An essay on Freudenthal-Tits polar spaces

Antonio Pasini Hendrik Van Maldeghem

Dedicated to the memory of Jacques Tits

Abstract

We study some properties of the nonembeddable polar spaces related to octonion division rings (and related to the index $\mathsf{E}_{7,3}^{28}$). More exactly, we classify its subspaces and show that its generating rank is equal to 5, whereas it was always believed to be at least 6. We also study self-projectivities of length 3 in the maximal singular subspaces, and these turn out to be polarities of octonion projective planes. We establish a connection between the conjugacy classes of such polarities and the orbits of the collineation group of the polar space on triples of opposite singular planes. Along the way, we classify all polar spaces for which each self-projectivity of length 3 in a maximal singular subspace is a polarity.

1 Introduction

The subject of this paper dedicated to the memory of Jacques Tits is chosen very carefully. Indeed, for Jacques, Hans Freudenthal was the first established mathematician (being 25 years older than Jacques) who took Jacques' work seriously in that he referred quite early to Jacques' papers in his own articles. For a while Jacques and Hans were working on similar topics, but they stimulated and respected each other. The outcome of this parallel work is nowadays referred to as the *Freudenthal-Tits Magic Square*, and one important ingredient of that square is the polar space with octonion projective planes. Freudenthal had written a cycle of papers on that geometry in the real case, mostly algebraic and computational, but however also containing some incidence geometry that was introduced by Jacques in his thesis [12]; that was Jacques' influence.

Also Jacques Tits dealt with these polar spaces in the early fifties of last century. However, the purposes of Jacques and Hans were quite different. Whereas Jacques was developing his general ideas about incidence geometry, diagram geometry, buildings, Witt indices, BN-pairs, even classifying polar spaces, Hans was particularly interested in the real form $\mathsf{E}_{8,4}^{28}$, and therefore also in $\mathsf{E}_{7,3}^{28}$, of the complex exceptional algebraic groups. In particular, he developed an axiom system for the geometry that we nowadays call "the nonembeddable real polar space". These axioms were later on the basis of Ferdinand Veldkamp's approach [15] to "polar geometry", which, strangely, did not include the nonembeddable polar spaces. Jacques Tits [13] closed the circle by simplifying and generalizing Ferdinand's axioms and classifying all resulting polar spaces of rank at least 3, including the nonembeddable ones. Once again this was done as part of a more general and monumental project, namely the classification of spherical buildings of rank at least 3. In fact, Jacques Tits had the axioms of polar spaces already when Veldkamp started his thesis, so the "simplification" just mentioned is purely technical, but not historically correct.

Before going on we have to clarify what exactly we mean with "polar space" and "nonembeddable polar space". Indeed, in the literature, the notion of a polar space ranges from any point-line geometry satisfying the one-or-all axiom to the precisely defined rank n geometry associated to any thick building of type B_n or D_n, $n \geq 2$. There are a lot of gradations in between, corresponding to notions like "thick", "thick-lined", "nondegenerate", "arbitrary or finite rank". In the present paper, we focus on the most strict form of polar spaces, namely those related to thick buildings of type B_n or D_n, $n \geq 2$. They have a finite rank, and when this rank is at least 3, they have been classified in [13]. The result is that either such a polar space arises from a "form" in a vector space (more exactly, an alternating or a pseudo-quadratic form), or it is the line Grassmannian of a projective space of dimension 3 over a noncommutative division ring, or it arises from the real form of relative rank 3 of an exceptional algebraic group of type E_7. In the latter case, the smallest splitting field is a subfield of an octonion division algebra; the subfield has dimension 2 over the centre of the octonion division algebra. Over that smallest splitting field, it is the fixed point structure of a semi-linear involution in a building of type E_7, see [9]. The latter polar spaces are commonly known as the "thick nonembeddable polar spaces". We will refer to them as the *Freudenthal-Tits polar spaces.* An explicit and elementary description of them can be found in [6]; we recall it briefly in Subsection 6.1.

The Freudenthal-Tits polar spaces play a somewhat isolated and special role in the theory of polar spaces. An important reason is that they always have an infinite number of points. Hence within finite geometry and combinatorics one is not concerned with them. Another reason is that they are less accessible just because they do not live in a projective space as most other polar spaces do. However, they also appear in the third row of the Freudenthal-Tits Magic Square, and as such they share some characteristic properties with the other polar spaces in that row, namely, the polar spaces of rank n embeddable in a projective space of dimension $2n-1$, different from the hyperbolic quadrics. These polar spaces appear as sub polar spaces of the Freudenthal-Tits polar spaces, and another reason to be unpopular is that, in characteristic 2, one of these sub polar spaces is related to a pseudo-quadratic form but cannot be described with an ordinary quadratic or Hermitian form. Nevertheless the Freudenthal-Tits polar spaces have some remarkable properties, and the purpose of this essay is to uncover some of them. More exactly, here is what we intend to do, with motivation.

Subspaces. In the embeddable case, all subspaces of a polar space arise from (intersecting the polar space with) subspaces of the ambient projective space in some or all of its embeddings, except possibly when the subspace is a family of $(d+1)$-dimensional singular subspaces sharing a common d-dimensional subspace, for some $d \geq -1$ (see [3]). This leaves us with the problem of classifying all subspaces of the nonembeddable polar spaces. In [5], Cohen & Shult determine all geometric hyperplanes of such polar spaces; in [11], we determine all subspaces of the line Grassmannians of projective 3-spaces. In the present paper we complete the job by classifying all subspaces of the Freudenthal-Tits polar spaces. We prove the following result in Section 3.

Main Result 1. *A proper subspace of a Freudenthal-Tits polar space is one of the following.*

(1) *An arbitrary set of noncollinear points.*
(2) *The union of an arbitrary set of noncoplanar lines through a given point.*
(3) *The union of an arbitrary set of planes through a given line.*
(4) *The set of all points collinear to a given point.*
(5) *The set of all points collinear to two given noncollinear points.*

This entails a shortcut in some arguments in [11], but more importantly, it triggered the next property, which really came as a surprise.

Generating rank. Usually, the generating rank of a geometry is investigated together with the embedding rank, because of the intimate connection between those. Nonembeddable geometries, however, have no embedding rank, and so the generating rank provides, so to speak, the moral embedding rank. For the Freudenthal-Tits polar spaces, it was long thought that their 'moral' embedding rank is 6, since 6 is the embedding rank of their siblings—the rank 3 polar spaces over quadratic associative division algebras with trivial anisotropic kernel, and since embeddable polar spaces of rank n have generating rank at least $2n$. However, we will show in Section 4 that

Main Result 2. *The generating rank of any nonembeddable polar space of rank* 3 *is* 5.

For our third main result, we go back to one of Freudenthal's axioms in [8]. With modern terminology (see below), the said axiom states that every self-projectivity of the generators of length 3 is a polarity.

Self-projectivities of length 3. Consider two opposite generators G_1, G_2 of a polar space. The projection map (see [13]) defines an anti-isomorphism $G_1 \to G_2$, mapping points to subgenerators, preserving incidence. For an ordered triple of opposite generators (G_1, G_2, G_3) of a polar space, the mapping $G_1 \to G_2 \to G_3 \to G_1$ is a duality of G_1, called a self-projectivity of length 3. Freudenthal's axiom states that every self-projectivity of length 3 is a polarity. One might wonder whether this axiom characterizes the Freudenthals-Tits polar spaces, or at least its siblings. However, Lemma 6.2 of [7] implies that this is also true for all polar spaces of hyperbolic type and even rank (where the polarity is always a symplectic one). We will show that every self-projectivity of length 3 in a polar space of rank $n \geq 3$ is a polarity if and only if the polar space either admits an embedding in $(2n-1)$-dimensional space, or is nonembeddable. Note that this includes the case where there are no triples of pairwise opposite generators. This allows us to prove our third main result.

Main Result 3. *The orbits in the full collineation group of a Freudenthal-Tits polar space of triples of pairwise opposite generators are in natural correspondence with the conjugacy classes of self-projectivities of length* 3*, which are polarities, with respect to the induced collineation group of a given generator.*

In fact the same holds for every polar space related to the third row of the Freudenthal-Tits Magic Square. See below for an explicit enumeration.

2 Preliminaries

The main mathematical notion of this paper is that of a polar space Δ of finite rank r, which we here introduce.

2.1 Abstract point-line geometries

A pair $\Delta = (P, \mathscr{L})$, with $\mathscr{L}$ a nonempty set of subsets of P, each one of size at least 2, is a(n abstract) point-line geometry (P is the set of points, $\mathscr{L}$ the set of lines).

Points $x, y \in X$ contained in a common line are called *collinear*, denoted $x \perp y$; the set of all points collinear to x is denoted by $x^\perp$; the set of all points collinear to all points of a set $X \subseteq P$ is denoted by $X^\perp$. We will always deal with situations where every point is contained in at least one line, so $x \in x^\perp$.

We say that Δ is *connected* if every pair of points can be joined with a finite sequence of consecutively nondisjoint lines. The point-line geometry Γ is called a *partial linear space* if each pair of distinct points is contained in at most one line.

A *subspace* of Δ is a subset S of P such that, if $x, y \in S$ are collinear and distinct, then all lines containing both x and y are contained in S. We will often treat a subspace as a point-line geometry, with naturally induced line set. A subspace S is *singular* if every pair of points in it is collinear. If A is a set of points, then $\langle A \rangle$ denotes the subspace generated by A (that is, the intersection of all subspaces containing A); if A consists of two distinct collinear points p and q, then, if $\langle A \rangle$ is a unique line, it is sometimes briefly denoted by pq. A proper subspace H is called a *geometric hyperplane* if each line of Δ has either one or all its points contained in H. A *subgeometry* Δ' of $\Delta = (P, \mathscr{L})$ is a point-line geometry $(P', \mathscr{L}')$ where $P' \subseteq P$ and the members of $\mathscr{L}'$ are intersections of members of $\mathscr{L}$ with P'. A subgeometry $\Delta' = (P', \mathscr{L}')$ is called *full* if $\mathscr{L}'$ is a subfamily of $\mathscr{L}$; it is called *complete* if it is full and no member of $\mathscr{L} \setminus \mathscr{L}'$ is fully contained in P'; it is called *ideal at the point* $p \in P'$ if no line of Δ through p intersects P' in only p (and p is called a deep point of Δ'); it is called *ideal* if it is ideal at all its points. A standard argument shows that an ideal and full subgeometry of a connected geometry Δ coincides with Δ itself.

2.2 Polar spaces

In what follows, a projective space $\mathrm{PG}(V)$ of dimension $d \geq 3$ is the poset of subspaces of a vector space V of dimension $d+1$. The 1-dimensional subspaces of V are called the *points* of $\mathrm{PG}(V)$, and the 2-dimensional subspaces the *lines*. These form the natural point-line geometry associated to $\mathrm{PG}(V)$. A projective space of dimension 2 is an axiomatic projective plane, and a projective space of dimension 1 is a set of size at least 3. We will usually make no distinction between a projective space, its point-line geometry, or even its point set, when it is clear from the context what is meant.

A *polar space* $\Delta = (P, \mathscr{L})$ is a point-line geometry satisfying the following four axioms, due to Buekenhout and Shult [2], which simplify the axiom system in [13].

(PS1) Every line contains at least three points, i.e., every line is *thick*.
(PS2) No point is collinear to every other point.
(PS3) Every nested sequence of singular subspaces is finite.
(PS4) The set of points incident with a given arbitrary line L and collinear to a given arbitrary point p is either a singleton or coincides with L.

