

ON WIEGOLD'S CONJECTURE FOR THE SMALL REE GROUPS

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ABSTRACT. The Wiegold conjecture holds for the small Ree groups for k -tuples where $k \geq 5$.

1. INTRODUCTION

We prove the following for the small Ree groups:

Main Theorem. *The small Ree groups ${}^2\mathbf{G}_2(3^{2e+1})$ satisfy the Wiegold conjecture for k -tuples, for all $e \geq 1$ and all $k \geq 5$.*

The original motivation of the Wiegold conjecture comes from actions of the automorphism group of the free group on group presentations.

1.1. T_k -systems and the Wiegold conjecture. Let G be a group. We call N a G -defining subgroup of a free group F_k if $N \trianglelefteq F_k$ and $F_k/N \cong G$. If we denote by $\Sigma_k(G)$ the set of G -defining subgroups of F_k we can consider an action of $\text{Aut}(F_k)$ on $\Sigma_k(G)$. The orbits of this action are T_k systems introduced by Bernhard and Hanna Neumann [25]. The group $\text{Aut}(F_k)$ is generated by Nielsen moves, transpositions and inversions (which we will all call Nielsen moves henceforth). These moves thus define a graph structure on $\Sigma_k(G)$. Thus, T_k -systems correspond to connected components of $\Sigma_k(G)$. The unpublished conjecture below, likely inspired by Gilman [14], is due to Wiegold. Graham and Diaconis [6] made a similar conjecture for $\text{Sym}(n)$.

Wiegold conjecture. *For every $k \geq 3$ and every nonabelian simple group G , there exists only one T_k -system, i.e. the graph $\Sigma_k(G)$ is connected.*

Thus far, the following was known on the above conjecture:

Theorem 1.1. *The Wiegold conjecture holds for k -tuples in these cases.*

- [14] $G = \text{PSL}(2, p)$, where $p \geq 5$ and $k \geq 3$
- [9] $G = \text{PSL}(2, 2^m)$ or $\text{Sz}(2^{2m-1})$, where $m \geq 2$ and $k \geq 3$
- [10] $G = \text{PSL}(2, q)$ where q is odd and $k \geq 4$.
- [5] $G = \text{Alt}(6), \text{Alt}(7)$ where $k = 3$.
- [4] $G = \text{Alt}(8), \text{Alt}(9), \text{Alt}(10), \text{Alt}(11)$ where $k = 3$.

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Note that in the survey paper [27], Alt(11) is not mentioned, but it is included in [4].

Avni and Garion [1] used deep results of Larsen and Pink [19] on finite subgroups of algebraic groups to prove that for finite simple groups of Lie type (possibly twisted, but not of Suzuki or Ree type), there exists a function c , only depending on the Lie rank r , such that the Wiegold conjecture holds for any $k \geq c(r)$. This is independent of the field but grows exponentially in r . Avni informed us that even if their methods were made to work for Ree groups the constant would be huge.

1.2. Connections of the Wiegold conjecture. Lubotzky [21] wrote a very interesting survey on the dynamics of $\text{Aut}(F_k)$ actions which is closely related to the Wiegold conjecture. Some connections include the following:

- Nielsen equivalence classes of generators of a group G parameterise free actions of G on handlebodies [21, Theorem 3.4].
- The Wiegold conjecture is equivalent to: for $k \geq 3$ and G a finite simple group the action of $\text{Aut}(F_k)$ on $\text{Epi}(F_k, G)$ is transitive. In [13] Gelander considered an analogous question for compact Lie groups, non-compact simple analytic groups and simple algebraic groups.
- Product replacement algorithm which we discuss in detail below.

1.3. Product replacement algorithm graph. The product replacement algorithm was designed by Leedham-Green and Soicher as mentioned in the first description of it by Celler, Leedham-Green, Murray, Niemeyer and O'Brien [3]. It very efficiently generates random elements in a finite group. Lubotzky and Pak [22] explained why the product replacement algorithm works so well using Kazhdan's property (T). We closely follow Igor Pak's survey paper [27].

Let $k \geq 2$ and let G be a finite simple group. Let

$$V_k(G) = \{(g_1, \dots, g_k) \in G^k : \langle g_1, \dots, g_k \rangle = G\}.$$

The set $V_k(G)$ is the set of all generating k -tuples of G . Such a k -tuple is called *minimal* if no proper subtuple generates G . Otherwise it is called *redundant*.

The *extended PRA graph* $\tilde{X}_k(G)$ has vertices given by $V_k(G)$ and edges corresponding to the *Nielsen moves* $R_{i,j}^\pm, L_{i,j}^\pm, P_{i,j}, I_i$, for $1 \leq i \neq j \leq k$, where

$$R_{i,j}^\pm: (g_1, \dots, g_i, \dots, g_k) \rightarrow (g_1, \dots, g_i g_j^{\pm 1}, \dots, g_k),$$

$$L_{i,j}^\pm: (g_1, \dots, g_i, \dots, g_k) \rightarrow (g_1, \dots, g_j^{\pm 1} g_i, \dots, g_k),$$

$$P_{i,j}: (g_1, \dots, g_i, \dots, g_j, \dots, g_k) \rightarrow (g_1, \dots, g_j, \dots, g_i, \dots, g_k),$$

and

$$I_i: (g_1, \dots, g_i, \dots, g_k) \rightarrow (g_1, \dots, g_i^{-1}, \dots, g_k).$$

