Dual polar spaces embedded in metasymplectic spaces

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Abstract. The main problem considered in this paper is "how does a dual polar space Γ of rank 3 embed in a metasymplectic space Δ?" The expected and generic answer is that Γ is isomorphic to a subgeometry of a point residual $\mathsf{Res}_\Delta(p)$ and that it arises as a subgeometry of a trace geometry, that is, $\Gamma \subseteq p^\perp \cap q^{\bowtie}$, for two opposite points p and q, where $q^{\bowtie}$ is the set of points special to q. However, this is not always the case, and we describe some counterexamples, even classify them for certain classes of metasymplectic spaces Δ. These results complement the analogous results for the exceptional geometries of diameter at most 3 arising from groups of types $\mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$ recently treated by Cooperstein and the second author.

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1. Introduction

Classifying subgeometries of a given geometry is very helpful to understand the structure of the given geometry. In particular if the subgeometry is, in some sense, large, and/or, in another sense, maximal. The latter would mean that it is not contained in a well-known subgeometry already; the former could for instance mean that lines of the subgeometry are full lines of the ambient geometry, or if there is a rank or dimension available, that these do not differ too much. Embeddings of geometries in projective or affine spaces has been thoroughly investigated, especially for polar spaces, but also for other, mostly finite, geometries such as (semi-)partial geometries, partial quadrangles, generalised hexagons, etc. Recently, there has been some interest to look at subgeometries of the standard exceptional geometries of Lie

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type, emphasising inclusions of exceptional geometries themselves. This was very fruitful for gaining insight into the structure of these geometries, as it revealed, for instance, beautiful properties of the so-called *equator geometries*, see for instance [9, 13]. Recently, Cooperstein and the second author [5] characterised so-called *trace geometries* in exceptional Lie incidence geometries of simply laced type as the only fully embedded geometries isomorphic to a point residual. The present paper investigates the analogue for metasymplectic spaces, that is, for the standard Lie incidence geometries related to spherical buildings of type F_4 (which is also an exceptional type).

There are three phenomena that complicate things for the metasymplectic spaces, and which are due to the fact that the diagram is not simply laced.

(*i*) Point residuals in metasymplectic spaces are dual polar spaces of rank 3. But that class of geometries is much larger than those that can appear as point residual. So, it is an additional complication that we do not know in advance which isomorphism class of geometries we are embedding.

(*ii*) In the simply laced case, the point residuals are Lie incidence geometries that are generated by the points of an apartment. This is not longer true in the metasymplectic case. Yet, this property was the basis of a fundamental technique in [5].

(*iii*) Finally, maybe the most prominent thing, not all (fully) embedded dual polar spaces in metasymplectic spaces are traces, even if they are isomorphic to the point residuals! It will also turn out that a dual polar space can be embedded in a symplecton.

Concerning the first complication mentioned above, it will turn out that in half of the cases, the isomorphism class and embedding will be determined by just assuming we have an embedded dual polar space of rank 3. In the remaining cases, we find subgeometries of embedded geometries isomorphic to point residuals. The second complication will be bypassed by using a different method. Concerning the third complication, we will show that in most metasymplectic spaces an embedding of a dual polar space must be "isometric" (precise definition see below) and we will describe isometric embeddings of dual polar spaces that do not arise as traces; sometimes we can even classify. Moreover we provide an example of a dual polar space isomorphic to a point residual that is embedded in a symplecton (and hence not isometrically embedded). Especially the isometric embeddings that are not traces provide more insight in the structure of metasymplectic spaces. We note, in particular, that this exceptional behaviour is not only for infinite cases, but also for those over a finite field in characteristic 2, in particular, the smallest, finite ones over $\mathbb{F}_2$ experience this. Roughly speaking, and referring forward for undefined notions, we prove the following:

Main Results—Informal statements.

(1) *Dual polar spaces fully embedded in metasymplectic spaces which do not admit central elations, are always traces;*

(2) *Dual polar spaces fully embedded in metasymplectic spaces the duals of which do not admit central elations are either contained in traces, or contained in a symp, or contained in the perp of a point, but not in a trace (and in the latter case the metasymplectic space has planes of order* 2 *and has* 5 *symps on each plane; the dual polar spaces have either 3 symps per line—and there is a unique non-trace example—or* 5 *symps per line—and again there is a unique non-trace example).*

(3) *If the metasymplectic space admits central elations and also its dual does, then a fully embedded dual polar space is either contained in a trace, in a symp, or in the perp of a point, but not in a trace.*

As a byproduct of our proof, we exhibit an alternative method to determine the embedding rank of the dual polar spaces of rank 3 with three points per line, a result due to Yoshiara [24], who made use of the Leech lattice to do so. We only use elementary (finite incidence) geometry, and the existence of metasymplectic spaces.

Let us mention two applications of our results. Firstly, one might wonder what about dual polar spaces of rank at least 4 embedded in metasymplectic spaces? We will show that, if a dual polar space of rank at least 4 is fully embedded in a metasymplectic space admitting no central elations, then it has rank 4 and is isomorphic to a so-called tropics geometry, see Proposition 2.19, hence essentially unique. Secondly, suppose the metasymplectic space Δ admits an embedding in a projective space; one can then consider the universal embedding of any point perp $p^{\perp}$ (a cone with vertex p over the point residual). Every hyperplane section not through p of this universal embedding gives rise to an embedded dual polar space. Our main results reveal in precisely which cases such a hyperplane section is always a trace, that is, arises from the geometry of the metasymplectic space itself. Moreover, in the non-embeddable case, our main results imply that every fully embedded dual polar space of rank 3 is a trace, and (so) nothing exceptionally happens due to the non-embeddability.

Alongside with our main results, and besides the alternatie proof for the universal embedding of the dual polar spaces of rank 3 with three points per line, we prove many other results that we hope will prove useful in other contexts. Most prominently, we for instance show that no nontrivial injective projection of the quadric Veronesean of any projective plane with at least 21 points, is contained in a nontrivial quadric, except if the underlying fiels has characteristic 2 and the projection is from a subspace of the nucleus plane, see Lemma 3.9.

We also provide some consequences of our results. One kind of corollary states that in the standard embedding of certain metasymplectic spaces Δ, every hyperplane of the projective subspace spanned by the points collinear to a given point p of Δ, not containing p itself, is the subspace spanned by the points collinear to p and not opposite some fixed point q, with q opposite p. Another consequence is the uniqueness of the so-called tropics geometry as fully embedded dual polar space of rank 4 in metasymplectic spaces not containing central elations. This is the geometric counterpart of the uniqueness of groups of type B_4 in groups of type F_4, or also of the uniqueness of root systems of type B_3 in root systems of type F_4.

We now introduce all notions needed to state our main results in detail. This is done in section 2. Before we proceed with the proof, we review some known properties of Veronese representations of projective planes and prove some new ones in section 3. We start the proofs with two general results: if a fully embedded dual polar space is not contained in a symp, then it is isometrically embedded and contained in the perp of a point p in such a way that every line through pv contains at most one point of the embedded dual polar space. This is the content of section 4. Then, in section 5 we prove our main results for the *separable case*, that is, for metasymplectic spaces either who do not admit central elations, or whose dual do not admit central elations. The inseparable case (both the metasymplectic space and its dual admit central elations) is then treated in section 6.

2. Preliminaries and statement of the Main Results

The main players in this paper are metasymplctic spaces. We will view these with the help of the language of parapolar spaces, which are point-line geometries satisfying certain axioms. Crucial substructures of parapolar spaces are polar spaces. Therefore, we start by briefly introducing point-line geometries, polar spaces and parapolar spaces, mainly to fix notation.

2.1. Point-line geometries

For the purpose of this paper, a *point-line geometry* is a pair $\Gamma = (X, \mathscr{L})$ consisting of a set X whose elements are called *points*, and a non-empty subset $\mathscr{L}$ of the set of subsets of X, each element of which is called a *line*. We assume every member of $\mathscr{L}$ has at least three elements. A *collineation* (or *automorphism*) of a point-line geometry $\Gamma = (X, \mathscr{L})$ is a bijection from X onto itself that induces a bijection on $\mathscr{L}$.

Let $\Gamma = (X, \mathscr{L})$ be a point-line geometry. We introduce some terminology and notation.

If every pair of distinct points is contained in at most one line, then we say that Γ is a *partial linear space*. From now on, assume that Γ is a partial linear space. We also assume that Γ is *thick*, that is, each line contains at least three points.

Two points $x, y \in X$ are *collinear*, denoted as $x \perp y$, if they are contained in a common line, which we usually denote by xy and which is unique. The set $x^\perp$ (the *perp* of x) is the set of points collinear to x and, more generally, for a subset $T \subseteq X$, the set $T^\perp$ is the set of points collinear to each point of T. A *subspace* S is a subset of X with the property that each member of $\mathscr{L}$ intersects S in either 0, 1 or all of its points. We will view subspaces as point-line geometries using the induced line set. A *singular* subspace is a subspace in which each pair of points is collinear. A *geometric hyperplane*, or briefly *hyperplane*, is a subspaces which is not disjoint from any line. A *subhyperplane* is a hyperplane of a hyperplane. The *point graph* of Γ is the graph with vertices the elements of X, adjacent when distinct and collinear. The *distance* between two points is the distance in the point graph. The *diameter* $\mathsf{diam}\,\Gamma$ of Γ is the diameter of its point graph. If the diameter is finite, then we call Γ *connected.* A *convex* set $C \subseteq X$ is a set of points at mutual finite distance with the property that all vertices of every minimal path in the point graph between any two arbitrary elements of C are contained in C.

Example 2.1. For an arbitrary right vector space V over some skew field $\mathbb{L}$, define the point set of the point-line geometry $\mathsf{PG}(V)$ as the set of 1-spaces of V, and the line set as the set of 2-spaces (seen as sets of the contained 1-spaces). If $\mathsf{dim}\,V = n < \infty$, then we denote $\mathsf{PG}(V)$ also as $\mathsf{PG}(n, \mathbb{L})$. Such a point-line geometry is called a *projective space (over* $\mathbb{L}$*).* Its *dimension* is $\dim V - 1$.

Projective spaces are *linear spaces*, that is, partial linear spaces in which every pair of points is contained in a line.

An arbitrary subspace of $\mathsf{PG}(V)$ is precisely the set of 1-spaces of a given subspace of V. In $\mathsf{PG}(V)$, we sometimes denote the line defined by two distinct points x, y by $\langle x, y\rangle$, and, more generally, the intersection of all subspaces containing a set A of points by $\langle A\rangle$.

For convenience we shall also call every abstract axiomatic projective plane (that is, a thick linear space with the property that each pair of lines intersects nontrivially) and every set of at least three elements with itself as unique line, a projective space (of dimension 2 and 1, respectively).

A special mention deserves the projective plane $\mathsf{PG}(2, \mathbb{O})$ obtained from a Cayley algebra $\mathbb{O}$ (see also below). It is obtained in the standard way from an affine plane (by adding points and a line at infinity) that can be described as the set of pairs $(x, y) \in \mathbb{O} \times \mathbb{O}$, where lines are the sets of points satisfying an equation of the form $y = mx + k$, or $x = x_0$, $m, k, x_0 \in \mathbb{O}$. It is called a *Cayley plane.*

We will also need the notion of a projective (sub)line.

Definition 2.2. Let $\mathbb{F}$ be a field. Then the set of 1-spaces of a two-dimensional vector space V over $\mathbb{F}$, together with the group $\mathsf{PGL}_2(\mathbb{F})$ acting naturally on that set, is called the *projective line over* $\mathbb{F}$. Let $\mathbb{K}$ be a subfield of $\mathbb{F}$. Then the vector lines over $\mathbb{F}$ defined by vectors which are a $\mathbb{K}$-linear combination of (two) given basis vectors, define a *standard projective subline over* $\mathbb{K}$, since the group $\mathsf{PGL}_2(\mathbb{K})$ acts naturally on that set of 1-spaces. Now let $\mathbb{F}$ have characteristic 2 and let V' be a subspace of the $\mathbb{F}^2$-vector space $\mathbb{F}$. Then the set of vector lines over $\mathbb{F}$ defined by vectors which are a V'-linear combination of (two) given basis vectors, is by definition a *mixed projective subline* $\mathsf{PG}(1, V')$ *over* V'. Using coordinates with respect to the given basis, $\mathsf{PG}(1, V')$ is given by the following set of projective points:

$$\{\mathbb{F}(1, x) \mid x \in V'\} \cup \{\mathbb{F}(0, 1)\}.$$

Given two arbitrary different 1-spaces in $\mathsf{PG}(1, V')$, we can always find two respective non-zero vectors e_1, e_2 on them with coordinates in V'. Then it is immediate that $\mathsf{PG}(1, V')$ is equal to $\{\mathbb{F}(e_1 + xe_2) \mid x \in V'\} \cup \{\mathbb{F}e_2\}$, and so a mixed projective subline is independent of the chosen basis. We now make a few remarks.

Remark 2.3. We note that, if $\mathbb{K}$ is a subfield of $\mathbb{F}$, then $\mathsf{PG}(2, \mathbb{K})$ is in a natural way a subplane of $\mathsf{PG}(2, \mathbb{F})$, and every line of $\mathsf{PG}(2, \mathbb{K})$ defines a standard projective subline over $\mathbb{K}$ of a unique line of $\mathsf{PG}(2, \mathbb{F})$. However, a mixed projective subline over V' of $\mathsf{PG}(1, \mathbb{F})$ only arises from a subplane in that way if V' has the structure of a field, which is not necessarily the case. For instance consider over $\mathbb{F}_2(t, u)$ the subspace generated by $1, t$ and u over $\mathbb{F}_2(t^2, u^2)$. This is not a subfield as tu does not belong to it.

Remark 2.4. Let $\mathbb{F}$ be a field and let σ be a Galois involution of $\mathbb{F}$. Then it is shown in [23] that the sets $\{x - x^\sigma \mid x \in \mathbb{F}\}$ and $\{y \in \mathbb{F} \mid y + y^\sigma = 0\}$ coincide; denote that set by S. We claim that the set $\{\mathbb{F}(1, y) \mid y \in S\} \cup \{\mathbb{F}(0, 1)\}$ of 1-spaces of a two-dimensional vector space $\mathbb{F} \times \mathbb{F}$ over $\mathbb{F}$ is a standard subline over $\mathbb{K}$, where $\mathbb{K}$ is the field of fixed elements of σ. Indeed, this follows from the fact that, if t is an arbitrary element satisfying $t + t^\sigma = 0$, then $S = t\mathbb{K}$ (and then we replace the standard basis (e_1, e_2) with (e_1, te_2)).

Remark 2.5. Let $\mathbb{F}$ be a field of characteristic 2 and let $\mathbb{K}$ be a subfield such that $\mathbb{F}/\mathbb{K}$ is separable and quadratic. Then $\mathbb{K}$ does not contain $\mathbb{F}^2$ and hence a projective subline over $\mathbb{K}$ cannot be a mixed projective subline. Indeed, let $x \in \mathbb{F}\backslash\mathbb{K}$ satisfy the separable quadratic equation $x^2 + ax + b = 0$, with $a, b \in \mathbb{K}^\times$. If x^2 belonged to $\mathbb{K}$, then so would $ax + b$, hence also $x \in \mathbb{K}$, a contradiction.

Definition 2.6. Let $\Gamma = (X, \mathscr{L})$ and $\Lambda = (Y, \mathscr{R})$ be two partial linear spaces. Then we say that Γ is *fully embedded* in Λ if $X \subseteq Y$ and $\mathscr{L} \subseteq \mathscr{R}$. For the purpose of the present paper, we call the embedding *isometric* if X is a subspace of Δ and every member of $\mathscr{R}$ contained in X belongs to $\mathscr{L}$ (so Γ is induced in X by Λ). If Λ is a projective space, then we call every full embedding of Γ in Λ a *projective (full) embedding* and say that Γ is *embeddable*. In this case a *secant* is a line of Λ that is not a line of Γ, but intersects the point set of Γ in at least two points. Recall that we denote the projective line defined by two points p, q of an embedded geometry as $\langle p, q\rangle$; it can be a secant!

A projective embedding of Γ into $\mathsf{PG}(V)$ is called *universal* if every other projective embedding, say in $\mathsf{PG}(V')$ is obtained from it by projection, that is, there exists a subspace S of $\mathsf{PG}(V)$, a complementary subspace V'' in V, and an isomorphism $\varphi : \mathsf{PG}(V'') \to \mathsf{PG}(V')$ such that the mapping $X \to \mathsf{PG}(V') : x \mapsto (\langle x, S\rangle \cap \mathsf{PG}(V''))^\varphi$ is injective and defines the given embedding in $\mathsf{PG}(V')$. If a point line-geometry has such a universal embedding in $\mathsf{PG}(V)$ for some vector space V, we call $\dim V$ the *embedding rank* of Γ. A projective embedding of Γ into $\mathsf{PG}(V)$ is called *homogeneous* if every collineation of Γ is induced by a collineation of $\mathsf{PG}(V)$. By the very definition of universal embedding the latter is always homogeneous.

2.2. Polar and parapolar spaces

A (thick) point-line geometry $\Gamma = (X, \mathscr{L})$ is a *polar space* if $x^\perp$ is a hyperplane distinct from X itself, for each $x \in X$. One shows that the singular subspaces of a polar space are either lines or projective spaces of dimension at least 2 (see for example Theorem 7.4.13 of [2]). We will only consider polar spaces of finite *rank* r, that is, polar spaces in which the maximum dimension of singular subspaces is finite and equal to $r-1$. Note that we included *non-degeneracy* (that is, the property that no point is collinear to all other points) in our definition of polar spaces; this is done for convenience..

Polar spaces have a number of nice properties and we refer to the recent book [23] for background and theory. We mention that $\mathsf{diam}\,\Gamma = 2$, for every polar space Γ, and that for every pair of non-collinear points x, y, the set $x^\perp \cap y^\perp$ is a subspace which defines a polar space of rank $r-1$ when endowed with the lines it contains. Also, the number of maximal singular subspaces (that is, singular subspaces of dimension $r-1$) containing a given *submaximal* singular subspace (that is, a singular subspace of dimension $r-2$) is a constant at least equal to 2. If this number is exactly 2, then we say that the polar space is *hyperbolic*. Otherwise it is non-hyperbolic.

Definition 2.7. For two points p, q of a polar space, we call the set $h(p,q) := (\{p,q\}^\perp)^\perp$ a *hyperbolic line*. We say it is *short* if $h(p,q) = \{p,q\}$.

Example 2.8. All the examples in the following paragraphs are non-hyperbolic.

(*i*) Let $\mathbb{A}$ be an alternative quadratic division algebra over some field $\mathbb{K}$, and we view $\mathbb{K}$ as the scalar multiples of the identity element 1 for the multiplication. Then $\mathbb{A}$ admits a standard involution $\mathbb{A} \to \mathbb{A} : x \mapsto \overline{x}$ with the property that $\mathsf{tr}(x) := x + \overline{x} \in \mathbb{K}$ and $\mathsf{n}(x) := x\overline{x} \in \mathbb{K}$. Every element $x \in \mathbb{A}$ satisfies a quadratic equation, namely $x^2 - \mathsf{tr}(x)x + \mathsf{n}(x) = 0$. The mapping $\mathsf{n} : \mathbb{A} \to \mathbb{K} : x \mapsto \mathsf{n}(x) = x\overline{x}$ is a quadratic form, called the *norm form* of $\mathbb{A}$. Let $V = \mathbb{K} \oplus \mathbb{K} \oplus \mathbb{K} \oplus \mathbb{A} \oplus \mathbb{K} \oplus \mathbb{K} \oplus \mathbb{K}$ and define the following quadratic form

$$q : V \to \mathbb{K} : (x_{-3}, x_{-2}, x_{-1}, x, x_1, x_2, x_3) \mapsto x_{-3}x_3 + x_{-2}x_2 + x_{-1}x_1 + \mathsf{n}(x).$$

Then the projective null set of q in $\mathsf{PG}(V)$, that is, the set of 1-spaces $v\mathbb{K}$ of V with $q(v) = 0$, endowed with the projective lines contained in it, constitute a polar space of rank 3 which we will denote by $\mathsf{B}_{3,1}(\mathbb{K}, \mathbb{A})$. If p, p' are non-collinear points of $\mathsf{B}_{3,1}(\mathbb{K}, \mathbb{A})$, then the polar space $p^\perp \cap p'^\perp$ is denoted as $\mathsf{B}_{2,1}(\mathbb{K}, \mathbb{A})$. On the other hand, the polar space with the property that for each pair of non-collinear points x, x', the subspace $x^\perp \cap x'^\perp$ is isomorphic to $\mathsf{B}_{3,1}(\mathbb{K}, \mathbb{A})$, is denoted as $\mathsf{B}_{4,1}(\mathbb{K}, \mathbb{A})$ (and has rank 4). All hyperbolic lines of the polar spaces $\mathsf{B}_{3,1}(\mathbb{K}, \mathbb{A})$ (and $\mathsf{B}_{2,1}(\mathbb{K}, \mathbb{A})$) are short, if $\mathbb{A}$ is not an inseparable multiple quadratic extension of $\mathbb{K}$ (including $\mathbb{A} = \mathbb{K}$) when $\mathsf{char}(\mathbb{K}) = 2$.

Note that in general the projective null set of a polynomial of which all terms have degree two is called a *quadric*. The maximal vector dimension of a projective subspace contained in the quadric is called the *Witt index* of the quadric. It belongs to folklore that in $\mathsf{PG}(3, \mathbb{K})$, with $\mathbb{K}$ a field, there is, up to isomorphism, only one quadric of Witt index 2 which defines a polar spaces. It is a hyperbolic polar space of rank 2, also known as a *grid*. It contains two systems of lines such that two lines belong to the same system if they are disjoint. Such a system will also be called a *regulus*; the other system is then referred to as the *opposite regulus.*

(ii) Let again $\mathbb{A}$ be an alternative quadratic division algebra over some field $\mathbb{K}$, but now assume that $\mathbb{A}$ is associative. Set $V := \mathbb{A}\oplus\mathbb{A}\oplus\mathbb{A}\oplus\mathbb{A}\oplus\mathbb{A}\oplus\mathbb{A}$ and consider this as a right vector space over $\mathbb{A}$. Consider the following mapping q' (which is a pseudo-quadratic form):

$$q' : V \to \mathbb{A} : (x_{-3}, x_{-2}, x_{-1}, x_1, x_2, x_3) \mapsto \overline{x}_{-3}x_3 + \overline{x}_{-2}x_2 + \overline{x}_{-1}x_1.$$

Then the set of points $v\mathbb{A}$ of $\mathsf{PG}(V)$ such that $q'(v) \in \mathbb{K}$, either endowed with all lines contained in it (if $\mathbb{A} \neq \mathbb{K}$), or endowed with the lines corresponding to the totally isotropic 2-spaces of the alternating form

$$((x_{-3}, x_{-2}, x_{-1}, x_1, x_2, x_3), (y_{-3}, y_{-2}, y_{-1}, y_1, y_2, y_3)) \mapsto x_{-3}y_3 + x_{-2}y_2 + x_{-1}y_1 - x_1y_{-1} - x_2y_{-2} - x_3y_{-3}$$

(if $\mathbb{K} = \mathbb{A}$), is a polar space of rank 3 which we denote as $\mathsf{C}_{3,1}(\mathbb{A}, \mathbb{K})$. If $\mathbb{A} \neq \mathbb{K}$ and $\mathbb{A}^2 \nsubseteq \mathbb{K}$, we call it a *Hermitian polar space*; if $\mathbb{A}^2 \subseteq \mathbb{K}$ and $\mathsf{char}\,\mathbb{K} = 2$, then we call it an *inseparable polar space*, and if $\mathbb{K} = \mathbb{A}$ we call it a *symplectic* polar space. Similarly as in (i), we denote by $\mathsf{C}_{2,1}(\mathbb{A}, \mathbb{K})$ the polar space of rank 2 obtained by intersecting $p^\perp$ with $p'^\perp$, for two non-collinear points p, p' of $\mathsf{C}_{3,1}(\mathbb{A}, \mathbb{K})$. It is well known that $\mathsf{B}_{2,1}(\mathbb{K}, \mathbb{A})$ is the dual of $\mathsf{C}_{2,1}(\mathbb{A}, \mathbb{K})$, this follows for example from 1.10 in [20] or explicitly 3.4.9, 3.4.11 and 3.4.13 in [22]. All hyperbolic lines of $\mathsf{C}_{3,1}(\mathbb{A}, \mathbb{K})$ are non-short and can be parametrised by $\mathbb{K} \cup \{\infty\}$ (that is, they are projective sublines over $\mathbb{K}$ of lines of $\mathsf{PG}(5, \mathbb{A})$), see Lemma 2.25 below.

(iii) For each Cayley algebra, that is, alternative non-associative quadratic division algebra $\mathbb{O}$, also sometimes called non-split octonion algebra, there exists a polar space, denoted as $\mathsf{C}_{3,1}(\mathbb{O}, \mathbb{K})$ of rank 3 whose planes (singular subspaces isomorphic to projective planes) are Cayley planes over $\mathbb{O}$, see Chapter 9 in [20] or [7] for an elementary description. The polar space $\mathsf{C}_{3,1}(\mathbb{O}, \mathbb{K})$ is referred to as a *non-embeddable polar space*, or a *Freudenthal-Tits polar space.* The polar space $p^\perp \cap p'^\perp$, with p, p' two non-collinear points of $\Delta(\mathbb{O})$, is the dual of $\mathsf{B}_{2,1}(\mathbb{K}, \mathbb{O})$, and therefore sometimes denoted by $\mathsf{C}_{2,1}(\mathbb{O}, \mathbb{K})$.

Recall that an alternative quadratic division algebra over the field $\mathbb{K}$ is one of the following: $\mathbb{K}$ itself, a separable quadratic extension of $\mathbb{K}$, a quaternion division algebra over $\mathbb{K}$ (sometimes also called a Hamiltonion division algebra over $\mathbb{K}$), an octonion division algebra over $\mathbb{K}$ (sometimes also called a Cayley division algebra over $\mathbb{K}$), or an inseparable multiple quadratic extension of $\mathbb{K}$ in characteristic 2. Recall also that, in the finite case, denoting the finite field with q elements as $\mathbb{F}_q$, the quadratic division algebras over $\mathbb{F}_q$

are the fields $\mathbb{F}_q$ and $\mathbb{F}_{q^2}$. We often replace a finite field with its order in the notation for the geometries. For instance, $\mathsf{C}_{3,1}(q,q)$ is the polar space arising from a non-degenerate alternating form (or symplectic polarity, see Section 2.4) in $\mathsf{PG}(5,q)$, the finite projective space over $\mathbb{F}_q$. In this paper, let $\mathbb{K}$ always be a field and $\mathbb{A},\mathbb{B}$ quadratic alternative division algebras over $\mathbb{K}$, unless specified otherwise.

