathscr{S}_1, \mathscr{S}_2, \mathscr{S}_3\}$, this induces a cyclic ordering on any spread. It is readily checked that this cyclic ordering arises in the way described in the first paragraph of this section. We call such an ordering a *tilde ordering* of the spreads. Clearly, every cyclic ordering satisfying the property outlined in the previous paragraph arises in the way just explained. In the following we will be working with the cyclic ordering $\mathscr{S}_1\to\mathscr{S}_2\to\mathscr{S}_3\to\mathscr{S}_1$.

With respect to the second construction of the Petersen graph (using the symmetric group $\mathbf{S}_5$), a tilde ordering is given by any ordering of any spread, and then taking conjugates with elements of $\mathbf{A}_5$. An element of $\mathbf{S}_5\backslash\mathbf{A}_5$ reverses the tilde ordering.

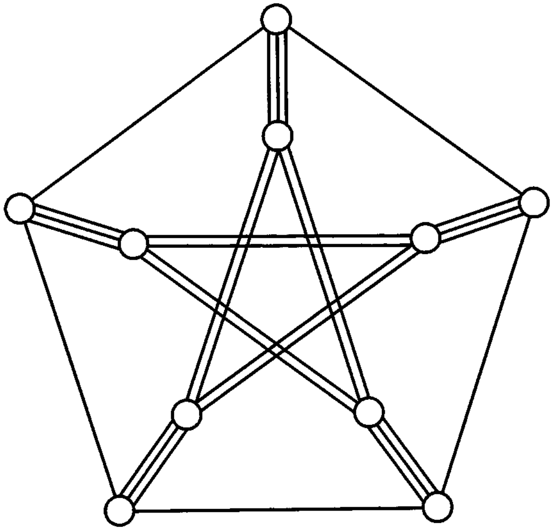

FIGURE 7

Construction 18. Let $G_i = (X_i, E_i)$, $i = 1, 2, 3$ be three copies of the Petersen graph and let $\varphi_{ij}: G_i \to G_j$, $i, j \in \{1, 2, 3\}$, be nine graph isomorphisms (identifications) with $\varphi_{ij}\varphi_{jk} = \varphi_{ik}$ (and hence $\varphi_{ii} = \mathrm{id}$), for all $i, j, k \in \{1, 2, 3\}$. Consider a fixed tilde ordering on the spreads of G_1. Let $\mathscr{P} = E_1 \cup E_2 \cup E_3$. Let $\mathscr{L} = X_1 \cup X_2 \cup X_3 \cup Y$, with Y the set of triples $\{e_1, e_2, e_3\}$, where $e_i \in E_i$ $(i = 1, 2, 3)$, $\{e_1, e_2^{\varphi_{21}}, e_3^{\varphi_{31}}\}$ is a spread of G_1, and $(e_1, e_2^{\varphi_{21}}, e_3^{\varphi_{31}})$ corresponds to the chosen tilde ordering. Then $\Gamma = (\mathscr{P}, \mathscr{L}, \mathrm{I}_{\mathrm{nat}})$ is a geometry of order $(2, 2)$ isomorphic to the tilde geometry.

6. PROOFS

In this section, we outline the proofs that the constructions above work. To this end, we use the following lemma, which belongs to folklore, but which we will proof for completeness' sake.

LEMMA 1. *Let $\Gamma = (\mathscr{P}, \mathscr{L}, \mathrm{I})$ be a rank 2 point-line geometry, such that each line of Γ is incident with $s+1$ points, and each point of Γ is incident with $t+1$ lines, for some (finite) constants $s, t \geqslant 1$. Suppose the number of points of Γ is $|\mathscr{P}| = (1+s)(1+st+(st)^2+\cdots+(st)^n)$, for some natural number $n \geqslant 1$. Suppose also that, given any line $L \in \mathscr{L}$, each point of Γ lies at distance $\leqslant 2n$ (measured in the incidence graph) from some point incident with L. Then Γ is a generalized $(2n+2)$-gon.*