By Proposition [27, 2.5.13] the Wiegold conjecture can be reformulated as:

Wiegold conjecture. *Let G be a nonabelian finite simple group. Then $\tilde{X}_k(G)$ is connected for every $k \geq 3$.*

By a celebrated recent result of Burness, Guralnick and Harper [2], finite simple groups have spread at least two and hence, the graph induced on the redundant k -tuples of $\tilde{X}_k(G)$, $k \geq 3$, is connected, see e.g. [27, Proposition 2.5.12]. Hence, in order to show that $\tilde{X}_k(G)$ is connected, it suffices to show that every vertex of $\tilde{X}_k(G)$ can be connected via a path to a redundant vector. In the rest of the paper, we will use the phrase *connected to a vector* to mean that two given vectors are in the same connected component of $\tilde{X}_k(G)$, $k \geq 3$ and G a small Ree group. Note that this means that we can change a given vector using only Nielsen moves into another given vector.

1.4. Wiegold and its variations for more general groups. Let $d(G)$ denote the minimum number of generators of G . Pak [27] asked whether there are finite groups such that the extended PRA graph $\tilde{X}_k(G)$ is disconnected for $k \geq d(G) + 1$. As $d(G) = 2$ for finite simple groups, conjecturing this does not happen is a generalisation of the Wiegold conjecture. This generalisation was proved for abelian groups by the Neumanns [25] and for solvable groups by Dunwoody [8]. Garion [10] also proved it for $\mathrm{PGL}(2, q)$ where q is an odd prime power. The next natural class of (almost) simple groups to look at are the projective special (general) unitary groups, which would conclude the rank one case of the Wiegold conjecture. Although a number of arguments here translate directly to this case extra complications arise from the richer subgroup structure and the fact that the characteristic is not uniform.

More broadly, Nielsen equivalence classes of generating tuples have also been studied extensively for infinite groups. For example Kapovich and Weidmann [17] provided a counterexample to a Wiegold type conjecture for word hyperbolic groups. If one wishes to disprove (variations of) the Wiegold conjecture for a given group one needs to show that certain generating k -tuples are not Nielsen equivalent. For concrete small groups this can, in principle, be done by exhaustive computation. For larger groups or families of groups a potential strategy consists in using Fox calculus which can provide obstructions to Nielsen equivalence [23, 24].

As pointed out by Kapovich and Weidmann [17], Nielsen equivalence is not decidable for finitely presented torsion-free small cancellation groups since they do not have a decidable subgroup membership problem (which is a special case) as shown by Rips [28].

1.5. Proof outline. We basically follow the same strategy as Evans [9] introduced to handle the Suzuki groups and as was also used later by Garion [10] for $\mathrm{PSL}_2(q)$. So, we start off with a generating k -tuple of $G := {}^2\mathrm{G}_2(q)$, $k \geq 3$ (we do as much as we can for these values, but most of the lemmas need $k \geq 5$). The eventual goal is to show that it is connected to a redundant vector. To accomplish that, there are three major steps. They all use the natural 2-transitive permutation representation of G on a set $U_R(q)$ with $q^3 + 1$ points. Unlike the situation for the Suzuki groups and $\mathrm{PSL}_2(q)$, this set is in fact a rank 2 geometry (that is a point-line geometry, thus not rank 1) owing to the existence of simple subgroups of Lie type over $\mathbb{F}_q$, namely $\mathrm{PSL}_2(q)$. The latter subgroups stabilise sets of $q + 1$ points on which

they act in the natural way (and so we can think of each such set of $q + 1$ points as a projective line over $\mathbb{F}_q$; we call it a block). The set of blocks turns $U_R(q)$ into a unital, that is, a $2 - (q^3 + 1, q + 1, 1)$ design [26]. Now, the method is to show first that we can connect to a k -tuple with the following property.

- (†) There is an element x in the k -tuple with the property that (1) all $(k - 1)$ -subtuples containing x generate subgroups isomorphic to a smaller Ree group, and (2) no subgroup generated by x and the conjugate of x by any other element of the vector fixes a point of $U_R(q)$ or stabilises a block of $U_R(q)$.

The first step to accomplish this consists in showing that we can connect to a vector containing an element x that does not fix exactly 1 or $q + 1$ points of the corresponding Ree unital. In this first step, understanding the structure of the point stabiliser is essential and for that, we use the explicit description using 7×7 matrices over $\mathbb{F}_q$ provided in [29]. We survey this description in the appendix, where we also prove some more details about it. Also, in Section 3, we show that (2) implies (1) in (†). In the second step, accomplished in Section 4, we show that (†) holds, first if the element x established above fixes exactly 2 points of $U_R(q)$, and then if the element x does not fix any point of $U_R(q)$. This establishes (†).

Then, in 5, we use an inductive procedure (inspired by Evans [9]) to connect to a redundant vector. This procedure in fact consists of showing that we may connect to a vector such that (†) holds under the additional assumption that all subgroups in (1) are isomorphic. And then, under this additional assumption, one can finally connect to a redundant vector.

Each of these steps needs some additional ideas, compared to [9, 10], to make the strategy work. The Ree groups are generally known to have a rather peculiar structure. For instance, Guralnick, Kantor, Kassabov and Lubotzky [15] provide short and small presentations for all finite simple groups—except for the Ree groups, the latter has been done recently by Hulpke, Kassabov, Seress and Wilson [16].