We will assume that the reader is familiar with the basic theory of polar spaces, see for instance [1]. Let us recall that every polar space, as defined above, is a partial linear space and has a unique *rank*, given by the length of the longest nested sequence of singular subspaces (including the empty set); the rank is always assumed to be finite (by (PS3)) and at least 2 since we always have a sequence $\emptyset \subseteq \{p\} \subseteq L$, for a line $L \in \mathscr{L}$ and a point $p \in L$. A maximal singular subspace of a polar polar space is also called a *generator*

Now let $\Delta = (P, \mathscr{L})$ be a polar space of rank $r \geq 2$. It is well known that the generators are mutually isomorphic projective spaces of dimension $r-1$ (and so two arbitrary points of Δ are contained in at most one line). Moreover, there is a (not necessarily finite) constant t such that every singular subspace of dimension $r-2$ (which we will call a *subgenerator*) is contained in exactly $t+1$ generators. If $t=1$, then we say that Δ is of *hyperbolic type*, or is a *hyperbolic* polar space. For each point $p \in P$, the set $p^\perp$ is a geometric hyperplane with deep point p. It is called a *singular hyperplane*, slightly confusing terminology since it is not singular as a subspace, but we let historical reasons overrule this objection. Axioms (PS2) and (PS4) together are equivalent to saying that for each point p, $p^\perp$ is a geometric hyperplane. A *Shult space* is a point-line geometry satisfying (PS4). If all singular subspaces are projective spaces, and the Shult space satisfies (PS3), then the *rank* $\mathrm{rk}(\Delta)$ of the Shult space Δ is the (projective) dimension of a maximal singular subspace minus one. A Shult space is *degenerate* if (PS2) is violated for some point.

A *partial ovoid* of a polar space Δ is a set of mutually noncollinear points. A partial ovoid is an *ovoid* if every generator intersects it in a unique point.

We will use some notions of the theory of buildings in polar spaces. For instance, two subspaces are called *opposite* if no point of their union is collinear to every point of this union; in particular two points are opposite if, and only if, they are not collinear and two maximal singular subspaces are opposite if, and only if, they are disjoint. Opposite subspaces necessarily have the same dimension. Also, the *residue* Δ_p *of a point* p in a polar space Δ of rank at least 3 is the point-line geometry $(\mathscr{L}_p, \mathscr{P}_p)$ with point set the set of lines containing p and line set the line pencils with vertex p in planes of Δ (planes are the 2-dimensional singular subspaces and are projective planes). The residue Δ_p is isomorphic (with the usual and obvious notion of isomorphism between geometries, using the usual symbol $\cong$) to the (point-line geometry induced by the) subspace $\{p, q\}^\perp = p^\perp \cap q^\perp$, for any point q opposite p.

Finally, the *dual* of a point-line geometry $\Delta = (P, \mathscr{L})$ is the point-line geometry $\Delta^* = (\mathscr{L}, \mathscr{P}^*)$, where a generic element of $\mathscr{P}^*$ is the set of all lines of Δ through a given point of Δ. If Δ is a polar space of rank 2 not of hyperbolic type, then the dual is again a polar space of rank 2, not of hyperbolic type. Polar spaces of rank 2 and their duals are nontrivial examples of *generalized quadrangles* (which are Shult spaces satisfying (PS2) and such that lines are the generators).

2.3 Some specific polar spaces

As noted in the introduction, any polar space of rank at least 3 either arises from a form on a vector space, or belongs to one of two classes of nonembeddable polar spaces. The ones arising from a form in a vector space V admit an embedding (or representation) in the projective space $\mathrm{PG}(V)$ in that they are full subgeometries of the point-line geometry associated to $\mathrm{PG}(V)$. We give a few examples that are relevant for the sequel. We use notation borrowed from the theory of Lie incidence geometries.

1. The symplectic polar space $\mathsf{C}_{n,1}(\mathbb{K})$ over a field $\mathbb{K}$ arises from a nondegenerate alternating form in a vector space of dimension $2n$. Hence its point set is $\mathrm{PG}(2n-1,\mathbb{K})$. A generic line is the line spanned by the points with coordinates $(x_{-n},\ldots,x_{-1},x_1,\ldots,x_n)$ and $(y_{-n},\ldots,y_{-1},y_1,\ldots,y_n)$ with $x_1y_{-1}-x_{-1}y_1+\cdots+x_ny_{-n}-x_{-n}y_n=0$.
2. For a field $\mathbb{K}$ of characteristic 2 and a proper subfield $\mathbb{K}'$ containing all squares of elements of $\mathbb{K}$, the complete subgeometry of $\mathrm{PG}(2n-1,\mathbb{K})$ defined in coordinates by the points satisfying $x_1x_{-1}+x_2x_{-2}+\cdots+x_nx_{-n}\in\mathbb{K}'$ is a subspace of $\mathsf{C}_{n,1}(\mathbb{K})$ defined above and denoted by $\mathsf{C}_{n,1}(\mathbb{K},\mathbb{K}')$. Usually one also denotes $\mathsf{C}_{n,1}(\mathbb{K})$ by $\mathsf{C}_{n,1}(\mathbb{K},\mathbb{K})$.
3. Let $\mathbb{A}$ be either a separable quadratic extension of $\mathbb{K}$, or a quaternion division algebra over $\mathbb{K}$. Let $x\mapsto\bar{x}$ be the standard involution in $\mathbb{A}$ as a quadratic algebra over $\mathbb{K}$. Then $\mathsf{C}_{n,1}(\mathbb{A},\mathbb{K})$ is the complete subgeometry of $\mathrm{PG}(2n-1,\mathbb{A})$ defined by all points whose coordinates satisfy $\bar{x}_1x_{-1}+\bar{x}_2x_{-2}+\cdots+\bar{x}_nx_{-n}\in\mathbb{K}$. When $\mathbb{A}$ is commutative, or $\mathbb{K}$ has characteristic different from 2, this point set coincides with the Hermitian variety with equation $\bar{x}_1x_{-1}-\bar{x}_{-1}x_1+\cdots+\bar{x}_nx_{-n}-\bar{x}_{-n}x_n=0$.
4. Let $\mathbb{A}$ be either a separable extension of $\mathbb{K}$, a quaternion division algebra over $\mathbb{K}$, or an octonion division algebra over $\mathbb{K}$. Set $d=\dim_{\mathbb{K}}\mathbb{A}$. Let $q(x)$ be the (anisotropic) norm of $x\in\mathbb{A}$ as a quadratic division algebra. Set $V=\mathbb{K}\times\mathbb{K}\times\mathbb{K}\times\mathbb{K}\times\mathbb{A}$, considered in the natural way as a vector space over $\mathbb{K}$, and define the complete subgeometry of $\mathrm{PG}(d+3,\mathbb{K})$ by the equation $x_2x_{-2}+x_1x_{-1}=q(x_0)$, where (x_{-2},x_{-1},x_1,x_2) are coordinates in $\mathbb{K}\times\mathbb{K}\times\mathbb{K}\times\mathbb{K}$ and $x_0\in\mathbb{A}$. Then this defines a polar space $\mathsf{B}_{2,1}(\mathbb{K},\mathbb{A})$ of rank 2. We will denote the dual by $\mathsf{C}_{2,1}(\mathbb{A},\mathbb{K})$, which is consistent with the previous example if $\mathbb{A}$ is associative by Propositions 3.4.9, 3.4.11 and 3.4.13 of [14].
5. For every skew field $\mathbb{K}$, the *line Grassmannian* of $\mathrm{PG}(3,\mathbb{K})$, denoted by $\mathsf{A}_{3,2}(\mathbb{K})$, is a polar space of rank 3, nonembeddabe in projective space if $\mathbb{K}$ is noncommutative. Its point set is the set of lines of $\mathrm{PG}(3,\mathbb{K})$, whereas its line set is the set of all planar line pencils. If $\mathbb{K}$ is commutative, then $\mathsf{A}_{3,2}(\mathbb{K})$ is isomorphic to the hyperbolic polar space given by the Klein quadric in $\mathrm{PG}(5,\mathbb{K})$.
6. For every octonion division algebra $\mathbb{O}$ there exists a unique polar space $\mathsf{C}_{3,1}(\mathbb{O},\mathbb{K})$ of rank 3 whose planes are the nondesarguesian projective planes over $\mathbb{O}$. A concrete description is given in Subsection 6.1. The residue at each point is the polar space $\mathsf{C}_{2,1}(\mathbb{O},\mathbb{K})$, and witnesses consistency of the notation despite the separate definition of $\mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$ for $\mathbb{A}=\mathbb{O}$ (however, the description in Subsection 6.1 holds for all quadratic alternative division algebras $\mathbb{A}$ explaining the notational consistency). We will call $\mathsf{C}_{3,1}(\mathbb{O},\mathbb{K})$ a *Freudenthal-Tits polar space (over $\mathbb{O}$).* If $\mathbb{K}\le\mathbb{L}\le\mathbb{H}\le\mathbb{O}$, with $\mathbb{L}$ a separable quadratic extension of $\mathbb{K}$ and $\mathbb{H}$ a quaternion algebra over $\mathbb{K}$, then we have the following natural inclusions, also

denoted with the symbol $\leq$, of geometries

$$\mathsf{C}_{3,1}(\mathbb{K}) \leq \mathsf{C}_{3,1}(\mathbb{L},\mathbb{K}) \leq \mathsf{C}_{3,1}(\mathbb{H},\mathbb{K}) \leq \mathsf{C}_{3,1}(\mathbb{O},\mathbb{K}).$$

2.4 Hyperbolic lines

Let $\Delta = (P, \mathscr{L})$ be a polar space, and let x, y be two opposite points. Then the *hyperbolic line* through x and y is the set $(\{x,y\}^{\perp})^{\perp} = \{x,y\}^{\perp\perp}$. A hyperbolic line is said to be *nontrivial* if its size is at least 3. The hyperbolic lines of $\mathsf{C}_{n,1}(\mathbb{K})$ are nontrivial and are just the lines of the ambient projective space that are not lines of the polar space. Also, the hyperbolic lines of $\mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$, with $\mathbb{A}$ either a separable extension of $\mathbb{K}$, a quaternion division algebra over $\mathbb{K}$, or an octonion division algebra over $\mathbb{K}$, are exactly the hyperbolic lines in all subgeometries $\mathsf{C}_{3,1}(\mathbb{K})$ naturally included (that is, by restricting $\mathbb{A}$ to $\mathbb{K}$ with respect to the descriptions we gave); these inclusions have the property that each plane of the large polar space through a line of the small polar space contains a unique plane of the small polar space.

3 Subspaces

Throughout this section $\Delta = \mathsf{C}_{3,1}(\mathbb{O},\mathbb{K}) = (P, \mathscr{L})$ is a Freudenthal-Tits polar space for some octonion division algebra over the field $\mathbb{K}$. It is known that all hyperplanes of Δ are singular (Cohen and Shult [5]). Turning to arbitrary subspaces of Δ, we shall prove the following:

Theorem 3.1 *Let X be a proper subspace of Δ. Then X is one of the following:*

(1) *a set of mutually opposite points;*
(2) *the union of a set of mutually noncoplanar lines through a given point of Δ;*
(3) *the union of a set of planes through a given line of Δ;*
(4) *a singular hyperplane $a^{\perp}$ of Δ;*
(5) *the common perp $\{a,b\}^{\perp}$ of two opposite points a, b of Δ.*

The next lemma, proved in [11], is the first step in the proof of Theorem 3.1.