A *parapolar space* is a connected partial linear space $\Delta = (X, \mathscr{L})$ with the following properties. First, the collection of convex subspaces of Δ isomorphic to a polar space is non-empty. Each member of that collection is called a *symplecton*, or *symp* for short. Secondly, every pair of points x, y with the property that $|\{x,y\}^\perp| > 1$ is contained in some symp. If all symplecta of Δ have the same rank r, then we say that Δ has *uniform symplectic rank* r. We will only work with parapolar space having uniform symplectic rank (but there exist many other parapolar spaces). By the convexity of symps, each symp is determined by any pair $\{p,q\}$ of its non-collinear points, and we denote the symp by $\xi(p,q)$. The pair $\{p,q\}$ is called a *symplectic pair*; also p *is symplectic to* q, and we write $p \perp\!\!\!\perp q$..

A parapolar space Δ is called *strong* if every pair of points at distance at most 2 is contained in a symp. In other words, if there do not exist points p, q such that $|p^\perp \cap q^\perp| = 1$. In a non-strong parapolar space, pairs $\{p,q\}$ with $p^\perp \cap q^\perp$ a singleton, are called *special* (and p is also said to be special to q, notation $p \bowtie q$). Note that every polar space of rank r is a strong parapolar space of uniform symplectic rank r.

Let $\Delta = (X, \mathscr{L})$ be a parapolar space of uniform symplectic rank r at least 3. Let $p \in X$ be arbitrary. Define the following geometry $\mathsf{Res}_\Delta(p)$. Its point set is the set of lines of Δ containing p. An arbitrary line of $\mathsf{Res}_\Delta(p)$ is the set of lines of Δ through p contained in a projective plane (a so-called *planar line pencil with vertex* p). The geometry $\mathsf{Res}_\Delta(p)$ is a strong parapolar space of uniform symplectic rank $r-1$. It is called the *point residual* at p.

A *dual polar space of rank* 3 is the point-line geometry obtained from a non-hyperbolic polar space of rank 3 by taking as point set the set of planes (maximal singular subspaces) and as lines the sets of planes containing a given line (submaximal singular subspace). The following is immediate.

Proposition 2.9. *A dual polar space of rank* 3 *is a strong parapolar space of uniform symplectic rank* 2.

Even though a dual polar space of rank 3 has a uniform symplectic rank smaller than 3, one can define a point residual as follows. Let $\Omega = (X, \mathscr{L})$ be a dual polar space of rank 3 and let $p \in X$ be arbitrary. Then the point set of $\mathsf{Res}_\Omega(p)$ is the set of lines of Ω containing p. An arbitrary line of $\mathsf{Res}_\Omega(p)$ is the set of lines of Ω through p contained in a symplecton of Ω. It follows immediately that $\mathsf{Res}_\Omega(p)$ is isomorphic to a projective plane.

The dual polar space of rank 3 associated to the polar spaces $\mathsf{B}_{3,1}(\mathbb{K},\mathbb{A})$ and $\mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$, will be denoted by $\mathsf{B}_{3,3}(\mathbb{K},\mathbb{A})$ and $\mathsf{C}_{3,3}(\mathbb{A},\mathbb{K})$, respectively. Note that the symps of these parapolar spaces are isomorphic to $\mathsf{C}_{2,1}(\mathbb{A},\mathbb{K})$

and $\mathsf{B}_{2,1}(\mathbb{K},\mathbb{A})$, respectively. If Ω is the dual polar space corresponding to the polar space Γ, then It is also convenient to call Γ the dual of Ω.

Now we introduce the metasymplectic spaces that we are concerned about in the present paper.

Definition 2.10. A parapolar space $\Delta = (X, \mathscr{L})$ is a *metasymplectic* space if it has uniform symplectic rank 3 and one of the following holds.

- (i) There exists a field $\mathbb{K}$ and an alternative quadratic division algebra $\mathbb{A}$ such that each point residual is isomorphic to $\mathsf{C}_{3,3}(\mathbb{A},\mathbb{K})$, whereas every symp is isomorphic to $\mathsf{B}_{3,1}(\mathbb{K},\mathbb{A})$. These metasymplectic spaces are denoted as $\mathsf{F}_{4,1}(\mathbb{K},\mathbb{A})$ and said to be *of type* $\mathsf{F}_{4,1}$, or of *long root type.*
- (ii) There exists a field $\mathbb{K}$ and an alternative quadratic division algebra $\mathbb{A}$ such that each point residual is isomorphic to $\mathsf{B}_{3,3}(\mathbb{K},\mathbb{A})$, whereas every symp is isomorphic to $\mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$. These metasymplectic spaces are denoted as $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$ and said to be *of type* $\mathsf{F}_{4,4}$, or of *short root type.*

If $\mathbb{A}$ is not an inseparable extension of $\mathbb{K}$ (this includes the assumption that $\mathbb{A}$ does not coincide with $\mathbb{K}$ if $\mathsf{char}\,\mathbb{K} = 2$), then we say that the metasymplectic space is *separable*; otherwise *inseparable.*

Remark 2.11. Note that the inseparable metasymplectic spaces are precisely those that are of both types $\mathsf{F}_{4,1}$ and $\mathsf{F}_{4,4}$.

Before we can state a precise version of our main results, we need some more terminology. We introduce that in the following proposition, which reviews the possible mutual positions of points and symps in a metasymplectic space, which can be deduced from [4].

Proposition 2.12. *Let $\Delta(X, \mathscr{L})$ be a metasymplectic space of type $\mathsf{F}_{4,1}$ or $\mathsf{F}_{4,4}$.*

[Point-Point] *Let p, q be two points of Δ. Then either $p \perp q$, or p and q are contained in a unique symp $\xi(p,q)$, or p and q are special, or p and q are at distance 3 from each other, in which case we call p and q opposite and denote this as $p \equiv q$ (if p is not opposite q we write $p \not\equiv q$).*

[Point-Symp] *Let p be a point and ξ a symp of Δ. Then either $p \in \xi$, or $p^\perp \cap \xi = L \in \mathscr{L}$, or $p^{\perp\!\!\perp} \cap \xi = \{x\}$, $x \in X$. In the second case, we say that p and ξ are close, the points of $(\xi \cap L^\perp)\backslash L$ are symplectic to p and all points of $\xi\backslash L^\perp$ are special to p. In the third case, we say that p and ξ are far, all points of $(\xi \cap x^\perp)\backslash\{x\}$ are special to p and all points of $\xi\backslash x^\perp$ are opposite p.*

[Symp-Symp] *Let ξ and ζ be two distinct symps of Δ. Then either $\xi \cap \zeta$ is a plane (and we say that ξ and ζ are adjacent), $\xi \cap \zeta$ is a point (and we say that ξ and ζ are symplectic), or $\xi \cap \zeta = \varnothing$. In the latter case either there exists a unique symp intersecting both ξ and ζ in respective planes (and we say that ξ and ζ are special), or each point of ξ is far from ζ (and we call ξ and ζ opposite).*

Furthermore, for a line L and a point p we either have that no point of L is opposite p, or all points except exactly one are. The non-opposite point is *then special to p. Consequently, points collinear to a point symplectic to p are never opposite p.*

We will use the symbols $\perp\!\!\!\perp, \bowtie, \equiv$ and $\not\equiv$ also in the exponent, with the obvious meaning. For example $p^{\not\equiv}$ denotes the set of points not opposite p.

Remark 2.13. The terminology for symps in (iii) of the previous proposition can be explained by the following property: Let Ξ be the set of symps of Δ and let $\mathfrak{L}$ be the set of *pencils of symps*, that is, the sets of symps containing a given plane. Then $\Delta^* = (\Xi, \mathfrak{L})$ is a metasymplectic space. If Δ is of type $\mathsf{F}_{4,1}$ or $\mathsf{F}_{4,4}$, then Δ^* is of type $\mathsf{F}_{4,4}$ or $\mathsf{F}_{4,1}$, respectively. We call Δ^* the *dual* of Δ.

Remark 2.14. We note that, if $\mathbb{A}$ is an inseparable extension of $\mathbb{K}$, then $\mathsf{B}_{3,1}(\mathbb{K}, \mathbb{A})$ is isomorphic to $\mathsf{C}_{3,1}(\mathbb{K}, \mathbb{A}^2)$, where $\mathbb{A}^2$ denotes the field of squares of $\mathbb{A}$. Consequently, $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A}) \cong \mathsf{F}_{4,4}(\mathbb{A}^2, \mathbb{K})$.

Remark 2.15. Note that the metasymplectic spaces that we have defined are in fact Lie incidence geometries, that is, geometries defined from Tits-buildings, namely from those of type F_4, as the notation suggests. We will not explicitly need this connection. Some terminology is borrowed from this, though: Opposite points and opposite symps in our sense are also opposite in the building-theoretic sense. Adjacent symps are symps that are contained in adjacent chambers in the building-theoretic sense.

By definition, point residuals in metasymplectic spaces are dual polar spaces. Such residues also give rise to embedded dual polar spaces as follows.

Definition 2.16. Let p and q be two opposite points of a metasymplectic space. Δ. Then the set of points $p^\perp \cap q^{\not\equiv}$ is called a *trace*. Endowed with all lines contained in it, we obtain a *trace geometry*.

Some interesting substructures of these metasymplectic spaces are equator geometries and imaginary lines.

Definition 2.17. Let $\Delta = (X, \mathscr{L})$ be a metasymplectic space. Let p, q be two opposite points of Δ. Then the set of points that are symplectic to both p and q is called the *equator* of p and q, and if we call every intersection of size at least 2 of the equator with a symp a *"line"*, then we talk about the *equator geometry* $E(p, q)$. It is well known that $E(p, q)$ is isomorphic to the polar space that is dual to any point residual (see for example Proposition 2.6.2 in [16]).

A hyperbolic line of an equator geometry through two opposite points a and b will be called an *imaginary line* of the corresponding metasymplectic space and denoted by $\mathscr{C}(a, b)$. These are short if, and only if, the metasymplectic space is separable of type $\mathsf{F}_{4,4}$.

In metasymplectic spaces of type $\mathsf{F}_{4,4}$ we can extend the equator geometry (we refer to Section 2.6 of [16] for many more properties of these subgeometries).

Definition 2.18. Let p and q be two opposite points of $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$. Then define the extended equator geometry $\widehat{E}(p,q)$ as the point-line geometry with point set

$$\bigcup\{E(x,y) \mid x,y \in E(p,q),\ x \text{ opposite } y\}$$

and line set the hyperbolic lines (of symps) contained in this point set.

We then have the following embedding result.

Proposition 2.19 (Section 2.7 of [16]). *Let Δ be the metasymplectic space $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$ and let p,q be opposite points. Then the set of points of Δ collinear to at least two points of $\widehat{E}(p,q)$ is the point set of a fully embedded polar space of rank 4 isomorphic to $\mathsf{B}_{4,1}(\mathbb{K},\mathbb{A})$.*

The fully embedded polar space of rank 4 of the previous proposition is called a *tropics geometry* of Δ.

2.3. Main Results

We can now state the main results.

Main Result A. *Let Ω be a dual polar space of rank 3 fully embedded in a separable metasymplectic space $\Delta \cong \mathsf{F}_{4,1}(\mathbb{K},\mathbb{A})$, with $\mathbb{A}$ an alternative quadratic division algebra over $\mathbb{K}$. Then one of the following possibilities occurs.*

- *(i)* *Ω is isomorphic to $\mathsf{C}_{3,3}(\mathbb{B},\mathbb{K})$, for some quadratic subalgebra $\mathbb{B}$ of $\mathbb{A}$, is isometrically embedded, and is contained in a trace geometry of Δ. If $\mathbb{B} = \mathbb{A}$, then Ω coincides with a trace geometry and conversely, every trace geometry is an isometrically fully embedded dual polar space isomorphic to $\mathsf{C}_{3,3}(\mathbb{A},\mathbb{K})$.*
- *(ii)* *Ω is isomorphic to $\mathsf{C}_{3,3}(\mathbb{B},\mathbb{K})$ and embedded in a symp, with $\mathbb{K}$ infinite and $\mathbb{B}$ a quadratic associative division algebra such that $\mathsf{dim}_{\mathbb{K}}(\mathbb{B}) < \mathsf{dim}_{\mathbb{K}}(\mathbb{A})$ or $\mathbb{B}$ an inseparable field extension different from $\mathbb{K}$ if $\mathsf{char}(\mathbb{K}) = 2$.*
- *(iii)* *$\mathbb{K} \cong \mathbb{F}_2$ and $\mathbb{A} \cong \mathbb{F}_4$, there is a point p such that each point of Ω is on a unique line of Δ through p and each line of Δ through p contains at most one point of Ω, but Ω is not contained in a trace geometry. Up to isomorphism, there are unique examples for $\Omega \in \{(\mathsf{C}_{3,3}(2,2), \mathsf{C}_{3,3}(4,2))\}$.*

Main Result B. *Let Ω be a dual polar space fully embedded in a separable metasymplectic space $\Delta \cong \mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$, with $\mathbb{A}$ an alternative quadratic division algebra over $\mathbb{K}$. Then Ω is isomorphic to $\mathsf{B}_{3,3}(\mathbb{K},\mathbb{A})$, isometrically embedded, and arises as a trace geometry of Δ. Conversely, every trace geometry is an isometrically fully embedded dual polar space isomorphic to $\mathsf{B}_{3,3}(\mathbb{K},\mathbb{A})$.*

Main Result C. *Let Ω be a dual polar space fully embedded in an inseparable metasymplectic space $\Delta \cong \mathsf{F}_{4,1}(\mathbb{K},\mathbb{K}')$, with $\mathbb{K}'$ an inseparable (possibly multiple or trivial) quadratic extension of $\mathbb{K}$, $\mathsf{char}\,\mathbb{K} = 2$. Then one of the following possibilities occurs.*

(*i*) *Ω is isomorphic to $\mathsf{C}_{3,3}(\mathbb{K}'', \mathbb{K})$, for some quadratic subalgebra $\mathbb{K}''$ of $\mathbb{K}'$, is isometrically embedded, and is contained in a trace geometry. Conversely, every trace geometry is an isometrically fully embedded dual polar space isomorphic to $\mathsf{C}_{3,3}(\mathbb{K}', \mathbb{K})$.*

(*ii*) *Ω is isomorphic to $\mathsf{C}_{3,3}(\mathbb{K}'', \mathbb{K})$, for some quadratic subalgebra $\mathbb{K}''$ of $\mathbb{K}'$, is isometrically embedded, there is a point p such that each point of Ω is on a unique line of Δ through p and each line of Δ through p contains at most one point of Ω, but Ω is not contained in a trace geometry.*

(*iii*) *$\mathbb{K} \neq \mathbb{K}'$ is infinite, Ω is isomorphic to $\mathsf{C}_{3,3}(\mathbb{B}, \mathbb{K})$, for some alternative division algebra $\mathbb{B} \neq \mathbb{K}$ over $\mathbb{K}$ such that either $\mathbb{B}$ is an inseparable multiple quadratic extension of $\mathbb{K}$ or $\dim_{\mathbb{K}}(\mathbb{B}) < \dim_{\mathbb{K}}(\mathbb{K}')$, and Ω is fully embedded in a symplecton.*

Remark 2.20. We provide examples of all cases except of Main Result A(*ii*). So whether this actually does occur, is still an open problem.

Remark 2.21. For each field $\mathbb{K}$, there are examples of Main Result C(*ii*). For finite fields, there is, up to isomorphism, a unique example for Main Result C(*ii*), except if $\mathbb{K} = \mathbb{F}_2$, then there are exactly two examples. We sketch a proof of this fact, along with some more remarks in the infinite case, in Remark 6.2.

We will also prove the following consequences of our main results.

Corollary A. *Every fully embedded dual polar space of rank at least rank 4 in a separable metasymplectic space of type $\mathsf{F}_{4,4}$ has rank 4 and is a tropics geometry.*

Finally, the next consequence is immediate. In the finite case it can also be proved directly with a counting argument, but in general it is a rather nice consequence of our main results.

Corollary B. *Let Δ be a separable metasymplectic space fully embedded in some projective space $\mathsf{PG}(V)$, and assume Δ is not isomorphic to $\mathsf{F}_{4,1}(2,4)$. Let p be an arbitrary point of Δ and let U be the subspace of $\mathsf{PG}(V)$ generated by $p^\perp$ (the points of Δ collinear to p in Δ). Let H be any hyperplane of U not containing p. Then $p^\perp \cap H$ is a trace.*

2.4. Some more preliminaries

Before we start proving those main results, we repeat some results from the literature which will prove useful many times. It mainly concerns properties of polar space, in particular of the symps of the metasymplectic spaces we are considering. We also explain the connection between the statement of the main results in the introduction, and the ones in the previous paragraphs.

Our definition of isometric embedding implies immediately that, if a polar space Σ is fully embedded in a parapolar space Δ, then this embedding is isometric if, and only if, non-collinear points of Σ are non-collinear (more exactly, have distance 2) in Δ. Hence distances between points are respected. Hence we can apply [9, Lemma 3.20].

Lemma 2.22 (Lemma 3.20 in [9]). *Let Σ be a polar space fully embedded in a parapolar space Δ. Then either Σ is completely contained in a singular subspace of Δ, or Σ is fully and isometrically embedded in a unique symplecton of Δ.*

Lemma 2.23. (*i*) *The embeddings of $\mathsf{B}_{3,1}(\mathbb{K},\mathbb{A})$ and $\mathsf{B}_{2,1}(\mathbb{K},\mathbb{A})$ in projective space as given in* Example 2.8 (*i*) *are universal. Even more, if $\mathbb{A}$ is not an inseparable field extension if $\mathsf{char}\,\mathbb{K}=2$, those embeddings are unique, i.e. there are no injective projections possible.*

(*ii*) *The embeddings of $\mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$ and $\mathsf{C}_{2,1}(\mathbb{A},\mathbb{K})$ in projective space as given in* Example 2.8 (*ii*) *are universal, except if $\mathsf{char}\,\mathbb{K}=2$ and $\mathbb{A}$ is an inseparable (multiple) extension of $\mathbb{K}$ (this includes $\mathbb{A}=\mathbb{K}$). Even more, those embeddings are unique if universal.*

Proof. The universality follows from 8.7 of [20]. The uniqueness in (*i*) follows from Proposition 3.18 of [18]. In (*ii*) the uniqueness follows from the fact that a projective plane does not contain disjoint lines, and a projective space of dimension at most 4 does not contain disjoint planes. □

We now provide some more background and explanation of the previous lemma, at the same time introducing some notions that we will need later on.

The polar spaces in Example 2.8 are defined using *forms*, which are associated to reflexive forms (symmetric bilinear, alternating and Hermitian forms), see [20, Chapter 8]. Exactly in the separable case, these, in turn, define a *polarity* of the projective space, that is, an involution (a permutation of order 2) of the subspaces reversing the inclusion relation. We refer to that polarity as the *defining polarity.* The image of a point p of the polar space (say, Σ) under that polarity is the *tangent hyperplane*, that is, the hyperplane spanned by all lines of Σ containing p. Alternatively, that hyperplane can be defined as the union of the set of lines of the projective space, going through p and either completely contained in Σ, or intersecting it in just $\{p\}$. Every subspace through p of the tangent hyperplane at p will be called a *tangent subspace.* We will mainly use this notion for "tangent lines". Since a polarity defines an isomorphism between the projective space to its dual (in all the examples, the dimension is finite), the global intersection of all tangent hyperplanes is the empty set.

In the inseparable case, the intersection of all tangent hyperplanes of the universal embedding is non-empty and corresponds to the radical of the associated reflexive form. Projections from subspaces of that radical yields different embeddings. Projection from the whole radical yields a *minimal embedding.* If $\mathbb{K}'$ is a multiple quadratic inseparabel extension of $\mathbb{K}$ in characteristic 2, then the embedding of $\mathsf{B}_{3,1}(\mathbb{K},\mathbb{K}')$ given by Example 2.8(*i*) is universal, and the one of $\mathsf{C}_{3,1}(\mathbb{K},\mathbb{K}'^2)$ given by Example 2.8(*ii*) is minimal. Hence the latter corresponds to a polarity again.

Now we relate to (in)separability of a metasymplectic space to the existence of so-called central elations. We first define the latter, also for polar spaces.

Definition 2.24. A *central elation* of a polar space with centre the point a is a collineation that fixes all points collinear with a. A *central elation* of a metasymplectic space with centre the point a is a collineation that fixes a and stabilises all the lines that have at least one point collinear to a.

Lemma 2.25. *Let ξ be the polar space $\mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$ or the polar space $\mathsf{C}_{2,1}(\mathbb{A},\mathbb{K})$ and let x, y be opposite points of ξ. Then the following holds:*

- *(i) The number of points on the hyperbolic line $h(x, y)$ is $|\mathbb{K}| + 1$;*
- *(ii) the group of central elations of ξ with centre x (see Definition 2.24) acts sharply transitive on $h(x, y)\setminus\{x\}$;*
- *(iii) if $\xi = \mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$, then $h(x, y) = L^\perp \cap M^\perp$ for all opposite lines $L, M \subseteq x^\perp \cap y^\perp$;*
- *(iv) if $\mathbb{A}$ is associative and ξ is minimally embedded in the projective space Π (which is at the same time universally in the separable case), then $h(x, y)$ is a subset of the projective line $\langle x, y\rangle$ through x and y in Π.*

Proof. If $\xi = \mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$, Lemma 2.6.9 of [16] yields (i) and (iii). However a similar argument shows that (i) also holds if $\xi = \mathsf{C}_{2,1}(\mathbb{A},\mathbb{K})$. If $\mathbb{A}$ is associative, Propositions 7.2.6 and 7.2.7 of [23] yield (i), (ii) and (iv). The fact that (ii) also holds in the non-associative case follows now from the comments after Proposition 7.2.7 in [23]. □

Now, one can deduce from [16] that a metasymplectic space admits central elations if, and only if, each equator geometry admits central elations. This, Lemma 2.25 and the fact that $\mathsf{B}_{3,1}(\mathbb{K},\mathbb{A})$ does not admit central elations in the separable case by [23, Proposirion 7.2.8], implies that the inseparable metasymplectic spaces Γ are precisely those for which both Γ and its dual admit central elations; the metasynplectic spaces of type $\mathsf{F}_{4,1}$ are precisely those that admit central elations, and the separable metasymplectic spaces of type $\mathsf{F}_{4,4}$ are precisely those that do not admit central elations. This now shows that the statements of the main results in the introduction and in the previous subsection are equivalent.

The following result belongs to folklore, but we provide an explicit proof for completeness.

Lemma 2.26. *Let $\mathbb{K}$ be a field of characteristic 2 and $\mathbb{K}'$ a (possibly trivial) inseparable (multiple) quadratic field extension of $\mathbb{K}$. Let $\mathsf{B}_{2,1}(\mathbb{K},\mathbb{K}')$ be embedded in $\mathsf{PG}(n,\mathbb{K})$. Then every secant intersects $\mathsf{B}_{2,1}(\mathbb{K},\mathbb{K}')$ in a mixed projective subline. The latter exists of only two points if, and only if, the embedding is universal.*

Proof. Let L be a secant. Denote $d := \dim_{\mathbb{K}}(\mathbb{K}')$. Note that the embedding must be a projection from the universal one as given in Example 2.8 (i) by Lemma 2.23 (i). So we may interpret $\mathsf{PG}(n,\mathbb{K})$ as a subspace of $\mathsf{PG}(3+d,\mathbb{K})$. Denote the coordinates of a point of the latter as $(x_{-2}, x_{-1}, (y_i)_{i\in I}, x_1, x_2)$

with I an index set of cardinality d. Without loss of generality, we may now suppose that L is given by the equations $x_{-1} = 0, x_1 = 0$ and $y_i = 0$ for all $i \in I$. As explained in paragraph 3.4 of [23], we may also suppose that $\mathsf{B}_{2,1}(\mathbb{K}, \mathbb{K}')$ is embedded in $\mathsf{PG}(3+d, \mathbb{K})$ as the null set of

$$x_{-2}x_2 + x_{-1}x_1 = \sum_{i \in I} a_i y_i^2,$$

for some a_i with $i \in I$ forming a base of $\mathbb{K}'^2$ over $\mathbb{K}^2$. Since the projection must be injective, we may suppose that $\mathsf{PG}(n, \mathbb{K})$ is the subspace given by the equations $y_j = 0$ for all $j \in J$ for some $J \subseteq I$ and that the projection is from the subspace given by the equations $x_k = 0$ for all $k \in \{-2, -1, 1, 2\}$ and $y_i = 0$ for all $i \in I \backslash J$. Then $\mathsf{B}_{2,1}(\mathbb{K}, \mathbb{K}')$ is given in $\mathsf{PG}(n, \mathbb{K})$ as the following set of points:

$$\{(x_{-2}, x_{-1}, (y_i)_{i \in I \backslash J}, x_1, x_2) \mid \exists (z_j)_{j \in J} \in \mathbb{K} : \\ x_{-2}x_2 + x_{-1}x_1 + \sum_{i \in I \backslash J} a_i y_i^2 = \sum_{j \in J} a_j z_j^2\}.$$

Now the intersection of L and this set of points is the set

$$\left\{ \left(1, 0, 0, 0, \sum_{j \in J} a_j \lambda_j^2\right) \mid (\lambda_j)_{j \in J} \in \mathbb{K} \right\} \cup \{(0, 0, 0, 0, 1)\},$$

which forms clearly a mixed projective subline.

The last claim follows now easily, noting that this corresponds to the case $J = \varnothing$. □

Finally we note down the classification of non-hyperbolic non-embeddable polar spaces as give by Tits [20].

Lemma 2.27 (9.1 of [20]**).** *Every non-hyperbolic non-embeddable polar space of rank* 3 *is isomorphic to* $\mathsf{C}_{3,1}(\mathbb{O}, \mathbb{K})$ *for some field* $\mathbb{K}$ *and some octonion division algebra* $\mathbb{O}$ *over* $\mathbb{F}$ *.*

For completeness (but irrelevant for us) we mention that the same source also implies that hyperbolic non-embeddable polar spaces are the line Grassmannians of projective spaces of dimension 3 over a non-commutative skew field.