Proof. Let L be any line of Γ, and suppose $1 \leqslant i \leqslant n$. Then there are at most $(st)^i$ points x of Γ at distance $2i$ from a given point p on L for which there exists a minimal path from p to x not containing L as second element. Varying i and p we see that there are at most $(1+s)(1+st+(st)^2+\cdots+(st)^n)$ points at distance $\leqslant 2n$ from some point on L. From our assumption that every point must be obtained in that way, we easily deduce that all points we counted so far are distinct (in other words, we did not count the same point twice). Hence there are no ordinary k-gons, with $k \leqslant 2n$, in Γ containing the line L. Since L was arbitrary, there are no ordinary k-gons in Γ, with $k \leqslant 2n$, at all. Now, consider one of the $(st)^n$ points x at distance $2n$ from a certain point p of the line L. Then x is at distance $2n+2$ from any other point of L (indeed, otherwise we counted x twice in the previous counting, a contradiction). Hence the diameter of the incidence graph is equal to $2n+2$ (noting that, by counting the number of lines, we may dualize all arguments). Now, consider two points y, y' collinear with x and such that x, y, y' are not collinear (since $t \geqslant 1$, such points can be found). If both y and y' are at distance $2n-2$ from p, then we counted x twice, a contradiction. None of them can be at distance $2n$ from p since otherwise (say y is at distance $2n$ from p) the point p is at distance $2n$ from two distinct point of the line xy, a contradiction. Hence we may assume that y is at distance $2n+2$ from p (and consequently at distance $2n$ from some point $q \neq p$ on L), and we obtain an ordinary $(2n+2)$-gon containing p, q, L, x, y. Hence the girth is $2n+2$ and the lemma is proved. ∎

Since all generalized quadrangles that we construct are uniquely determined by their order, it suffices to show that the geometries are generalized quadrangles. Moreover, since it is readily checked that each time we have the right number of points and lines, it suffices, by the above lemma for $n=1$, to show that for each line L, we obtain the full set of points by considering all points collinear to at least one point incident with L. In all constructions, this is an easy exercise.

Note that Construction 1 is equivalent to Construction 2 by applying the polarity related to the conic.

That Construction 3 works follows directly form the definition of the Heawood graph and Section 5 of Coxeter [2].

Concerning the constructions of the generalized hexagons $\mathbf{H}(2)$ and its dual, it is, similar to the case of quadrangles, easy to show that the geometries under consideration are generalized hexagons of order $(2,2)$. One again counts the number of points and shows that, for every line L, the full point set is obtained by considering the set of points at distance $\leqslant 4$ from at least one point incident with L. Then apply Lemma 1 with $n=2$. Such an explicit proof related to Construction 8 is worked out in [7, 1.3.12]. Now, since there are essentially two different generalized hexagons of order (2, 2), one the dual of the other (see Tits [5, Appendix] or, for an explicit proof, Cohen and Tits [1], we proceed to identify which of the two hexagons we are dealing with in the different constructions.

It is clear that, by Construction 3, the incidence graph of $\mathbf{PG}(2, 2)$, or equivalently, the Heawood graph, is a subgeometry of the generalized hexagon Γ of Construction 4. Since the dual of $\mathsf{H}(2)$ does not admit such a subgeometry (see for instance [7, 6.3.5]), Γ must be isomorphic to $\mathsf{H}(2)$.

Concerning Construction 9, it is easily seen that the set of affine lines through p forms a spread of the generalized hexagon (indeed, every other line has a point in common with a unique line of the spread). Since the dual of $\mathsf{H}(2)$ does not admit spreads (by Thas [4]), the hexagon in question must be isomorphic to $\mathsf{H}(2)$.

Construction 10 is a direct translation of the previous construction into the language of quadrangles. We only note that the affine points together with the affine symplectic lines and the affine lines through p form the quadrangle of order (2, 4).

In Construction 15, the geometry induced on the set of varieties is exactly the incidence graph of $\mathbf{PG}(2, 2)$ (see the comments on the construction of the Coxeter graph). Hence we again have $\mathsf{H}(2)$.

In Construction 16 it is clear that the Heawood graph is a subgeometry of the hexagon, hence, again, we have $\mathsf{H}(2)$. This also follows directly from Construction 12.