Lemma 3.2 *Let X be a proper subspace of Δ. Then X is either as in cases* (1)–(4) *of Theorem* 3.1 *or the induced geometry is a generalized quadrangle, closed under taking hyperbolic lines.*

Proof. This statement is indeed Lemma 2.16 of [11]. The following is a simplified version of the proof we gave in [11] for that lemma. Let X be a proper subspace of Δ. So, X is a Shult space or a partial ovoid. Suppose that X is not a partial ovoid, then its rank is at least 2. The residue of a point of Δ admits no proper full subquadrangle [14, Proposition 5.9.4]. Consequently, if X is degenerate then it falls into one of the cases (2), (3) or (4) of Theorem 3.1. Suppose now that X is nondegenerate and $2 \leq \mathrm{rk}(X) \leq 3$. If $\mathrm{rk}(X) = 3$ then the fact that, as said above, the point-residues of Δ admit no proper full subquadrangles, implies that X is an ideal subspace of Δ. Hence $X = P$. However $X \subset P$ by assumption. Therefore $\mathrm{rk}(X) = 2$. We shall now show that if a, b are two opposite points of X then $X \supseteq \{a, b\}^{\perp\perp}$.

Given two opposite points a, b of X, let x_1, x_2 be opposite points in $\{a, b\}^{\perp} \cap X$ and let L_1, L_2 be opposite lines in $\{a, b\}^{\perp}$ containing x_1 and x_2 respectively. Clearly, $L_i \cap X = \{x_i\}$, since $\mathrm{rk}(X) = 2$. Put $y_i = x_j^{\perp} \cap L_i$ for $\{i, j\} = \{1, 2\}$. Then y_1 and y_2 are opposite and neither of them belongs to X. The intersection $\{y_1, y_2\}^{\perp} \cap X$ contains the points a, b, x_1, x_2, which form an ordinary quadrangle. Hence $\{y_1, y_2\}^{\perp} \cap X$ is a full subquadrangle of $\{y_1, y_2\}^{\perp}$. However $\{y_1, y_2\}^{\perp}$ admits no proper full subquadrangles by [14, Proposition 5.9.4]. Hence $\{y_1, y_2\}^{\perp} \subseteq X$. In particular, $\{a, b\}^{\perp\perp} \subset X$, as claimed. □

In view of Lemma 3.2, proving Theorem 3.1 amounts to proving the following:

Lemma 3.3 *Let Q be a polar subspace of Δ of rank 2, closed under taking hyperbolic lines. Then $Q = \{a, b\}^{\perp}$ for two noncollinear points $a, b \in P$.*

3.1 Proof of Lemma 3.3

With Q as in the hypotheses of Lemma 3.3, let a be a point of Δ exterior to Q. If $a^{\perp} \supseteq Q$ then we say that a is *bright* (with respect to Q), otherwise we say that a is *faint*. If a is faint then the intersection $a^{\perp} \cap Q$ is a (geometric) hyperplane of Q, hence it is either a proper full subquadrangle of Q, or the perp $c^{\perp} \cap Q$ in Q of a point $c \in Q$, or an ovoid of Q. We say that a is of *quadrangular*, *singular* or *ovoidal type* according to whether the first, the second or the third one of these three cases occurs.

Note that, as the residue Δ_a at a point $a \in P$ admits no proper full subquadrangles [14, Proposition 5.9.4], we have $\{a, b\}^{\perp} = \langle L, L' \rangle$ for any two opposite lines L, L' of $\{a, b\}^{\perp}$.

Lemma 3.4 *Let $a \notin Q$ be either bright or faint of quadrangular type. Then every line through a meets Q in a point, every plane through a meets Q in a line and we have $a^{\perp} \cap Q \cong \Delta_a$.*

Proof. As $a^{\perp} \cap Q$ is a full subquadrange of Q, possibly equal to Q, the lines of a which meet Q nontrivially form a full subquadrangle of Δ_a. Since, however, Δ_a admits no proper full subquadrangles, the assertions follow. □

Corollary 3.5 *If a bright point exists then Lemma 3.3 holds true.*

Proof. Let $a \notin Q$ be bright. Then $Q \cong \Delta_a$ by Lemma 3.4. We know that any two opposite lines of Δ_a span Δ_a. Hence the same is true for Q. Let L, L' be two opposite lines of Q. A point $b \not\perp a$ exists such that $L \cup L' \subseteq b^{\perp}$ (choose a plane α on L not containing a and put $b := \alpha \cap L'^{\perp}$). So, $L \cup L' \subseteq \{a, b\}^{\perp}$. However $\{a, b\}^{\perp} = \langle L, L' \rangle$, as we know. Hence $Q = \{a, b\}^{\perp}$. □

Lemma 3.6 *If there exist a faint point of singular type, then a bright point also exists.*

Proof. Let $b \notin Q$ be such that $b^{\perp} \cap Q = c^{\perp} \cap Q$ for some $c \in Q$. Let $d \in Q$ be not collinear with c. Let a be the unique point on $\langle b, c\rangle$ collinear to d. Then $a^{\perp} \cap Q$ contains $b^{\perp} \cap Q = c^{\perp} \cap Q$ and the additional point d. Hence $a^{\perp} \cap Q$ is a subquadrangle of Q, it is ideal at c, and since it is also full, it coincides with Q by Propositions 1.8.1 and 1.8.2 of [14]. $\square$

In view of Lemma 3.6 and Corollary 3.5, we can focus on the following situation:

($*$) all points exterior to Q are faint and none of them has singular type.

If we show that this is impossible, then Lemma 3.3 is proved.

Lemma 3.7 *Assume ($*$) and let a be a faint point of quadrangular type. Then all points exterior to Q and collinear with a are faint of quadrangular type.*

Proof. If $b \perp a$ then all planes through $\langle a, b\rangle$ meet Q in a line, by Lemma 3.4. Hence b cannot be of ovoidal type. As no points of singular type exist, b is of quadrangular type. $\square$

Corollary 3.8 *Assume ($*$). Then all points exterior to Q are faint of quadrangular type.*

Proof. Pick a point a of quadrangular type. Such a point always exists. Indeed, given any two opposite lines L, L' of Q, every point $a \in (L \cup L')^{\perp}$ is (faint because of ($*$) and) of quadrangular type. By Lemma 3.7, all points in the same connected component of $\Delta \setminus Q$ as a are faint of quadrangular type. If $\Delta \setminus Q$ is connected then we are done. Assuming that $\Delta \setminus Q$ is not connected, let $b \in \Delta \setminus Q$ belong to a connected component different from the one containing a. Then $\{a, b\}^{\perp} \subseteq Q$. Therefore $b^{\perp} \cap Q$ contains $\{a, b\}^{\perp}$. Hence b is of quadrangular type. $\square$

End of the proof. Assume ($*$). By Corollary 3.8, all points exterior to Q are faint of quadrangular type. Hence every line either is contained in Q or all of its points but at most one are faint of quadrangular type. However, by Lemma 3.4, every line through a faint point of quadrangular type meets Q in a point. So, every line meets Q nontrivially. It follows that Q is a hyperplane. However we know from [5] that all hyperplanes of Δ are singular. This contradiction shows that the situation described in ($*$) is impossible. $\square$

3.2 Comments on the nonembeddable hyperbolic case

As proved in [11] (Lemma 2.15), the statement of Theorem 3.1 also holds true if $\Delta \cong \mathsf{A}_{3,2}(\mathbb{K})$, with $\mathbb{K}$ noncommutative. The proof we offered in [11] for that statement is fairly different from the one we have given here for the Freudenthal-Tits polar spaces. However the present proof suits that hyperbolic case as well, even word-by-word. The following is the unique difference between these two cases: while in the Freudenthal-Tits case, for a full subquadrangle of Δ, being closed under taking hyperbolic lines is a significant property, in the hyperbolic case this property comes down to a triviality. Apart from this, all the rest is the same. In particular, the proof we have exposed here pivots on the following two facts: all hyperplanes are singular and the residue of a point admits no proper full subquadrangles. Both these properties also hold true in the hyperbolic case: the first one is proved by Cohen and Shult [5] for all nonembeddable polar spaces, while the latter is trivial in the hyperbolic case.

4 Generating rank

We recall that the *generating rank* of a point-line geometry Γ is the minimum size of a set of points of Γ which generates Γ. Obviously, the generating rank of an embeddable polar space of rank n is at least $2n$ and every generalized quadrangle has generating rank at least 4. Thus, one would quite naturally conjecture that a nonembeddable polar space of rank 3 has generating rank at least 6. In this section we disprove this conjecture. Indeed, with the help of Theorem 3.1 and its hyperbolic analogue we shall prove the following:

Theorem 4.1 *The generating rank of a nonembeddable polar space of rank* 3 *is equal to* 5*.*

Before turning to the proof of Theorem 4.1, we mention a characterization of symplectic polar spaces. Of course, nonembeddable polar spaces do not satisfy the hypothesis. The proof of Theorem 4.1 will pivot on this remark.

Given a polar space $\Delta = (P, \mathscr{L})$, let $\mathscr{H}$ be the set of hyperbolic lines of Δ and let $\mathfrak{L}(\Delta) = (P, \mathscr{L} \cup \mathscr{H})$ be the point-line geometry with the same points as Δ but $\mathscr{L} \cup \mathscr{H}$ as the family of lines. It is folklore that this geometry is a linear space (any two distinct points are joined by a unique line) and the polar space Δ is symplectic if and only if $\mathfrak{L}(\Delta)$ is a projective space. If this is the case then we can choose $\mathfrak{L}(\Delta)$ as Σ. The symplectic polarity of $\Sigma = \mathfrak{L}(\Delta)$ associated to Δ maps every point $x \in P$ onto the hyperplane $x^\perp$ of Δ, which turns out to be a hyperplane of $\mathfrak{L}(\Delta)$ too. The property that $x^\perp$ is geometric hyperplane of the point-line geometry $\mathfrak{L}(\Delta)$ for every $x \in P$ is in turn equivalent to Δ being symplectic. Indeed, as proved in [4] we have:

Proposition 4.2 *The linear space $\mathfrak{L}(\Delta)$ is projective if and only if $x^\perp \cap h \neq \emptyset$ for every point $x \in P$ and every hyperbolic line $h \in \mathscr{H}$.*

Proof of Theorem 4.1. Let Δ be a nonembeddable polar space of rank 3. Let c_1, c_2, c_3, c_4 be four points of Δ forming an ordinary quadrangle, say $c_1 \perp c_2 \perp c_3 \perp c_4 \perp c_1$, $c_1 \not\perp c_3$ and $c_2 \not\perp c_4$. We can also assume to have chosen these points in such a way that the lines $\langle c_1, c_2 \rangle$ and $\langle c_3, c_4 \rangle$ are mutually opposite. With c_1, c_2, c_3 and c_4 chosen in this way, we have $\langle c_1, c_2, c_3, c_4 \rangle = \{a, b\}^\perp$ for two opposite points a and b of Δ by Theorem 3.1 (if Δ is Freudenthal-Tits) or its hyperbolic analogue (when $\Delta \cong \mathsf{A}_{3,2}(\mathbb{K})$, $\mathbb{K}$ not commutative). As Δ is not a symplectic polar space, Proposition 4.2 implies that there exists a point c_0 such that $c_0^\perp \cap \{a, b\}^{\perp\perp} = \emptyset$. Let $X := \langle c_0, c_1, c_2, c_3, c_4 \rangle$. Then X properly contains $\{a, b\}^\perp$. By Theorem 3.1 (or its hyperbolic analogue) either $X = x^\perp$ for a point $x \in P$ or $X = P$, which would prove the theorem. So we can assume that $X = x^\perp$ and $\{a,b\}^\perp \subseteq x^\perp$. This inclusion forces $x \in \{a,b\}^{\perp\perp}$. However $x \in c_0^\perp$, since $c_0 \in x^\perp$. Hence $c_0^\perp$ meets $\{a,b\}^{\perp\perp}$ in a point, namely x. This contradicts the choice of c_0. Therefore $X = P$.