3. Veronese varieties

Since several arguments to prove our main results will involve Veronese varieties and Veronese representations of projectie planes, we review the main properties of these objects that we will use, and prove some new ones. A good reference is [15].

The reason why Veroneseans like that turn up is that, if a dual polar space is embedded universally in a projective space, then the point residual is a Veronese variety (see for example [8], or Remark 4.8).

Definition 3.1. Let $x \mapsto \overline{x}$ be the standard involution of $\mathbb{A}$ as a quadratic alternative division algebra over $\mathbb{K}$. Then $\mathbb{A}$ can be seen as a vector space over $\mathbb{K}$ and we can consider the projective space $\mathsf{PG}(V)$, with $V = \mathbb{K} \oplus \mathbb{K} \oplus \mathbb{K} \oplus \mathbb{A} \oplus \mathbb{A} \oplus \mathbb{A}$. Note $\dim V = 3\dim_{\mathbb{K}}(\mathbb{A}) + 2$. Consider the following points of $\mathsf{PG}(V)$, given by their coordinates:

$$\begin{cases} p(x,y) = (1, x\overline{x}, y\overline{y}; x\overline{y}, y, \overline{x}), & x, y \in \mathbb{A}; \\ p(y) = (0, 1, y\overline{y}; \overline{y}, 0, 0), & y \in \mathbb{A}; \\ p(\infty) = (0,0,1;0,0,0). \end{cases}$$

Then we call $\mathscr{V}_2(\mathbb{K},\mathbb{A}) = \{p(x,y) \mid x, y \in \mathbb{A}\} \cup \{p(y) \mid y \in \mathbb{A}\} \cup \{p(\infty)\}$ a *Veronese variety*, or briefly a *Veronesean* (associated to $\mathsf{PG}(2,\mathbb{A})$).

We now consider $p(x,y), p(y)$ and $p(\infty)$ as points of $\mathsf{PG}(2,\mathbb{A})$ with respective coordinates $(x,y,1), (1,y,0)$ and $(0,1,0)$. Then the lines are given as follows: the line $L_{m,q}$ with equation $y = mx + q$, $m, q \in \mathbb{A}$ consist of the points $p(x, mx+q)$ $(x \in \mathbb{A})$ and $p(m)$; the line L_a with equation $x = a$, $a \in \mathbb{A}$, consist of the points $p(a,y)$ $(y \in \mathbb{A})$ and $p(\infty)$; and the line L_∞ consists of the points $p(y)$ $(y \in \mathbb{A})$ together with $p(\infty)$. These lines then correspond to quadrics of Witt index 1 in the subspaces of $\mathsf{PG}(V)$ that they generate. We will refer to such quadrics as *the ovoids of the Veronesean*.

When $\mathbb{A}$ is associative, we can directly and homogeneously define the Veronese variety from the projective plane $\mathsf{PG}(2,\mathbb{A})$ as

$$\rho(x,y,z) = (x\overline{x}, y\overline{y}, z\overline{z}; y\overline{z}, z\overline{x}, x\overline{y}).$$

By Theorem 3.2 of [15], every translation of $\mathsf{PG}(2,\mathbb{A})$ is induced by a collineation of $\mathsf{PG}(V)$ stabilising $\mathscr{V}_2(\mathbb{K},\mathbb{A})$ (a *translation* of a projective plane is a collineation with fixed points precisely the set of points of some line). Considering the group generated by the translations, we see that the stabiliser in $\mathsf{PG}(V)$ of $\mathscr{V}_2(\mathbb{K},\mathbb{A})$ acts transitively on the set of ordered triples of non-collinear points, and, likewise, on the set of ordered triples of non-concurrent lines of $\mathsf{PG}(2,\mathbb{A})$ (in other words, on ordered triples of ovoids not having a common point).

Furthermore we will need the following results.

Lemma 3.2 (Theorem 3.3 of [15]**).** *Let O_1, O_2 be two ovoids of $\mathscr{V}_2(\mathbb{K},\mathbb{A})$ (embedded in $\mathsf{PG}(V)$ as above), for some alternative quadratic division algebra $\mathbb{A}$ over $\mathbb{K}$. Then the intersection $\langle O_1\rangle \cap \langle O_2\rangle$ is a point p of $\mathscr{V}_2(\mathbb{K},\mathbb{A})$.*

We will also use Main Result 4.3 of [12], which says the following.

Proposition 3.3 (Main Result 4.3 of [12]**).** *Let the point set of a projective plane Π be a set of points spanning a projective space $\mathsf{PG}(W)$, for some vector space W (not necessarily finite-dimensional), such that the lines of Π correspond to quadrics of Witt index 1 in subspaces of $\mathsf{PG}(W)$ of given uniform dimension d such that any two such subspaces intersect in a unique point (necessarily belonging to Π). Then Π is a Veronese variety and the dimension of W is $3d$, except possibly if $\Pi \cong \mathsf{PG}(2,2)$ (and necessarily $d = 2$), or if $\Pi \cong \mathsf{PG}(2,4)$ and $d = 3$.*

In [12] also the case $\Pi \cong \mathsf{PG}(2,q)$, $(q,d) = (2,2),(4,3)$, is considered. The arguments in [12, §7.3.1,§7.3.2] easily imply the following statements.

Lemma 3.4. *Let the point set of a projective plane $\Pi \cong \mathsf{PG}(2,q)$, $q = 2,4$, be a set of points of a projective space $\mathsf{PG}(n,2)$, for some $n \geqslant 2$, such that the lines of Π correspond to quadrics of Witt index 1 in subspaces of $\mathsf{PG}(n,2)$ of dimension 2 if $q = 2$, and of dimension 3 if $q = 4$, such that any two such subspaces intersect in a unique point (necessarily belonging to Π). Suppose also that the natural action of $\mathsf{PGL}_3(q)$ on Π is induced by $\mathsf{PGL}_{n+1}(2)$. Then either Π is a Veronese variety, or $q = 2$, $n = 6$ and Π is a set of seven points generating $\mathsf{PG}(6,2)$, or $q = 4$, $n = 10$ and there is a unique such example. In the last case, let, for each point $p \in \Pi$, T_p be the subspace generated by the tangent planes at p to the quadrics of Witt index 1 corresponding to the lines of $\mathsf{PG}(2,4)$ through p. Then, for each $p \in \Pi$, $\dim T_p = 6$, each triple of tangent planes at p to the quadrics of Witt index 1 through p generates T_p. The intersection of all these T_p is a line L the projection of Π from which is a Veronesean $\mathscr{V}_2(2,4)$ and the stabilisator of Π in $\mathsf{PG}(10,4)$ acts transitively on L.*

The next result probably belongs to folklore, but we provide a proof for completeness.

Lemma 3.5. *Let C be a plane conic completely contained in $\mathscr{V}_2(\mathbb{K},\mathbb{A})$, for some alternative quadratic division algebra $\mathbb{A}$ over $\mathbb{K}$, and suppose $|\mathbb{K}| > 2$. Then C is contained in an ovoid of $\mathscr{V}_2(\mathbb{K},\mathbb{A})$.*

Proof. By the properties of the full collineation group of $\mathsf{PG}(V)$ stabilising $\mathscr{V}_2(\mathbb{K},\mathbb{A})$ mentioned above, we may assume that, if three points of C are not contained in a common ovoid of $\mathscr{V}_2(\mathbb{K},\mathbb{A})$, then they have respective coordinates $(1,0,0;0,0,0)$, $(0,1,0;0,0,0)$ and $(0,0,1;0,0,0)$. Clearly, no nontrivial linear combination of these points, except for the points themselves, belongs to $\mathscr{V}_2(\mathbb{K},\mathbb{A})$, as $\mathbb{A}$ has no zerodivisors. □

A *skeleton* of a projective plane is an ordered quadruple of points such that no three are on a line.

Lemma 3.6. *Let C_1, C_2 be two intersecting conics contained in distinct ovoids of $\mathscr{V}_2(\mathbb{K},\mathbb{A})$. Then there exists a unique Veronese variety that is isomorphic to $\mathscr{V}_2(\mathbb{K},\mathbb{K})$ containing both C_1 and C_2 and itself contained in $\mathscr{V}_2(\mathbb{K},\mathbb{A})$.*

Proof. Since the collineation group of $\mathsf{PG}(2,\mathbb{A})$ acts transitively on skeletons of $\mathsf{PG}(2,\mathbb{A})$ (see Satz 4.1.2 and 7.3.14 of [19]), we may assume that C_1 contains the points $a_1 = (1,0,0;0,0,0)$, $a_2 = (0,1,0;0,0,0)$ and $a_{12}(1,1,0;0,0,1)$, and C_2 contains the points $a_1 = (1,0,0;0,0,0)$, $a_3 = (0,0,1;0,0,0)$ and $a_{13} = (1,0,1;0,1,0)$. If $\mathscr{V}_2$ is a Veronese variety isomorphic to $\mathscr{V}_2(\mathbb{K},\mathbb{K})$ containing $C_1 \cup C_2$, then it should also contain a conic C through a_2 and a_3, and a conic C' through a_{12} and a_{13}. Moreover, the conics C and C' should meet in a point. Since conics lie on unique ovoids, the intersection point p of C and C' is the intersection of the ovoid O of $\mathscr{V}_2(\mathbb{K},\mathbb{A})$ through a_2 and a_3 with the ovoid

O' through a_{12} and a_{13}. Since, with above notation, O corresponds to the line L_∞ and O' to the line $L_{-1,1}$ of $\mathsf{PG}(2,\mathbb{A})$, it follows that $p = p(-1)$ and hence has coordinates $(0,1,1;-1,0,0)$. Since the six points $a_1, a_2, a_3, a_{12}, , a_{13}$ and $p(-1)$ uniquely define the six-dimensional subspace $\mathbb{K}\oplus\mathbb{K}\oplus\mathbb{K}\oplus\mathbb{K}\oplus\mathbb{K}\oplus\mathbb{K}$ of V (with each $\mathbb{K}$ of the last three components the natural inclusion of $\mathbb{K}$ in $\mathbb{A}$), the lemma is proved. □

In the case $\mathscr{V}_2(\mathbb{K},\mathbb{K})$, the ovoids are plane conics. When $\mathsf{char}\,\mathbb{K} = 2$, then each conic has a *nucleus*, which is, geometrically, the intersection of all tangent lines, or, algebraically, the radical of the bilinear form associated to the quadratic form defining the conic. Now we note the following, which is known in the finite case, but we need it in full generality:

Lemma 3.7. *Assume* $\mathsf{char}\,\mathbb{K} = 2$. *The set of nuclei of conics of* $\mathscr{V}_2(\mathbb{K},\mathbb{K})$ *is the plane consisting of the points* $(0,0,0;a,b,c)$, $a,b,c \in \mathbb{K}$, *not all zero.*

Proof. The lemma is proved if we show that the point $(0,0,0;a,b,c)$ is the nucleus of the conic C determined by the image of the line of $\mathsf{PG}(2,\mathbb{K})$ with equation $ax + by + cz = 0$. Without loss of generality, we may assume $c = 1$. Then an arbitrary point of the conic is $(x^2, y^2, a^2x^2 + b^2y^2, axy + by^2, ax^2 + bxy, xy)$, and all points lie in the plane α given by the equations

$$\begin{cases} aX_0 + X_4 + bX_5 = 0, \\ bX_1 + X_3 + aX_5 = 0, \\ X_2 + bX_3 + aX_4 = 0, \end{cases}$$

which is clearly disjoint from the plane π with equations $X_0 = X_1 = X_5 = 0$. Projecting α from π onto the plane β with equations $X_2 = X_3 = X_4 = 0$, we obtain as projection of C the conic with equations $X_0X_1 + X_5^2 = X_2 = X_3 = X_4 = 0$, which has nucleus $n' = (0,0,0;0,0,1)$. The projection of n' from π onto α is $(0,0,0;a,b,1)$, what we had to prove. □

We will refer to this plane as the *nucleus plane* of $\mathscr{V}_2(\mathbb{K},\mathbb{K})$.

Lemma 3.8. *Let* $\mathscr{V}_2(\mathbb{K},\mathbb{K})$ *be the quadric Veronesean naturally embedded in* $\mathsf{PG}(5,\mathbb{K})$, *and given by the points with coordinates* $(X^2, Y^2, Z^2, YZ, ZX, XY)$, $X, Y, Z \in \mathbb{K}$. *Then every point of* $\mathsf{PG}(5,\mathbb{K})$ *not contained in the nucleus plane if* $\mathsf{char}\,\mathbb{K} = 2$, *is equivalent to some point with last three coordinates zero.*

Proof. First note that the nucleus plane is the plane consisting of the points with coordinates $(0,0,0,a,b,c)$ with $a,b,c \in \mathbb{K}$. Each such point is the nucleus of a conic on the Veronesean, in particular $(0,0,0,a,b,c)$ is the nucleus of the conic that comes from the line $aX + bY + cZ = 0$ in $\mathsf{PG}(2,\mathbb{K})$.

A linear transformation of $\mathsf{PG}(2,\mathbb{K})$ with matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

and seen as an automorphism of $\mathscr{V}_2(\mathbb{K},\mathbb{K})$, extends to (or is induced by) a linear transformation of $\mathsf{PG}(5,\mathbb{K})$ with matrix

$$\begin{pmatrix} a_{11}^2 & a_{12}^2 & a_{13}^2 & 2a_{12}a_{13} & 2a_{11}a_{13} & 2a_{11}a_{12} \\ a_{21}^2 & a_{22}^2 & a_{23}^2 & 2a_{22}a_{23} & 2a_{21}a_{23} & 2a_{21}a_{22} \\ a_{31}^2 & a_{32}^2 & a_{33}^2 & 2a_{32}a_{33} & 2a_{31}a_{33} & 2a_{31}a_{32} \\ a_{21}a_{31} & a_{22}a_{32} & a_{23}a_{33} & a_{22}a_{33}+a_{23}a_{32} & a_{21}a_{33}+a_{23}a_{31} & a_{21}a_{32}+a_{22}a_{31} \\ a_{11}a_{31} & a_{12}a_{32} & a_{13}a_{33} & a_{12}a_{33}+a_{13}a_{32} & a_{11}a_{33}+a_{13}a_{31} & a_{11}a_{32}+a_{12}a_{31} \\ a_{11}a_{21} & a_{12}a_{22} & a_{13}a_{23} & a_{22}a_{13}+a_{23}a_{12} & a_{21}a_{13}+a_{23}a_{11} & a_{21}a_{12}+a_{22}a_{11} \end{pmatrix}$$

Let $p=(A,B,C,D,E,F)$ be an arbitrary point of $\mathsf{PG}(5,\mathbb{K})$, not in the nucleus plane if $\mathsf{char}\,\mathbb{K}=2$. Clearly, if $\mathsf{char}\,\mathbb{K}=2$, one of A,B,C is nonzero. We can also assume this if $\mathsf{char}\,\mathbb{K}\neq 2$, since if then $A=B=C=0$ at least one of D,E,F must be nonzero. Let for example $D\neq 0$, then the linear transformation inducing

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

maps p to (A',B',C',D',E',F') with C' nonzero. So we may suppose without loss of generality in both cases that $C\neq 0$.

If $p\in\mathscr{V}_2(\mathbb{K},\mathbb{K})$, then regardless of the characteristic of $\mathbb{K}$, the transformation defined by

$$\begin{pmatrix} 1 & 0 & -\frac{E}{C} \\ 0 & 1 & -\frac{D}{C} \\ 0 & 0 & 1 \end{pmatrix}$$

maps p to a point with last three coordinates zero.

So from now on we assume that $p\notin\mathscr{V}_2(\mathbb{K},\mathbb{K})$. Suppose first that $BC-D^2\neq 0$. Now one calculates that the transformation defined by

$$\begin{pmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{pmatrix} \text{ with } \begin{cases} a_{23} = -\frac{D}{C}, \\ a_{12} = \frac{DE-CF}{BC-D^2}, \\ a_{13} = \frac{DF-BE}{BC-D^2} \end{cases}$$

maps p to a point with last three coordinates zero.

Let now $\mathsf{char}\,\mathbb{K}\neq 2$. We claim that, then we may always assume that $BC-D^2\neq 0$ and the above transformation does the job. Indeed, suppose not. By cyclic permutation, we then have, besides $BC-D^2=0$, also $AC-E^2=0$. If also $CF-DE=0$, then $p\in\mathscr{V}_2(\mathbb{K})$, hence $CF-DE\neq 0$. Then the linear transformation inducing

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

maps p to $(A'',B'',C'',D'',E'',F'')=(A,A+B+2F,C,D+E,E,A+F)$ with $B''C''-D''^2=AC+BC+2CF-D^2-E^2-2DE=(AC-E^2)+(BC-D^2)+2(CF-DE)\neq 0$. Hence the claim.

So let now $\mathsf{char}\,\mathbb{K} = 2$. Then, similarly as above, the assertion follows if $BC - D^2 \neq 0$ or $AC - E^2 \neq 0$. Hence suppose $BC = D^2$ and $AC = E^2$. Then $AB \neq F^2$, as otherwise $p \in \mathscr{V}_2(\mathbb{K})$. By cyclic permutation, we may therefore assume $A = B = 0$. Then we have the point $(0, 0, C, 0, 0, F)$, which is mapped onto a point with coordinates $(*, *, *, 0, 0, 0)$ by the linear transformation defined by

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{F}{C} \\ 1 & \frac{C}{F} & 1 \end{pmatrix}.$$

The lemma is proved. □

Lemma 3.9. *Let $\mathbb{K}$ be a field distinct from $\mathbb{F}_2$ and $\mathbb{F}_3$. Then no proper injective projection from a subspace not contained in the nucleus plane of the quadric Veronesean $\mathscr{V}_2(\mathbb{K})$ is contained in a (proper) quadric.*

Proof. Suppose for a contradiction that a proper projection of $\mathscr{V}_2(\mathbb{K})$ (from a subspace not contained in the nucleus plane) is contained in a quadric. Then $\mathscr{V}_2(\mathbb{K})$ is contained in a degenerate quadric Q with at least one vertex v not contained in the nucleus plane of $\mathscr{V}_2(\mathbb{K})$. The point v is not contained in any secant line of $\mathscr{V}_2(\mathbb{K})$. We now determine all quadrics that contain $\mathscr{V}_2(\mathbb{K})$.

We write a generic point of $\mathscr{V}_2(\mathbb{K})$ in coordinates as $(X^2, Y^2, Z^2, YZ, ZX, XY)$. Let

$$\sum_{0 \leqslant i < j \leqslant 5} a_{ij} x_i x_j$$

be an equation of a generic quadric containing $\mathscr{V}_2(\mathbb{K})$. Then, for all $X, Y, Z \in \mathbb{K}$, we have

$$\begin{aligned} a_{00}X^4 + (a_{05}Y + a_{04}Z)X^3 + ((a_{01} + a_{55})Y^2 + (a_{03} + a_{45})YZ + (a_{02} + a_{44})Z^2)X^2 + \\ + (a_{15}Y^3 + (a_{14} + a_{35})Y^2Z + (a_{34} + a_{25})YZ^2 + a_{24}Z^3)X + \\ + a_{11}Y^4 + a_{13}Y^3Z + (a_{12} + a_{33})Y^2Z^2 + a_{23}YZ^3 + a_{22}Z^4 = 0 \end{aligned}$$

Setting $Y = Z = 0$, we obtain $a_{00} = 0$ and similarly $a_{11} = a_{22} = 0$. Fixing arbitrary Y and Z, we get a cubic equation that must admit all field elements X as roots. Since the field has at least four elements, all coefficients of the cubic have to be zero. Hence, we have the following identities in Y, Z.

$$\begin{cases} a_{05}Y + a_{04}Z = 0, \\ (a_{01} + a_{55})Y^2 + (a_{03} + a_{45})YZ + (a_{02} + a_{44})Z^2 = 0, \\ a_{15}Y^3 + (a_{14} + a_{35})Y^2Z + (a_{25} + a_{34})YZ^2 + a_{24}Z^3 = 0, \\ a_{13}Y^3Z + (a_{12} + a_{33})Y^2Z^2 + a_{23}YZ^3 - 0. \end{cases}$$

The first identity clearly yields $a_{05} = a_{04} = 0$. The second one yields (putting $Y = 0$ and Z arbitrary, and $Z = 0$ and Y arbitrary)

$$\begin{cases} a_{01} + a_{55} = 0, \\ a_{02} + a_{44} = 0, \\ a_{03} + a_{45} = 0. \end{cases}$$

Cyclic permutation $X \to Y \to Z \to X$ quickly yields $a_{15} = a_{13} = a_{24} = a_{23} = 0$, and then it is easy to see that the third and fourth identities above yield

$$\begin{cases} a_{14} + a_{35} = 0, \\ a_{25} + a_{34} = 0, \\ a_{12} + a_{33} = 0. \end{cases}$$

Hence a generic quadric containing $\mathscr{V}_2(\mathbb{K})$ has equation

$$A(x_1x_2 - x_3^2) + B(x_0x_2 - x_4^2) + C(x_0x_1 - x_5^2) \\ +D(x_0x_3 - x_4x_5) + E(x_1x_4 - x_3x_5) + F(x_2x_5 - x_3x_4) = 0.$$

Hence we may assume that this is the equation of Q, for some constants A, B, C, D, E, F.

Since almost every point of the projective space is equivalent to some point with coordinates $(*, *, *, 0, 0, 0)$ by Lemma 3.8, we may assume that v has coordinates $(k, \ell, m, 0, 0, 0)$, with $k, \ell, m \in \mathbb{K}\backslash\{0\}$ as otherwise v is contained in the plane of a conic of $\mathscr{V}_2(\mathbb{K})$, contradicting the assumption that v is not a nucleus and the embedding is injective. The fact that v is a vertex is equivalent to v lying on Q and the tangent space at v being the whole projective space. Since an equation of the tangent space at v is

$$(C\ell + Bm)x_0 + (Ck + Am)x_1 + (Bk + A\ell)x_2 + Dkx_3 + E\ell x_4 + Fmx_5 = 0,$$

we have the identities

$$\begin{cases} \frac{A}{k} + \frac{B}{\ell} + \frac{C}{m} = 0, \\ \frac{B}{\ell} + \frac{C}{m} = 0, \\ \frac{A}{k} + \frac{C}{m} = 0, \\ \frac{A}{k} + \frac{B}{\ell} = 0, \\ D = E = F = 0, \end{cases}$$

which implies $A = B = C = D = E = F = 0$, a contradiction. □

4. Isometricity and locality

First we take a closer look at trace geometries in metasymplectic spaces. One easily observes the following for these geometries.

Lemma 4.1. *Let Δ be a metasymplectic space. Then any trace geometry is isomorphic to a dual polar space of rank 3 and is isometrically embedded. In particular, if $\Delta \cong \mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$ then the trace geometry is isomorphic to $\mathsf{C}_{3,3}(\mathbb{A}, \mathbb{K})$, and if $\Delta \cong \mathsf{F}_{4,4}(\mathbb{K}, \mathbb{A})$ then the trace geometry is isomorphic to $\mathsf{B}_{3,3}(\mathbb{K}, \mathbb{A})$.*

Proof. The fact that the trace geometries are of the described form follows immediately from the observation that a trace geometry is always isomorphic to a point residual. The isometricity follows from the fact that, if two points of a trace geometry $p^\perp \cap q^{\not\equiv}$ (for some opposite points p and q of Δ) are collinear in Δ, they must be contained in a plane with p and consequently the line containing them is the intersection of $q^{\not\equiv}$ with this plane. □

In the rest of the paper we will try to prove that in general the converse of this lemma is true. In this section we will prove that the embedding of a dual polar space of rank three in a metasymplectic space is in general isometric and contained in the perp of a point of the metasymplectic space. The fact that they are also (exactly) the points not opposite another point, will be proved in the next sections (and it will turn out not to be true in all cases).

4.1. Isometricity

First we start by proving an equivalent definition of "isometric" for dual polar spaces fully embedded in metasymplectic spaces. It is actually intuitively closer to what one expects from something "isometric".

Lemma 4.2. *The embedding of a dual polar space Ω in a metasymplectic space Δ is isometric if, and only if,*

- `[Col]` *two points collinear in Ω are also collinear in Δ;*
- `[Sym]` *two points at distance two in Ω are symplectic in Δ;*
- `[Spe]` *two points at distance three in Ω are special in Δ.*

Proof. If the embedding has the mentioned properties, then it is clearly isometric (points of Ω collinear in Δ are also collinear in Ω and hence joined by a line).

Now suppose the embedding of Ω in Δ is isometric. Clearly, `[Col]` holds. Also, two points at distance two in Ω must be symplectic or special in Δ. The latter is impossible since there are no special pairs of points in a dual polar space of rank three. This implies that `[Sym]` holds. Finally, let p, q be two points at distance 3 in Ω and let x, y be points of Ω such that $p \perp x \perp y \perp q$ (in Ω). If $q \in \xi(p, y)$, then q is collinear in Δ to a point z of $\langle p, x\rangle \subseteq \Omega$. This produces a line $\langle q, z\rangle$ in Δ through two points of Ω which does not belong to Ω, a contradiction. So $q \notin \xi(p, y)$ and then by Proposition 2.12 [Point-Symp] the point q is special to p, proving `[Spe]`. □

The following lemma is an immediate consequence of Lemma 2.22. Since we will however use this often, we formulate (and prove) this here.

Lemma 4.3. *Let Ω be a dual polar space of rank 3 fully embedded in a metasymplectic space Δ. Let ξ be a symplecton of Ω. Then ξ is isometrically embedded in a unique symplecton $\xi^{\blacktriangle}$ of Δ.*

Proof. By Lemma 2.22, ξ must be isometrically embedded in a symplecton $\xi^{\blacktriangle}$ of Δ, since ξ does contain two disjoint lines while a maximal singular subspace of Δ, i.e. a projective plane, does not. Clearly this symplecton is unique, since it is determined by two non-collinear points. □

From now on, we will often use this $\blacktriangle$-notation $\xi^{\blacktriangle}$ for the symplecton of Δ containing the symplecton ξ of Ω. Now we are in a position to prove the general result of this paragraph.

Proposition 4.4. *Let Ω be a dual polar space of rank 3, fully embedded in a metasymplectic space. Then the embedding is either contained in a symplecton, or is isometric.*

Proof. By Lemma 4.3 we immediately have that two collinear (resp. symplectic) points in Ω are also collinear (resp. symplectic) in Δ. Let now a and d be two opposite points in Ω. Then there exists a path $a \perp b \perp c \perp d$ in Ω. By Proposition 2.12 [Point-Symp], a is contained in or close to the symplecton $\xi(b,d)$. In the latter case we get immediately that a is special to d.