On the positive side, the list of maximal subgroups of ${}^2G_2(q)$ is rather small and can be compared to that of the Suzuki groups.

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2. THE SMALL REE GROUPS

2.1. General notation. Given a finite group G and a prime p , we denote $\text{Syl}_p(G)$ as the collection of Sylow p -subgroups. For a natural number m , we denote the cyclic subgroup of order m by C_m . For a subgroup $H \leq G$, we denote the centraliser of H in G as $C_G(H)$ and the normaliser of H in G as $N_G(H)$. We denote the commutator subgroup of G by $[G, G]$, by $Z(G)$ the center of G , and by $\Phi(G)$ the Frattini subgroup of G . Given a group, we denote $|G|$ as its order and $|x|$ as the order of the element $x \in G$. We denote the greatest common divisor of two integers n and m as $\gcd(m, n)$.

A *Frobenius group* G is a transitive permutation group on a finite set X , such that no nontrivial element fixes more than one point and some nontrivial element fixes a point. A subgroup $H \leq G$ fixing a point of X is called a *Frobenius complement*. The identity element together with all elements not in any conjugate of H form a normal subgroup called the *Frobenius kernel* K . In particular, $G = K \rtimes H$. There is a unique Frobenius group of order 21 and we denote it by $\text{Frob}(21)$. Given a finite group G , a subgroup $H \leq G$ is called a *Hall subgroup* if $\gcd(|H|, [G : H]) = 1$.

Finally we consider actions on the left and we use the standard convention that $x^y = y^{-1}xy$ and ${}^yx = yxy^{-1}$.

2.2. Generalities. There are three types of Suzuki–Ree groups: type ${}^2\text{B}_2$, usually called *Suzuki groups*, types ${}^2\text{G}_2$ and ${}^2\text{F}_4$, usually called *small* and *large Ree groups*, respectively. Henceforth, *Ree group* will mean one of type ${}^2\text{G}_2$. The Ree group $G = {}^2\text{G}_2(q)$, sometimes also denoted as $\text{R}(q)$, has order $q^3(q+1)(q-1)$ where $q = 3^{2e+1}$. We denote by θ the field automorphism $x \mapsto x^{3^{e+1}}$; note $x^{\theta^2} = x^3$. Hence, θ is the square root of the Frobenius automorphism. The subgroup structure of the Ree groups was described by Levchuk and Nuzhin [20], see also Kleidman's paper [18] and [29, Section 9.2.4]. The Ree groups are one of the families of exceptional finite simple groups of Lie type. From now on we fix $q = 3^{2e+1}$, $e \geq 1$, and denote the Ree group ${}^2\text{G}_2(q)$ by G . Note that $e = 0$ corresponds to $\text{PSL}_2(8) \rtimes C_3 \cong \text{P}\Gamma\text{L}_2(8)$.

The Ree group G has a natural doubly transitive action on its set $\Omega := \text{Syl}_3(G)$ of Sylow 3-subgroups. The stabiliser G_P of $P \in \Omega$ is given by $N_G(P)$. The stabiliser $G_{P,Q}$ of two members of $\text{Syl}_3(G)$ is a cyclic subgroup C_{q-1} .

A Sylow 2-subgroup P of G is elementary abelian of order 8. Moreover $C_G(P) = P$ and $N_G(P) \cong 2^3 \rtimes \text{Frob}(21)$, which has order $8 \cdot 7 \cdot 3 = 168$. As a result, the 2-subgroups of equal order are conjugate. In particular, all involutions are conjugate to one another. The centraliser of an involution η is isomorphic to $\text{PSL}_2(q) \times \langle \eta \rangle$.

The group G has cyclic Hall subgroups M_i ($i = \pm 1$) of orders $q+1+i \cdot \sqrt{3q}$. Each subgroup M_i coincides with $C_G(x)$ for all nontrivial $x \in M_i$. The subgroups $N_G(M_i)$ are Frobenius groups with kernel M_i and cyclic complement of order 6. For each subgroup V of order 4, there exists a cyclic Hall subgroup M_0 of order $\frac{q+1}{4}$ and an element t of order 6 such that $N_G(V) = N_G(M_0) = V \rtimes (M_0 \rtimes \langle t \rangle) \cong (V \times (M_0 \rtimes C_2)) \rtimes C_3$, where $M_0 \rtimes C_2$ is dihedral of order $\frac{q+1}{2}$. All subgroups of order $\frac{q-1}{2}$, $\frac{q+1}{4}$, $q + \sqrt{3q} + 1$, or $q - \sqrt{3q} + 1$ are conjugate in G . The same holds for subgroups of order $q-1$ and $q+1$.

A Sylow 3-subgroup P of G has order q^3 . The following properties can also be checked directly on the model presented in Section A (but are well known). The center $Z(P)$ of P is an abelian subgroup of order q , P has nilpotent step length 3, and $[P, P] = \Phi(P)$ is an elementary abelian subgroup of order q^2 containing $Z(P)$. The elements of $P \setminus [P, P]$ have order 9 where their cubes are forming $Z(P) \setminus \{1\}$. The involutions of $N_G(P)$ are conjugate in $N_G(P)$. We collect some of the above information in a lemma.