So, five points are enough to generate Δ. Clearly, four points cannot do the job: this is trivial if the four points are pairwise non-collinear; if two of them are collinear, say they generate the line L, then the other two points are collinear to at least one common point x of an arbitrary plane through L and so the span of the four points is contained in the singular hyperplane $x^\perp$. Hence 5 is indeed the generating rank of Δ. □

5 Self-projectivities of length 3

5.1 Some terminology and notation for dualities of projective spaces

We recall that a *duality* of a finite-dimensional projective space Σ is an anti-automorphism of the poset of all subspaces of Σ, namely a bijection of that poset which reverses the inclusion relation. A duality θ of Σ is a *polarity* precisely when $\theta = \theta^{-1}$ (equivalently, θ^2 is the identity mapping).

A subspace X of Σ is *absolute* for a duality θ if $\{X, \theta(X)\}$ is a flag (possibly $X = \theta(X)$). We denote by $A(\theta)$ the set of subspaces which are absolute for θ. Clearly, $A(\theta) = A(\theta^{-1})$ and $\theta(A(\theta)) = A(\theta)$.

We say that a subspace X of Σ is θ-*stable* (also *stable*, for short, when the reference to θ is clear from the context) when $\theta^2(X) = X$ (equivalently, $\theta(X) = \theta^{-1}(X)$). Let $S(\theta)$ be the set of θ-stable subspaces of Σ. Then $S(\theta) = S(\theta^{-1})$ and $\theta(S(\theta)) = S(\theta)$. Moreover, $S(\theta)$ is closed under taking arbitrary intersections and spans.

Obviously, θ is a polarity if and only if all subspaces of Σ are stable; equivalently all points of Σ are stable (equivalently, all hyperplanes of Σ are stable).

5.2 Dualities from self-projectivities of length 3

We recall that when Δ is not hyperbolic, then for any pair of opposite generators of Δ a third generator always exists which is opposite both of them. The same holds true if Δ is hyperbolic but $\mathrm{rk}(\Delta)$ is even. When Δ is hyperbolic of odd rank, then Δ admits no triple of mutually opposite generators.

For the rest of this subsection Δ is a polar space of rank $n \geq 2$, with n even when Δ is hyperbolic, and M_1, M_2, M_3 are three pairwise opposite generators of Δ. For $i \neq j$ let θ_i^j be the projection from M_i onto M_j, which maps every projective subspace X of M_i onto $X^\perp \cap M_j$. For $\{i,j,k\} = \{1,2,3\}$ put $\theta_{i,j,k} = \theta_k^i \circ \theta_j^k \circ \theta_i^j$. Note that $\theta_{i,k,j} = \theta_{i,j,k}^{-1}$. So, $\theta_{i,j,k}$ is a polarity if and only if $\theta_{i,j,k} = \theta_{i,k,j}$.

Set $\theta := \theta_{1,2,3}$. According to the notation introduced in the previous subsection, $A(\theta)$ and $S(\theta)$ are the families of subspaces of M_1 which are absolute for θ and θ-stable, respectively.

Proposition 5.1 *Let X be a projective subspace of M_1 and let $d = \dim(X)$ be its dimension.*

(1) *Let $d \leq n/2-1$. Then $X \in A(\theta)$ if and only if Δ admits a singular subspace Y of dimension $\dim(Y) = 2d+1$ such that Y contains X and meets each of M_2 and M_3 in a d-dimensional subspace. Moreover, if $X \in A(\theta)$ then just one singular subspace Y exists which satisfies these properties and, if Y is that subspace, then $\theta(X) = Y^\perp \cap M_1$ and $\theta_2^3(\theta_1^2(X)) = Y \cap M_3$.*

(2) *Let $d \geq n/2-1$. Then $X \in A(\theta)$ if and only if Δ admits a singular subspace Y of dimension $\dim(Y) = 2(n-d)-3$ such that Y is contained in $X^\perp$ and meets each of X, M_2 and M_3 in a $(n-d-2)$-dimensional subspace. If $X \in A(\theta)$ then just one singular subspace Y exists with these properties and we have $\theta(X) = Y \cap M_1$ and $\theta_1^2(X) = Y \cap M_2$.*

Proof. Put $X_2 := \theta_1^2(X)$ and $X_3 := \theta_2^3(X_2)$, for ease of notation. Let $d \leq n/2-1$ and suppose that $X \in A(\theta)$, so $\theta(X) \supseteq X$. Then $\theta(X) = X_3^\perp \cap M_1$ and $X \perp X_3$, since $X \subseteq \theta(X)$ by assumption. However $X_3 \perp X_2$. Hence X_3 is contained in the generator $M := \langle X, X_2\rangle$. Accordingly, $Y := \langle X, X_3\rangle$ is a $(2d+1)$-dimensional subspace of M (recall that $\dim(X_3) = \dim(X) = d$ and $X_3 \cap X = \emptyset$). As $\dim(X_2) = n-d-2$, the Grassmann formula for dimensions yields $\dim(Y \cap X_2) = d$. So, Y enjoys the properties required in (1). In particular, $\theta(X) \perp Y$ because $Y = \langle X, X_3\rangle$ and $\theta(X) \perp X_3$. Hence $\theta(X) = X_3^\perp \cap M_1 = Y^\perp \cap M_1$.

Conversely, suppose that a singular subspace Y as in (1) exists. Then $Y = \langle X, Y \cap M_2\rangle$, $X_2 \supseteq Y \cap M_2$ and $X_2 \perp Y$. Accordingly, $X_3 = Y \cap M_3$. Hence $\theta(X) = \theta_3^1(X_3) = (Y \cap M_3)^\perp \cap M_1 \supseteq X$. Note also that $Y = \langle X X_3\rangle$. The uniqueness of Y is also proved.

Let now $d \geq n/2-1$ and put $X' := \theta(X)$. Then $\dim(X') = d' := n-d-2 \leq n/2-1$. We have $X \in A(\theta)$ if and only if $X' \in A(\theta)$. Claim (2) on X follows from claim (1) on X'. □

Corollary 5.2 *We have $A(\theta) \subseteq S(\theta)$.*

Proof. The conditions which characterize Y in (1) and (2) of Proposition 5.1 are symmetric with respect to M_2 and M_3. Moreover, $\theta(X) = Y^\perp \cap M_1$ in case (1) and $\theta(X) = Y \cap M_1$ in case (2) of Proposition 5.1. The inclusion $A(\theta) \subseteq S(\theta)$ follows. □

Proposition 5.1 also implies the following:

Corollary 5.3 *A subspace X of M_1 with $\dim(X) \leq n/2-1$ belongs to $A(\theta)$ only if $X \subseteq \langle M_2, M_3\rangle$. In particular, all absolute points of θ belong to $\langle M_2, M_3\rangle$. Accordingly, they span M_1 only if $M_1 \subseteq \langle M_2, M_3\rangle$.*

Example 5.4 Let Δ be hyperbolic of even rank n. Then for any three mutually opposite generators M_1, M_2, M_3 of Δ, the duality $\theta_{1,2,3}$ is a symplectic polarity (the identity when $n = 2$). The special case $n = 6$ of this claim is a step of the proof of Lemma 6.2 of [7] but the argument used in [7] to prove that claim can easily be generalized to any even n, as follows. For $x \in M_1$, put $y = \theta_2^3\theta_1^2(x)$. Then $\langle x, \theta_1^2(x)\rangle$ and $\langle y, \theta_1^2(x)\rangle$ coincide, as both of them meet M_2 in $\theta_1^2(x)$ and the latter is a hyperplane of M_1. Therefore $y \perp x$, namely $x \in \theta_3^1(y)$. In other words,

$x \in \theta_{1,2,3}(x)$. It follows that all points of M_1 are absolute for $\theta_{1,2,3}$. Hence $\theta_{1,2,3}$ is a symplectic polarity.

The converse also holds true: if $\theta_{1,2,3}$ is a symplectic polarity for any choice of mutually opposite generators M_1, M_2 and M_3 then Δ is hyperbolic and n is even; but we are not going to prove this here.

5.3 Regularity

Henceforth Δ is a polar space of rank $n \geq 2$. We say that a pair $\{a, b\}$ of noncollinear points is *regular* precisely when $(N \cup N')^\perp = \{a, b\}^{\perp\perp}$ for any pair of opposite subgenerators $N, N' \subseteq \{a, b\}^\perp$. We say that Δ is *regular* if every pair of noncollinear points of Δ is regular.

Lemma 5.5 *A pair $\{a, b\}$ of noncollinear points of Δ is regular if and only if*

(R) *for every generator M of Δ, the equality $a^\perp \cap M = b^\perp \cap M$ implies $M \cap \{a, b\}^{\perp\perp} \neq \emptyset$.*

Proof. Let $\{a, b\}$ be regular. Given M as in the hypotheses of (R), let $N = a^\perp \cap M = b^\perp \cap M$ and let N' be a subgenerator of Δ contained in $\{a, b\}^\perp$ and opposite N. Put $c = N'^\perp \cap M$. Then $c \in N^\perp \cap N'^\perp$. Hence $c \in \{a, b\}^{\perp\perp}$, since $\{a, b\}$ is regular. So, M meets $\{a, b\}^{\perp\perp}$ in a point (namely c), as claimed in (R).

Conversely, suppose that (R) holds. Let N and N' be opposite subgenerators of Δ contained in $\{a, b\}^\perp$. Let $x \in (N \cup N')^\perp$ and put $M := \langle N, x\rangle$ and $M' := \langle N', x\rangle$. Then M and M' are generators of Δ and $M \cap M' = \{x\}$. Moreover $M \cap \{a, b\}^\perp = N$. Hence M contains a point $c \in \{a, b\}^{\perp\perp}$, by (R). We have $c \perp N \cup N'$ because $c \in \{a, b\}^{\perp\perp}$. Moreover $c \perp x$, as $c, x \in M$. Therefore $c \perp M'$. This forces $c \in M'$. Hence $c = x$. So, $x \in \{a, b\}^{\perp\perp}$. Thus we have proved that $N^\perp \cap N'^\perp \subseteq \{a, b\}^{\perp\perp}$, consequently $\{a, b\}$ is regular. □

Corollary 5.6 *If Δ is hyperbolic, then it is regular.*

Proof. The hypotheses of property (R) of Lemma 5.5 are vacuous when Δ is hyperbolic. Hence (R) trivially holds true in this case. □

Lemma 5.7 *Suppose that $n > 2$ and let a, c, b, d be four distinct points forming an ordinary quadrangle in Δ, more exactly $a \perp c \perp b \perp d \perp a$, $a \not\perp b$ and $c \not\perp d$. Then*

$$\{a, b\}^{\perp\perp} = \{a, b, c, d\}^{\perp\perp} \cap \{c, d\}^\perp.$$

In other words, the hyperbolic line of $\{c, d\}^\perp$ through a and b is just the same as $\{a, b\}^{\perp\perp}$.