We claim that, in the former case, Ω is contained in $\xi(b,d)$. Remark that we may assume that a is symplectic to d as all the points of $ab\backslash\{b\}$ are opposite d in Ω and at most one of them could be collinear to d in Δ. So we may assume that $\xi(b,d) = \xi(a,d)$. Every point in Ω collinear to a (or to d) is contained in this symplecton by the following reasoning. Every line of Ω through a (or through d) contains a point b' at distance 2 from d (or from a) in Ω and this point must then be symplectic to d (or to a) in Δ. But a (or d) must then by the above argument also be contained in $\xi(b',d)$ (or $\xi(b',a)$) and consequently $\xi(b',d) = \xi(a,d) = \xi(b,d)$. The claim is now proved using a connectivity argument and the fact that for every point $a' \in \Omega$ collinear to a (in Ω), there exists a point $d' \in \Omega$ collinear to d (in Ω), such that a' and d' are opposite in Ω and not collinear in Δ.

Note now that if our chosen a and d were special, we get that this must be the case for every pair of opposite points in Ω by the previous paragraph. So the embedding is isometric. □

In the rest of this subsection, we discuss when both situations (contained in a symp and isometric) can occur. It will follow that in the separable $\mathsf{F}_{4,4}$-case the embedding must be isometric, while in the inseparable case there are examples of embeddings in symplecta. In the separable $\mathsf{F}_{4,1}$-case, it is not clear if there could be embeddings in a symplecton. However those could never be isomorphic to a point residual, while they can be in the inseparable case. We start by studying the separable $\mathsf{F}_{4,4}$-case.

Lemma 4.5. *Let Ω be a dual polar space of rank 3, fully embedded in a separable metasymplectic space $\Delta \cong \mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$. Then every symplecton ξ of Ω arises as the common perp of two opposite points in a symplecton $\xi^{\blacktriangle}$ of Δ.*

Proof. First remark that ξ is isometrically embedded in a symplecton $\xi^{\blacktriangle}$ of Δ by Lemma 4.3.

Suppose now that $\mathbb{A}$ is not an octonion division algebra over $\mathbb{K}$. Let $\xi^{\blacktriangle} \cong \mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$ be universally (by Lemma 2.23) embedded in $\mathsf{PG}(5,\mathbb{A})$ as in Example 2.8 (ii) and denote with ρ the defining polarity. By the Main Theorem of [3], ξ is the intersection of the polar space $\xi^{\blacktriangle}$ with a subspace U of $\mathsf{PG}(5,\mathbb{A})$. As ξ contains two opposite lines, but no planes, the dimension of U is 3 or 4.

We claim that the dimension of U is 3. Suppose first that the polar space Ω^* dual to Ω is not embeddable. From Lemma 2.27 we infer that $\Omega^* \cong \mathsf{C}_{3,1}(\mathbb{O},\mathbb{F})$ (and consequently $\Omega \cong \mathsf{C}_{3,3}(\mathbb{O},\mathbb{F})$) for some field $\mathbb{F}$ and some octonion division algebra $\mathbb{O}$ over $\mathbb{F}$. Then $\xi \cong \mathsf{C}_{2,2}(\mathbb{O},\mathbb{F}) \cong \mathsf{B}_{2,1}(\mathbb{F},\mathbb{O})$, which has a unique embedding in $\mathsf{PG}(11,\mathbb{F})$ by Lemma 2.23 (i) and is consequently not embeddable in $\mathsf{PG}(3,\mathbb{A})$ or $\mathsf{PG}(4,\mathbb{A})$, a contradiction. So Ω^* must be embeddable and consequently also the dual ξ^* of ξ is embeddable. With Proposition 10.10 of [20] we then get that ξ (and ξ^*) is of the form $\mathsf{B}_{2,1}(\mathbb{F},\mathbb{B})$ or $\mathsf{C}_{2,1}(\mathbb{B},\mathbb{F})$, with $\mathbb{F}$ a field and $\mathbb{B}$ a quadratic associative division algebra over $\mathbb{F}$. Suppose first that $\xi \cong \mathsf{C}_{2,1}(\mathbb{B},\mathbb{F})$ and $\mathbb{B}$ is not an inseparable multiple quadratic extension over $\mathbb{F}$ if $\mathsf{char}(\mathbb{F}) = 2$ (including $\mathbb{F} = \mathbb{B}$), then U has dimension 3, by Lemma 2.23 (ii). So it suffices to exclude that ξ is isomorphic to $\mathsf{B}_{2,1}(\mathbb{F},\mathbb{B})$ for some field $\mathbb{F}$ and some quadratic associative division algebra $\mathbb{B}$ (taking Remark 2.14 into account). Suppose now for a contradiction that ξ is of this form. Then ξ has a universal embedding in $\mathsf{PG}(3+n,\mathbb{F})$ with $n = \dim_{\mathbb{F}}(\mathbb{B})$ and must also be embedded in $\mathsf{PG}(5,\mathbb{A})$. So we get that $\mathbb{A} = \mathbb{F}$ must be a field, so either $\mathbb{A} = \mathbb{K}$ and $\mathsf{char}(\mathbb{K}) \neq 2$ or $\mathbb{A} = \mathbb{L}$ is a separable quadratic field extension of $\mathbb{K}$. Recall that the hyperbolic line of $\mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$ through two opposite points x and y is the intersection of the projective line $\langle x, y\rangle$ of $\mathsf{PG}(5,\mathbb{A})$ with $\mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$ (see Lemma 2.25 (iv)). First suppose that the induced embedding of ξ in the subspace X of $\mathsf{PG}(5,\mathbb{A})$ that it spans is the universal one, as given in Example 2.8 (i). Then the projective line through two opposite points x and y of ξ intersects ξ in only those two points, a contradiction since ξ arises as the intersection of X with $\xi^\blacktriangle$ by [3]. So we may suppose that ξ is not universally embedded in X. This implies by Lemma 2.23 (i) that $\mathsf{char}(\mathbb{F}) = 2$, $\mathbb{B}$ is an inseparable multiple quadratic extension over $\mathbb{F}$ (including $\mathbb{F} = \mathbb{B}$) and $\mathbb{A} = \mathbb{F} = \mathbb{L}$ is a separable quadratic field extension of $\mathbb{K}$. Let now L be a secant to ξ in this embedding. Then L intersects ξ in a mixed projective subline of at least three points over $\mathbb{L}^2$ by Lemma 2.26. However, by Remark 2.4, L intersects $\xi^\blacktriangle$ in a projective subline over $\mathbb{K}$. This yields that $\mathbb{L}^2 \leqslant \mathbb{K}$, contradicting Remark 2.5.

So U is spanned by two opposite lines L and M of ξ. Let now π, π' be two (locally opposite) planes through L, then each of these planes contains a unique point (p, p' respectively) collinear to M. Applying ρ to the projective line $\langle p, p'\rangle$ we obtain a three-dimensional subspace of $\mathsf{PG}(5,\mathbb{A})$ containing the span of L and M. Consequently $\xi = p^\perp \cap p'^\perp$ in $\xi^\blacktriangle$.

Suppose now that $\mathbb{A}$ is an octonion division algebra, then we can apply Main Result 1 from [17], which gives us that ξ is the common perp of two points of $\xi^\blacktriangle$. □

Proposition 4.6. *Let Ω be a dual polar space of rank 3, fully embedded in a separable metasymplectic space $\Delta \cong \mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$. Then the embedding is isometric.*

Proof. By Proposition 4.4 it suffices to prove that Ω cannot be contained in a symplecton. Suppose for a contradiction that Ω is contained in a symplecton $\xi^\blacktriangle$ of Δ. Let p be a point of Ω. The point residual $\mathsf{Res}_{\xi^\blacktriangle}(p)$ of p in $\xi^\blacktriangle$ is a generalised quadrangle $\mathsf{C}_{2,1}(\mathbb{A},\mathbb{K})$ (where points correspond to lines of $\xi^\blacktriangle$ through p and lines correspond to planes of $\xi^\blacktriangle$ through p). However the point residual of p in Ω, notation $\mathsf{Res}_\Omega(p)$, is a projective plane (where points correspond to lines of Ω through p and lines correspond to symplecta of Ω through p). We interpret now the lines of $\mathsf{Res}_\Omega(p)$ in $\mathsf{Res}_{\xi^\blacktriangle}(p)$. Let ζ be a symplecton of Ω through p. Then by Lemma 4.5, ζ is the common perp of two points a, b in $\xi^\blacktriangle \setminus \Omega$. These points determine lines pa and pb and we denote by a', b', respectively, the corresponding points in $\mathsf{Res}_{\xi^\blacktriangle}(p)$. Now the symplecton ζ seen as line in $\mathsf{Res}_\Omega(p)$ corresponds to all points of $\mathsf{Res}_{\xi^\blacktriangle}(p)$ collinear to a' and b' in $\mathsf{Res}_{\xi^\blacktriangle}(p)$.

We now dualise $\mathsf{Res}_{\xi^\blacktriangle}(p)$. This is an orthogonal generalised quadrangle $Q \cong \mathsf{B}_{2,1}(\mathbb{K}, \mathbb{A})$, which has a unique (by Lemma 2.23 (i)) embedding as a quadric in $\Pi = \mathsf{PG}(3+n, \mathbb{K})$ with $n = \dim_{\mathbb{K}}(\mathbb{A})$ (described in Example 2.8(i)).

The points of $\mathsf{Res}_\Omega(p)$ are certain lines on Q. Note that two such disjoint lines, say L, M, span a three-dimensional subspace S of Π, which intersects the quadric necessarily in a hyperbolic quadric, i.e. a grid, since this is the only non-degenerate quadric of Witt index 2 embedded in three dimensions. By the above interpretation of the lines of $\mathsf{Res}_\Omega(p)$, the regulus of this hyperbolic quadric containing L and M, notation $\mathscr{R}(L, M)$, corresponds to a line of $\mathsf{Res}_\Omega(p)$. Note further that every projective line in S intersecting all lines of the opposite regulus of $\mathscr{R}(L, M)$ is itself a line of $\mathscr{R}(L, M)$.

Now we express that the lines and reguli corresponding to the points and lines of $\mathsf{Res}_\Omega(p)$ form a projective plane, since $\mathsf{Res}_\Omega(p)$ is a projective plane. Take three such "non-collinear" lines X, Y, Z, i.e. three disjoint lines on Q (corresponding to points of $\mathsf{Res}_\Omega(p)$) not contained in the same regulus. These must span the projective plane, so this plane is contained in the five-dimensional subspace T of Π spanned by these three lines. The latter can not be four-dimensional since then Z must intersect the subspace $\langle X, Y \rangle$ in a point (necessarily on the quadric) of a line U of $\mathscr{R}(X, Y)$. This would contradict the fact that there must be a regulus through the two lines Z, U of the projective plane. Let $y \in Y$ be a point and denote by x, z the respective unique points on X, Z collinear to y. We now claim that x must be collinear to z, contradicting the fact that Q does not contain planes.

Therefore let Y' be a line on $\mathscr{R}(X, Z)$ different from X and Z and let Z' vary over $\mathscr{R}(X, Y)$. Then $\langle Y', Z' \rangle$ has dimension 3 and must intersect $\langle Y, Z \rangle$ in a line X' of $\mathscr{R}(Y, Z)$, since every two lines in a projective plane intersect. Now also the three-dimensional space $\langle xy, Y' \rangle$ must intersect $\langle Y, Z \rangle$ in a line and by the above intersections, this line must intersect each line of the regulus $\mathscr{R}(Y, Z)$. So this must be a line of the opposite regulus. Since y is clearly contained in this intersection, this intersection is the line yz. Then the plane $\langle x, Y' \rangle$ intersects $\langle Y, Z \rangle$ in the point z. Since this plane intersects Q at least in the line Y', it must be a tangent plane. Since the points x and z are not contained in Y', they must be contained in the other line of Q in this tangent plane and are consequently collinear. □

Now we'll take a closer look at the $\mathsf{F}_{4,1}$-case.

Lemma 4.7. *Let Ω be a dual polar space of rank 3, fully embedded in a metasymplectic space $\Delta \cong \mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$. Then $\Omega \cong \mathsf{C}_{3,3}(\mathbb{B}, \mathbb{K})$ for some quadratic alternative division algebra $\mathbb{B}$ over $\mathbb{K}$.*

Proof. Let Ω^* be the dual of Ω. If Ω^* is non-embeddable, it must be $\mathsf{C}_{3,1}(\mathbb{O},\mathbb{K})$ with $\mathbb{O}$ an octonion division algebra over $\mathbb{K}$ by Lemma 2.27.

So we may from now on assume that Ω^* is embeddable. Let ξ be a symplecton of Ω, then ξ is embeddable as it must, by Lemma 4.3, be isometrically embedded in a symplecton of Δ which is isomorphic to the embeddable polar space $\mathsf{B}_{3,1}(\mathbb{K},\mathbb{A})$. Since Ω^* is also embeddable, also the dual ξ^* of ξ must be embeddable. With Proposition 10.10 of [20] we then get that ξ (and ξ^*) is of the form $\mathsf{B}_{2,1}(\mathbb{F},\mathbb{B})$ or $\mathsf{C}_{2,1}(\mathbb{B},\mathbb{F})$, with $\mathbb{F}$ a field and $\mathbb{B}$ a quadratic associative division algebra over $\mathbb{F}$. If $\mathbb{B}$ is not an inseparable field extension, we claim that the latter is impossible. Suppose for a contradiction that $\mathsf{C}_{2,1}(\mathbb{B},\mathbb{F})$ is isometrically embedded in $\mathsf{B}_{3,1}(\mathbb{K},\mathbb{A})$, which is universally (by Lemma 2.23) embedded in $\mathsf{PG}(5,\mathbb{A})$ as described in Example 2.8 (i). By Theorem 1 of [3], $\mathsf{C}_{2,1}(\mathbb{B},\mathbb{F})$ then arises as an intersection of $\mathsf{B}_{3,1}(\mathbb{K},\mathbb{A})$ with a subspace U of $\mathsf{PG}(5,\mathbb{A})$. However, this induced embedding of $\mathsf{C}_{2,1}(\mathbb{B},\mathbb{F})$ in U must be as described in Example 2.8 (ii) by Lemma 2.23 (ii). This contradicts the fact that in these embeddings secants intersect $\mathsf{C}_{2,1}(\mathbb{B},\mathbb{F})$ in more than two points, while they intersect $\mathsf{B}_{3,1}(\mathbb{K},\mathbb{A})$ in only two points. So ξ is isomorphic to $\mathsf{B}_{2,1}(\mathbb{F},\mathbb{B})$ for some field $\mathbb{F}$ and some quadratic associative division algebra $\mathbb{B}$ (possibly inseparable by Remark 2.14). Then one concludes that $\mathbb{F}=\mathbb{K}$ by looking at the universal embeddings (Lemma 2.23) of $\mathsf{B}_{3,1}(\mathbb{K},\mathbb{A})$ and $\mathsf{B}_{2,1}(\mathbb{F},\mathbb{B})$ and consequently Ω is isomorphic to $\mathsf{C}_{3,3}(\mathbb{B},\mathbb{K})$. □

An explicit description of the dual polar spaces of rank 3 isomorphic to $\mathsf{C}_{3,3}(\mathbb{B},\mathbb{K})$, with $\mathbb{B}$ an alternative quadratic division algebra over $\mathbb{K}$, via an embedding in a projective space $\mathsf{PG}(V)$, can be found in [8]; we will call this embedding the *standard embedding.* Explicitly, using the version of [11, Definition 10.1], set

$$V=\mathbb{K}\oplus\mathbb{K}\oplus\mathbb{K}\oplus\mathbb{K}\oplus\mathbb{B}\oplus\mathbb{B}\oplus\mathbb{B}\oplus\mathbb{K}\oplus\mathbb{K}\oplus\mathbb{K}\oplus\mathbb{B}\oplus\mathbb{B}\oplus\mathbb{B}\oplus\mathbb{K},$$

and, for $|\mathbb{K}|>2$, define the embedding parametric as the projective closure (that is, the smallest overset of points the complement of which intersects no lne in exactly one point) of the set of all points

$$(1,k,\ell,m,x,y,z,x\overline{x}-\ell m,y\overline{y}-km,z\overline{z}-k\ell,k\overline{x}-yz,\ell\overline{y}-zx,m\overline{z}-xy,$$
$$kx\overline{x}+\ell y\overline{y}+mz\overline{z}-(xy)z-\overline{z}(\overline{y}\,\overline{x})-k\ell m,$$

with $k,\ell,m\in\mathbb{K}$ and $x,y,z\in\mathbb{B}$, and $x\mapsto\overline{x}$ is the standard involution in $\mathbb{B}$. If $|\mathbb{K}|=2$, one has first to consider a suitable overfield, or explicitly enumerate the other points, as done in [8] (however, we will not need this).

It is shown in [8] that this embedding is always homogeneous and even universal if $\mathbb{K}\neq\mathbb{F}_2$.

Remark 4.8. Taking $p=(1,0,\dots,0)$ and $q=(0,0,\dots,0,1)$, the set $p^\perp\cap q^{\perp\!\!\!\perp}$ is given by

$$\{(0,k,\ell,m,x,y,z,0,\dots,0)\ |x\overline{x}-\ell m=y\overline{y}-km=z\overline{z}-k\ell=$$
$$k\overline{x}-yz=\ell\overline{y}-zx=m\overline{z}-xy=0\}.$$

If $k \neq 0$, we can set $k = 1$; if $k = 0$ and $\ell \neq 0$, we can set $\ell = 1$. This way we obtain

$$\{(0, 1, z\overline{z}, y\overline{y}, \overline{z}\,\overline{y}, y, z, 0, \ldots, 0) \mid y, z \in \mathbb{B}\} \cup$$
$$\{(0, 0, 1, x\overline{x}, x, 0, 0, \ldots, 0)\} \cup \{(0, 0, 0, 1, 0, 0, 0, \ldots, 0)\},$$

which, after some easy recoordinatising, is seen to coincide with a Veronese variety as in Definition 3.1.

Remark 4.9. In the above representation, a (standard) symp of $\mathsf{C}_{3,3}(\mathbb{B}, \mathbb{K})$ is given by the intersection with the subspace $(*, 0, *, *, *, 0, 0, *, 0, \ldots, 0)$, and appears in its universal embedding, as one easily checks.

The following lemmas combine some observations about these dual polar spaces and their embeddings and will turn out to be convenient to know in the sequel of our paper. The first lemma holds for all dual polar spaces of rank 3.

Lemma 4.10. *Let Ω be a dual polar space of rank 3 and let x and y be opposite points in Ω. Then Ω is spanned by $x^\perp$ and $y^\perp$.*

Proof. Let S be the subspace of Ω spanned by $x^\perp$ and $y^\perp$, i.e. the intersection of all subspaces of Ω containing $x^\perp$ and $y^\perp$. It suffices to prove that all points of Ω are contained in S. By dualising to Ω^* the polar space of rank 3 dual to Ω, it suffices to prove that S^*, the set of planes corresponding to points contained in S, contains all planes of Ω^*. Let π_x, π_y be the planes of Ω^* corresponding with x and y. It is then clear that those planes are opposite in Ω. Note further that the definition of S as a subspace translates to the the fact that if two planes of Ω^* intersecting in a line are contained in S^*, then all planes of Ω^* through that line must be contained in S^*.

Let π be a plane of Ω. If π equals π_x or π_y, or intersects one of both in a line, it is contained in S^* by definition. Suppose now that π intersects both, π_x, π_y, in a point. Then we can project these points on the other plane (project $\pi \cap \pi_x$ onto π_y and $\pi \cap \pi_y$ onto π_x) to get two planes of the previous type intersecting π in the same line. This yields by the previous observations that $\pi \in S^*$. So suppose that π intersects only one of the planes π_x, π_y in a point a and is disjoint from the other. Without loss of generality, we may assume that $a \in \pi_x$. Let π_a be the plane spanned by a and its projection onto π_y. Suppose first that there exists a plane α through a intersecting both π_x and π in a line, but intersecting π_a only in a. Then $\alpha \cap \pi_y = \varnothing$ and the plane β spanned by $\alpha \cap \pi$ and its projection on π_y is distinct from α. Both planes α and β belong to S^*, and hence by a previous consideration, also π does. Suppose now that every plane through a intersecting π_x and π in a line, intersects also π_a in a line. Then this is not true for any plane π' intersecting π in a line through a. Hence every plane intersecting π in a line through a belongs to S^*, and with that, $\pi \in S^*$. Finally let π be a plane disjoint from π_x and π_y and let L be a line of π. Projecting this line to π_x and to π_y yields two planes contained in S^* through this line, which concludes the proof if these two planes are really distinct, But if for each choice of L, these planes coincide, then the duality of π given by projecting first to π_x, then π_x to π_y and this back to π maps each line L to a point $p \in L$, a contradiction since there do not exist null polarities in a plane. □

Lemma 4.11. *The only homogeneous embedding of* $\mathsf{C}_{3,3}(2,2)$ *in* $\mathsf{PG}(13,2)$ *is the standard embedding.*

Proof. The standard embedding is homogeneous as is shown in [8]. Suppose now that there is another homogeneous embedding in 13 dimensions. Since the universal embedding of $\mathsf{C}_{3,3}(2,2)$ is in $\mathsf{PG}(14,2)$ by [24, §6.4], both embeddings in 13 dimensions arise as the projection of the universal embedding from a point onto a hyperplane. By the homogeneity of the embeddings, this point must be fixed under the extensions to $\mathsf{PG}(14,2)$ of the automorphisms of $\mathsf{C}_{3,3}(2,2)$. Suppose now that $\mathsf{C}_{3,3}(2,2)$ is embedded in the standard way in $\mathsf{PG}(13,2)$ and denote $(X_0, X_1, \ldots, X_{13})$ for the coördinates of the embedding given above (note $\mathbb{K} = \mathbb{B} = \mathbb{F}_2$). Then there is a projection of this $\mathsf{C}_{3,3}(2,2)$ onto $\mathsf{PG}(7,2)$ from the so called nucleus space. It is the natural projection from the 5-dimensional subspace of $\mathsf{PG}(13,2)$ given by $X_0 = X_1 = X_2 = X_3 = X_7 = X_8 = X_9 = X_{13} = 0$ onto the 7-dimensional subspace given by $X_4 = X_5 = X_6 = X_{10} = X_{11} = X_{12} = 0$. This projection is, as point set, the hyperbolic quadric in 7 dimensions, in particular given by the equations $X_0X_{13} + X_1X_7 + X_2X_8 + X_3X_9 = 0$. The second homogeneous embedding gives rise to a point of $\mathsf{PG}(13,2)$ fixed under the extensions to $\mathsf{PG}(13,2)$ of the automorphisms of $\mathsf{C}_{3,3}(2,2)$. The latter can clearly not be contained in the nucleus subspace $\mathsf{C}_{3,3}(2,2)$ in $\mathsf{PG}(13,2)$ by the automorphisms given in Table 2 of [8] (the automorphism group acts on the nucleus subspace as a symplectic group). So we still have a fixed point under the extensions of the automorphisms of $\mathsf{C}_{3,3}(2,2)$ to $\mathsf{PG}(7,2)$. That is a contradiction since through each point of $\mathsf{PG}(7,2)$ goes at least one secant to the hyperbolic quadric, giving rise to an imprimitive action of the automorphism group of $\mathsf{C}_{3,3}(2,2)$. □

Lemma 4.12. *Suppose* $\Omega' \cong \mathsf{C}_{3,3}(\mathbb{B},\mathbb{K})$ *is isometrically embedded in* $\Omega \cong \mathsf{C}_{3,3}(\mathbb{A},\mathbb{K})$. *Then* $\mathbb{B}$ *must be a subalgebra of* $\mathbb{A}$. *If furthermore the dimensions of* $\mathbb{A}$ *and* $\mathbb{B}$ *over* $\mathbb{K}$ *are finite and* Ω *is universally embedded (resp. embedded in a standard way) in the projective space* Π, *then* Ω' *is universally embedded or embedded in a standard way (resp. embedded in a standard way) in the subspace of* Π *spanned by its points.*

Proof. The first statement follows immediately by looking at a point residual which yields a projective subplane $\mathsf{PG}(2,\mathbb{B})$ of $\mathsf{PG}(2,\mathbb{A})$.

Suppose now that $d := \mathsf{dim}_{\mathbb{K}}(\mathbb{A})$ and $d' := \mathsf{dim}_{\mathbb{K}}(\mathbb{B})$ are finite. Let then Ω be homogeneously embedded in the projective space Π and denote by Π' the subspace of Π spanned by the points of Ω'. Let x, y be opposite points of Ω'. Then they must also be opposite in Ω by the isometricity and similar arguments as in the proof of Lemma 4.2. Denote with $\mathscr{V}$ the set of points of Ω in $x^\perp \cap y^{\perp\!\!\!\perp}$ and with $\mathscr{V}'$ those of Ω' in $x^\perp \cap y^{\perp\!\!\!\perp}$.

Assume first that $\mathbb{K} \neq \mathbb{F}_2$. In this case the standard and universal embedding coincide, so suppose Ω is embedded in this way in $\Pi = \mathsf{PG}(6d+7, \mathbb{K})$. It suffices then to prove that the dimension of Π' over $\mathbb{K}$ equals $6d'+7$. Note that in this case $\mathscr{V}$ is the Veronesean $\mathscr{V}_2(\mathbb{K}, \mathbb{A})$ and $\mathscr{V}'$ is a projection of the Veronesean $\mathscr{V}_2(\mathbb{K}, \mathbb{B})$. We claim that this is not a proper projection.

Let ξ_1' and ξ_2' be symps of Ω' through x and let ξ_1 and ξ_2 be the respective corresponding symps in Ω. From now on let $i = 1, 2$. Denote by O_i the intersection of ξ_i with $\mathscr{V}$, and by O_i' the intersection of ξ_i' with $\mathscr{V}'$, note that both are the quadrics of Witt index 1. By Lemma 3.2, we get that the intersection $\langle O_1 \rangle \cap \langle O_2 \rangle$ is a point. This implies immediately that also $\langle O_1' \rangle \cap \langle O_2' \rangle$ is a point. Note however that since ξ_i must be universally embedded in its span, also ξ_i' must be, by combining Lemma 2.26 with the Main result of [3]. Consequently $\langle O_i' \rangle$ has dimension $d'+1$, and the claim now follows from Proposition 3.3.