Lemma 2.1. *The maximal subgroups of G are exhausted up to conjugacy, by the following*

- $N_G(P) \cong P \rtimes C_{q-1}$, the normaliser of a Sylow 3-subgroup P ;
- $C_G(\eta) \cong \text{PSL}_2(q) \times \langle \eta \rangle$, the centraliser of an involution η ;
- ${}^2\text{G}_2(q_0)$, $q = q_0^\delta$, δ being a prime;
- $N_G(M_i)$, the normaliser of a cyclic Hall subgroup M_i , $i = -1, 0, 1$, of order $q - \sqrt{3q} + 1$, $\frac{q+1}{4}$ and $q + \sqrt{3q} + 1$, respectively.

2.3. The Ree unital $U_R(q)$. As mentioned earlier, G has a natural 2-transitive action on its set Ω of Sylow 3-subgroups. We can give Ω the structure of a *unital*, that is, a $2 - (q^3 + 1, q + 1, 1)$ design by defining, for each involution g , the *block* $C_\Omega(g) = \{\omega \in \Omega : g.\omega = \omega\}$. No block is (pointwise) fixed by two distinct involutions; hence, there are

$$\frac{|G|}{2 \cdot |\text{PSL}_2(q)|} = q^2(q^2 - q + 1)$$

blocks. Every pair of Sylow 3-subgroups is contained in a unique block, which we sometimes call the *join* of the two corresponding elements of Ω . In the sequel we will refer to the elements of Ω as *points*, and we will denote the points by Greek lower case letters. The unital will be denoted by $U_R(q)$, and consists of points and blocks. The unique block containing two given distinct points α and β will be denoted as $B(\alpha, \beta)$.

If $\alpha \in \Omega$, it is in fact a Sylow 3-subgroup, but we see it merely as a point. When we want to emphasise the group structure of α , we sometimes denote it also as P_α . This notation is justified by the fact that the “group α ” stabilises the “point α ”.

We list some properties of the permutation group $(G, U_R(q))$. For a point α we denote the set of blocks through α by $\mathcal{B}_\alpha$. We say that an element $x \in G$ acts freely on $U_R(q)$ if $\langle x \rangle$ does.

Lemma 2.2. *Let $P \in \text{Syl}_3(G)$. Then the following hold.*

- (a) *P fixes a unique point $\alpha \in \Omega$ and acts sharply transitively on $\Omega \setminus \{\alpha\}$. In particular, it acts transitively on $\mathcal{B}_\alpha$.*
- (b) *Each element of $P \setminus [P, P]$, which has order 9, acts freely on the set of blocks, in particular on $\mathcal{B}_\alpha$.*
- (c) *Each element of $[P, P] \setminus Z(P)$ stabilises exactly q blocks, all belonging to $\mathcal{B}_\alpha$, and acts freely on the set of all other blocks.*
- (d) *Each member of $Z(P)$, which is the third power of an element of order 9, acts freely on the set of blocks.*

Proof. The first two assertions are easy and immediate, noting $P = P_\alpha$. Now let $x \in P$ have order 9. Suppose x^i stabilises a block B , $i \in \{1, 2, \dots, 8\}$. Since $|x^i| \in \{3, 9\}$ and $|B| = q + 1$, x^i fixes a point on B , and hence, $\alpha \in B$ (by the first assertion). Also, x acts freely on the set of blocks through α if, and only if, x^3 does. Now we use the notation of Section A. Up to conjugacy, we may assume $x^3 = (0, 0, 1)_\infty$. Then one calculates $x^3 \cdot B_{b,b'} = B_{b,b'+1}$, for all $b, b' \in \mathbb{F}_q$. It follows that both x and x^3 act freely on the set of blocks through α . Now let $x = (0, a', 0)_\infty$ be, up to conjugacy, an arbitrary member of $[P, P] \setminus Z(P)$. Then x maps the block $B_{b,b'}$ to the block $B_{b,b'-a'b}$.

It follows that the q blocks $B_{0,b''}$, $b'' \in \mathbb{F}_q$, are stabilised, and none of the other blocks is. A similar argument as for $|x| = 9$ shows that every element of P of order 3 acts freely on the set of blocks not containing α .

The lemma is proved. $\square$

We need one more property of the Sylow 3-subgroups.

Lemma 2.3. *Let $P \in \text{Syl}_3(G)$, and let $\alpha \in \Omega$ be the unique point fixed by P . Then P acts imprimitively on $\mathcal{B}_\alpha$. Each imprimitivity class is the set of fixed blocks of some element of P . Each element of $[P, P]$ stabilises each imprimitivity class. Finally, no element of $P \setminus [P, P]$ stabilises any such imprimitivity class, so every element of $P \setminus [P, P]$ acts on the set of imprimitivity classes with all orbits of size 3.*

Proof. We again use the notation of Section A and put $\alpha = \infty$. Using the explicit form of $[P_\infty, P_\infty]$, where $P = P_\infty$, given there, it is easy to see that that the system of imprimitivity is

$$\{\{B_{a,a''} \mid a'' \in \mathbb{F}_q\} \mid a \in \mathbb{F}_q\}.$$

The lemma is now obvious from the fact that each member of $P \setminus [P, P]$ has the form $(b, b', b'')_\infty$ with $b \neq 0$. $\square$

Orders and fixed points We will use the following fixed point properties on a few occasions. Although they must be well-known, we sketch short proofs.