Proof. Clearly, $\{a, b\}^{\perp\perp} \subseteq \{a, b, c, d\}^{\perp\perp} \cap \{c, d\}^\perp$. We shall prove that the reverse inclusion also holds. Let $x \in \{a, b, c, d\}^{\perp\perp} \cap \{c, d\}^\perp$. Then $x^\perp$ contains both $\langle \{a, b, c, d\}^\perp \cup \{c\}\rangle = \{c, a, b\}^\perp$

and $\langle \{a,b,c,d\}^\perp \cup \{d\}\rangle = \{d,a,b\}^\perp$. However $\{c,a,b\}^\perp$ and $\{d,a,b\}^\perp$ are distinct singular hyperplanes of $\{a,b\}^\perp$ and the latter is a polar space of rank $n-1 \geq 2$. Therefore $\{a,b\}^\perp$ is spanned by $\{c,a,b\}^\perp \cup \{d,a,b\}^\perp$. Consequently $x^\perp \supseteq \{a,b\}^\perp$, that is $x \in \{a,b\}^{\perp\perp}$. $\square$

Recall that Δ_p stands for the residue of Δ at a point p of Δ. We extend this notation to singular subspaces: for a singular subspace X of Δ of dimension $m < n-1$ we denote by Δ_X the star of Δ at X, that is the residue of a flag $F = \{X_i\}_{i=0}^m$ of Δ, where $X_0 \subset X_1 \subset ... \subset X_{m-1} \subset X_m = X$.

Proposition 5.8 *Let $n > 2$. Then Δ is regular if and only if Δ_p is regular for every point p.*

Proof. Suppose that Δ is regular and let p be a point of Δ. Then $\Delta_p \cong \{p,q\}^\perp$ for any point $q \not\perp p$. So, we can switch from Δ_p to $\{p,q\}^\perp$. We shall prove that property (R) holds in $\{p,q\}^\perp$. For two noncollinear points $a,b \in \{p,q\}^\perp$ let N be a generator of $\{p,q\}^\perp$ such that $a^\perp \cap N = b^\perp \cap N$. Let h be the hyperbolic line of $\{p,q\}^\perp$ through a and b. We must prove that N meets h in a point. However, $\{a,b\}^{\perp\perp} \subseteq h$ (in fact $\{a,b\}^{\perp\perp} = h$, as proved in Lemma 5.7, but we do not need this fact here). By (R), which holds in Δ by assumption, $\{a,b\}^{\perp\perp}$ meets $M = \langle N,p\rangle$ in a point, say c, which necessarily belongs to h as $\{a,b\}^{\perp\perp} \subseteq h$. The 'only if' part is proved.

Turning to the 'if' part, suppose that Δ_p is regular for every point p of Δ. For a generator M and two noncollinear points a,b of Δ, suppose that $M \cap a^\perp = M \cap b^\perp$. So, $X := M \cap a^\perp = M \cap b^\perp$ is a subgenerator of Δ. Choose a point $p \in \{a,b\}^\perp \setminus X^\perp$, a point $q \in X \setminus p^\perp$ and let $N := p^\perp \cap M$. Then $a,b \in \{p,q\}^\perp$, N is a generator of $\{p,q\}^\perp$ and $N \cap a^\perp = N \cap b^\perp = N \cap X$. Let h be the hyperbolic line of $\{p,q\}^\perp$ through a and b. By property (R), which holds in $\{p,q\}^\perp \cong \Delta_p$ by assumption, N meets h in a point. However $h = \{a,b\}^{\perp\perp}$ by Lemma 5.7. Hence $N \cap \{a,b\}^{\perp\perp} \neq \emptyset$ and, therefore, $M \cap \{a,b\}^{\perp\perp} \neq \emptyset$. $\square$

Corollary 5.9 *Let $n > 2$. Then all the following are equivalent:*

(1) Δ *is regular;*
(2) Δ_X *is regular for every singular subspace X of dimension* $\dim(X) \leq n-3$*;*
(3) Δ_X *is regular for every d-dimensional singular subspace X, for some $d \in \{0,1,...,n-3\}$;*
(4) Δ_X *is regular for some d-dimensional singular subspace X, for some $d \in \{0,1,...,n-3\}$.*

Proof. Obviously (2) implies (3), and (3) implies (4). If Δ is regular then Δ_p is regular for every point p of Δ, by Proposition 5.8. If Δ_p is regular for every point p then, for every line $L = \langle p,q\rangle$, the star $\Delta_L \cong (\Delta_p)_q$ is regular, by Proposition 5.8 applied to Δ_p. Claim (2) follows by iterating this argument as many times as we can. So, (1) implies (2). By reversing the above argument we can prove that (3) implies (1). For instance, if Δ_L is regular for every line $L = \langle p,q\rangle$, then Δ_p is regular for every point p by Proposition 5.8; therefore Δ is regular, again by Proposition 5.8. Finally, it is well known that all point residues of a polar space of rank at least 3 are mutually isomorphic; an obvious inductive argument then shows that (4) implies (3). $\square$

The following is also worth a mention. We refer to [10] for the proof.

Theorem 5.10 *Let Δ be embeddable. Then Δ is regular if and only if it admits a $(2n-1)$-dimensional embedding.*

Proposition 5.11 *Let $n = 3$ and suppose that Δ is nonembeddable. Then Δ is regular.*

Proof. We have already noticed that Δ is regular whenever it is hyperbolic (Corollary 5.6). If Δ is a Freudenthal-Tits polar space, Proposition 5.9.4 of [14] yields the conclusion. □

5.4 Regularity and the Three Generators Property

Given three mutually opposite generators M_1, M_2 and M_3 of a polar space Δ, let θ_{M_1,M_2,M_3} be the duality called $\theta_{1,2,3}$ in Section 5.2, now keeping track of M_1, M_2 and M_3 in our notation. Consider the following property of a polar space Δ:

(3G) (**Three Generators Property**) The self-projectivity θ_{M_1,M_2,M_3} of length 3 is a polarity, for any choice of mutually opposite generators M_1, M_2 and M_3 of Δ.

This is Freudenthal's 'Axiom C' in Section 17 of [8] (stated in the real context there). After Corollary 5.13 below, we discuss in a bit more detail its role in Freudenthal's approach.

The following is a rather unexpected connection.

Theorem 5.12 *The polar space Δ is regular if and only if* (3G) *holds in it.*

Proof. Suppose first that Δ is hyperbolic. Then Δ is regular (Corollary 5.6). Turning to (3G), when n is odd then Δ admits no triple of mutually opposite generators. In this case the assumption on the generators in property (3G) is vacuous, hence (3G) trivially holds. If n is even then (3G) holds true, as shown in Example 5.4. So, when Δ is hyperbolic, both terms of the equivalence we want to prove are true; hence the equivalence as well holds true.

For the rest of this proof we assume that Δ is not hyperbolic. Let Δ be regular and let $\theta = \theta_{M_1,M_2,M_3}$, for three pairwise opposite generators M_1, M_2, M_3. Recall that $\theta^{-1} = \theta_{M_1,M_3,M_2}$. By contradiction, suppose that $\theta \neq \theta^{-1}$ and let N_1 be a subgenerator contained in M_1 such that $\theta(N_1) \neq \theta^{-1}(N_1)$ (hence N_1 cannot be absolute, by Corollary 5.2). For $\{i, j\} = \{2, 3\}$ set $p_i := N_1^\perp \cap M_i$ and $N_j := p_i^\perp \cap M_j$, $\{i, j\} = \{1, 2\}$. So $p_1 := \theta(N_1) = N_3^\perp \cap M_1$ and $p_1' := \theta^{-1}(N_1) = N_2^\perp \cap M_1$ are assumed to be distinct. Moreover, neither of them belongs to N_1. Accordingly, $p_3 \notin N_3$ and $p_2 \notin N_2$. Hence $p_2 \not\perp p_3$. Moreover, $\{p_2, p_3\}^\perp \cap M_1 = N_1$. By property (R), M_1 meets $\{p_2, p_3\}^{\perp\perp}$ in a point, say p_0. Note that $N_2 \cup N_3 \subseteq \{p_2, p_3\}^\perp$. Hence $N_2 \cup N_3 \subseteq p_0^\perp$. Consequently, since $p_0, p_1' \in N_2^\perp$ and $p_0, p_1 \in N_3^\perp$, and both $N_2^\perp$ and $N_3^\perp$ meet M_1 only in a point, we conclude $p_1' = p_0 = p_1$. This contradiction shows that $\theta = \theta^{-1}$, as claimed in (3G).

Conversely, assume property (3G). Let M_1 be a generator of Δ and let a, b be noncollinear points such that $M_1 \cap a^\perp = M_1 \cap b^\perp =: N_1$, as in the hypotheses of (R). Select a generator M_2 containing

a and opposite M_1, let $N_2 = b^\perp \cap M_2$ and $p := M_1 \cap N_2^\perp$. So, $p = \theta_{M_1,M_3,M_2}(N_1) = \theta^{-1}_{M_1,M_2,M_3}(N_1)$ for any choice of a generator M_3 opposite both M_1 and M_2 and containing b. According to (3G), we also have $p = \theta_{M_1,M_2,M_3}(N_1)$ for every choice of M_3 as above. On the other hand, every point $x \in \{a,b\}^\perp \setminus (N_1 \cup N_2)$ belongs to at least one such generator M_3 (consider the residue Δ_b to see this), hence it also belongs to $N_3 := a^\perp \cap M_3$. However $p \perp N_3$, as $p = \theta_{M_1,M_2,M_3}(N_1)$. Therefore $x \perp p$. Consequently $p^\perp \supseteq \{a,b\}^\perp$, that is, $p \in \{a,b\}^{\perp\perp}$. So, $M_1 \cap \{a,b\}^{\perp\perp} \neq \emptyset$, as claimed in (R). □

By Theorem 5.12, Proposition 5.8, Theorem 5.10 and Corollary 5.9 we get the following:

Corollary 5.13 *Let Δ be any polar space of rank $n > 2$. Then the following are equivalent:*

(1) *Property* (3G) *holds in Δ;*
(2) *Property* (3G) *holds in Δ_X for every singular subspace X of dimension* $\dim(X) \leq n-3$*;*
(3) *for some nonnegative integer $d \leq n-3$ and every d-dimensional singular subspace X, property* (3G) *holds in Δ_X;*
(4) *for some nonnegative integer $d \leq n-3$ and some d-dimensional singular subspace X, property* (3G) *holds in Δ_X;*
(5) *either Δ is nonembeddable, or Δ admits a $(2n-1)$-dimensional embedding.*

Digression on the role of Freudenthal's Axiom C. In [8], Freudenthal states three axioms, called A, B and C, satisfied by the polar spaces $\mathsf{C}_{3,1}(\mathbb{R})$, $\mathsf{C}_{3,1}(\mathbb{C},\mathbb{R})$, $\mathsf{C}_{3,1}(\mathbb{H},\mathbb{R})$ and $\mathsf{C}_{3,1}(\mathbb{O},\mathbb{R})$ (with $\mathbb{H}$ the real quaternion and $\mathbb{O}$ the real octonion division rings). Briefly, the first two axioms A and B boil down to the axioms of a polar space of rank 3 with planes $\mathrm{PG}(2,\mathbb{R})$, $\mathrm{PG}(2,\mathbb{C})$, $\mathrm{PG}(2,\mathbb{H})$ and $\mathrm{PG}(2,\mathbb{O})$, respectively, and his Axiom C is equivalent to (3G). In view of (5) of Corollary 5.13, the latter axiom is necessary for the first three polar spaces, but not sufficient. This is best illustrated by the fact that Freudenthal needs the octonions to show that the polar space is not of hyperbolic type (and indeed, in the other cases polar spaces of hyperbolic type exist). For $\mathsf{C}_{3,1}(\mathbb{O},\mathbb{R})$, the axiom (3G) is of course not necessary, as we know from Tits' classification [13]. Nevertheless, it may be considered a tour-de-force that Freudenthal succeeds in showing that his axioms with the octonions characterize the real Freudenthal-Tits polar space. Note also that Freudenthal, although not stating it explicitly in an axiom, uses the assumption that there are disjoint planes, which is a non-degeneracy condition equivalent to (PS2).