By interchanging the roles of x and y above, we find that the subspaces spanned by $x^\perp$ and $y^\perp$ in Π' have both dimension at least $3d'+3$. However they must be disjoint as well, since they are contained in the disjoint subspaces $x^\perp$ and $y^\perp$ in Π. This shows Ω' spans a subspace of at least (and hence precisely) dimension $6d'+7$.

So assume now that $\mathbb{K} = \mathbb{F}_2$. If $\mathbb{A} = \mathbb{B}$, the claim follows immediately since the finiteness implies $\Omega = \Omega'$. So we suppose that $\mathbb{A} = \mathbb{F}_4$ and $\mathbb{B} = \mathbb{F}_2$. It is known that the embedding rank of Ω is then 22 (see [24, §7.1] or Corollary 5.12, proved independently of the current lemma) and the one of Ω' is 15 (see [24, §6.4]). Note that the assumptions of Lemma 3.4 are satisfied for both $\mathscr{V}$ and $\mathscr{V}'$, by Lemma 2.23(i) and Remark 4.8 so the induced embedding of $\mathscr{V}$ is in $\mathsf{PG}(8,2)$ or $\mathsf{PG}(10,2)$ and the one of $\mathscr{V}'$ is in $\mathsf{PG}(5,2)$ or $\mathsf{PG}(6,2)$.

Let first Ω be embedded universally in $\Pi = \mathsf{PG}(21,2)$. Then $\mathscr{V}$ must be embedded in 10 projective dimensions, by Lemma 4.10. Now $\mathscr{V}'$ must be embedded in 6 dimensions, since for the Veronesean $\mathscr{V}_2(2,2)$ the tangent lines to the ovoids through a point lie in a plane (see for example [14, p.152]), which is not compatible with the last statement of Lemma 3.4. Similarly as before, we get that the subspaces of Π spanned by $x^\perp$ and $y^\perp$ have both dimension 11 and intersect in a line, while those of Π' spanned by $x^\perp$ and $y^\perp$ have dimensions 7. The latter intersect of course in at most a line, which yields similarly as before that Π' has at least dimension 13. The embedding of Ω' in Π' is however still homogeneous. So with Lemma 4.11 we get that it must be the universal embedding, as in the standard embedding, $\mathscr{V}'$ is clearly embedded in 5 dimensions.

Let now the embedding of Ω in $\Pi = \mathsf{PG}(19,2)$ be the standard embedding. It follows immediately from the description in [8], see also Remark 4.8, that $\mathscr{V}$ is then embedded as a Veronesean in 8 dimensions. The subspaces of Π spanned by $x^\perp$ and $y^\perp$ have both dimension 9 and are disjoint. This implies that also the subspaces of Π' spanned by $x^\perp$ and $y^\perp$ must be disjoint. Consequently they cannot be 7-dimensional and so the embedding of Ω' in

Π' must be in 13 dimensions, which implies it is also the standard embedding by Lemma 4.11. □

The following lemma does not immediately yield conditions for our dual polar space in order that it is embedded isometrically, but this follows then easily when combined with previous results such as Proposition 4.4 and Lemma 4.7, as will be done in the Proof of Main Result A (see Section 5.3.4).

Lemma 4.13. *Let $\mathbb{A}$ and $\mathbb{B}$ be quadratic alternative division algebras over $\mathbb{K}$ such that one of the following is satisfied*

(i) $\mathbb{K}$ *is finite;*
(ii) $\mathbb{A} = \mathbb{B} = \mathbb{K}$*;*
(iii) $\dim_{\mathbb{K}}(\mathbb{A}) \leqslant \dim_{\mathbb{K}}(\mathbb{B})$ *and* $\mathbb{B}$ *is not an inseparable (multiple) quadratic field extension* $\mathbb{K}$*.*

Then there does not exist a full embedding of $\Omega \cong \mathsf{C}_{3,3}(\mathbb{B},\mathbb{K})$ *in* $\zeta \cong \mathsf{B}_{3,1}(\mathbb{K},\mathbb{A})$*.*

Proof. First assume that $\mathbb{K}$ is finite and denote $k := |\mathbb{K}|, a := |\mathbb{A}|, b := |\mathbb{B}|$. One easily calculates that the number of points in ζ is $\alpha = ak^4 + ak^3 + ak^2 + k^2 + k + 1$. In the same way one calculates that the number of planes in a polar space of type $\mathsf{C}_{3,1}(\mathbb{B},\mathbb{K})$ is $\beta = b^3k^3 + b^3k^2 + b^2k^2 + bk^2 + b^2k + bk + k + 1$. This is exactly the number of points in the dual polar space Ω. Now clearly $\alpha < \beta$, keeping in mind that $0 < k \leqslant a, b \leqslant k^2$ (since there are no quaternions over nor inseparable extensions of finite fields), which proves that Ω cannot be contained in ζ.

From now on we may assume that $\mathbb{K}$ is infinite, so $|\mathbb{K}| > 3$ and denote $n := \dim_{\mathbb{K}}(\mathbb{A})$ and $m := \dim_{\mathbb{K}}(\mathbb{B})$. Suppose for a contradiction that Ω is fully embedded in ζ and let the latter be universally embedded in $\mathsf{PG}(n+5,\mathbb{K})$. Take a point p in Ω and let q be an opposite point in ζ. Let W be the projection of the point residual $\mathsf{Res}_{\Omega}(p)$ onto $p^{\perp} \cap q^{\perp}$. Then W is a projection of the Veronesean $\mathscr{V} := \mathscr{V}_2(\mathbb{K},\mathbb{B})$ embedded in $\mathsf{PG}(n+3,\mathbb{K})$. By the assumptions on the dimensions of the algebras, W must now be a proper projection of $\mathscr{V}$ onto $p^{\perp} \cap q^{\perp} \cong \mathsf{B}_{2,1}(\mathbb{K},\mathbb{A})$.

We now prove that no conic of $\mathscr{V}$ is mapped to a line under this projection. Let C be a conic of $\mathscr{V}$. Since, by Lemma 3.5, every conic on a Veronesean is the intersection of a plane with one of the ovoids and every ovoid corresponds in W to the intersection of W with a symplecton of Ω through p, the projection of C is contained in a unique symplecton ξ of Ω through p. Note now that $\xi \cong \mathsf{B}_{2,1}(\mathbb{K},\mathbb{B})$ is embedded universally in ζ by Lemma 2.22. Consequently the projection of C cannot be a line, since the intersection of W with this symplecton does not contain lines.

If now $\mathbb{B} = \mathbb{K}$, we get a contradiction with Lemma 3.9, since the projection is from a subspace disjoint from the nucleus plane if $\mathsf{char}\,\mathbb{K} = 2$ by the previous paragraph. So suppose now that $\mathbb{B} \neq \mathbb{K}$. Let O_1 and O_2 be the projections of two ovoids of $\mathscr{V}$ onto W and let from now on $i = 1, 2$. Denote by ξ_i the unique symplecton of Ω through p containing O_i and by X_i the subspaces of $\mathsf{PG}(n+5,\mathbb{K})$ such that $\xi_i = X_i \cap \zeta$ (this exists by [3]). Then X_i has dimension $m+3$ by Lemma 2.23 and O_i spans a subhyperplane of X_i since it is contained in $p^\perp \cap q^\perp$ (in ζ), so the dimension of $\langle O_i \rangle$ equals $m+1$. Since both O_1 and O_2 are contained in $p^\perp \cap q^\perp$, the dimension of $\langle O_1, O_2 \rangle$ is at most $n+3$. Now the dimension formula $\dim\langle O_1\rangle + \dim\langle O_2\rangle = \dim\langle\langle O_1\rangle, \langle O_2\rangle\rangle + \dim(\langle O_1\rangle \cap \langle O_2\rangle)$ yields $\dim(\langle O_1\rangle \cap \langle O_2\rangle) \geqslant (m+1)+(m+1)-(n+3) = 2m-n-1 > 0$. Let x be the unique point in $O_1 \cap O_2$ and let L be a line in $\langle O_1\rangle \cap \langle O_2\rangle$. Taking now the intersection of a plane through L in $\langle O_i\rangle$ delivers a conic C_i. Lemma 3.6 yields a (unique) projection W' of a Veronesean $\mathscr{V}_2(\mathbb{K},\mathbb{K})$ containing $C_1 \cup C_2$. Since L is the tangent line at x to both C_1 and C_2, W' is a proper projection from a subspace disjoint from the nucleus plane by the previous paragraph if $\mathsf{char}\,\mathbb{K} = 2$. This contradicts again Lemma 3.9. □

Example 4.14. Define the following function fields

$$\mathbb{K} := \mathbb{F}_2(\ldots, t_{-2}^2, t_{-1}^2, t_0^2, t_1^2, t_2^2, t_3^2, t_4^2, t_5^2, t_6^2, \ldots),$$
$$\mathbb{B} := \mathbb{F}_2(\ldots, t_{-2}^2, t_{-1}^2, t_0^2, t_1^2, t_2^2, t_3^2, t_4^2, t_5, t_6, \ldots),$$
$$\mathbb{A} := \mathbb{F}_2(\ldots, t_{-2}^2, t_{-1}^2, t_0^2, t_1, t_2, t_3, t_4, t_5, t_6, \ldots),$$

and let $V \cong (\mathbb{K}\oplus\mathbb{K}\oplus\mathbb{K}\oplus\mathbb{K}\oplus\mathbb{K}\oplus\mathbb{K}\oplus\mathbb{K}\oplus\mathbb{K}) \oplus (\mathbb{A}\oplus\mathbb{A}\oplus\mathbb{A}\oplus\mathbb{A}\oplus\mathbb{A}\oplus\mathbb{A})$ (as vector spaces over $\mathbb{K}$). We now prove that in this case the dual polar space $\mathsf{C}_{3,3}(\mathbb{B},\mathbb{K})$ can fully be embedded in $\mathsf{B}_{2,1}(\mathbb{K},\mathbb{A})$. The former is clearly isomorphic to $\mathsf{C}_{3,3}(\mathbb{A},\mathbb{K})$ since $\mathbb{A}$ and $\mathbb{B}$ are isomorphic as algebra's. So this example will show that in the inseparable case it is possible that a geometry isomorphic to a trace geometry of a metasymplectic space is fully embedded in a symp of the metasymplectic space. We use therefore the description of $\mathsf{C}_{3,3}(\mathbb{B},\mathbb{K})$ as given in [8], in particular the one given in Proposition 3.9 where it is given as the set of points in the projective space $\mathsf{PG}(V)$ satisfying 26 equations. We will take a linear combination of six of these equations and show that they determine a polar space of rank 2. Clearly $\mathsf{C}_{3,3}(\mathbb{B},\mathbb{K})$ is then also contained in this polar space.

Denote a point of $\mathsf{PG}(V)$ as

$$(Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7, Y_8, Y_9, Y_{10}, Y_{11}, Y_{12}, Y_{13}, Y_{14}),$$

with $Y_1, Y_2, Y_3, Y_4, Y_8, Y_9, Y_{10}, Y_{14} \in \mathbb{K}$ and $Y_5, Y_6, Y_7, Y_{11}, Y_{12}, Y_{13} \in \mathbb{B}$. Define the quadric

$$\begin{aligned} 0 = {} & [Y_5^2 + Y_3Y_4 + Y_1Y_8] + t_1^2 \cdot [Y_6^2 + Y_2Y_4 + Y_1Y_9] \\ & + t_2^2 \cdot [Y_7^2 + Y_2Y_3 + Y_1Y_{10}] + (t_1^2t_3^2 + t_2^2t_4^2) \cdot [Y_{11}^2 + Y_9Y_{10} + Y_2Y_{14}] + \\ & + t_3^2 \cdot [Y_{12}^2 + Y_8Y_{10} + Y_3Y_{14}] + t_4^2 \cdot [Y_{13}^2 + Y_8Y_9 + Y_4Y_{14}]. \end{aligned}$$

(The six quadratic forms between brackets are obtained from [8, Proposition 3.9].) So after the coördinate transformation

$$\begin{aligned} & Y_1 = t_4^2Z_2 + t_3^2Z_3 + X_{-2}, \quad Y_2 = Z_1, \quad Y_3 = t_1^2Z_1 + t_4^2Z_4 + X_{-1}, \\ & Y_4 = t_2^2Z_1 + t_3^2Z_4 + X_1, Y_5 = Z_5, Y_6 = Z_6, Y_7 = Z_7, Y_8 = t_1^2Z_2 + t_2^2Z_3 + X_2, \\ & Y_9 = Z_2, \quad Y_{10} = Z_3, \quad Y_{11} = Z_8, \quad Y_{12} = Z_9, \quad Y_{13} = Z_{10}, \quad Y_{14} = Z_4, \end{aligned}$$

we get the quadric with equation

$$\begin{aligned} X_{-2}X_2 + X_{-1}X_1 = {} & t_1^2t_2^2Z_1^2 + t_1^2t_4^2Z_2^2 + t_2^2t_3^2Z_3^2 + t_3^2t_4^2Z_4^2 \\ & + Z_5^2 + t_1^2Z_6^2 + t_2^2Z_7^2 + (t_1^2t_3^2 + t_2^2t_4^2)Z_8^2 + t_3^2Z_9^2 + t_4^2Z_{10}^2 \end{aligned}$$

in $\mathsf{PG}(6n+7,\mathbb{K})$ with $X_i \in \mathbb{K}$ for $-2 \leqslant i(\neq 0) \leqslant 2$, $Z_j \in \mathbb{K}$ for $1 \leqslant j \leqslant 4$ and $Z_l \in \mathbb{B}$ for $5 \leqslant l \leqslant 10$. This quadric is clearly embedded in the nondegenerate quadric $\mathsf{B}_{2,1}(\mathbb{K},\mathbb{A})$.

4.2. Collinear to a point

Now we take a closer look at the isometric case. In this case all points of the embedded dual polar space are collinear to one point of the metasymplectic space.

Proposition 4.15. *Let Ω be a dual polar space of rank 3 isometrically embedded in a metasymplectic space Δ. Then there exists a unique point $p \in \Delta$ with $\Omega \subseteq p^\perp$ such that every line through p contains at most one point of Ω.*

Proof. Let a and d be two opposite points in Ω. Then these points are special in Δ. Let p be the unique point collinear to both in Δ. Let L be a line through a in Ω. Then there is a unique point l on L at distance two from d in Ω, since the latter is a dual polar space of rank 3. Note that, by the isometricity, l is symplectic to d in Δ. Let ζ be the symp of Δ containing l and d. Then a is close to ζ and hence $a^\perp \cap \zeta =: K$ is a line containing p. It follows that all other points of L are special to d and collinear to p. Consequently all points collinear to a in Ω (and similarly to d) are collinear to p in Δ. Since every point in $a^\perp$ has an opposite point in $d^\perp$, it follows now by a connectivity argument that every point of Ω is collinear to p.

It is clear that every line through p contains at most one point of Ω, since $p \notin \Omega$. □

In the separable case the "at most" in the previous proposition can often be exchanged by "exactly", as is shown in the rest of this subsection.

Proposition 4.16. *Let Ω be a dual polar space of rank 3 isometrically embedded in a separable metasymplectic space $\Delta \cong F_{4,4}(\mathbb{K},\mathbb{A})$. Then there exists a unique point $p \in \Delta$ with $\Omega \subseteq p^\perp$ such that every line through p contains exactly one point of Ω. Consequently Ω must be isomorphic to $\mathsf{B}_{3,3}(\mathbb{K},\mathbb{A})$.*

Proof. Denote with p the unique point of Δ such that $\Omega \subseteq p^\perp$ from Proposition 4.15. It suffices then to prove that every line through p contains a point of Ω. We prove that a geometry $\Gamma \cong \mathsf{B}_{3,3}(\mathbb{K},\mathbb{A})$ (with dual $\Gamma^* \cong \mathsf{B}_{3,1}(\mathbb{K},\mathbb{A})$) does not contain a proper full subgeometry Ω isomorphic to the dual of a polar space Ω^* of rank 3. Let $x \perp y \perp z \perp\!\!\!\perp x$ be three points of Ω, corresponding to the planes α, β, γ in Ω^*, respectively. Then α and γ intersect β in two different lines. Denote the intersection of these lines as c. As the embedding of Ω in Γ is full, every plane in Γ^* through a line of Ω^* is a plane of Ω^*. So, using the terminology of [22], the point residual of c in Ω^* is an ideal subquadrangle of the point residual of c in Γ^*. Hence by Proposition 5.9.4 of [22] these point residuals coincide. Let now π be a plane through c in Γ^*, then it must be contained in Ω^* and consequently it contains a second point c' of Ω^*. As the same reasoning can be applied to c', all lines through c or through c' are contained in Ω^* and so are their intersection points as the embedding is isometric. So every point of π is contained in Ω^*. By a connectivity argument, one now proves that $\Omega^* = \Gamma^*$ and consequently also $\Omega = \Gamma$. □

Lemma 4.17. *Let $\Omega \cong \mathsf{C}_{3,3}(\mathbb{A}, \mathbb{K})$ be isometrically embedded in a separable metasymplectic space $\Delta \cong \mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$. Then every symplecton ξ of Ω arises as the common perp of two opposite points in a symplecton $\xi^\blacktriangle$ of Δ. These points are also the unique points of Δ collinear to all points of ξ.*

Proof. Denote for every symplecton ξ in Ω by $\xi^\blacktriangle$ the corresponding symplecton in Δ and $E_\xi := \{x \in \xi^\blacktriangle \mid x \in \xi^\perp\}$. Then by our assumptions, ξ is isomorphic to $\mathsf{B}_{2,1}(\mathbb{K}, \mathbb{A})$ and $\xi^\blacktriangle$ is isomorphic to $\mathsf{B}_{3,1}(\mathbb{K}, \mathbb{A})$. Suppose that the latter is universally embedded in the projective space Π as in Example 2.8(i) and denote with ρ the defining polarity. By the uniqueness of the embeddings of these spaces (see Lemma 2.23(i)), the codimension of the projective space U spanned by the points of ξ in Π is 2. So U^ρ is a projective line K. Clearly the point p of Proposition 4.15 is contained in this line. Suppose now that this is the only point of $\xi^\blacktriangle$ on K. Then K is contained in the tangent hyperplane p^ρ which implies that $p \in U$. But this contradicts the fact that p is collinear to all points of ξ, that ξ does not contain planes, and that $U \cap \xi^\blacktriangle = \xi$ (by the main result of [3]). It is immediately clear that K is not completely contained in $\xi^\blacktriangle$, by the non-degeneracy of ξ. So K intersects $\xi^\blacktriangle$ in exactly two points, E_ξ consists of two points and the common perp of these two points is exactly ξ. □

Proposition 4.18. *Let $\Omega \cong \mathsf{C}_{3,3}(\mathbb{A}, \mathbb{K})$ be isometrically embedded in a separable metasymplectic space $\Delta \cong \mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$. Then there exists a unique point $p \in \Delta$ with $\Omega \subseteq p^\perp$ such that every line through p contains exactly one point of Ω.*

Proof. Denote with p the unique point of Δ such that $\Omega \subseteq p^\perp$ from Proposition 4.15. It suffices to prove that every line through p intersects Ω. By Lemma 4.17, we see that for every symp ξ in Ω there exists a unique point in Δ different from p collinear to all points of ξ. Denote this point by q_ξ and let again $\xi^\blacktriangle = \xi(p, q_\xi)$ be the unique symp of Δ containing ξ. Let now $x \in \Omega$ be random. We will prove that the cone with p as vertex and $\mathsf{Res}_\Omega(x)$ as base coincides with $\mathsf{Res}_\Delta(x)$ (with as point set the lines through x, as line set the planes through x and inclusion as incidence relation). Then a connectivity argument concludes the proof.

Note first that $\mathsf{Res}_\Delta(x)$ is isomorphic to a dual polar space of rank 3 and $\mathsf{Res}_\Omega(x)$ is isomorphic to a projective plane. The points of this plane correspond to the points in $\mathsf{Res}_\Delta(x)$ collinear to p' and to some q'_ξ (with ξ a symp in Ω containing p), where p' and q'_ξ are the points of $\mathsf{Res}_\Delta(x)$ corresponding to the lines px and xq_ξ respectively. Every line in this plane corresponds to such a q'_ξ and consists exactly of the points in the common perp of p' and q'_ξ. By dualising $\mathsf{Res}_\Delta(x)$, we see that $(p')^*$ is a plane in a polar space of rank 3 and $(q'_\xi)^*$ is a plane intersecting this plane in a point. Now the points on the line of $\mathsf{Res}_\Omega(x)$ corresponding to q'_ξ are exactly the planes intersecting both planes $(q'_\xi)^*$ and $(p')^*$ in a line in the dual of $\mathsf{Res}_\Delta(x)$. Since we are working in a polar space, every line through $(q'_\xi)^* \cap (p')^*$ in $(p')^*$ corresponds to such a plane. So we can interpret the dual of $\mathsf{Res}_\Omega(x)$ as an ideal subplane of the plane $(p')^*$. Since clearly proper ideal subplanes do not exist, we find that every plane through px intersects Ω in a line, which concludes the proof. □

We now have almost immediately the following corollary. However since it uses a lot of previous results we provide a proof for completeness.

Corollary 4.19. *Let $\Omega \cong \mathsf{C}_{3,3}(\mathbb{A},\mathbb{K})$ be isometrically embedded in a separable metasymplectic space $\Delta \cong \mathsf{F}_{4,1}(\mathbb{K},\mathbb{A})$ and let p be a point of Δ such that $\Omega \subseteq p^\perp$. Then every symplecton $\xi^\blacktriangle$ of Δ through p intersects Ω in a symplecton ξ.*

Proof. Let L be a line of Δ through p contained in $\xi^\blacktriangle$, let α and β be two planes through L contained in $\xi^\blacktriangle$. By 4.18, α and β intersect Ω in two intersecting lines; denote those as L_α and L_β, respectively, and let x be the intersection point. Let a and b be points of $L_\alpha \backslash x$ and $L_\beta \backslash x$, respectively. Then, by Lemma 4.2, a and b determine a symplecton ξ of Ω. The latter is contained in $\xi^\blacktriangle$ by Lemma 4.3. Now $\xi^\blacktriangle \cap \Omega$ cannot contain a point not contained in ξ, since that point would then be at distance at most 2 from all points in ξ, by Lemma 4.2, which is impossible in a dual polar space of rank 3. □

We give one more interesting lemma about this connection between symplecta of a metasymplectic space and those of a dual polar space isometrically embedded in it.

Lemma 4.20. *Let Ω be a dual polar space of rank* 3 *isometrically embedded in a metasymplectic space Δ. Let ξ and ζ be two opposite symplecta in Ω. Then the corresponding symplecta in Δ are locally opposite through p, with p the unique point collinear to all points of Ω as in* Proposition 4.15.

Proof. Note that by Lemma 4.3 and Proposition 4.15, ξ and ζ are isometrically embedded in symplecta $\xi^\blacktriangle$ and $\zeta^\blacktriangle$, respectively, of Δ through p. Since ξ and ζ are opposite in Ω (a dual polar space of rank 3), we know that for every point q of ξ there is a unique point in ζ collinear to q in Ω, but also at least one point opposite q in Ω. Suppose now that $\xi^\blacktriangle$ and $\zeta^\blacktriangle$ are not locally opposite through p. They cannot coincide, since then there cannot be a point in ζ opposite to q in Ω, or in other words, special to q in Δ by the isometricity of the embedding. So $\xi^\blacktriangle$ and $\zeta^\blacktriangle$ must intersect in a plane. Then all the points of $\zeta^\blacktriangle$ collinear to q are contained in this plane, but this plane is disjoint from Ω (since ξ and ζ are disjoint). This means that ζ cannot contain a point collinear to q, a contradiction. □

5. The separable case

In this section we prove that almost all dual polar spaces isometrically embedded in a separable metasymplectic space are contained in a trace geometry, and classify the exceptions. Together with the previous section, this allows us to prove Main Results A and B. We start with the metasymplectic spaces of type $\mathsf{F}_{4,4}$, since the results and proofs are more concise in this case.

5.1. Separable metasymplectic spaces of short root type

Proposition 5.1. *Let Ω be a dual polar space of rank 3, isometrically embedded in a metasymplectic space $\Delta \cong \mathsf{F}_{4,4}(\mathbb{K}, \mathbb{A})$. Then Ω is contained in a trace geometry, i.e. $\Omega \subseteq p^\perp \cap q^{\not\equiv}$ for some opposite points $p, q \in \Delta$.*

Proof. By Proposition 4.15 we know that $\Omega \subseteq p^\perp$ for some $p \in \Delta$ and every line through p intersects Ω in at most one point. Denote for every symplecton ξ in Ω by $\xi^\blacktriangle$ the corresponding symplecton in Δ and define $E_\xi := \{x \in \xi^\blacktriangle \mid x \in \xi^\perp\}$, which is a hyperbolic line in $\xi^\blacktriangle$ by Lemma 4.5.

Let ξ_1 and ξ_2 be two opposite symplecta in Ω. By Lemma 4.20 the symplecta $\xi_1^\blacktriangle$ and $\xi_2^\blacktriangle$ are locally opposite through p and consequently every point $a \in E_{\xi_1} \backslash p$ is opposite every point $b \in E_{\xi_2} \backslash p$. Fix such an a and b arbitrarily and let q be a point of $E(a, b)$ opposite p.

We claim that $E_\zeta \subseteq \widehat{E}(p, q)$ for every symplecton ζ in Ω. Suppose first that $\zeta = \xi_1$. Since $a \in \widehat{E}(p, q)$ and $\zeta^\blacktriangle$ intersects $\widehat{E}(p, q)$ in a hyperbolic line h by [16, Lemma 2.6.18], we conclude $h = h(p, a) = E_{\xi_1}$. In the same way one shows that also $E_{\xi_2} \subseteq \widehat{E}(p, q)$. Suppose now that ζ intersects ξ_1 and ξ_2 in a line. Then it is obvious that $E_\zeta \subseteq E(a, b) \subseteq \widehat{E}(p, q)$. If ζ intersects ξ_1 in a line, but is disjoint from ξ_2, then ζ corresponds in Ω^* to a point ζ^* collinear to ξ_1^* and not collinear to ξ_2^* . Two locally opposite planes through the line $\langle \xi_1^*, \zeta^* \rangle$ of Ω^* now yield two opposite points in $(\xi_1^*)^\perp \cap (\xi_2^*)^\perp$. Denote the corresponding symplecta as ξ_1', ξ_2' and note that each of them fulfil the assumptions for ζ of the previous case. So we already know that $E_{\xi_1'}, E_{\xi_2'} \subseteq \widehat{E}(a, b)$ and we can take opposite a', b' in $E_{\xi_1'}, E_{\xi_2'}$ respectively, so that $E_\zeta \subseteq E(a', b') \subseteq \widehat{E}(p, q)$ (the latter inclusion follows by again combining Lemma 2.6.18 and Proposition 2.6.15 of [16]).