Lemma 2.4. *Let $x \in G$ have order $k > 1$. Then one of the following holds.*

- $k = 2$ and the points fixed by x are the $q + 1$ points of a block.
- $k \in \{3, 6, 9\}$ and x fixes a unique point.
- $k \mid \frac{q-1}{2}$ and x fixes exactly two points.
- $k \mid \frac{q+1}{4}$ and x acts freely on Ω , hence does not have any fixed points.
- $k \mid q^2 - q + 1$ and x acts freely on Ω , in particular it has no fixed points.
- $k \neq 2$ is even and $k \mid q - 1$, then x fixes exactly two points.
- $k \neq 2$ is even and $k \mid \frac{q+1}{2}$, then x has no fixed points.

Proof. If x fixes at least one point, then it is contained in $P \rtimes C_{q-1}$ for some $P \in \text{Syl}_3(G)$. From the structure of this point stabiliser then follow the first three bullet points and the second last. This can also be verified with the description given in Section A. Hence, in all other cases there are no fixed points. The free actions follow from the fact that, in the cases under consideration, each divisor d of k is again relatively prime with $q^3(q - 1)$, and hence, x^d has no fixed points, too. Finally, the fact that no other orders show up is due to the structure of the maximal subgroups, in particular the normalisers of the cyclic Hall subgroups as given above. $\square$

Lemma 2.5. *An element x fixing at least three points not contained in a common block is the identity. An element x fixing exactly two points α and β stabilises a unique block, namely $B(\alpha, \beta)$.*

Proof. The first assertion follows straight from Observation A.2. Now assume that an element $x \in G_{\alpha,\beta}$, which fixes exactly two points, stabilises a second block $B \neq B(\alpha, \beta)$. By the first assertion of Observation A.2, $|x|$ divides $q - 1$. By the second assertion of Observation A.2, $|x|$ divides $|B \setminus B(\alpha, \beta)|$. Since two distinct points determine a unique block, $|B \cap B(\alpha, \beta)| \leq 1$. If $|B \cap B(\alpha, \beta)| = 1$, then $|B \setminus B(\alpha, \beta)| = q$, and as $\gcd(q-1, q) = 1$, the above divisibility conditions forces $|x| = 1$, a contradiction. Hence $B \cap B(\alpha, \beta) = \emptyset$, so $|B \setminus B(\alpha, \beta)| = q+1$; as $\gcd(q-1, q+1) = 2$, we conclude $|x| = 2$. Then x is an involution and fixes $B(\alpha, \beta)$ pointwise by Lemma 2.4(1), a contradiction. $\square$

Lemma 2.6. *Let $\alpha, \beta \in \Omega$ be distinct. Let P_α and P_β be the corresponding Sylow 3-subgroups. Let $x \in [P_\alpha, P_\alpha] \setminus Z(P_\alpha)$ and $y \in [P_\beta, P_\beta] \setminus Z(P_\beta)$. Suppose $\langle x, y \rangle$ does not stabilise $B(\alpha, \beta)$. Then x does not stabilise both blocks $B(\alpha, y \cdot \alpha)$ and $B(\alpha, y^{-1} \cdot \alpha)$ simultaneously.*

Proof. We again use the notation of Section A and put $\alpha = \infty$ and $\beta = O$. Clearly, the statement is trivial if x does not stabilise $B(\infty, O)$ and y does since $B(\infty, y \cdot \infty) = B(\infty, O)$. Hence, we may assume that y does not stabilise $B(\infty, O)$. So, up to conjugation, we may assume that $y = (0, b', 1)_O$, and $x = (0, a', a'')_\infty$. Then

$$y \cdot \infty = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b'^\theta & 1 & 0 & 0 & 0 & 0 & 1 \\ -1+b' & 0 & 1 & -b' & 0 & -1 & b'^\theta \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1+b'^{\theta+1} & b' & 0 & 1 & 1 & -b'^\theta & 1+b' \\ -b' & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ b'^\theta \\ -1+b' \\ -1 \\ 1+b'^{\theta+1} \\ -b' \\ 0 \end{pmatrix}$$

$$= p \left(\frac{1-b'}{1+b'^{\theta+1}}, \frac{-b'^\theta}{1+b'^{\theta+1}}, \frac{1}{1+b'^{\theta+1}} \right).$$

Now, $y^{-1} = (0, -b', -1)$ and so we find that

$$y^{-1} \cdot \infty = p \left(-\frac{1+b'}{1+b'^{\theta+1}}, \frac{b'^\theta}{1+b'^{\theta+1}}, \frac{1}{1+b'^{\theta+1}} \right).$$

On the other hand, one easily calculates that x maps the block $B_{c,c''}$ to the block $B_{c,c''+a''-a'c}$, and hence, the blocks stabilised by x are $B_{a''a'^{-1},c''}$. Clearly at most one of the values $\frac{1-b'}{1+b'^{\theta+1}}$ or $-\frac{1+b'}{1+b'^{\theta+1}}$ is equal to $a''a'^{-1}$, which proves the lemma. $\square$

Lemma 2.7. *Let $\alpha, \beta \in \Omega$ be distinct. Let P_α and P_β be the corresponding Sylow 3-subgroups. Let $x \in [P_\alpha, P_\alpha] \setminus Z(P_\alpha)$ and $y \in [P_\beta, P_\beta] \setminus Z(P_\beta)$. Suppose both x and y stabilise $B(\alpha, \beta)$. Then each block distinct from $B(\alpha, \beta)$ and fixed by x intersects a unique block distinct from $B(\alpha, \beta)$ and fixed by y , necessarily in a unique point.*