However, Freudenthal was slightly too optimistic in stating a consequence of his axiom C, here called (3G). Indeed, in Paragraph 16.22 of [8], it is claimed that (3G) implies that every self-projectivity of length 3 of a line is an involution. But this is not true. The purpose of the next subsection is to show this and to propose an alternative (given by $(3\mathrm{S}_1)$ below).

5.5 Controlled self-projectivities of length 3 of subspaces

Let Δ be a polar space of rank $n > 2$. Given three mutually opposite singular subspaces X_1, X_2 and X_3 of Δ, of (the same) positive dimension $d < n$, let θ_{X_1,X_2,X_3} be, with self-explaining notation, the self-projectivity $X_1 \to X_2 \to X_3 \to X_1$ of length 3. The following property is analogous to and a generalisation of the Three Generators Property.

(3S$_d$) The duality θ_{X_1,X_2,X_3} is a polarity for any choice of mutually opposite d-dimensional singular subspaces X_1, X_2, X_3 such that $X_1 \cup X_2 \cup X_3 \subseteq Y^\perp$ for at least one $(n-d-2)$-dimensional singular subspace Y.

Here, we might say that Y *controls* $\{X_1, X_2, X_3\}$. Also, clearly, (3S$_{n-1}$) is the same as (3G).

Proposition 5.14 *The following are equivalent:*

(1) *Property* (3G) *holds in* Δ*;*
(2) *Property* (3S$_d$) *holds in* Δ *for any* $d \in \{1, 2, ..., n-1\}$*;*
(3) *Property* (3S$_d$) *holds in* Δ *for some* $d \in \{1, 2, ..., n-2\}$*.*

Proof. Clearly (2) implies (3). Assume (3). Let Y and Y' be two mutually opposite $(n-d-2)$-dimensional singular subspaces. Then $(Y \cup Y')^\perp \cong \Delta_Y$. Property (3S$_d$) implies that (3G) holds in Δ_Y. By Corollary 5.13, property (3G) holds in Δ. So, (3) implies (1). We shall now prove that (1) implies (2).

Assume (3G). Let Y be an $(n-d-2)$-dimensional singular subspace and X_1, X_2, X_3 three mutually opposite d-dimensional singular subspaces contained in $Y^\perp$. As X_1, X_2, X_3 are mutually opposite, we have $X_i \cap Y = \emptyset$ for $i = 1, 2, 3$ (equivalently, $\langle X_i, Y\rangle$ is a generator of Δ) and $\langle X_i, Y\rangle \cap \langle X_j, Y\rangle = Y$ for $1 \leq i < j \leq 3$. Choose an $(n-d-2)$-dimensional singular subspace Y' opposite Y. If Z is a singular subspace contained in $Y^\perp \setminus Y$ then the intersection $\langle Z, Y\rangle \cap Y'^\perp$ is a singular subspace, it is still contained in $Y^\perp \setminus Y$ and has the same dimension as Z. In particular, if x is a point of $Y^\perp \setminus Y$ then $Y'^\perp$ meets $\langle x, Y\rangle$ in a point. In the sequel, for a singular subspace $Z \subseteq Y^\perp \setminus Y$ we put $Z' := \langle Z, Y\rangle \cap Y'^\perp$. In particular, if x is a point of $Y^\perp \setminus Y$ then $x' := \langle x, Y\rangle \cap Y'^\perp$.

With this notation, X'_1, X'_2, X'_j are mutually opposite d-dimensional singular subspaces in $(Y \cup Y')^\perp$. Moreover, for $x \in X_i$, we have $x'^\perp \cap X'_j = \langle x^\perp \cap X_j, Y\rangle \cap Y'^\perp$. Therefore $\theta_{X'_1,X'_1,X'_2}(x') = \langle \theta_{X_1,X_2,X_3}(x), Y\rangle \cap Y'^\perp$ for every point $x \in X_1$. However $\theta_{X'_1,X'_2,X'_3}$ is a polarity, since (3G) holds in $(Y \cup Y')^\perp \cong \Delta_Y$ by Corollary 5.13 (recall that (3G) holds in Δ by assumption). Accordingly, θ_{X_1,X_2,X_3} is also a polarity. Thus, we have proved that (1) implies (2). $\square$

5.6 Free self-projectivities of length 3 for lines

For three mutually opposite lines L_1, L_2 and L_3 of Δ, let θ_{L_1,L_2,L_3} be the permutation of L_1 defined as in Section 5.5. If Δ is regular and $(L_1 \cup L_2 \cup L_3)^\perp$ contains an $(n-3)$-dimensional singular subspace then θ_{L_1,L_2,L_3} is an involution, by Theorem 5.12 and Proposition 5.14. When $n = 2$ the hypothesis that the triple $\{L_1, L_2, L_3\}$ is controlled by an $(n-3)$-dimensional singular subspace is vacuous. In contrast, when $n > 2$ that hypothesis severely restricts the class of triples to be considered and it cannot be removed. Indeed:

Theorem 5.15 *Let* Δ *be a polar space of rank* $n > 2$*. When* $n = 3$*, suppose moreover that* $\Delta \ncong \mathsf{A}_{3,2}(\mathbb{F}_2)$*. Then a triple of mutually opposite lines* L_1, L_2, L_3 *always exists such that* θ_{L_1,L_2,L_3} *is not an involution.*

Proof. By contradiction, suppose that

(3L) the permutation θ_{L_1,L_2,L_3} is an involution for any choice of mutually opposite lines L_1, L_2, L_3.

Given two opposite lines L_2, L_3 and a point $x_2 \in L_2$, let $x_3 := x_2^\perp \cap L_3$. Choose a point $x_1 \in x_2^\perp \setminus x_3^\perp$ in such a way that $x_1^\perp \cap L_2 = \{x_1\}$ and put $y_3 := x_1^\perp \cap L_3$ and $y_2 := y_3^\perp \cap L_2$. We have $y_3 \neq x_3$ and $y_2 \neq x_2$ by the choice of x_1. Choose any line X through x_3 different from both L_3 and $\langle x_2, x_3\rangle$ and let $y_1 = x_1^\perp \cap X$. Put $L_1 := \langle x_1, y_1\rangle$. Assume that L_1 is opposite both L_2 and L_3. Then (3L) forces $y_2 \perp y_1$. In other words, if $y_1 \in \{x_1, x_3\}^\perp$ and $L_1 = \langle x_1, y_1\rangle$ is opposite both L_2 and L_3, then $y_1 \in y_2^\perp$. We shall prove that this forces $y_2 \in \{x_1, x_3\}^{\perp\perp}$, but before to come to that we must consider the case where L_1 is opposite either L_2 or L_3.

Suppose first that $L_2 \subseteq z^\perp$ for a point $z \in L_1$. Then $x_2 \perp z$. Also $z \neq x_1$ by the choice of x_1. Hence $x_2^\perp \supseteq L_1$. In particular, $x_2 \perp y_1$. On the other hand, let L_1 contain a point z such that $z^\perp \supseteq L_3$. Then $z \perp y_3$. However, $y_3 \perp x_1 \neq z$ (recall that $x_1 \not\perp x_3$ by the choice of x_1). Hence $y_3^\perp \supseteq L_1$. In particular, $y_3 \perp y_1$.

Thus, we have proved that $y_2 \perp x$ for every point $x \in \{x_1, x_3\}^\perp \setminus (x_2^\perp \cup y_3^\perp)$. However, as $\Delta \ncong \mathsf{A}_{3,2}(\mathbb{F}_2)$, the set $\{x_1, x_3\}^\perp \setminus (x_2^\perp \cup y_3^\perp)$ is large enough to generate the whole of $\{x_1, x_3\}^\perp$. Therefore $y_2^\perp \supseteq \{x_1, x_3\}^\perp$, that is, $y_2 \in \{x_1, x_3\}^{\perp\perp}$, as claimed.

We have obtained this conclusion only exploiting the fact that $y_2^\perp$ contains two points of $\{x_1, x_3\}^\perp$, namely x_2 and y_3. It follows that, for any choice of three noncollinear points a, b and c, if $|c^\perp \cap \{a, b\}^\perp| > 1$ then $c \in \{a, b\}^{\perp\perp}$. However this is impossible, since $n > 2$. A final contradiction has been reached. □

Remark 1 The case $\Delta \cong \mathsf{A}_{3,2}(\mathbb{F}_2)$ is a true exception to the conclusion of Theorem 5.15, because it is easy to see that, for any skew field $\mathbb{K}$, three pairwise opposite lines in $\mathsf{A}_{3,2}(\mathbb{K})$ intersect at least one common generator of each kind.

6 Orbits on triples of opposite singular planes

Throughout this section $\Delta = (P, \mathscr{L})$ is the Freudenthal-Tits polar space over the octonion division ring $\mathbb{O}$, with standard involution $x \mapsto \bar{x}$.

6.1 A description of Δ by means of coordinates

We introduce the explicit description of Δ as given in [6]. First we define six kind of points. Then the point set P is the union of these six sets.

(A) We denote by (∞) a unique point of Δ and call it the *point of Type A*.

(B) For each $x \in \mathbb{O}$ define the point (x) and call it a *point of Type B*.

(C) For each $x_1, x_2 \in \mathbb{O}$, we define the point (x_1, x_2) and call it a *point of Type C*.

(D) For each $x_1, x_2 \in \mathbb{O}$ and each $k \in \mathbb{K}$, we define the point $(x_1, x_2; k)$ and call it a *point of Type D.*

(E) For each $x_1, x_2, x_3 \in \mathbb{O}$ and each $k \in \mathbb{K}$, we define the point $(x_1, x_2, x_3; k)$ and call it a *point of Type E.*

(F) For each $x_1, x_2, x_3, x_4 \in \mathbb{O}$ and each $k \in \mathbb{K}$, we define the point $(x_1, x_2, x_3, x_4; k)$ and call it a *point of type F.*

We now define eight families of subsets of P which we call *planes.*

(I) We denote by $[\infty]$ the set consisting of the points

$$\begin{aligned} p_1(a,b) &:= (a,b), \\ p_2(s) &:= (s), \\ p_3^* &:= (\infty), \end{aligned}$$

where $a, b, s \in \mathbb{O}$. We call $[\infty]$ the *plane of Type I.*

(II) For every $k \in \mathbb{K}$, we denote by $[k]$ the set consisting of the points

$$\begin{aligned} p_1(a,b) &:= (a,b;k), \\ p_2(s) &:= (s), \\ p_3^* &:= (\infty), \end{aligned}$$

where $a, b, s \in \mathbb{O}$. We call $[k]$ a *plane of Type II.*

(III) For every $x \in \mathbb{O}$ and every $k \in \mathbb{K}$, we denote by $[x;k]$ the set consisting of the points

$$\begin{aligned} p_1(a,b) &:= (x,a,b;k), \\ p_2(s) &:= (-\bar{x},s), \\ p_3^* &:= (\infty), \end{aligned}$$

where $a, b, s \in \mathbb{O}$. We call $[x;k]$ a *plane of Type III.*

(IV) For every $x \in \mathbb{O}$ and all $k, \ell \in \mathbb{K}$, we denote by $[x;k,\ell]$ the set consisting of the points

$$\begin{aligned} p_1(a,b) &:= (a, x+\ell a, b; k+\bar{x}a+\bar{a}x+\ell a\bar{a}), \\ p_2(s) &:= (\bar{x},s;\ell), \\ p_3^* &:= (\infty), \end{aligned}$$

where $a, b, s \in \mathbb{O}$. We call $[x;k,\ell]$ a *plane of Type IV.*

(V) For all $x_1, x_2 \in \mathbb{O}$ and every $k \in \mathbb{K}$, we denote by $[x_1, x_2; k]$ the set consisting of the points

$$\begin{aligned} p_1(a,b) &:= (-\bar{x}_2, -\bar{x}_1, a, b; k), \\ p_2(s) &:= (s, x_1 + x_2 s), \\ p_3^* &:= (x_2), \end{aligned}$$

where $a, b, s \in \mathbb{O}$. We call $[x_1, x_2; k]$ a *plane of Type V.*

(VI) For all $x_1, x_2 \in \mathbb{O}$ and all $k, \ell \in \mathbb{K}$, we denote by $[x_1, x_2; k, \ell]$ the set consisting of the points

$$\begin{aligned} p_1(a,b) &:= (-\bar{x}_2, a, \bar{x}_1 + ka, b; \ell + x_1 a + \bar{a}\bar{x}_1 + ka\bar{a}), \\ p_2(s) &:= (s, x_1 + x_2 s; k), \\ p_3^* &:= (x_2), \end{aligned}$$

where $a, b, s \in \mathbb{O}$. We call $[x_1, x_2; k, \ell]$ a *plane of Type VI.*