Now we get similarly as in the previous case that $E_\zeta \subseteq \widehat{E}(a', b')$. Suppose finally that ζ is disjoint from both ξ_1 and ξ_2. Then in Ω^* the point ζ^* is not collinear to either ξ_1^* or ξ_2^*. Let L, M be locally opposite lines through ζ^*. Both contain a point collinear to ξ_1^*, which are contained in $\widehat{E}(a, b)$ by the previous cases. Then we can conclude in a similar way as in the previous case that $E_\zeta \subset \widehat{E}(p, q)$.

Now q is symplectic to some point of E_ζ for every symplecton ζ in Ω, by Proposition 2.6.15 of [16], and, by Proposition 2.12 [Point-Symp], we see that $\Omega \subseteq q^{\not\equiv}$. Thus Ω is contained in a trace geometry. □

Note that we now have proved all components of Main Result B. We combine them in the proof below.

Proof of Main Result B. From Proposition 4.6 it follows that the embedding must be isometric. Then Proposition 4.16 combined with Proposition 5.1 delivers the first statement. The second statement follows immediately from Lemma 4.1. □

5.2. A consequence

Proof of Corollary A. Let Δ be isomorphic to $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$, separable, and let first Ω be a dual polar space of rank 4 fully embedded in Δ. We denote by Ω^* the corresponding polar space (the "dual"). Each point x of Ω^* defines a dual polar space Ω_x of rank 3 fully embedded in Δ by considering all maximal singular subspaces of Ω^* through x. Then Proposition 5.1 yields a unique point p_x of Δ collinear to each point of Ω_x. If y is a point of Ω^* collinear to x, then Ω_x and Ω_y intersect in a symp ξ_{xy} of Ω. Since $p_x, p_y \in \xi_{xy}^{\perp}$, we find $p_x \perp\!\!\!\perp p_y$, $p_x \perp p_y$ or $p_x = p_y$. However, the latter two are impossible since $p_x \perp p_y$ would imply that Δ does contain three-dimensional subspaces and $p_x = p_y$ would imply that Ω_x and Ω_y would intersect in a hyperplane by Proposition 4.16, a contradiction since they only share a symp.

Now suppose $z \perp y$ in Ω^*, but z not collinear to x. Then ξ_{xy} and ξ_{yz} are opposite symps of Ω_y. Hence, the unique symps $\xi_{xy}^{\blacktriangle}$ and $\xi_{yz}^{\blacktriangle}$ of Δ containing ξ_{xy} and ξ_{yz}, respectively, are locally opposite at p_y by Lemma 4.20. This implies that p_x and p_z are opposite in Δ.

Define $\widehat{E} = \{p_x \mid x \text{ a point of } \Omega^*\}$. The previous paragraph implies that $\widehat{E}$ only contains symplectic and opposite pairs of points. Let x and z be as above, then we claim $\widehat{E} = \widehat{E}(p_x, p_z)$. Indeed, let $\xi^{\blacktriangle}$ be any symp of Δ through p_x, then it must intersect Ω_x in a symplecton ξ. However, the latter must be contained in another dual polar space of rank 3, say $\Omega_{y'}$, corresponding to some point y' of Ω^* collinear to x. Let now y be the point on the line xy' of Ω^* collinear to z. Then Ω_y must intersect Ω_x in ξ and consequently we may assume without loss of generality that $\xi^{\blacktriangle} = \xi_{xy}^{\blacktriangle}$. So $\xi^{\blacktriangle}$ contains a point, p_y, of $\widehat{E}$ symplectic to p_z. This shows $E(p_x, p_z) \subseteq \widehat{E}$. Now let $u \in \widehat{E}(p_x, p_z) \backslash E(p_x, p_z)$. Then there exist opposite points $u_1, u_2 \in E(p_x, p_z)$ symplecic to u and the previous argument shows $u \in \widehat{E}$. Hence $\widehat{E}(p_x, p_y) \subseteq \widehat{E}$. Conversely, if w is a point of Ω^* collinear to both x and z, then $p_x \perp\!\!\!\perp p_w \perp\!\!\!\perp p_z$ and hence $p_w \in \widehat{E}(p_x, p_z)$. If w is a point of Ω^* not collinear to both x and z, then there are non-collinear points w_1, w_2 of Ω^* collinear to x, z and w, and the previous arguments imply $p_w \in E(p_{w_1}, p_{w_2}) \subseteq \widehat{E}(p_x, p_y)$. Hence $\widehat{E} \subseteq \widehat{E}(p_x, p_z)$, and thus $\widehat{E} = \widehat{E}(p_x, p_z)$, which proves the claim.

Note that every point of Ω is contained in some $\xi_{x'y'}$ for $x', y' \in \Omega^*$. Furthermore $p_x^{\perp} \cap p_y^{\perp} = \xi_{xy}$ by Lemma 4.5, so every point collinear to two points of $\widehat{E}$ is contained in Ω. So we conclude that, by definition, the tropics geometry associated to $\widehat{E}$ is precisely Ω.

Next, let Γ be a dual polar space of rank at least 5 fully embedded in Δ, and let Γ' be a subspace of Γ isomorphic to a dual polar space of rank 4. If ξ is a symp of Γ', then every point of $\xi^{\perp}$ occurs as a point of the extended equator geometry corresponding to Γ'. Let now Γ'' be another subspace of

Γ isomorphic to a dual polar space of rank 4, containing ξ and let p be the unique point of Δ collinear to ξ. Then Γ' and Γ'' intersect $p^\perp$ respectively in Σ' and Σ'', which are dual polar spaces of rank 3. These should only intersect in ξ, but as above $\Sigma' \cap \Sigma''$ is a geometric hyperplane in both Σ' and Σ'', a contradiction as there are lines in Σ' disjoint from ξ.

The proof of Corollary A is complete. □

5.3. Separable metasymplectic spaces of long root type

In this case we have no longer the extended equator geometries as a tool. However we are able to define some other interesting geometries. We start by recalling some lemmas and notation from the previous sections. We will eventually have to distinguish between $|\mathbb{K}| = 2$ and $|\mathbb{K}| \geqslant 3$, but we start with general considerations.

5.3.1. General considerations. Let $\Omega \cong \mathsf{C}_{3,3}(\mathbb{A}, \mathbb{K})$ be an isometrically fully embedded subgeometry of a separable metasymplectic space $\Delta \cong \mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$. Let p be the unique point such that $\Omega \subseteq p^\perp$ (exists by Proposition 4.15). Denote for every symplecton ξ in Ω the corresponding symplecton in Δ as $\xi^\blacktriangle$ (which exists by Lemma 4.3); denote by q_ξ the unique point in $\xi^\blacktriangle$ such that $\xi = p^\perp \cap q_\xi^\perp$ (exists by Lemma 4.17). Now we can define the following geometries.

Definition 5.2 (The point-line geometry $\Gamma_\Omega = (X_\Omega, \mathscr{L}_\Omega)$). The geometry Γ_Ω is the point-line geometry $(X_\Omega, \mathscr{L}_\Omega)$ with point set $X_\Omega = \{q_\xi \mid \xi \text{ symp of } \Omega\}$ and line set $\mathscr{L}_\Omega = \{\{q_\xi \in X_\Omega \mid L' \in \xi\} \mid L' \text{ line of } \Omega\}$.

Note that this geometry clearly is isomorphic to the dual of the point residual of p in Δ, i.e. $\mathsf{C}_{3,1}(\mathbb{A}, \mathbb{K})$. We now define in this polar space an analogue for the equator geometries in metasymplectic spaces (see Definition 2.17).

Definition 5.3 (The point-line geometry $E_{\Gamma_\Omega}(q, q')$). Let q and q' be non-collinear points of Γ_Ω. Then $E_{\Gamma_\Omega}(q, q')$ is the equator geometry defined by q and q' in Γ_Ω, i.e. the point-line geometry with point set the points collinear to q and q' in Γ_Ω, and line set the lines of Γ_Ω included in this point set.

Clearly the latter could be defined in all polar spaces of rank at least 3 and will always be isomorphic to a point residual. Also, the points of $E_{\Gamma_\Omega}(q, q')$ form a subset of those of $E(q, q')$. We will make use of this inclusion of these geometries later on. First we prove some more properties about this inclusion.

Lemma 5.4. *Let $\Omega \cong \mathsf{C}_{3,3}(\mathbb{A}, \mathbb{K})$ be an isometrically fully embedded subgeometry of a separable metasymplectic space $\Delta \cong \mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$. Then two points of Γ_Ω are collinear in Γ_Ω if, and only if, they are symplectic in Δ; they are non-collinear in Γ_Ω if, and only if, they are opposite in Δ. Consequently if q and q' are opposite points in Γ_Ω, then two points of $E_{\Gamma_\Omega}(q, q')$ are collinear in $E_{\Gamma_\Omega}(q, q')$ if, and only if, they are collinear in $E(q, q')$.*

Proof. It is clear that, if two points are contained in a line of Γ_Ω, then they are symplectic in Δ, since the corresponding symplecta in Δ intersect in a plane and both points are collinear in Δ to the same line of this plane. Conversely if two points x, y of Γ_Ω are symplectic in Δ, then p is close to the corresponding symplecton $\xi(x, y)$ and the symplecta $\xi(p, x), \xi(p, y)$ contain the line $\xi(x, y) \cap p^\perp$, which is contained in Ω as it is contained in $p^\perp \cap x^\perp$. If now two points of Γ_Ω are not contained in a line of Γ_Ω, then the corresponding symplecta in Ω are opposite. By Lemma 4.20 the corresponding symplecta in Δ are locally opposite through p and so the points are opposite in Δ. The converse follows from the same lemma.

The last statement follows now immediately from the definition of lines in an equator geometry of a metasymplectic space. □

In the next proof, we make use of central elations (see Definition 2.24).

Lemma 5.5. *Let $\Omega \cong \mathsf{C}_{3,3}(\mathbb{A}, \mathbb{K})$ be an isometrically fully embedded subgeometry of a separable metasymplectic space $\Delta \cong \mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$ and let q and q' be opposite points in Γ_Ω. Then the hyperbolic lines in $E_{\Gamma_\Omega}(q, q')$ are subsets of hyperbolic lines in $E(q, q')$.*

Proof. By Lemmas 2.10.5 and 6.5.1 of [16] a hyperbolic line in $E(q, q')$ through two opposite points a, b is the set $\{b^\theta \mid \theta \text{ central elation of } \Delta \text{ with centre } a\} \cup \{a\}$. So we have to prove that, for a and b not collinear in E_{Γ_Ω}, the group of central elations with centre a in Δ acts transitively on the set of points of the hyperbolic line through a and b (except a) in $E_{\Gamma_\Omega}(q, q')$. Note that every central elation θ of Δ with centre a induces by restriction a central elation of $E_{\Gamma_\Omega}(q, q')$ with centre a by the following reasoning. Since a is collinear with a symplecton ξ_a of Ω and every other point of Ω is collinear with at least one point of ξ_a, we see that θ stabilises Ω. Consequently θ stabilises X_Ω and even fixes q and q', since the corresponding symps of Ω intersect ξ_a in a line. Let now c be an arbitrary point of this hyperbolic line through a and b in $E_{\Gamma_\Omega}(q, q')$ different from a. Then by Lemma 2.25(ii) there exists a central elation θ' of $E_{\Gamma_\Omega}(q, q')$ with centre a that maps b to c. First note that this extends unambiguously to a central elation of Γ_Ω, since the hyperbolic lines through a and b coincide in both geometries (this follows for example from Lemma 2.25(iii)) and Lemma 2.25(ii) also holds in Γ_Ω. Suppose we can show that this is the restriction of a central elation in Δ; then the symplecton $\xi_a = p^\perp \cap a^\perp$ in Ω is pointwise fixed and every line intersecting this symp in a point is stabilised. Let $t \in \Omega$ be a point on such a line (not contained in ξ_a) and let t' be the image. Then these correspond to planes in Γ_Ω through a line L' collinear with a. It follows from Lemma 2.25(ii) that the central elations of Γ_Ω with centre a act sharply transitively on these planes. So there is a unique central elation with centre a in Γ_Ω that could extend to one in Δ mapping t to t'. Since there is a unique central elation in Δ with centre a that maps t to t' by Lemma 6.5.1 of [16], we can indeed extend θ' unambiguously to a collineation of Δ. □

Lemma 5.6. *Let* $\mathsf{C}_2 \cong \mathsf{C}_{2,1}(\mathbb{A},\mathbb{K})$ *be isometrically fully embedded in* $\mathsf{C}_3 \cong \mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$ *with* $\mathbb{A}$ *over* $\mathbb{K}$ *separable. Let* L_1 *and* L_2 *be two disjoint lines of* C_2*. Then* $L_1^\perp \cap L_2^\perp$ *is a hyperbolic line* h *in* C_3 *and every point of* h *is collinear to all points of* C_2*.*

Proof. Since the embedding is isometric, L_1 and L_2 are opposite in C_3. Then $L_1^\perp \cap L_2^\perp$ is a hyperbolic line h of C_3 by Lemma 2.25(*iii*). Let now x be an arbitrary point of h. Since x is collinear to both L_1 and L_2, it must be collinear to the subquadrangle spanned by them in C_2. However the latter has no proper full subquadrangles by the dual of [22, Proposition 5.9.4] (keeping in mind that $\mathbb{A}$ is separable over $\mathbb{K}$). □

The following lemma allows us to recognise subgeometries of equator geometries in Δ.

Lemma 5.7. *Let* $E \cong \mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$ *be a point-line geometry* $(X,\mathscr{L})$ *with* X *a subset of points of a separable metasymplectic space* $\Delta \cong \mathsf{F}_{4,1}(\mathbb{K},\mathbb{A})$ *and* $\mathscr{L}$ *a set of lines of the form* $\{x \in \xi \mid x \in L^\perp \cap M^\perp\}$ *with* ξ *a symplecton in* Δ *and* L, M *two opposite lines in* ξ*, such that two points are collinear in* E *if they are symplectic in* Δ *and not collinear in* E *if they are opposite in* Δ*. Then* E *is contained in an equator geometry* $E(p,q)$ *for some opposite points* $p, q \in \Delta$*, with* $\mathscr{C}(p,q)$ *independent of* p *and* q*.*

Proof. Let $a \in X$ be arbitrary. Denote with $\mathsf{Res}_E(a)$ the point residual of a in E; then it is clear that $\mathsf{Res}_E(a) \cong \mathsf{C}_{2,1}(\mathbb{A},\mathbb{K})$. We now claim that this can be interpreted as an isometric subgeometry of the dual point residual of a in Δ, $\mathsf{Res}_\Delta(a)^* \cong \mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$. Note that the points of $\mathsf{Res}_E(a)$ correspond to lines of E through a, while those of $\mathsf{Res}_\Delta(a)^*$ correspond to symplecta of Δ through a; and furthermore the lines of $\mathsf{Res}_E(a)$ correspond to planes of E through a, while those of $\mathsf{Res}_\Delta(a)^*$ correspond to planes of Δ through a.

To make this identification, we first prove that every symplecton contains at most one line of E (note that the converse is trivially true: every line of E is contained in exactly one symplecton). Suppose for a contradiction that the symplecton ζ does contain two lines of E. If these lines are not contained in a plane, this would imply that ζ contains a pair of opposite points by the assumptions, a contradiction. However, if these lines would span a plane π, one gets a contradiction by taking a point r of $E\backslash\pi$, which must then be opposite and symplectic to multiple points of this symplecton (impossible by Proposition 2.12 [Point-Symp]). So we can unambiguously identify the set of points of $\mathsf{Res}_E(a)$ with a subset of the set of points of $\mathsf{Res}_\Delta(a)^*$.

We now prove that the same can be done for the sets of lines. Consider a line in $\mathsf{Res}_E(a)$ corresponding to a plane π of E through a. Let $L', M' \in \pi$ be two lines of E through a and denote with ξ_L, ξ_M the corresponding symplecta. A point of $L'\backslash\{a\}$ must be collinear in Δ to a line K of ξ_M which is collinear in Δ to all points of M'. Note that $a \in L'$ and consequently K is also contained in ξ_L. So ξ_L and ξ_M intersect in the plane $\langle a, K\rangle$ of Δ. Furthermore all points of L' must be collinear in Δ to this line K (by Proposition 2.12 [Point-Symp]). Let now x be an arbitrary point of π different from a. Then x lies on a line

lm of E with $l \in L'\backslash\{a\}$ and $m \in M'\backslash\{a\}$. But then x is collinear to K as a is close to $\xi(l,m)$, a is collinear to the line K of $\xi(l,m)$ and x is symplectic to a. Consequently all symplecta through a corresponding to lines of E through a in π contain the plane $\langle a,K\rangle$. Fix some $l \in L'\backslash\{a\}$ and $m \in M'\backslash\{a\}$. It suffices now to prove that every symplecton through $\langle a,K\rangle$ intersects the line lm of E in a point. Let ξ be an arbitrary symplecton through $\langle a,K\rangle$. Then it intersects $\xi(l,m)$ in a plane of Δ through K. So it suffices to prove that every plane α through K in $\xi(l,m)$ contains a point of the line lm of E. Let α be such a plane of Δ through K and let L, M be opposite lines of Δ in $\xi(l,m)$ such that $lm = L^\perp \cap M^\perp$ (these exist by our assumptions on the lines of E). Denote with s the unique point of α collinear in Δ with all points of M and note that $s \notin K$ since all points of lm must be collinear to K and M. Denote with t the unique point of $\langle M,s\rangle$ collinear in Δ with all points of L. Then t must be collinear with K in Δ by the previous reasoning, since it is contained in the line lm of E. Since $\langle M,s\rangle$ has a unique point collinear with all points of K in Δ, $t = s$ and this point is contained in the intersection $\alpha \cap lm$.

It follows now easily that the above identification leads to $\mathsf{Res}_E(a)$ being isometrically fully embedded in $\mathsf{Res}_\Delta(a)^*$. So with Lemma 5.6 we find two locally opposite symplecta ξ,ζ of Δ through a that both intersect each symplecton through a which corresponds to a point of $\mathsf{Res}_E(a)$ (in some plane). Now let b be an arbitrary point of E opposite a in E. Then clearly b is also opposite a in Δ and defines two opposite points $p := \xi \cap b^{\perp\!\!\!\perp}$ and $q := \zeta \cap b^{\perp\!\!\!\perp}$. We claim that $E \subseteq E(p,q)$. So let c be an arbitrary point of E. We show that $c \in E(p,q)$. If $c = a$ or $c = b$, this is trivial; so we suppose that $c \neq a,b$.

Suppose first that c is collinear to a and b in E. Set $\alpha := \xi\cap\xi(a,c)$. Then by the possible point-line relations in Δ (Corollary 2.5.2 in [16]), b is special to all points of a unique line B in α and consequently p and c are collinear to this line (by Proposition 2.12 [Point-Symp] and the fact that $p \perp\!\!\!\perp b \perp\!\!\!\perp c$). So c is symplectic or collinear to p. But note that c is clearly not collinear to p as the only points collinear to c in $\xi(a,p)$ are contained in α.

Suppose now that c is collinear to b but not to a in E. Denote $c' := \xi(b,c) \cap a^{\perp\!\!\!\perp}$. Let L'_1 and L'_2 be two lines of E through c' contained in $a^{\perp\!\!\!\perp} \cap b^{\perp\!\!\!\perp}$, denote by ζ_1,ζ_2 the respective corresponding symplecta of Δ and let from now on $i \in \{1,2\}$. Then these symplecta must be locally opposite as there are no planes of E that are collinear to a and b in E. Denote by π_i the intersection of ζ_i with $\xi(b,c)$. As all points of the line L'_i of E are symplectic in Δ to all points of the line bc of E, the plane π_i contains a line K_i collinear with all points of both lines bc and L'_i of E. Note that ξ is adjacent to $\xi(a,a_i)$, for every $a_i \in L'_i$. Hence p is not opposite any point of ξ_i, which implies that ζ_i and $\xi(p,c')$ are adjacent. Looking now, for fixed i, at the pairwise adjacent symplecta $\zeta_i,\xi(b,c),\xi(p,c')$ through c', these correspond to three pairwise collinear points in $\mathsf{Res}_\Delta(c')^*$. So, the intersection planes of the symplecta must have one line M_i of Δ through c' in common (if they would have a plane in common, then replacing p with q, that plane has a point collinear to q, contradicting the fact that no point of that plane is special to p). Let now $m_i \in M_i$ be the unique point collinear to p. Then $m_i = K_i \cap M_i$, as p is symplectic to all points of L_i' by the previous case. So we constructed two points $m_1, m_2 \in p^\perp \cap c^\perp$ and c is again symplectic or collinear to p. Note that clearly c is again not collinear to p.

Suppose now that c is not collinear in E to either a or b, but there exists a point b' of E collinear in E to b and c, but not to a. We can then replace b by b' in the previous two paragraphs to conclude respectively that all points of E in $a^{\perp\!\!\!\perp} \cap b'^{\perp\!\!\!\perp}$ and all points of E in $b'^{\perp\!\!\!\perp}$ are symplectic to p. So c is symplectic to p.

So the only case that is left is a point c of E such that all points collinear in E to both b and c are also collinear in E to a, or, in other words, c is contained in the hyperbolic line through a and b in E. Let K' be some line through c in E, let ξ' be the unique symplecton containing K' and let x and y be two points of K' different from c. Then p is symplectic to x and y by the previous cases, since there are no collinear points on a hyperbolic line. Now Lemma 2.25(iii) applied to $\xi' \cong \mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$ implies that also c is symplectic to p (choose as L in the statement of that lemma the line $p^\perp \cap \xi'$). □

Even though we use similar techniques, we make from now on a distinction between fields with at least three elements and the field with two elements. We start with the fields with at least three elements.

5.3.2. Separable metasymplectic spaces of long root type over a field of at least three elements.

Lemma 5.8. *Let* $\mathsf{C}_2 \cong \mathsf{C}_{2,1}(\mathbb{A}',\mathbb{K})$ *be embedded in a polar space* $\mathsf{C}_3 \cong \mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$ *with* $\mathbb{K}$ *a field of order at least* 3 *and* $\mathbb{A},\mathbb{A}'$ *separable quadratic alternative division algebras over* $\mathbb{K}$*, such that*

(i) *the points of* C_2 *form a subset of the set of points of* C_3*;*
(ii) *two points in* C_2 *are collinear if and only if they are collinear in* C_3*;*
(iii) *the hyperbolic lines of* C_2 *are subsets of hyperbolic lines of* C_3*.*

Then the lines of C_2 *are subsets of lines of* C_3*.*

Proof. We prove this by contradiction. Suppose that a, b, c are three collinear points in C_2 which are not collinear in C_3. Since they are however pairwise collinear by (ii), they must be contained in a plane of C_3. Then a, b, c are contained in a(n ideal) sub polar space $\mathsf{C}_2' \cong \mathsf{C}_{2,1}(\mathbb{K},\mathbb{K})$ of C_2. The latter still satisfies the assumptions of this lemma for C_2, since hyperbolic lines are preserved under taking this sub polar space, as all lines of C_2 through a point of C_2' are also contained in C_2'. Let C_2' be universally embedded as in Example 2.8(ii) in $\Pi_2 \cong \mathsf{PG}(3,\mathbb{K})$. Then the lines through a point correspond to all lines through that point in the tangent hyperplane of Π_2. Furthermore, by Lemma 2.25(iv), the hyperbolic line through two opposite points in this embedding is just the projective line of Π_2. Let L be the line in C_2' through a, b, c and let b'' be a point of C_2' collinear to a but not contained in L. Let $\mathsf{C}_3' \cong \mathsf{C}_{3,1}(\mathbb{K},\mathbb{K})$ be a sub polar space of C_3 containing a, b, c and b''. Let C_3' now as well be universally embedded as in 2.8(ii) in $\Pi_3 \cong \mathsf{PG}(5,\mathbb{K})$. The hyperbolic line through two opposite points is then again the projective line of Π_3 containing these points, by Lemma 2.25(iv).

Remark that C_2' is no longer necessarily embedded in C_3'. However, we claim that still all points collinear to a in C_2' are contained in C_3'. Indeed, note first that hyperbolic lines are also preserved under taking the sub polar space C_3' of C_3, by a similar argument as for C_2'. Denote by $H := \langle a, b, b'' \rangle$ the tangent hyperplane of Π_2 to C_2' at a. Let p be an arbitrary point in H not contained in L. Then $h(b, b'')$ must intersect $h(c, p)$ in a point q, since both correspond to projective lines in H. This point q is contained in $h(b, b'') \subseteq C_3'$ and consequently p is contained in $h(c, q) \subseteq \mathsf{C}_3'$. A similar argument now shows that also all points of L must be contained in C_3'.

Denote by L'' the line of C_2' through a and b'' and let L' be a line of C_2' different from L and L'' through a. By the above observations the hyperbolic line $h(b, b'')$ must intersect L' in a point, say b'. Similarly we find a point c' as the intersection of $h(c, b'')$ and L'. In C_3' the line L corresponds to a plane π_L containing a, b, c. First we prove that L' and L'' correspond to planes $\pi_{L'}$ and $\pi_{L''}$, respectively, of C_3' which intersect π_L both in the same line A. Suppose for a first contradiction that a, b' and c' are collinear in C_3'. Then the plane π of Π_3 spanned by $h(b, b')$ and $h(c, c')$ must contain a, which contradicts the fact that a, b and c are not collinear in C_3'. So L' corresponds to a plane $\pi_{L'}$ in C_3'. Similarly one shows that also L'' corresponds to a plane $\pi_{L''}$ of C_3'. Then in π the projective lines $\langle b, c \rangle$ and $\langle b', c' \rangle$ must intersect in a point different from a (since $a \notin \pi$), so the planes π_L and $\pi_{L'}$ intersect in a line of C_3'. Similarly one shows that also the planes π_L and $\pi_{L''}$ and the planes $\pi_{L'}$ and $\pi_{L''}$ intersect in a line of C_3'. Since C_3' does not contain three-dimensional projective spaces, we get that the three planes intersect in the same projective line A of Π_3.