Proof. We use the notation of Section A. Clearly, the elements $(0, a, 0)_\infty$, which are precisely those elements of P_∞ stabilising $B(\infty, O)$, and which, apart from the identity, all belong to $[P_\infty, P_\infty] \setminus Z(P_\infty)$, stabilise the blocks

$B_{(0,a'')}$, containing points of the form $p(0, a', a'')$. Likewise, $(0, a, 0)_O$ stabilises all lines containing points of the form $q(0, b', b'')$. Observation A.3 implies that the common points of the sets

$$\{p(0, a', a'') \mid a' \in \mathbb{F}_q, a'' \in \mathbb{F}_q^\times\} \text{ and } \{q(0, b', b'') \mid b' \in \mathbb{F}_q, b'' \in \mathbb{F}_q^\times\}$$

are the points $p(0, a''^{\theta-1}, a'')$. It follows that each block $B_{(0,a'')}$, $a'' \neq 0$, contains a unique such point. This proves the assertion. $\square$

A *central element* x of order 3 in G is by definition one that is contained in $Z(P)$, for some Sylow 3-subgroup P of G .

Lemma 2.8. *A central element of order 3 of G does not stabilise any block, and is not contained in the normaliser of any (cyclic) Hall subgroup.*

Proof. The first assertion follows from Lemma 2.2. From the structure of $N_G(M_i)$, $i \in \{-1, 0, 1\}$ one sees that each element x of order 3 of $N_G(M_i)$ is contained in a cyclic subgroup of order 6. Hence, there must exist an involution η such that $x\eta$ has order 6. Clearly η has to fix the fixed point of x which by Lemma 2.4 is unique. Hence, with the notation of Section A, we can write $\eta = h(-1)$ and $x = (0, 0, b'')_\infty$, for some $b'' \in \mathbb{F}_q$. But then one calculates that $x\eta$ maps $p(a, a', a'')$ to $p(-a, a', -a'' - b'')$, and hence, $x\eta$ has order 2, a contradiction. $\square$

We now show that the maximal subgroups $N(M_0)$ are the counterparts of the stabilisers of (real) triangles in the unitary groups $U_3(q)$.

Lemma 2.9. *The maximal subgroup $N(M_0)$ of G , for a Hall subgroup M_0 , stabilises three mutually disjoint blocks, each of them stabilised by every member of $N(M_0)$ with an order not divisible by 3, and permuted around in a cycle by each subgroup of order 3 of $N(M_0)$.*

Proof. The subgroup $N(M_0)$ is the normaliser of a Klein four-group $V =: \{1, \eta_1, \eta_2, \eta_3\}$. As such, each member of it permutes the three fix blocks B_i of the η_i , respectively, $i = 1, 2, 3$. Each η_i itself, $i = 1, 2, 3$, centralises V and hence stabilises each block B_i , $i = 1, 2, 3$. It follows that $B_1 \cap B_2 = \emptyset$ as otherwise η_1 fixes a unique point of B_2 , leading to $2|q$, a contradiction. Let $g \in N(M_0)$ be an arbitrary member that does not stabilise all of B_1, B_2, B_3 . Suppose first that g stabilises B_1 and interchanges B_2 and B_3 . Then $|g|$ is even. By the structure of $N(M_0)$, we see that $|g| = 2k$, for k odd. Hence, $g^k = \eta_1$ and g^k stabilises both B_1, B_2 , a contradiction. Now assume that g acts as an order 3 permutation on $\{B_1, B_2, B_3\}$. Then clearly $|g| \in \{3, 6\}$ by the structure of $N(M_0)$ again (alternatively, note that any element whose order divides 3 is contained in the centraliser of an order 3 element, which is on its turn contained in a point stabiliser, from which the claim also follows). If $|g| = 3$, then g has a unique fixed point α by Lemma 2.4, which we claim is not contained in $B_1 \cup B_2 \cup B_3$. Indeed, suppose for a contradiction that $\alpha \in B_1$. Then by the disjointness of the blocks B_1, B_2, B_3 , the block B_1 is stabilised by g . But we proved above that in this case also B_2 and B_3 have to be stabilised, contradicting Lemma 2.2 and the fact that B_2 and B_3 are disjoint from B_1 . Hence, the claim follows and Lemma 2.2 implies that g

cycles B_1, B_2, B_3 around. If $|g| = 6$, then g^2 cycles these blocks around, and hence also g . $\square$

2.4. Structural subgroups. Structural subgroups are subgroups of the maximal subgroups (i) $N_G(P)$, for $P \in \text{Syl}_3(G)$, and (ii) $C_G(\eta)$, where η is an involution.

(i): The normalisers of Sylow 3-subgroups each stabilise a point. Sylow 3-subgroups consist of *unipotent* elements, have order q^3 , their center is elementary abelian of order q , they have class 3 and the Frattini subgroups equal the commutator subgroups and contain the center.

The stabiliser of two points α, β is a cyclic group W of order $q - 1$ and $N_G(P) = P \rtimes W$, where $P \in \text{Syl}_3(G)$ and fixes the point α . The group W acts by conjugation on the set of elements of $[P, P]$ that stabilise $B(\alpha, \beta)$ in exactly two orbits, mapped onto each other by taking inverses.