(VII) For all $x_1, x_2, x_3 \in \mathbb{O}$ and all $k, \ell \in \mathbb{K}$, we denote by $[x_1, x_2, x_3; k, \ell]$ the set consisting of the points

$$\begin{aligned} p_1(a,b) &:= (a, -\bar{x}_3 + x_1 a, b, \bar{x}_2 + ka - \bar{x}_1 b; \ell + x_2 a + \bar{a}\bar{x}_2 + ka\bar{a}), \\ p_2(s) &:= (x_1, s, x_2 + x_3 s; k), \\ p_3^* &:= (-\bar{x}_1, x_3), \end{aligned}$$

where $a, b, s \in \mathbb{O}$. We call $[x_1, x_2, x_3; k, \ell]$ a *plane of Type VII.*

(VIII) For all $x_1, x_2, x_3 \in \mathbb{O}$ and all $k, \ell, m \in \mathbb{K}$, we denote by $[x_1, x_2, x_3; k, \ell, m]$ the set consisting of the points

$$\begin{aligned} p_1(a,b) &:= (a, b, \bar{x}_3 + \ell b + x_1 a, \bar{x}_2 + ka + \bar{x}_1 b; \\ &\qquad m + x_2 a + \bar{a}\bar{x}_2 + x_3 b + \bar{b}\bar{x}_3 + ka\bar{a} + \ell b\bar{b} + (\bar{a}\bar{x}_1)b + \bar{b}(x_1 a)), \\ p_2(s) &:= (s, x_1 + \ell s, x_2 + x_3 s; k + \bar{x}_1 s + \bar{s} x_1 + \ell s\bar{s}), \\ p_3^* &:= (\bar{x}_1, x_3; \ell), \end{aligned}$$

where $a, b, s \in \mathbb{O}$. We call $[x_1, x_2, x_3; k, l, m]$ a *plane of Type VIII.*

Now, by definition, the set $\mathscr{L}$ of lines of Δ consists of the intersection of two distinct planes sharing at least two points. Alternatively, for each plane in the list above, the lines are given by the sets

$$\begin{cases} \{p_3^*\} \cup \{p_2(s) \mid s \in \mathbb{O}\}, & \\ \{p_3^*\} \cup \{(p_1(a,b) \mid b \in \mathbb{O}\}, & \text{for all } a \in \mathbb{O}, \\ \{(s)\} \cup \{a, sa + k) \mid a \in \mathbb{O}\}, & \text{for all } s, k \in \mathbb{O}. \end{cases}$$

6.2 Explicit calculation of a self-projectivity of length 3 in Δ

Let π_i, $i = 1, 2, 3$, be three pairwise opposite planes of Δ. We may choose coordinates so that $\pi_1 = [\infty]$ and $\pi_2 = [0, 0, 0; 0, 0, 0]$. Let $\theta = \theta_{1,2,3}$ be the duality of π_1 defined as in Section 5.2. We recall that θ is obtained by projecting π_1 onto π_2, π_2 in turn onto π_3 and finally π_3 onto π_1. We know from Proposition 5.11 and Theorem 5.12 that θ is a polarity. By well known properties of plane polarities (which the readers can easily check for themselves), there exists a triangle p_1, p_2, p_3 in π_1 such that $\theta(p_1) = \langle p_2, p_3\rangle$ and $\theta(p_2) = \langle p_1, p_3\rangle$.

With p_1, p_2 and p_3 as above, we can assume to have chosen coordinates in such a way that $p_1 = (\infty)$, $p_2 = (0)$ and $p_3 = (0, 0)$. So, $\theta((\infty)) = \langle (0), (0, 0)\rangle$ and $\theta((0)) = \langle (\infty), (0, 0)\rangle$.

As in Section 5.2, let θ_1^2 be the projection of π_1 onto π_2. Then $\theta_1^2((\infty)) = \langle (0, 0; 0), (0, 0, 0; 0)\rangle$ and $\theta_1^2((0)) = \langle (0, 0; 0), (0, 0, 0, 0; 0)\rangle$ (the latter can be seen inside the plane $[0, 0; 0, 0]$ of Type VI; the former inside the plane $[0; 0, 0]$ of Type IV). Hence π_3 contains a point collinear to (0), $(0, 0)$, $(0, 0; 0)$ and $(0, 0, 0; 0)$, and also a point collinear to $(\infty), (0, 0), (0, 0; 0)$ and $(0, 0, 0, 0; 0)$.

An arbitrary plane through (0) and $(0, 0)$ distinct from π_1 is of Type V and has coordinates $[0, 0; k]$, for some $k \in \mathbb{K}$, as can readily be deduced. Likewise, an arbitrary plane through $(0, 0; 0)$ and $(0, 0, 0; 0)$ is either of Type IV and coincides with $[0; 0, 0]$, or is of Type VIII and has coordinates $[0, 0, 0; 0, 0, \ell]$, for some $\ell \in \mathbb{K}$. Type IV and Type V planes are disjoint, and so are Type I and Type VIII planes; Type V and Type VIII planes only have Type F points in common. The points of Type F of $[0, 0; k]$ are $(0, 0, a, b; k)$, and those of $[0, 0, 0; 0, 0, \ell]$ are $(c, d, 0, 0; \ell)$, $a, b, c, d \in \mathbb{O}$. Type I and Type IV planes only have (∞) in common. Hence we conclude $\{(0), (0, 0)\}^\perp \cap \{(0, 0; 0), (0, 0, 0; 0)\}^\perp = \{(0, 0, 0, 0; m) \mid m \in \mathbb{K}\} \cup \{(\infty)\}$.

An arbitrary plane through (∞) and $(0, 0)$ distinct from π_1 is of Type III and has coordinates $[0; k]$, for some $k \in \mathbb{K}$, as can readily be deduced. Likewise, an arbitrary plane through $(0, 0; 0)$ and $(0, 0, 0, 0; 0)$ is either of Type VI and coincides with $[0, 0; 0, 0]$, or is of Type VIII and has coordinates $[0, 0, 0; \ell, 0, 0]$, for some $\ell \in \mathbb{K}$. Type III and Type VI planes are disjoint, and so are Type I and Type VIII planes; Type III and Type VIII planes only have Type E points in common. The points of Type E of $[0; k]$ are $(0, a, b; k)$, and those of $[0, 0, 0; \ell, 0, 0]$ are $(c, 0, 0; \ell)$, $a, b, c \in \mathbb{O}$. The Type I plane π and Type VI planes only have (0) in common. Hence we conclude $\{(\infty), (0, 0)\}^\perp \cap \{(0, 0; 0), (0, 0, 0, 0; 0)\}^\perp = \{(0, 0, 0; k) \mid k \in \mathbb{K}\} \cup \{(0)\}$.

Now it is easy to check that the only planes of Type VIII (other types are not opposite π_1) containing the points $(0, 0, 0; k)$ and $(0, 0, 0, 0; m)$ are $\alpha_{k,\ell,m} := [0, 0, 0; k, \ell, m]$, $k, \ell, m \in \mathbb{K}$. Moreover, $\alpha_{0,\ell,m}$ has $(0, 0, 0; 0)$ in common with π_2; $\alpha_{k,0,m}$ has $(0, 0; 0)$ in common with π_2 and $\alpha_{k,\ell,0}$ has $(0, 0, 0, 0; 0)$ in common with π_2. Hence $k, \ell, m \in \mathbb{K}^\times$.

Similarly we now calculate that $\theta((0, 0)) = \langle (\infty), (0)\rangle$. We now calculate the image of an arbitrary point $(x, y) \in \pi_1$ under the action of θ. That point lies in the plane $[-\bar{x}, 0, y; 0, 0]$, which intersects π_2 in the line $\{(a, -\bar{y} - \bar{x}a, 0, 0; 0) \mid a \in \mathbb{O}\} \cup \{(-\bar{x}, 0, 0; 0)\}$. This line is contained in the planes $\pi_{\ell'}$ of Type VIII with coordinates $[\ell'\bar{x}, \ell'y\bar{x}, \ell'y; \ell'x\bar{x}, \ell', \ell'y\bar{y}]$, $\ell \in \mathbb{K}$. Since we are looking at a generic point, we find the points of Type F which are the intersection of $\pi_{\ell'}$ and $\pi_3 = [0, 0, 0; k, \ell, m]$. Such a point has coordinates $(a, b, *, *; *)$, and comparing the entries blanked out by $*$ in the expressions of the points belonging to both planes, we obtain the following equalities.

$$\begin{aligned}
\ell b &= \ell'\bar{y} + \ell' b + \ell'\bar{x}a, && (1)\\
ka &= \ell' x\bar{y} + \ell' x\bar{x}a + \ell' xb, && (2)\\
m + ka\bar{a} + \ell b\bar{b} &= \ell' y\bar{y} + \ell'(y\bar{x})a + \ell'\bar{a}(x\bar{y}) + \ell' yb + \ell'\bar{b}\bar{y}\\
&\quad + \ell' x\bar{x}a\bar{a} + \ell' b\bar{b} + \ell'(\bar{a}x)b + \ell'\bar{b}(\bar{x}a). && (3)
\end{aligned}$$

Note that $\ell' \neq 0$ otherwise (1) and (2) force $a = b = 0$, hence $m = 0$ from (3), while $m \in \mathbb{K}^\times$. Equations (1) and (2) yield $\ell xb = ka$. This in particular implies that b belongs to the skew field generated by x and a, and, substituting xb by $\ell^{-1}ka$ in (2), a is a $\mathbb{K}$-multiple of $x\bar{y}$. Hence the four triple products in (3) are associative. Multiplying (1) at the left with $\bar{b}$, and (2) at the left with $\bar{a}$, and substituting this in (3), we obtain

$$m = \ell' y\bar{y} + k^{-1}\ell\ell' x\bar{x}yb + \ell' yb.$$

This implies

$$(y\bar{y})b \quad = \quad k(m - \ell' y\bar{y})(\ell\ell' x\bar{x} + k\ell')^{-1}\bar{y}. \qquad (4)$$

We can safely assume that $\ell x\bar{x} + k \neq 0$. Indeed we aim at a description of $\theta(p)$ for $p = (x, y)$ a *generic* point of π_1 of type **C**. So, we may freely assume that whatever we need to be nonzero is indeed such, even if we will miss a tiny set of points because of this. In fact, we will miss precisely the absolute points of type **C** but once we know how θ acts on nonabsolute points, we also know its action on the absolute ones too.