The lines induced on the elements of E'' of Construction 17 form a spread of the hexagon, hence $\mathsf{H}(2)$ arises there (again by [4]).

The proof that Construction 17 works will be complete if we show the following lemma.

LEMMA 2. *Let $G = (X, E)$ be the Pappus graph. Then, up to left and right compositions with isomorphisms of G, there exist exactly two hexagon permutations τ and τ^{-1} (and, as the notation indicates, they can be chosen inverse to each other). Hence, if $G' = (X', E')$ is another copy of the Pappus graph, then, up to automorphisms of G and G', and up to a permutation of $\{G, G'\}$, there exists a unique bijection $E \to E'$ preserving opposition and mapping half opposite edges to adjacent ones.*

Proof. Let G be the Pappus graph and label the vertices as indicated in Fig. 2.

We will denote the edge corresponding to the vertices k, $k \in \{1, 2, ..., 9\}$ and x, $x \in \{a, b, ..., i\}$ by xk. Further, we will call two edges at distance 3 from each other which are not half opposite, *almost opposite*, and we will call edges at distance 2 from each other *almost half opposite*. We search for a bijection τ as stated in the lemma. Since the Pappus graph is the incidence graph of the biaffine plane of order 3, it has a large automorphism group, and we may assume that τ fixes the edges $a1$, $f3$ and $h5$. The eight edges half opposite $f3$ and $h5$ are the eight edges almost half opposite $a1$, and the four edges adjacent to $f3$ and $h5$ are exactly the four edges almost opposite $a1$. Hence τ will map almost half opposite edges to almost opposite edges. Now, consider the two half spreads of any edge half opposite $a1$. One such half spread contains $a1$ and two edges half opposite $a1$; the other half spread contains two edges almost opposite $a1$. The first half spread is mapped under τ to a clique of edges containing $a1$; the second half spread must hence be mapped onto a clique of edges not containing $a1$, but containing an edge adjacent to $a1$. It follows that the two edges (of that half spread) almost opposite $a1$ are mapped onto two edges amost half opposite $a1$. Hence almost opposite edges are mapped onto almost half opposite ones. Hence adjacent edges are mapped onto half opposite ones.

Therefore $(i1)^\tau \in \{b8, c9, d7, e6\}$. There is an automorphism of G fixing $1, a, g, i$ and interchanging the half spreads $\{a1, d7, e6\}$ and $\{a1, b8, c9\}$. So, up to conjugation, we may assume that $(i1)^\tau \in \{d7, e6\}$. We will show that τ is determined by the image of $i1$. The arguments are similar for the cases $(i1)^\tau = d7$ and $(i1)^\tau = e6$. To fix the ideas, we set $(i1)^\tau = e6$. Hence $(g1)^\tau = d7$. It suffices to show that the images of the edges opposite $i1$ and of those adjacent to $i1$ are determined (because then we let $i1$ play the role of $a1$ and we can continue). The two edges opposite $i1$ are $b3$ and $d5$. They must be mapped onto edges opposite $e6$, and these are $c4$ and $g8$. But $b3$ is adjacent to $f3$ (which is fixed), hence $b3$ must also be mapped onto an edge half opposite $f3$. Since $g8$ qualifies for this and $c4$ does not, we have $(b3)^\tau = g6$ and $(d5)^\tau = c4$.

Since the edges containing the vertex 1 are mapped onto the half spread $\{a1, d7, e6\}$, and $i1$ is mapped onto $e6$, the edges containing the vertex i will be mapped onto the half spread $\{b3, e6, h9\}$. But $i6$ is half opposite $h5$, hence $(i6)^\tau$ must be adjacent to $(h5)^\tau = h5$. This implies $(i6)^\tau = h9$. We conclude that τ is completely determined. In order to show existence, it suffices to continue the construction, and we obtain that τ acts as follows:

$$a1 \mapsto a1, \qquad f3 \mapsto f3, \qquad h5 \mapsto h5,$$

$$a2 \mapsto c9 \mapsto a4 \mapsto b8 \mapsto a2, \qquad b2 \mapsto d5 \mapsto c4 \mapsto e5 \mapsto b2,$$

$$b3 \mapsto g8 \mapsto c3 \mapsto i9 \mapsto b3, \qquad d4 \mapsto f7 \mapsto e2 \mapsto f6 \mapsto d4,$$

$$d7 \mapsto i1 \mapsto e6 \mapsto g1 \mapsto d7, \qquad g7 \mapsto h8 \mapsto i6 \mapsto h9 \mapsto g7.$$