(ii): The centraliser of an involution r is isomorphic to $\langle r \rangle \times \text{PSL}_2(q)$ and these are the block stabilisers.

The following table complements Lemma 2.1, lists the maximal subgroups of G up to conjugacy, but also their geometric properties.

Maximal subgroup	Geometric prop.	Orders
$N_G(P), P \in \text{Syl}_3(G)$	Point stabiliser	2, 3, 6, 9
$C_G(r), r$ involution	Block stabiliser	2, 3, 6, divisors of $q - 1, (q + 1)/2$
${}^2\text{G}_2(q_0), q_0^p = q, p$ prime	Subunital stabiliser	
$N_G(M_{\pm 1})$		2, 3, 6, divisors of $q \pm \sqrt{3q} + 1$
$N_G(M_0)$	Triangle stabiliser	2, 3, 6, divisors of $(q + 1)/2$

TABLE 1. Maximal subgroups of ${}^2\text{G}_2(3^{2e+1})$.

Subgroups of the maximal subgroups in the first two rows will be referred to as *structural subgroups*. Aligning with [9], non-trivial subgroups of maximal subgroups in the last two rows which are not structural, will be referred to as *small subgroups*. Finally, subgroups of the form ${}^2\text{G}_2(q_0)$, with $q_0 > 1$ such that $\mathbb{F}_{q_0} < \mathbb{F}_q$, or of the form ${}^2\text{G}_2(3)'$, are called *subunital stabilisers* or *subfield subgroups*. An immediate consequence of this terminology is the following.

Lemma 2.10. *All small subgroups of G are at most 3-generated. Moreover, if a small subgroup is 3-generated but not 2-generated, then it is a subgroup of the normaliser of a Klein four-group (or, equivalently, of a cyclic Hall subgroup of order $\frac{q+1}{4}$) and it contains an element of order 3.*

Proof. Small subgroups of G lie, by their very definition, in $N_G(M_i)$, which is the normaliser of the cyclic Hall subgroup M_i , $i = -1, 0, 1$, of order $q - \sqrt{3q} + 1$, $\frac{q+1}{4}$, and $q + \sqrt{3q} + 1$, respectively. For $i \in \{-1, 1\}$ the subgroup

$N_G(M_i)$ is cyclic-by-cyclic hence its subgroups are either cyclic or at most 2-generated. The structure of $N_G(M_0)$ is given by $(V \times M_0) \rtimes \langle t \rangle$ where V is an elementary abelian group of order 4, M_0 has odd order $\frac{q+1}{4}$, and t has order 6. We note that t acts on V by cyclically interchanging the three involutions. If H is a subgroup of $N_G(M_0)$ not containing an element of order 3, then, by Lemma 2.9, it is structural and hence not small. Suppose now that H contains an element of order 3. Then either it contains an element of order 6, call it ρ , or it does not, and then we let ρ be an element of order 3. We multiply each element of H with a suitable power of ρ to obtain a subgroup of H of index 6 or 3, respectively, contained in $V \times M_0$, which is isomorphic to the direct product of a group of order 2b and a cyclic group of order $\frac{q+1}{2}$, hence 2-generated. Hence, H is at most 3-generated. We conclude that all subgroups of $N_G(M_0)$ are at most 3-generated. $\square$

2.5. Classification of elements in ${}^2G_2(q)$ and their normalisers. A list of conjugacy classes in G is given by the following table, where the left column denotes either the order of an element, or a multiple m of that order, and the right column denotes either the number of conjugacy classes of elements of that given order, or the total number of conjugacy classes of elements whose order divides m , respectively.

Order n	Number of conjugacy classes
9	3
6	2
3	3
2	1
1	1
divisor of $q + \sqrt{3q} + 1$	$\frac{q + \sqrt{3q}}{6}$
divisor of $q - \sqrt{3q} + 1$	$\frac{q - \sqrt{3q}}{6}$
divisor of $\frac{q+1}{2}, n \neq 2$	$\frac{q-3}{6}$
divisor of $\frac{q-1}{2}$	$\frac{q-3}{2}$

TABLE 2. Number of conjugacy classes in ${}^2G_2(q)$

Given this table, the facts listed in Section 2.2 and the table of maximal subgroups of G , the following proposition is obvious.

Proposition 2.11. *Let x be a nontrivial element of G . Then $C_G(x)$ is given by the following subgroups.*

$$C_G(x) = \begin{cases} C_{q-1} & |x| \mid \frac{q-1}{2} \\ C_2^2 \times C_{\frac{q+1}{4}} & |x| \mid \frac{q+1}{2}, |x| \neq 2 \\ C_{q \pm \sqrt{3q} + 1} & x \in M_i, i = \pm 1 \\ P & x \in Z(P), P \in \text{Syl}_3(G) \\ [P, P] \times \langle r \rangle, |r| = 2 & x \in [P, P] \setminus Z(P), P \in \text{Syl}_3(G) \\ Z(P) \cdot \langle x \rangle & x \in P \setminus [P, P], P \in \text{Syl}_3(G) \\ \langle x \rangle \times \text{PSL}_2(q) & |x| = 2 \end{cases}$$

Corollary 2.12. *Let $x \neq 1 \in G$ be an element whose order divides $\frac{q+1}{2}$, but x is not an involution. Then x is contained in a unique maximal subgroup isomorphic to the normaliser of a cyclic Hall subgroup of order $\frac{q+1}{4}$.*