On the other hand, substituting $a = k^{-1}\ell xb$ in (1), we obtain

$$(k\ell - k\ell' - \ell\ell' x\bar{x})b \quad = \quad k\ell'\bar{y}. \qquad (5)$$

Comparing (4) and (5), we calculate

$$\ell'^{-1} \quad = \quad \ell^{-1} + k^{-1}x\bar{x} + m^{-1}y\bar{y}, \qquad (6)$$

and substituting this back in (5) we finally obtain

$$b = m\ell^{-1}y^{-1} \text{ and } a = mk^{-1}xy^{-1}.$$

As above, we can safely assume that $mk + \ell(mx\bar{x} + ky\bar{y}) \neq 0$. Hence the sought intersection point is

$$(mk^{-1}xy^{-1}, m\ell^{-1}y^{-1}, mxy^{-1}, my^{-1}; m^2(y\bar{y})^{-1}(\ell^{-1} + k^{-1}x\bar{x} + m^{-1}y\bar{y})).$$

From the formulae of planes of Type V follows that the line of $[\infty]$ containing all points collinear to that point has equation $Y = -m\ell^{-1}\bar{y}^{-1} - mk^{-1}(\bar{y}^{-1}\bar{x})X$. Its slope (point at infinity) is

$(-mk^{-1}\bar{y}^{-1}\bar{x})$. If we denote by $[[\infty]]$ the line at infinity of π_1 (we use double brackets to avoid confusion with the planes of Δ), by $[[a,b]]$ the line with equation $Y = aX + b$ and by $[[a]]$ the line with equation $X = a$, then with a standard method we can now determine all images under θ:

$$
\begin{array}{rcl}
(\infty) & \mapsto & [[0,0]], \\
(0) & \mapsto & [[0]], \\
(x) & \mapsto & [[-mk^{-1}\bar{x}^{-1}, 0]], \quad x \neq 0, \\
(0,0) & \mapsto & [[\infty]], \\
(x,0) & \mapsto & [[-k\ell^{-1}\bar{x}^{-1}]], \quad x \neq 0, \\
(x,y) & \mapsto & [[-mk^{-1}\bar{y}^{-1}\bar{x}, -m\ell^{-1}\bar{y}^{-1}]], \quad y \neq 0, \\
[[\infty]] & \mapsto & (0,0), \\
[[0]] & \mapsto & (0), \\
[[a]] & \mapsto & (-k\ell^{-1}\bar{a}^{-1}, 0), \quad a \neq 0, \\
[[0,0]] & \mapsto & (\infty), \\
[[a,0]] & \mapsto & (-mk^{-1}\bar{a}^{-1}), \quad a \neq 0, \\
[[a,b]] & \mapsto & (k\ell^{-1}\bar{a}\bar{b}^{-1}, -m\ell^{-1}\bar{b}^{-1}), \quad b \neq 0.
\end{array}
$$

The set of absolute points for θ is given by

$$\{(x,y) \mid k^{-1}x\bar{x} + m^{-1}y\bar{y} + \ell^{-1} = 0\} \cup \{(x) \mid m^{-1}x\bar{x} + k^{-1} = 0\}.$$

We can now state the main theorem of this section. Recall that two polarities θ and θ' of a projective plane π are said to be *equivalent*, in symbols $\theta \sim \theta'$, if $\theta' = \gamma^{-1}\theta\gamma$ for a collineation γ of π.

Theorem 6.1 *Given two opposite planes π_1 and π_2 of Δ, let π_3 and π_3' be planes of Δ opposite both π_1 and π_2 and let $\theta := \theta_{\pi_1,\pi_2,\pi_3}$ and $\theta' := \theta_{\pi_1,\pi_2,\pi_3'}$ be the polarities of π_1 defined by the self-projectivities of length 3 associated with $\{\pi_1, \pi_2, \pi_3\}$ and $\{\pi_1, \pi_2, \pi_3'\}$ respectively. Let G_{π_1,π_2} be the stabilizer of π_1 and π_2 in $G := \mathrm{Aut}(\Delta)$. Then $\theta \sim \theta'$ if and only if π_3 and π_3' belong to the same orbit of G_{π_1,π_2}.*

Proof. It follows from the definition of θ and θ' that, if $\pi_3' = \tau(\pi_3)$ for some $\tau \in G_{\pi_1,\pi_2}$ and γ_τ is the type-preserving automorphism (namely collineation) of π_1 induced by γ, then $\theta' = \gamma_\tau^{-1}\theta\gamma_\tau$. The 'if' part of the theorem is proved. Turning to the 'only if' part, recall first that G_{π_1,π_2} induces on π_1 its full collineation group. Accordingly, we only need to prove that, if $\theta' = \theta$, then $\pi_3' = \tau(\pi_3)$ for a suitable $\tau \in G_{\pi_1,\pi_2}$.

Suppose that $\theta' = \theta$. Assuming as above that $\pi_1 = [\infty]$ and $\pi_2 = [0,0,0;0,0,0]$, we have $\pi_3 = [0,0,0;k,\ell,m]$ and $\pi_3' = [0,0,0;k',\ell',m']$ for $k,m,\ell,k',m',\ell' \in \mathbb{K}^\times$. The previous description of θ shows that $\theta' = \theta$ if and only if $(k',\ell',m') = (tk,t\ell,tm)$ for some $t \in \mathbb{K}^\times$. The following automorphism of Δ indeed maps $\pi_1 = [0,0,0;k,\ell,m]$ onto $\pi_3' = [0,0,0;k',\ell',m']$.

$$\begin{array}{rclrcl} & & & [x_1,x_2,x_3;k,\ell,m] & \mapsto & [tx_1,tx_2,tx_3;tk,t\ell,tm],\\ (\infty) & \mapsto & (\infty), & [x_1,x_2,x_3;k,\ell] & \mapsto & [x_1,tx_2,x_3;tk,t\ell],\\ (x) & \mapsto & (x), & [x_1,x_2;k,\ell] & \mapsto & [tx_1,x_2;tk,t\ell],\\ (x_1,x_2) & \mapsto & (x_1,x_2), & [x_1,x_2;k] & \mapsto & [x_1,x_2;tk],\\ (x_1,x_2;k) & \mapsto & (tx_1,tx_2;tk), & [x;k,\ell] & \mapsto & [tx;tk,t\ell],\\ (x_1,x_2,x_3;k) & \mapsto & (x_1,tx_2,tx_3;tk), & [x;k] & \mapsto & [x;tk],\\ (x_1,x_2,x_3,x_4;k) & \mapsto & (x_1,x_2,tx_3,tx_4;tk), & [k] & \mapsto & [tk],\\ & & & [\infty] & \mapsto & [\infty]. \end{array}$$

The fixed point set of τ is exactly the union of the planes $\pi_1 = [\infty]$ and $\pi_2 = [0,0,0;0,0,0]$. Consequently, τ belongs to the kernel of the action of G_{π_1,π_2} on π_1 (and π_2 accordingly). □

Example 6.2 Over the reals, there are essentially two kinds of polarities (involving the standard involution): one without absolute points (this happens if all of k,ℓ,m have the same sign), and one with absolute points. Clearly, when there are absolute points we can arrange it so that $[[\infty]]$ and $[[0]]$ contain absolute points. This happens if $m\ell < 0$ and $mk < 0$. All such polarities are clearly conjugate (recoordinatize by absorbing $\sqrt{|k^{-1}\ell|}$ and $\sqrt{-m^{-1}\ell}$ in x and y, respectively). Hence over the reals, there are two orbits on triples of opposite planes.

Recall that, in view of Proposition 5.1, the set of absolute points is precisely the set of points of π_1 through which a line exists that intersects both π_2 and π_3. So over the reals, the triples that admit such "transversals" and the triples that don't just make up for the two orbits. A quite neat situation.

Remark 2 The above computations and consequences are valid in any polar space of rank 3 involving a quadratic alternative division ring, that is, in the polar space $\mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$, with $\mathbb{A}$ a quadric alternative division algebra over the field $\mathbb{K}$. Accordingly, Theorem 6.1 holds for all these polar spaces, and most certainly also for their higher rank analogous. In the finite case, as there only exists one conjugacy class of semi-linear polarities (related to the Hermitian curves), and also only one class of linear polarities, this implies that the automorphism group of $\mathsf{C}_{3,1}(\mathbb{F}_q)$ and the one of $\mathsf{C}_{3,1}(\mathbb{F}_{q^2},\mathbb{F}_q)$, act transitively on ordered triples of pairwise opposite planes, for every prime power q.

References

[1] F. Buekenhout & A. M. Cohen, Diagram Geometry, Related to Classical Groups and Buildings, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, 3. Folge **57** (2013).

[2] F. Buekenhout & E. E. Shult, On the foundations of polar geometry, *Geom. Dedicata* **3** (1974) 155–170.

[3] I. Cardinali, L. Giuzzi and A. Pasini. Nearly all subspaces of a classical polar space arise from its universal embedding, *Lin. Alg. Appl.* **627** (2021), 287–307.

[4] I. Cardinali, H. Cuypers, L. Giuzzi and A. Pasini. Characterizations of symplectic polar spaces, *Adv. Geom.* **23** (2023), 281–293.

[5] A.M. Cohen and E.E. Shult. Affine polar spaces, *Geom. Dedicata* **35** (1990), 43–76.

[6] B. De Bruyn and H. Van Maldeghem. Non-embeddable polar spaces, *Münster J. Math.* **7** (2014), 557–588.

[7] A. De Schepper, N. S. N. Sastry and H. Van Maldeghem. Buildings of exceptional type in buildings of type E_7, *Dissertationes Math.* **573** (2022), 1–80.

[8] H. Freudenthal. Beziehung der E_7 un E_8 zur Oktavenebene, I-XI, *Proc. Kon. Ned. Akad. Wet. Ser. A*, **57** (1954), 218-230, 363-368: **57** (1955), 151-157, 277-285; **62** (1959), 165-201, 447-474; **66** (1963), 457–487.

[9] B. Mühlherr, A geometric approach to non-embeddable polar spaces of rank 3, *Bull. Soc. Math. Belg. Ser.* A **42** (1990), 463–814.

[10] A. Pasini. Synthetic and projective properties of embeddable polar spaces, to appear in *Innovations in Incidence Geometry.*

[11] A. Pasini and H. Van Maldeghem. Ovoidal maximal subspaces of polar spaces, to appear in *Algebraic Combinatorics and the Monster: from Moonshine to Majorana* (ed. A.A. Ivanov et. al.).

[12] J. Tits, Sur certaines classes d'espaces homogènes de groupes de Lie, *Acad. Roy. Belg. Cl. Sci. Mém. Collect. 8^o (2)* **29** (1955), 5–268.

[13] J. Tits. *Buildings of Spherical Type and Finite BN-Pairs*, Springer Lecture Notes in Math. **386** (1974).

[14] H. Van Maldeghem. *Generalized Polygons*, Birkhäuser, Basel 1998.

[15] F. D. Veldkamp, Polar geometry, I–V, *Indag. Math.* **21** (1959) 512–551, *Indag. Math.* **22** (1960) 207–212.

Department of Information Engineering and Mathematics, University of Siena, Via Roma 56, I–53100 *Siena* `antonio.pasini@unisi.it`

Department of Mathematics: Algebra and Geometry, Ghent University, Krijgslaan 281, S25, B–9000 *Gent.* `Hendrik.VanMaldeghem@UGent.be`