We have depicted τ in Fig. 8, using the graph as in Fig. 3, but deleting the edges for clarity.

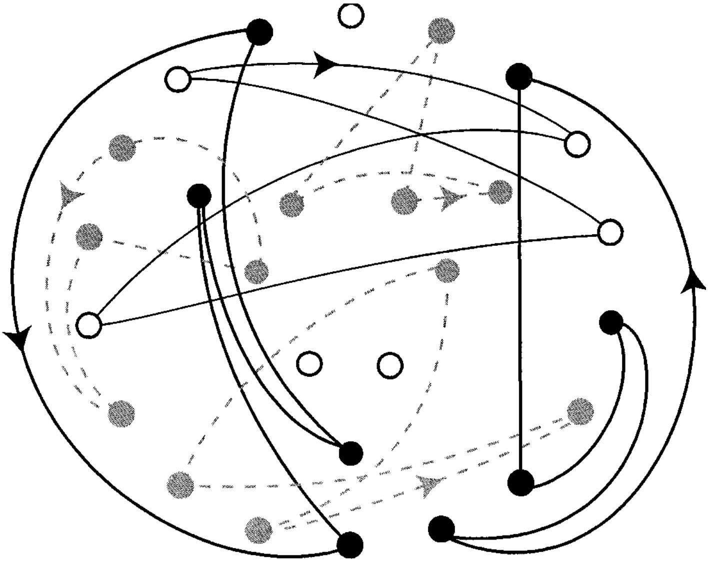

FIGURE 8

It is clear that τ^{-1} also satisfies the hypotheses, and that it is obtained by assuming that $i1$ is mapped onto $d7$.

The lemma is proved. ∎

Now consider Construction 18. It is easy to check that, given a vertex $v \in X_i$, $i \in \{1, 2, 3\}$, the set of points of Γ at distance 8 in the incidence graph of Γ is precisely $\{v^{\varphi_{ij}}, v^{\varphi_{ik}}\}$, with $\{i, j, k\} = \{1, 2, 3\}$. It follows that every automorphism of Γ preserves the set of orbits of the φ_{ij} on $X_1 \cup X_2 \cup X_3$. Hence the pointwise stabilizer of this set of 10 orbits is a normal subgroup of the full automorphism group of Γ. This stabilizer is clearly generated by the maps $\varphi_{12}, \varphi_{23}, \varphi_{31}$, hence it is a group of order 3. The quotient geometry Q under this group is now clearly isomorphic to $\mathbf{W}(2)$, see Construction 13. We now claim that the type-preserving automorphism group of Γ is the non-trivial threefold cover of the symmetric group $\mathbf{S}_6$ (and so Γ is isomorphic to the tilde geometry). Indeed, let θ be an automorphism of Q. Then we must lift θ to Γ in order to obtain an automorphism $\bar{\theta}$ of Γ. Now Q can be identified with the Petersen graph G_1, together with the five spreads. If θ preserves G_1, then there are two possibilities: either θ belongs to $\mathbf{A}_5$ (in this case we define $x^{\bar{\theta}} = x^{\varphi_{i1}\theta\varphi_{1i}}$, for any vertex $x \in X_i$, $i = 1, 2, 3$; and this naturally extends to E_1, E_2, E_3 and to Y, because θ preserves the tilde ordering), or θ belongs to $\mathbf{S}_5 \backslash \mathbf{A}_5$ (and then, for all $x \in X_i$, $i = 1, 2, 3$, we define $x^{\bar{\theta}} = x^{\varphi_{i1}\theta\varphi_{1\sigma(i)}}$, where σ fixes 1 and interchanges 2 and 3; again this can be naturally extended to E_1, E_2, E_3