Denote by d the intersection of the projective line $\langle b, c \rangle$ with A in π_L. We prove now that every point from $L \backslash \{a\}$ is contained in $\langle b, c \rangle$. Let p be such a point. In Π_2 we see immediately that $h(b, b')$ must intersect $h(p, c')$ in a point of C_2', let's say q, collinear to a. In Π_3 we first get that $q \in h(b, b') \subseteq \pi$ and consequently also $p \in h(c', q) \subseteq \pi$. But p is also contained in π_L and consequently must be contained in the intersection of these two planes, i.e. the projective line $\langle b, c \rangle$ of Π_3. So all points of L except a are contained in one line of C_3', repeating the argument switching the roles of a and b for example, we find that a must also lie on this line since $|\mathbb{K}| > 2$, a contradiction. □

Proposition 5.9. *Let* $\Omega \cong \mathsf{C}_{3,3}(\mathbb{A}, \mathbb{K})$ *be an isometric subgeometry of a separable metasymplectic space* $\Delta \cong \mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$ *with* $\mathbb{K}$ *a field of order at least* 3. *Then* Ω *is a trace geometry, i.e.* $\Omega = p^{\perp} \cap q^{\not\equiv}$ *for some opposite points* $p, q \in \Delta$.

Proof. Recall the point-line geometry Γ_Ω from Definition 5.2. We will show that this geometry satisfies the conditions of Lemma 5.7, which will almost complete the proof. With Lemma 5.4 it suffices to prove that the lines of Γ_Ω correspond to sets of points of the form $\{x \in \xi \mid x \in K^{\perp} \cap K'^{\perp}\}$ with ξ a symplecton in Δ and K, K' two opposite lines in ξ. Let L be a line of Γ_Ω.

Let $q_{\xi_1}, q_{\xi_2}, q_{\xi_3} \in L$ be points of Γ_Ω with respective corresponding symplecta ξ_1, ξ_2 and ξ_3 in Ω and denote with L' the line of Ω contained in these three symplecta. First we claim that $q_{\xi_3} \in \xi(q_{\xi_1}, q_{\xi_2})$. Indeed, let ζ, ζ' be two symps of Ω intersecting L' in two different points. Then these correspond to opposite points $q_\zeta, q_{\zeta'}$ of Γ_Ω both collinear to $q_{\xi_1}, q_{\xi_2}, q_{\xi_3}$ in Γ_Ω. So $q_{\xi_1}, q_{\xi_2}, q_{\xi_3}$ are contained in $E_{\Gamma_\Omega}(q_\zeta, q_{\zeta'}) \cong \mathsf{C}_{2,1}(\mathbb{A}, \mathbb{K})$ which is clearly embedded in the equator geometry defined by q_ζ and $q_{\zeta'}$ in Δ, that is, $E(q_\zeta, q_{\zeta'}) \cong \mathsf{C}_{3,1}(\mathbb{A}, \mathbb{K})$. Translating our claim to this setting, it suffices to prove that every line of $E_{\Gamma_\Omega}(q_\zeta, q_{\zeta'})$ is contained in a line of $E(q_\zeta, q_{\zeta'})$. Taking Lemma 5.4 and Lemma 5.5 into account, we see that we can apply Lemma 5.8. So, we obtain $q_{\xi_3} \in \xi(q_{\xi_1}, q_{\xi_2})$ and denote $\xi_L := \xi(q_{\xi_1}, q_{\xi_2})$.

Let now π, π' be two locally opposite planes through L in Γ_Ω, with corresponding points k, k', respectively, on L', and let $q_r, q_{r'}$ be arbitrary points of $\pi \backslash L$, $\pi' \backslash L$, respectively. Note that q_r and $q_{r'}$ are opposite in Δ, since they cannot be collinear in Γ_Ω as there are no 3-dimensional subspaces in Γ_Ω and non-collinear points of Γ_Ω are opposite in Δ by the first paragraph. However q_r and $q_{r'}$ are close to ξ_L, as they are collinear to a point of ξ_L (i.e. the points k, k', respectively), but not contained in ξ_L (since otherwise they are not mutually opposite in Δ). So, by Corollary 2.5.4 of [16], they define opposite lines K, K' in ξ_L and all points of the line L of Γ_Ω are collinear in Δ to these lines, as they must be symplectic to q_r and $q_{r'}$ in Δ. We now prove that an arbitrary point d in ξ_L collinear to K, K' is actually contained in L. Note that d and p are symplectic, since $k \in K$ and $k' \in K'$ (d is not collinear to p, since this would contradict the opposition of K and K'). Now the symplecton $\xi(p, d)$ of Δ intersects Ω in a symp of Ω through L by Corollary 4.19. So $\xi(p, d)$ must contain a point of $L \subseteq \Gamma_\Omega$, which must by the previous argument be collinear with K, K'. If these lines are not contained in the symplecton, there can at most be one point of $\xi(p, d)$ collinear to both. Since this is the case for d, we can then conclude that d is indeed contained in L. So we need to exclude that K is contained in $\xi(p, d)$ (a similar argument shows the same for K'). Suppose that K is contained in $\xi(p, d)$. Then q_r is close to or contained in this symplecton. The latter is impossible, since $q_{r'}$ is not far from this symp as it is collinear to k'. So q_r is close to $\xi(p, d)$ and symplectic to p. Then p must be collinear to K, which contradicts L being the only line of ξ_L collinear to p.

So we finally checked all the assumptions to apply Lemma 5.7 and get a point $q \in \Delta$ opposite p symplectic to all points of Γ_Ω. Consequently $\Omega \subseteq q^{\not\equiv}$ by [16, Corollary 2.5.3] and so Ω is contained in a trace geometry. With Proposition 4.18 we get that Ω must coincide with the trace geometry $p^\perp \cap q^{\not\equiv}$. □

5.3.3. Separable metasymplectic spaces of long root type over the field with two elements. So the only separable case left are the dual polar spaces isometrically embedded in $\mathsf{F}_{4,1}(\mathbb{F}_2, \mathbb{F}_4)$. Contrary to expectations, we will encounter here dual polar spaces embedded isometrically that are not contained in a trace geometry. Since this is a finite case, some counts will be done, such as in the following lemma.

Lemma 5.10. *Let* $\mathsf{C}_3 \cong \mathsf{C}_{3,1}(\mathbb{F}_4, \mathbb{F}_2)$ *be a polar space and let* ∞ *be a point of* C_3*. Then* C_3 *has* 2^{10} *embedded subgeometries isomorphic to* $\mathsf{C}_{2,1}(\mathbb{F}_4, \mathbb{F}_2)$ *such that hyperbolic lines of both geometries coincide, points in the subgeometry are collinear if, and only if, they are collinear in* C_3 *and all the points in the subgeometry are collinear to* ∞*. In* 2^8 *of these subgeometries the lines coincide with lines of* C_3*, but in the other* $3 \cdot 2^8$ *subgeometries no three collinear points are contained in a line of* C_3*.*

Proof. Consider the universal embedding of C_3 in $\mathsf{PG}(5,4)$ as given in Example 2.8(ii) and let C_2 be embedded in C_3 such that hyperbolic lines of both geometries coincide and points in C_2 are collinear if, and only if, they are collinear in C_3.

Let

$$\begin{cases} L = \{x_1, x_2, x_3, x_4, x_5\}, \\ L' = \{x_1', x_2'.x_3', x_4', x_5\}, \\ L'' = \{x_1'', x_2'', x_3'', x_4'', x_5\} \end{cases}$$

be three different lines of C_2 through the point x_5. Writing x_i as i, x_i' as i' and x_i'' as i'', we may assume that the following are hyperbolic lines (with obvious notation):

$$\begin{array}{cccc} 11'1'' & 12'2'' & 13'3'' & 14'4'' \\ 21'2'' & 22'1'' & 23'4'' & 24'3'' \\ 31'3'' & 32'4'' & 33'1'' & 34'2'' \\ 41'4'' & 42'3'' & 43'2'' & 44'1'' \end{array}$$

Suppose first that L contains three points x_1, x_2, x_3 of a line M of C_3. We claim that then also x_4 belongs to M. Let π_0 be the plane of $\mathsf{PG}(5,4)$ containing x_1, x_2, x_3 and x_1'. Since hyperbolic lines of both geometries coincide and those of C_3 are projective lines of $\mathsf{PG}(5,4)$ by Lemma 2.25(iv), we conclude that π_0 also contains x_1'', x_2'' and x_3''. The hyperbolic line $12'2''$ yields $x_2' \in \pi_0$, and similarly the hyperbolic line $42'3''$ yields $x_4 \in \pi_0$. Now $x_4 \in \langle x_1, x_2 \rangle = M$, since otherwise $\pi_0 = \langle x_1, x_2, x_4 \rangle$ must be a plane of C_3, while it contains hyperbolic lines. Similarly, $x_5 \in M$. Note that, by the above, the line L' is also contained in π_0 and since its points are pairwise collinear, we conclude that also L' is a line of C_3. By connectivity, C_2 is an ordinary embedding in C_3, i.e. lines of C_2 are lines of C_3. Combining the main result of [3] with Lemma 2.23(ii), we get that C_2 is the intersection of a three-dimensional subspace U of $\mathsf{PG}(5,4)$ with C_3. Suppose now that all points of C_2 must be collinear to the point ∞ of C_3, then $\infty \in U^\rho$ with ρ the defining polarity of C_3 in $\mathsf{PG}(5,4)$. However the line U^ρ intersects C_3 in three points, since it is no tangent line. Denote by x, x' the points contained in this intersection different from ∞, then $\mathsf{C}_2 = x^\perp \cap \infty^\perp = x'^\perp \cap \infty^\perp$. Clearly each such point x opposite ∞ gives rise to such a C_2. So, since C_3 contains 2^9 points opposite ∞, there are 2^8 such embeddings.

So, from now on we may assume that no 3 points of any line of C_2 are contained in a line of C_3. It follows (by [6, Theorem 4.9] for example) that each line L of C_2 is a non-degenerate conic in some plane π_L of C_3. Denote by π_L, π'_L and π''_L the planes of C_3 spanned by the points of L, L' and L'', respectively. We claim that π_L, π'_L, π''_L intersect in a common line K and that the nuclei of the corresponding conics coincide with a single point n on K. Indeed, the plane $\langle x_1, x_2, x'_1\rangle$ of $\mathsf{PG}(5,4)$ contains the points x''_1, x''_2 and x'_2; hence the lines $\langle x_1, x_2\rangle$ and $\langle x''_1, x''_2\rangle$ intersect in a point $z_{12} \neq x_5$. Similarly the line $\langle x'_1, x'_2\rangle$ must intersect both $\langle x_1, x_2\rangle$ and $\langle x''_1, x''_2\rangle$. Since C_3 has rank three, we get that this is also in the point z_{12}. Hence the line $K := \langle x_5, z_{12}\rangle$ is contained in each of the planes π_L, π'_L, π''_L. If we define z_{ij} as $\{z_{ij}\} = K \cap \langle x_i, x_j\rangle$, $1 \leqslant i < j \leqslant 4$, then, similarly as above, we have $\{z_{1j}\} = K \cap \langle x'_1, x'_j\rangle = K \cap \langle x''_1, x''_j\rangle$, $2 \leqslant j \leqslant 4$. Hence the nucleus of the conic L in π_L is the unique point n of $K \backslash \{x_5, z_{12}, z_{13}, z_{14}\}$, and the same holds for the nucleus of L' in π'_L and the nucleus of L'' in π''_L. The claim is proved. By connectivity the nucleus of the conic corresponding to any line of C_2 inside the plane it spans on C_3 is n. Hence all points of C_2 are collinear to a common point n of C_3, and n corresponds to the point ∞ in the statement of this lemma. Let Π be the 4-space of $\mathsf{PG}(5,4)$ spanned by $\infty^\perp$. Then Π contains C_2.

Now consider at the embedding of C_2 in $\mathsf{PG}(3,4)$ as in Example 2.8(ii). Let π be a plane in $\mathsf{PG}(3,4)$ that is not a tangent plane to C_2. Then π intersects C_2 in an ovoid O of C_2 (a set of points intersecting each line of C_2 in exactly one point), which forms an affine plane $\mathsf{AG}(2,3)$ of order 3 when structured with its hyperbolic lines. Consequently also in Π the points of O are contained in a plane π'. Now let h, h', h'' be parallel lines of $\mathsf{AG}(2,3)$. Define then the following hyperbolic lines of C_2: $g := h^\perp$, $g' := h'^\perp$ and $g'' := h''^\perp$. Let x be a point of g (and consequently also of C_2). We now claim that the 3-space $\Sigma := \langle O, x\rangle$ of Π intersects C_2 exactly in the 18 points of C_2 contained in the hyperbolic lines h, h', h'', g, g' and g'' and that these form a 2-ovoid O_2 (a set of points intersecting each line of C_2 in exactly two points). First we prove that these 18 points are contained in Σ. This is clear for the points of h, h' and h''. Note that in $\mathsf{PG}(3,4)$ the projective lines containing g, g' and h'' are contained in the (projective) plane $(h \cap h')^\perp$. Clearly this is also a non-tangent plane intersecting C_2 in an ovoid and consequently these three hyperbolic lines are also coplanar in Π and contained in Σ. Similar one proves that g'' is contained in Σ. The fact that every line of C_2 intersects this set of points O_2 in exactly two points follows immediately by the fact that π is a hyperplane of $\mathsf{PG}(3,4)$ and the definition of the lines g, g', g''. So suppose for a contradiction that there exists a point a in $(\mathsf{C}_2 \cap \Sigma)\backslash O_2$. By the above we then find two other points b, c of $\mathsf{C}_2 \cap \Sigma$ on a line of C_2 with a, so the points a, b, c, ∞ are coplanar in Π. This implies that also ∞ is contained in Σ. This is impossible since then the projective line $\langle \infty, x\rangle$ must intersect the plane π', necessarily in a point of h, contradicting the fact that x and this point must lie on an conic with nucleus ∞.

Now let L_1, L_2, L_3, L_4 and L_5 be different lines of C_3 through ∞, such that the first three are pairwise locally opposite, L_4 is coplanar with L_2 and with L_3, and L_5 is contained in the plane $\langle L_3, L_4\rangle$. We claim that C_2 is completely determined by its points on these lines. Clearly, every C_2 gives rise to exactly one point on each line. We now prove that given 5 points $y_i \in L_i$, with $1 \leqslant i \leqslant 5$, such that $y_5 \notin \langle y_3, y_4\rangle$, there exists at most one possible C_2 embedded in C_3 such that the assumptions of the lemma are satisfied and no three collinear points of C_2 are contained in a line of C_3. We first show that y_1, y_2, y_3 and y_4 determine the only 18 points of C_2 in the 3-dimensional space spanned by them. First note that, with similar reasonings as before, C_3 intersects the plane $\pi_1 := \langle y_1, y_2, y_3\rangle$ of Π in exactly nine points forming an $\mathsf{AG}(2,3)$ when structured with its hyperbolic lines. Since the hyperbolic lines of C_2 and C_3 coincide and an $\mathsf{AG}(2,3)$ is spanned by three non-collinear points, these nine points must be contained in C_2 (and no other of π_1). Now applying the same reasoning to the planes $\langle y_4, l\rangle$ and $\langle y_4, l'\rangle$, with l, l' the two lines parallel to, but disjoint from the line through y_2 and y_3 in this $\mathsf{AG}(2,3)$, one finds 8 more points of C_2 in the 3-dimensional space $\langle y_1, y_2, y_3, y_4\rangle$. One shows easily that these 18 points are in the same configuration as in the previous paragraph, and those are then all the points of C_2 in this 3-dimensional space and form a 2-ovoid in C_2 by the reasoning at the end of that paragraph. Now all the other points are determined by the fact that four points of $\mathsf{PG}(2,4)$ no three on a line determine a unique hyperoval of $\mathsf{PG}(2,4)$ (a set of six points no three of which collinear) and the geometry of 27 points and 27 lines obtained from $\mathsf{C}_2(4,2)$ by removing the points of a 2-ovoid is connected. The latter is the case since three points of $\mathsf{C}_2(4,2)$ on a line not contained in the 2-ovoid are collinear to 12 other points of C_2 not contained in the 2-ovoid, so each connected component of the new geometry contains at least 15 points, which is more than half, and hence there is only one component.

For the points y_1, y_2, y_3, y_4 and y_5 there are $3 \cdot 4^4$ possible choices. If we can prove that there exists at least one geometry $\mathsf{C}_2' \cong \mathsf{C}_{2,1}(4,2)$ satisfying the assumptions in the statement of the lemma while no three collinear points of C_2' are contained in a line of C_3, we can conclude that there are exactly $3 \cdot 2^8$ such subgeometries in C_3 by the following reasoning. We already proved that for given y_i with $1 \leqslant i \leqslant 5$ there exists at most one C_2 through them in the previous paragraph. However we can transform our example to one through these points, so there is at least one as well. This transformation goes as follows. Let y_i' denote the point on L_i of C_2' for $1 \leqslant i \leqslant 5$. First apply the translation of Π with center ∞ that maps the hyperplane $\langle y_1', y_2', y_3', y_4'\rangle$ to the hyperplane $\langle y_1, y_2, y_3, y_4\rangle$ and denote by y_5'' the image of y_5' under this map. Then apply the homology of Π with center ∞ and axis $\langle y_1, y_2, y_3, y_4\rangle$ that maps y_5'' to y_5. As a composition of collineations, this is a collineation of Π fixing ∞ and preserving the necessary conditions for C_2.

To conclude the proof, we construct such an example C_2'. Let C_3 be given as in Example 2.8(ii) and let ∞ be the point with coördinates $(0,0,0,0,0,1)$. Then Π has equation $x_{-3} = 0$ and the points of $\infty^\perp$ in this space are given by $x_{-2}^2x_2 + x_{-1}^2x_1 + x_1^2x_{-1} + x_2^2x_{-2} = 0$. After a coördinate transformation, we can denote the points of Π with coördinates $(z_1, z_2, z_3, z_4, z_5)$ such that ∞ has coördinates $(0, 0, 0, 0, 1)$ and the points of $\infty^\perp$ are exactly those with $z_1^3 + z_2^3 + z_3^3 + z_4^3 = 0$. We then choose the points of C_2' to be those with $z_1^3 + z_2^3 + z_3^3 + z_4^3 = 0$ and $z_5 = z_1z_2z_3z_4$. These are well-defined since $\epsilon^4 = \epsilon$, for each $\epsilon \in \mathbb{F}_4$. By explicitly enumerating all points (an elementary exercise which we shall not do), one checks that the set of these points together with the line set existing of sets of points contained in the same plane of C_3 through ∞ form a point-line geometry isomorphic to $\mathsf{C}_{2,1}(4, 2)$. Points are then collinear in C_2' if, and only if, they are in C_3. Furthermore there is at least one line of C_2' not contained in a line of C_3, namely $\{(1, 1, 1, 1, 1), (1, 1, 0, 0, 0), (0, 0, 1, 1, 0), (1, 1, \epsilon, \epsilon, \epsilon^2), (1, 1, \epsilon^2, \epsilon^2, \epsilon)\}$, so by the second paragraph of this proof, we get that all lines are like this. To finally conclude that C_2' is indeed an example of a geometry described in the previous paragraph, one only has to check that it is closed under taking hyperbolic lines in C_3. So suppose we have two noncollinear points of C_2'. Note that the geometry is preserved under the following automorphisms of Π: multiplying the last coördinate and one other coördinate with a non-zero element; permuting the four first coördinates; the field automorphism of $\mathbb{F}_4$. If one of the points has only nonzero coördinates, we can suppose without loss of generality by the above that it is $(1, 1, 1, 1, 1)$. Then we can again by the above suppose that the other point is one of the following three: $(1, \epsilon, 0, 0, 0)$, $(1, 1, \epsilon, \epsilon^2, 1)$ and $(1, 1, 1, \epsilon, \epsilon)$ where $\mathbb{F}_4 = \{0, 1, \epsilon, \epsilon^2\}$. One sees immediately that the third point on the hyperbolic line through these two is now contained in C_2', since it has coördinates $(\epsilon^2, \epsilon, 1, 1, 1)$, $(0, 0, \epsilon^2, \epsilon, 0)$ and $(1, 1, 1, \epsilon^2, \epsilon^2)$ respectively. If now both points have coördinates equal to zero, we can suppose without loss of generality that the first one has coördinates $(1, 1, 0, 0, 0)$ and the second $(1, \epsilon, 0, 0, 0)$ or $(1, 0, 1, 0, 0)$. Then the third point on the hyperbolic line is also contained in C_2' since it is $(1, \epsilon^2, 0, 0, 0)$ or $(0, 1, 1, 0, 0)$, respectively. □

Lemma 5.11. *Let $\Delta \cong \mathsf{F}_{4,1}(\mathbb{F}_2, \mathbb{F}_4)$ be a separable metasymplectic space and let ∞ be a point of Δ. Then $\infty^\perp$ contains 2^{22} subgeometries $\Omega \cong \mathsf{C}_{3,3}(\mathbb{F}_4, \mathbb{F}_2)$ embedded isometrically in Δ. For 2^{20} of these subgeometries, each line in Γ_Ω is contained in a symp of Δ; while for the other $3 \cdot 2^{20}$ subgeometries no three points on a line of Γ_Ω are contained in a symp of Δ.*

Proof. Let $\Omega \cong \mathsf{C}_{3,3}(4, 2)$ be isometrically embedded in $\infty^\perp$. By Proposition 4.18, ∞ is the only point of Δ collinear to all points of Ω and each line through ∞ contains exactly one point of Ω. Let now $\xi_0^\blacktriangle$ and $\xi_1^\blacktriangle$ be two locally opposite symps of Δ through p and define $\xi_i := \xi_i^\blacktriangle \cap \Omega$. Then ξ_i is a symp of Ω by Corollary 4.19. Let finally (q_0, q_1) be the pair of points such that $\infty^\perp \cap q_i^\perp = \Omega \cap \xi_i^\blacktriangle$ for $i = 0, 1$, which exists by Lemma 4.17.

Note that there are in total $2^6 \cdot 2^6 = 2^{12}$ possibilities for (q_0, q_1) as pair of points with $q_i \in \xi_i^\blacktriangle$ opposite ∞.

Fix now such a pair (q_0, q_1). We determine how many isometric subgeometries $\Omega \cong \mathsf{C}_{3,3}(4, 2)$ of $\infty^\perp$ correspond to this pair, such that $\infty^\perp \cap q_i^\perp =$

$\Omega \cap \xi_i^{\blacktriangle}$. By Lemma 5.4 and Lemma 5.5, $E_{\Gamma_\Omega}(q_0, q_1)$ is an embedded subgeometry of $\mathsf{E}(q_0, q_1)$ with the properties described in Lemma 5.10. We claim that Ω is completely determined by this $E_{\Gamma_\Omega}(q_0, q_1)$. Note that, since each plane of $\Gamma_\Omega \cong \mathsf{C}_{3,1}(4, 2)$ has at least one point in common with the intersection of two point-perps, the point set of Ω coincides with the union of all sets $\infty^\perp \cap q^\perp$, with $q \in E_{\Gamma_\Omega}(q_0, q_1)$. Now we determine the line set of Ω. There are two different types of lines, so the line set is the union of the two sets described in the rest of this paragraph. First we have the lines that are contained in some $\infty^\perp \cap q^\perp$, with $q \in E_{\Gamma_\Omega}(q_0, q_1)$. These correspond with lines of Γ_Ω intersecting or contained in $E_{\Gamma_\Omega}(q_0, q_1)$. They are determined in the same way as we determined the point set: it are the lines contained in $\infty^\perp \cap q^\perp$, for some $q \in E_{\Gamma_\Omega}(q_0, q_1)$. The second type of lines are those not contained in any such $\infty^\perp \cap q^\perp$, with $q \in E_{\Gamma_\Omega}(q_0, q_1)$. So they correspond to lines of Γ_Ω disjoint from $E_{\Gamma_\Omega}(q_0, q_1)$. Each such line L' existing of the points x', y', z' in Ω corresponds in Γ_Ω to three planes π_x, π_y, π_z respectively intersecting in a line L of Γ_Ω. However each of these planes intersects $E_{\Gamma_\Omega}(q_0, q_1)$ in a point, let's say x, y, z respectively, forming a hyperbolic line of $E_{\Gamma_\Omega}(q_0, q_1)$. The latter can be seen as follows: look at the universal embedding of Γ_Ω in $\mathsf{PG}(5, 4)$, then $E_{\Gamma_\Omega}(q_0, q_1) = q_0^\perp \cap q_1^\perp$ and $\langle \pi_x, \pi_y, \pi_z \rangle = L^\perp$ are intersections of Γ_Ω with 3-dimensional subspaces by the main result of [3] combined with Lemma 2.23(ii), so these subspaces intersect in a projective line. Let now ξ_x, ξ_y, ξ_z, respectively, be the symps in Ω corresponding to these points. Then L' must be a line intersecting those symps. Now by Lemma 5.5, keeping in mind that in this case the number of points is finite, we get that the hyperbolic lines of $E_{\Gamma_\Omega}(q_0, q_1)$ are just those of $E(q_0, q_1)$ completely contained in the point set of $E_{\Gamma_\Omega}(q_0, q_1)$. Furthermore the number of lines collinear to x and z in Γ_Ω (which is the number of lines in $\mathsf{C}_{2,1}(4, 2)$) is exactly the same as the number of lines intersecting ξ_x, ξ_y and ξ_z, (which is the number of points in $\mathsf{B}_{2,1}(2, 4)$). So we can reconstruct the lines of the second type by taking all lines intersecting each three symps of Ω corresponding to the three points on a hyperbolic line of $E(q_0, q_1)$ contained in $E_{\Gamma_\Omega}(q_0, q_1)$.

Let now $\mathsf{C}_2 \cong \mathsf{C}_{2,1}(4, 2)$ be embedded in $E(q_0, q_1)$ such that hyperbolic lines of C_2 coincide with hyperbolic lines of $E(q_0, q_1)$, points are collinear in C_2 if and only if they are in $E(q_0, q_1)$ and all points of C_2 are collinear to ∞. We claim that using the methods described in the previous paragraph to construct a point set and a line set (with two types of lines), we get an Ω satisfying the conditions in the beginning of the previous paragraph. Note first that we reconstruct the correct amount of points and lines by the previous paragraph. Furthermore these lines are clearly full. If we can now prove that every line through ∞ contains at most one point of the constructed point set, we get that the embedding is also isometric and isomorphic to $\mathsf{C}_{3,3}(4, 2)$. Suppose for a contradiction that one line M through ∞ contains two points of the constructed set, let's say m and m'. Then there exist points n and n' in C_2 such that $m \in \infty^\perp \cap n^\perp$ and $m' \in \infty^\perp \cap n'^\perp$. But then the symplecta

$\xi(\infty, n)$ and $\xi(\infty, n')$ of Δ intersect in a plane and n and n' are symplectic. This yields $m = m'$ by Proposition 2.12 [Point-Symp], a contradiction.

So the number of possibilities for Ω given ∞ is equal to 2^{12} times 2^{10} by Lemma 5.10. Let now Ω be such a subgeometry. We claim that as soon as some symplecton of Δ contains three points of a line of Γ_Ω, each line of Γ_Ω is contained in a symplecton. Recall therefor that lines of an equator geometry of a metasymplectic space are given by the intersection of the point set with symplecta. Suppose that three points of the line N of Γ_Ω are contained in one symplecton of Δ and let M be another arbitrary line of Γ_Ω. Let π, π' be two locally opposite planes in Γ_Ω through N. Suppose first that M is opposite N in Γ_Ω and denote by r, r' the respective projections of M onto π and π'. Then both N and M are contained in $E_{\Gamma_\Omega}(r, r')$ which is again by Lemma 5.4 and Lemma 5.5 a subgeometry of $E(r, r')$ as described in Lemma 5.10. Since three points of N are contained in a line of $E(r, r')$ by assumption, the latter lemma implies that both N and M are lines of $E(r, r')$ and consequently each contained in a symplecton. If now M is not opposite N, then we find a line M' of Γ_Ω opposite both by Proposition 1.6.16 of [23] and can apply the previous reasoning twice to get the same conclusion, which proves the claim. It is now clear that the last statement of this lemma now also follows from Lemma 5.10. □

The following result was already known and proved by Yoshiara in [24, §7.2]. However we give here an alternative proof that only uses elementary geometry and the existence of the metasymplectic space $\mathsf{F}_{4,1}(2,4)$.

Corollary 5.12. *The embedding rank of* $\mathsf{C}_{3,3}(4,2)$*, the dual polar space related to the unique non-degenerate Hermitian variety in* $\mathsf{PG}(5,4)$ *(also denoted by* $\mathsf{DH}(5,4)$*), is* 22.

Proof. Let $\Omega \cong \mathsf{C}_{3,3}(4,2)$ be a dual polar space related to the unique non-degenerate Hermitian variety in $\mathsf{PG}(5,4)$ and denote by n its embedding rank. Let Λ be the $\mathbb{F}_2$-cone over Ω. More exactly, let ∞ be an additional point, and define for each point p of Ω an additional point p' (not belonging to Ω). Then the points of Λ are ∞, all points p of Ω and all points p'. The lines are all lines of Ω, the subsets $\{\infty, p, p'\}$, with p a point of Ω, and, for each line $\{p, q, r\}$ of Ω, the subsets $\{p, q', r'\}, \{p', q, r'\}$ and $\{p', q', r\}$.

We now show that the number of isometrically embedded geometries isomorphic to Ω in Λ equals 2^n. It is obvious that the embedding rank of Λ is $n+1$. Suppose now that we have a universal embedding of Λ in $\mathsf{PG}(n,2)$. Then every hyperplane of $\mathsf{PG}(n,2)$ not containing ∞ intersects Λ in an isometrically embedded subgeometry isomorphic to Ω. Conversely, let d be the (projective) dimension of the subspace of $\mathsf{PG}(n,2)$ spanned by an isometrically embedded geometry in Λ isomorphic to Ω. Then, by the definition of embedding rank, we have $d \leqslant n-1$. This inequality must be an equality, since adding the point ∞ gives a set spanning the whole space $\mathsf{PG}(n,2)$ (note that by the finiteness every line of Λ through ∞ must contain exactly one point of the subgeometry). It follows that the number of fully embedded geometries in Λ

isomorphic to Ω and not containing ∞ is equal to the number of hyperplanes in $\mathsf{PG}(n,2)$ not containing ∞, i.e. 2^n.

Let now Δ and ∞ be as in Lemma 5.11, then clearly $\infty^\perp \cong \Lambda$. It is obvious that a subgeometry of $\infty^\perp$ is embedded isometrically in Δ if, and only if, it is in $\infty^\perp$. So when we combine the result in the previous paragraph with Lemma 5.11, we get that $n = 22$. □

The following is an immediate consequence of the previous two results.

Corollary 5.13. *Let Λ be the $\mathbb{F}_2$-cone over $\Omega = \mathsf{C}_{3,3}(4,2)$. If Λ is embedded in $\Pi = \mathsf{PG}(22,2)$ such that Ω is embedded universally, then there is a bijection between the hyperplanes of Π not containing the vertex of the cone and the isometric subgeometries of the cone isomorphic to Ω.*

Lemma 5.14. *Let $\Delta \cong \mathsf{F}_{4,1}(\mathbb{F}_2,\mathbb{F}_4)$ be a separable metasymplectic space and let p,p' be two opposite points of Δ. Then there exists a full embedding of Δ in a projective space such that $p^\perp \cap p'^{\not\equiv}$ is embedded in the standard way in $\mathsf{PG}(19,2)$.*

Proof. Let q be a prime power. Look at the following chain of full embeddings:

$$\mathsf{C}_{3,3}(q^2,q) \subseteq \mathsf{F}_{4,1}(q,q^2) \subseteq \mathsf{E}_{6,2}(q) \subseteq \mathsf{PG}(77,q).$$

The last embedding here is due to section 4.3 of [1] and this is homogeneous. The middle embedding is by Galois descent due to [21]. Note now that Galois descent preserves homogeneity, again by [21].

Let now $q = 8$. Then the embedding of $\mathsf{C}_{3,3}(64,8)$ induced by the above chain is the standard (and universal) one in $\mathsf{PG}(19,8)$, since this is the only homogeneous embedding by [8].

Applying now Galois descent from $\mathbb{F}_8$ to $\mathbb{F}_2$ (by using the irreducible polynomial x^2+x+1) to the whole chain above gives an embedding of $\mathsf{F}_{4,1}(2,4)$ inducing the standard embedding of $\mathsf{C}_{3,3}(4,2)$ in $\mathsf{PG}(19,2)$, by homogeneity. □

Finally we are now able to prove Main Result A.

5.3.4. Proof of Main Result A. Lemma 4.7 yields $\Omega \cong \mathsf{C}_{3,3}(\mathbb{B},\mathbb{K})$ for some quadratic alternative division algebra $\mathbb{B}$ over $\mathbb{K}$. Combining then Proposition 4.4 and Lemma 4.13, yields that the embedding is isometric or we are in case (ii). So from now on we may assume isometricity. Then with Proposition 4.15, we find a unique point p of Δ such that $\Omega \subseteq p^\perp$ and each line through p contains at most one point of Ω.

Suppose now that $\mathbb{K} \neq \mathbb{F}_2$ and let q be a point of Δ opposite p. Then $p^\perp \cap q^{\not\equiv}$ is isomorphic to $\mathsf{C}_{3,3}(\mathbb{A},\mathbb{K})$ by Lemma 4.1 and can be universally embedded in $\mathsf{PG}(6n+7,\mathbb{K})$ by [8], where $n = \mathsf{dim}_{\mathbb{K}}(\mathbb{A})$. Consequently the cone with vertex p and base this $p^\perp \cap q^{\not\equiv}$ can be embedded in $\Pi := \mathsf{PG}(6n+8,\mathbb{K})$. Let Ω' be the projection of Ω from p onto $p^\perp \cap q^{\not\equiv}$. Then $\Omega' \cong \mathsf{C}_{3,3}(\mathbb{B},\mathbb{K})$ and if we denote by S' the subspace of Π spanned by the points of Ω', we get that Ω' is embedded universally in S' by Lemma 4.12 and $\mathbb{B}$ is a subalgebra of $\mathbb{A}$. So the dimension of S' equals $6d+7$ with $d = \mathsf{dim}_{\mathbb{K}}(\mathbb{B})$. This implies that also the dimension of the subspace S of Π spanned by the points of Ω has this dimension and that S does not contain p. So we can extend S to a hyperplane H of Π not containing p which intersects the cone in a geometry $\Omega_{\mathbb{A}} \cong \mathsf{C}_{3,3}(\mathbb{A},\mathbb{K})$. The latter is clearly isometrically embedded in Δ. Now Proposition 5.9 yields that $\Omega_{\mathbb{A}}$ is a trace geometry, which leads immediately to case (i). Note that the last statement of that case is just Lemma 4.1.

If now $\mathbb{K} = \mathbb{F}_2$, then $\mathbb{A} = \mathbb{F}_4$ by the separability. Suppose first that $\Omega \cong \mathsf{C}_{3,3}(4,2)$. Then we get by Lemma 5.11 that it is possible that Ω is (contained in) no trace geometry, since for trace geometries, E_{Γ_Ω} must be an equator geometry and must consequently have lines contained in symplecta. This also follows from the following counting argument. One counts that p has 2^{21} opposite points in Δ. Each such point gives rise to a trace geometry in $p^\perp$. Let now q be such a point opposite p. Then $\mathscr{C}(p,q)$ contains one more point, let's say q'. Now $E(p,q) = E(p,q')$ by Lemma 2.10.4 of [23] and consequently the trace geometries $p^\perp \cap q^{\not\equiv}$ and $p^\perp \cap q'^{\not\equiv}$ coincide as well. So we have at most 2^{20} trace geometries contained in $p^\perp$, which shows that case (iii) does occur for $\Omega \cong \mathsf{C}_{3,3}(4,2)$ when combined with Lemma 5.11.

Suppose now that $\Omega \cong \mathsf{C}_{3,3}(2,2)$. It is clear that Ω can be contained in a trace geometry. We now show that it is also possible that Ω is not contained in a trace geometry. Let q be a point opposite p and let Δ be embedded as in Lemma 5.14 such that $p^\perp \cap q^{\not\equiv}$ is embedded in a standard way in $\mathsf{PG}(19,2)$. We claim that the cone over this geometry with vertex p is embedded in $\mathsf{PG}(20,2)$. Indeed, the only other option is $\mathsf{PG}(19,2)$, in which case p is contained in the subspace spanned by all points of $q^{\not\equiv}$, contradicting the observation in [1] that the embedding of Δ is *polarized*, that is, for each point x of Δ, the subspace spanned by all points of Δ not opposite x is a proper subspace and hence does not contain any point of Δ opposite x. Since $p^\perp \cap q^{\not\equiv}$ is the intersection of this cone with a hyperplane, since the embedding of Δ is homogeneous, since the stabilisator of p in Δ acts transitively on the points opposite p by the BN-pair property due to Tits (see [20, Theorem 5.2]) and since there are exactly 2^{20} trace geometries in the cone and hyperplanes in $\mathsf{PG}(20,2)$ not through p, every trace geometry corresponds to such a hyperplane.

We now take a look at the projection of the cone with vertex p over the universal embedding of $p^\perp \cap q^{\not\equiv}$ in $\Pi_{22} \cong \mathsf{PG}(22,2)$ (let's say K') onto the cone with vertex p over the standard embedding (let's say K). Then K is the projection of K' onto a subspace $\Pi_{20} \cong \mathsf{PG}(20,2)$ from a line L (disjoint from K' and Π_{20}). By Corollary 5.13 and the previous paragraph, the hyperplanes through L not containing p intersect K' in a trace geometry, while those not through L not containing p intersect K' in an isometric subgeometry isomorphic to $\mathsf{C}_{3,3}(4,2)$ that is no trace geometry. Let $\tilde{\Omega}' \cong \mathsf{C}_{3,3}(4,2)$ be such an isometric subgeometry of $\mathbb{K}'$ that is not a trace geometry, denote with Π_{21} the subspace it spans and denote with L' the intersection of Π_{21} with $\langle p, L\rangle$ (L' is a line not through p different from L). Note that then $\Pi_{20} \subseteq \Pi_{21} \cong \mathsf{PG}(21,2)$. Let $\Omega' \cong \mathsf{C}_{3,3}(2,2)$ be an isometric subgeometry of $\tilde{\Omega}'$. By Lemma 4.12, Ω' spans a 14-dimensional projective space Π_{14}, which must intersect Π_{20} in a subspace not containing p, so we can take a hyperplane Π_{19} of Π_{20} containing $\Pi_{14} \cap \Pi_{20}$ but not containing p. Now projecting Ω' onto this Π_{19} from $\langle p, L\rangle$ yields the standard embedding by Lemma 4.12. This implies that $\Pi_{14} \cap \Pi_{20}$ is 13-dimensional and the unique point p' of Π_{14} from which we can project Ω' onto its standard embedding, is contained in L'. By combining the last statement of Lemma 3.4 with the homogeneity of the universal embedding, the case $p' \neq L \cap L'$ does occur. In that case, there are no hyperplanes containing Π_{14} and L not through p and consequently Ω' is not contained in a trace geometry. So this Ω' is projected onto some $\Omega \cong \mathsf{C}_{3,3}(2,2)$ isometrically embedded in Δ not contained in a trace geometry. Note that by the previous reasoning, it is clear that exactly half of the $\mathsf{C}_{3,3}(2,2)$ that are isometrically embedded in $\mathsf{F}_{4,1}(2,4)$, are not contained in any trace geometry.

The uniqueness up to isomorphism of the embeddings described in Main Result A(iii) follows from some transitivity arguments. Note first that the group of automorphisms of Π_{22} induced by an automorphism of K (i.e. the extension of the corresponding automorphism on K'), are exactly the automorphisms of Π_{22} induced by automorphisms of K' that stabilise L. We will restrict ourselves to these automorphisms for the rest of this proof. It follows similarly as before from Lemma 3.4 that they act transitively, more exactly cyclically, on the set of three lines of $\langle p, L\rangle$ not containing p and different from L. So for the case $\Omega \cong \mathsf{C}_{3,3}(4,2)$ it suffices to prove that this group also acts transitively on the set of hyperplanes through such a line. This is the case since the group acts transitively on the set of hyperplanes through L as these correspond to hyperplanes of Π_{20}, but at the same time, the automorphisms fixing such a hyperplane through L act transitive, more exactly, cyclically, on (the set of points of) L. So the automorphism group G pointwise fixing L (and consequently $\langle p, L\rangle$), also acts transitively on the set of hyperplanes through L. This now implies that G also acts transitively on the set of hyperplanes through any other line of $\langle p, L\rangle$ not containing p by a counting argument, taking into account that the stabilisers of such hyperplanes all have the same size.

So suppose now that $\Omega \cong \mathsf{C}_{3,3}(2,2)$. By the arguments of the previous paragraph we only have to show that it can be mapped to any other $\Omega' \cong \mathsf{C}_{3,3}(2,2)$ that has the same projection point p_0 in $\langle p, L\rangle$ and is contained in the same $\Omega_0 \cong \mathsf{C}_{3,3}(4,2)$, by an automorphism of K' stabilising L. Now, all of Ω, Ω' and Ω_0 are embedded in K'. Inside the hyperplane H_0 containing Ω_0, there clearly exists an automorphism of Ω_0 mapping Ω to Ω' and one can extend that automorphism uniquely to one, say φ, of Π_{22} by assuming that also p is fixed. Then K' is stabilised. We only have to show that also L is stabilised. Clearly, φ stabilises the unique line L_0 of $\langle p, L\rangle$ contained in H_0, and it fixes $p_0 \in L_0$ by assumption. Since the stabiliser of Ω_0 inside H_0 acts cyclically on the points of L_0, φ pointwise fixes L_0. Since it aso fixes p, it pointwise fixes $\langle p, L\rangle$ and the proof is complete. □

Remark 5.15. Note that similar arguments as in the last part of the above proof, could also be used to proof that there are indeed geometries isomorphic to $\mathsf{C}_{3,3}(4,2)$ isometrically embedded in $\mathsf{F}_{4,1}(2,4)$ not contained in a trace geometry. In this way, Lemma 5.10 and Lemma 5.11 would be redundant, but then the embedding rank of $\mathsf{C}_{3,3}(4,2)$ is needed and so one uses [24, §6.5]. The authors found it more interesting to give an explicit proof that this can be done without using that and even prove that fact. However, for the geometries isomorphic to $\mathsf{C}_{3,3}(2,2)$ isometrically embedded in $\mathsf{F}_{4,1}(2,4)$ not contained in a trace geometry, we did make use of the results in [24] to limit the length of the article.

6. The inseparable case

Let Δ be a metasymplectic space $\mathsf{F}_{4,1}(\mathbb{K},\mathbb{K}')$ with $\mathbb{K}$ a field of characteristic 2 and $\mathbb{K}'$ a (possibly trivial) inseparable (multiple) quadratic field extension of $\mathbb{K}$, i.e. $(\mathbb{K}')^2 \leqslant \mathbb{K} \leqslant \mathbb{K}'$). Then as for the metasymplectic space $\mathsf{F}_{4,1}(\mathbb{F}_2,\mathbb{F}_4)$, Δ contains many dual polar spaces of rank 3 fully and isometrically embedded, but not contained in a trace geometry. Since one can no longer count in this case and our interest does not lie in classifying exactly those, we provide an example of such an embedding. Note that by Proposition 4.15 this geometry will still be contained in the perp of a point. Also, by our observations in the introduction, it should arise from the universal embedding of a point perp. The difficulty is to explicitly exhibit such an embedding and show it is not contained in a trace. Our technique consists in choosing a symp that is already not embeddable in a trace.

Example 6.1. Let p,q be two opposite points of Δ and let Ω be the point-line geometry consisting of the points in $p^\perp \cap q^{\not\equiv}$ and the lines completely contained in this set. Then $\Omega \cong \mathsf{C}_{3,3}(\mathbb{K}',\mathbb{K})$, since it is isomorphic to the point residual of p. Let $\zeta^\blacktriangle$ be a symp of Δ through p. This intersects Ω in a quadrangle $\zeta = p^\perp \cap q_\zeta^\perp$ with q_ζ the unique point of $\zeta^\blacktriangle$ symplectic to q.

We now take a look at the universal embeddings of $\zeta^\blacktriangle, \zeta$ and Ω. Let $\zeta^\blacktriangle$ be given by the equation

$$x_{-3}x_3 + x_{-2}x_2 + x_{-1}x_1 = x_0^2$$

with $(x_{-3},x_{-2},x_{-1},x_0,x_1,x_2,x_3) \in \mathbb{K}\times\mathbb{K}\times\mathbb{K}\times\mathbb{K}'\times\mathbb{K}\times\mathbb{K}\times\mathbb{K}$ where x_0 can be written as $(x_{0,0},x_{0,1},\ldots)$ in $\mathbb{K}' \cong \mathbb{K}\times\mathbb{K}\times\cdots$ ($\dim_{\mathbb{K}}(\mathbb{K}')$ factors). Further assume that p has coördinates $(1,0,0,0,0,0,0)$ and q_ζ has coördinates $(0,0,0,0,0,0,1)$. Then ζ is given in this projective space by the system of equations

$$\begin{cases} x_{-2}x_2 + x_{-1}x_1 = x_0^2, \\ x_3 = 0, \\ x_{-3} = 0. \end{cases}$$

However, by Remark 4.9, we can embed Ω in a projective space Π' over $\mathbb{K}$, such that ζ (as subspace of Ω) is universally embedded. Let the cone with vertex p over Ω be embedded in a projective space Π over $\mathbb{K}$ of one dimension more than Π' such that the induced embedding of Ω is the one described in

[8]. Then the universal embedding of the cone with vertex p over ζ arises as the intersection of a subspace S of Π with this embedding.

Now let ζ' be the intersection of $\zeta^\blacktriangle$ with the subhyperplane $x_3 = x_{-3} + x_{0,0} = 0$. Then ζ' is isomorphic to $\mathsf{B}_{2,1}(\mathbb{K}, \mathbb{K}')$. It is clearly contained in the cone with vertex p over ζ and it arises in the universal embedding of the latter as the intersection with a hyperplane H of S not containing p. Consequently it is embedded in the cone with vertex p and base Ω as the intersection of a subspace H of Π not containing p. So we can extend this H to a hyperplane H' of Π not containing p. Denote by Ω' the intersection of the cone with H'. This is clearly an isometric subgeometry of Δ isomorphic to $\mathsf{C}_{3,3}(\mathbb{K}', \mathbb{K})$.

So suppose now for a contradiction that it would be contained in a trace geometry, let's say $\Omega' \subseteq p^\perp \cap q'^{\not\equiv}$ with q' some point of Δ opposite p. Then $\Omega' \cap \zeta^\blacktriangle = \zeta'$ must be contained in $p^\perp \cap q'^\perp_\zeta$ with q'_ζ the unique point of $\zeta^\blacktriangle$ symplectic to q'. So $q'_\zeta = (y_{-3}, y_{-2}, \dots, y_3) \in \zeta^\blacktriangle$ is collinear to all points of ζ' in $\zeta^\blacktriangle$. All points $p_{a,b,c} = (a, 1, b, a, c, a^2 + bc, 0)$ with $a, b, c \in \mathbb{F}_2$ are contained in ζ'. Expressing that $q'_\zeta \perp p_{0,0,0}$ yields $y_2 = 0$, that $q \perp p_{0,1,0}$ yields $y_1 = 0$, that $q'_\zeta \perp p_{0,0,1}$ yields $y_{-1} = 0$, that also $q'_\zeta \perp p_{0,1,1}$ yields $y_{-2} = 0$ and that $q'_\zeta \perp p_{1,0,0}$ yields $y_3 = 0$. Finally expressing that q'_ζ is contained in $\zeta^\blacktriangle$, yields $y_0 = 0$, which means $q'_\zeta = p$, a contradiction.

Note that the following proof could be given before this example. However the example concludes that all cases of Main Result C do occur.

Proof of Main Result C. With Lemma 4.7, we get that $\Omega \cong \mathsf{C}_{3,3}(\mathbb{B}, \mathbb{K})$ for some quadratic alternative division algebra $\mathbb{B}$ over $\mathbb{K}$. Combining then Proposition 4.4 and Lemma 4.13, yields that the embedding is isometric or we are in case (iii). Note that the latter indeed occurs sometimes by Example 4.14. So from now on we may assume isometricity. Then with Proposition 4.15, we find a unique point p of Δ such that $\Omega \subseteq p^\perp$ and each line through p contains at most one point of Ω. Let now q be a point of Δ opposite p. Then $p^\perp \cap q^{\not\equiv}$ is isomorphic to $\mathsf{C}_{3,3}(\mathbb{A}, \mathbb{K})$ by Lemma 4.1 so after projection of Ω onto $p^\perp \cap q^{\not\equiv}$, we get a full embedding of $\mathsf{C}_{3,3}(\mathbb{B}, \mathbb{K})$ into $\mathsf{C}_{3,3}(\mathbb{K}', \mathbb{K})$. This implies by Lemma 4.12 that $\mathbb{B}(=: \mathbb{K}'')$ is a subalgebra of $\mathbb{K}'$. Finally both (i) and (ii) occur by Lemma 4.1 (which also proves the last statement of (i)) and Example 6.1, respectively. □

Remark 6.2. Call an isometric embedding of a dual polar space Ω in a metasymplectic space Δ *maximal* if every line through the unique point p of Δ collinear to each point of Ω contains a point of Ω. Using arguments similar to the ones in the proof of Main Result A for the case $(\mathbb{K}, \mathbb{A}) = (\mathbb{F}_2, \mathbb{F}_4)$, one can refine the statements of Main Result C(i) and (ii) for maximal embeddings as follows. First suppose $\mathbb{K} \neq \mathbb{F}_2$. Then the universal embedding of $\mathsf{C}_{3,3}(\mathbb{K}', \mathbb{K})$ is the standard one, say in $\mathsf{PG}(V)$, with V as defined just before Remark 4.8. This has a nucleus space N of codimension 7, just like in the case $\mathbb{K} = \mathbb{F}_2$, cf. the proof of Lemma 4.11. Now embed $\mathsf{PG}(V)$ in a projective space $\mathsf{PG}(V')$ as a hyperplane and consider the cone with base $\mathsf{C}_{3,3}(\mathbb{K}', \mathbb{K}) \subseteq \mathsf{PG}(V)$ and vertex some point p in $\mathsf{PG}(V')$ not in $\mathsf{PG}(V)$. Then there is a natural bijective correspondence between the maximal isometric embeddings of $\mathsf{C}_{3,3}(\mathbb{K}', \mathbb{K}$ and the hyperplanes H of $\mathsf{PG}(V')$ not containing p. Moreover, the hyperplanes through N precisely correspond to the traces.

In particular, for a perfect field, the nucleus subspace N has dimension 5 and the automorphism group acts transitively on the hyperplanes. This implies that, in the perfect case, there are, up to isomorphism, exactly two maximal isometric embeddings of a dual polar space. In the finite case, say $\mathbb{K} = \mathbb{F}_q$, this implies that, for a given point p of Δ, the number of embedded dual polar spaces in $p^\perp$ is equal to q^{14}, from which 2^8 are a trace. If we call a symp ξ of an isometric embedded dual polar space *straight* if it arises as the intersection of two point perps in its ambient symp $\xi^\blacktriangle$, then every embedding of $\mathsf{C}_{3,3}(\mathbb{F}_q, \mathbb{F}_q)$ in Δ that is not a trace contains precisely q^5 symps that are not straight (that is, the number of points of $N \backslash H$).

If $\mathbb{K} = \mathsf{F}_2$, the universal embedding of $\mathsf{C}_{3,3}(\mathbb{K}, \mathbb{K})$ happens in $\mathsf{PG}(14, 2)$ and hence, similarly as before, there are 2^{15} embedded dual polar spaces in $p^\perp$, fr a given point p of Δ. This time, the nucleus subspace of the universal embedding has the structure of an orthogonal (parabolic) space (meaning that the set of nuclei of the symps forms a parabolic quadric Q), which itself has a nucleus point $n \in N$. This implies that, up to isomorphism, there are exactly three embeddings of dual polar spaces: 2^8 traces (the hyperplane H, with above notation, contains N), $2^{14} - 2^8$ embeddings with exactly 2^5 symps that are not straight (the hyperplane H contains n but not N), and 2^{14} embeddings with exactly $2^5 - 2^2 = 28$ symps that are not straight (the hyperplane H does not contain n and hence intersects Q in a hyperbolic quadric). We omit the details of the proofs.

It now also follows that, in the standard embedding of $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{K})$ in $\mathsf{PG}(25, \mathbb{K})$, every hyperplane section in the subspace spanned by $p^\perp$, for any point p of $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{K})$, is a trace. This now holds for every field $\mathbb{K}$.

Data availability statement

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