and to Y, the map σ entering the scene because θ does not preserve the tilde ordering). Since $\mathbf{S}_5$ is a maximal subgroup of the automorphism group $\mathbf{S}_6$ of Q, we only need to show how one element $\theta \in \mathbf{S}_6 \backslash \mathbf{S}_5$ can be lifted to Γ. Therefore consider the labeling of edges of G_i, $i = 1, 2, 3$, as in Fig. 9. The label ixj of an edge tells us that the edge belongs to G_i and also belongs to a spread with cyclic ordering $(ix1, ix2, ix3)$, where j denotes the weight of the corresponding edge in Fig. 7 (this cyclic ordering coincides with the one chosen in Section 5). Further, $ixj = (kxj)^{\varphi_{ki}}$, $i, j, k \in \{1, 2, 3\}$, $x \in \{a, b, c, d, e\}$. It follows that, for all $x \in \{a, b, c, d, e\}$, the triple $\{1xi, 2xj, 3xk\}$ belongs to Y, for $(i, j, k) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$. We choose θ to be the unique involution of Q that fixes the edges $1a1$, $1b1$, $1c3$, $1d3$ and $1e1$. Figure 9 then shows exactly how $\bar{\theta}$ acts on the point set $E_1 \cup E_2 \cup E_3$ of Γ: the thick edges are fixed, edges of the same graph that are interchanged have a double arrow between them, the rest of the involution is written in between the graphs by specifying which edges are interchanged. One can verify that this induces an automorphism of Γ (in other words, $\bar{\theta}$ preserves collinearity of points of Γ). Note that there are three choices to define $\bar{\theta}$: one that fixes $1a1$ (as in Fig. 9), one that fixes $2a1$, and

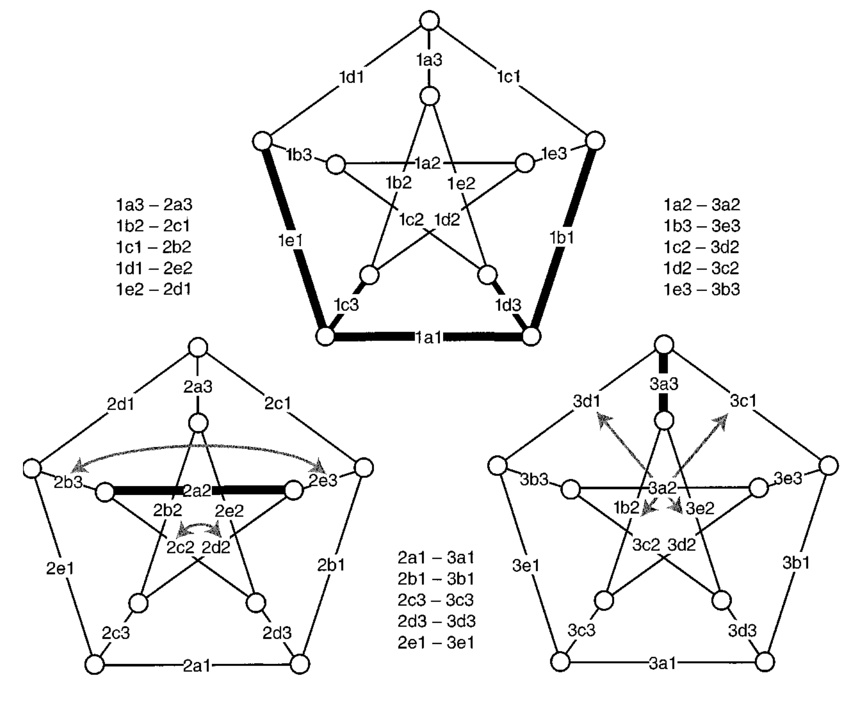

FIGURE 9

one that fixes $3a1$. Each of them is an involution, showing that we indeed have a *non-trivial* triple cover of $\mathbf{S}_6$.

This completes the proofs that all constructions work.

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