Proof. By Proposition 2.11, there is a Klein four-group K centralising x . Consequently, x is contained in the normaliser $N(K)$, which is isomorphic to the normaliser of a cyclic Hall subgroup of order $\frac{q+1}{4}$. This shows existence. Uniqueness follows from the fact that the above Klein four-group K is unique in $C_G(x)$. $\square$

Corollary 2.13. *Let $x \neq 1 \in G$ be an element whose order divides $\frac{q+1}{2}$, but x is not an involution. Then x stabilises exactly three mutually disjoint blocks of $U_R(q)$. Hence the three blocks mentioned in Lemma 2.9 are unique.*

Proof. Suppose x stabilises the block B , and suppose B is pointwise fixed by the involution η . Then η^x also pointwise fixes B and hence coincides with η . It follows that η centralises x . Since, by Proposition 2.11 the centraliser of x only contains 3 involutions, the assertion follows. $\square$

Lemma 2.14. *Suppose that $x, y \in G$, where x is not an involution, and that $\langle x, x^y \rangle$ is not a structural subgroup and is not contained in a subgroup isomorphic to ${}^2\text{G}_2(3)' \cong \text{PSL}_2(8)$. Then x and x^y are conjugate in $\langle x, x^y \rangle$.*

Proof. This follows readily from the description of the conjugacy classes. For the Hall subgroup stabilisers, this is a consequence of the fact that they are normalised by a cyclic group of order 6, whereas the corresponding denominator in the right column of Table 2 is exactly 6. For subfield subgroups H , this follows from the observation that the number of conjugacy classes of elements of H of a given order in H is equal to the number of such conjugacy classes in G , together with the fact that, if this number is strictly larger than 1, which happens only if $|x| \in \{3, 6, 9\}$, then the following holds. If $|x| = 3$, then one class is given by the centre $Z(P')$ of a Sylow 3-subgroup P' of H , where $Z(P') \leq Z(P)$, for some Sylow 3-subgroup P of G , and the two other classes are related by inverse (so all three classes are present in every subfield subgroup). If $|x| = 6$, then the two classes are related by inverse. For $|x| = 9$, two of the classes are related by inverse and the third class has, with the notation of Section A, representative $(1, -1, 0)_\infty$ and hence exists in every Ree subgroup, except in the smallest derived case ${}^2\text{G}_2(3)' \cong \text{PSL}_2(8)$. $\square$

Lemma 2.15. *Let x be an element which is not an involution in G , and suppose that $x \in R_1 \cap R_2$ where R_1 and R_2 are Ree subgroups of G , distinct from ${}^2\text{G}_2(3)'$. If $|C_{R_1}(x)| \leq |C_{R_2}(x)|$, then $|R_1| \leq |R_2|$.*

Proof. This follows from Proposition 2.11 by noting that the orders of the centralisers depend proportionally on the order of the Ree subgroup. Also, elements of a given Ree subgroup R_1 stay “in the same kind of class” when considered in another Ree subgroup R_2 ; that is, if $C_{R_1}(x)$ for some member $x \in R_1$ is read on the n th line of the displayed cases in Proposition 2.11, then on the same line one reads $C_{R_2}(x)$. This follows from the fact that, if R_i is defined over the field $\mathbb{F}_{q_i}$, the order $|x|$ of any member $x \in R_1 \cap R_2$ divides

$q_i + 1$ if, and only if, it divides $q + 1$; it divides $q_i - 1$ if, and only if, it divides $q - 1$, and it divides $q_i^2 - q_i + 1$ if, and only if, it divides $q^2 - q + 1$. (These follow from easy divisibility conditions possibly using elementary cyclotomic polynomial theory.) $\square$

2.6. Basic group theory lemmas. We conclude this section by stating a special case of a theorem by Gaschütz [12]. It also follows from [10, Lemma 4.1] and induction.

Lemma 2.16. *Let K be a finite cyclic group generated by $\{x_1, x_2, \dots, x_k\}$. Then there exists natural numbers $m_1, \dots, m_{k-1}$ so that*

$$K = \langle x_1^{m_1} \dots x_{k-1}^{m_{k-1}} x_k \rangle.$$

We now embark on the proof of Theorem 1. We show that every vertex of $\tilde{X}_k(G)$, for $G = {}^2G_2(q)$ and $k \geq 5$, is connected to a redundant vector. We prove as much as we can for shorter vectors, in particular for vectors of length 3. Globally we show that we can transform any vector $(x, y, z, v, w) \in V_5(G)$ with a finite number of Nielsen moves to a redundant vector. Obviously we may assume that (x, y, z, v, w) is not redundant itself, in particular, none of x, y, z, v, w is the identity.

3. REDUCTION AWAY FROM INVOLUTIONS, UNIPOTENT ELEMENTS AND ELEMENTS OF ORDER 6

We first prove that we may connect a generating k -tuple $(x, y, z, \dots, w)$, $k \geq 3$, to a generating k -tuple $(x, y', z', \dots, w')$ such that $y', z', \dots, w'$ are not involutions and none of them normalises the cyclic subgroup $\langle x \rangle$.

Lemma 3.1. *Let G be a finite simple group, and let $(x, y, z, \dots, w) \in V_k(G)$, $k \geq 3$. Then $(x, y, z, \dots, w)$ is connected to $(x, y', z', \dots, w')$, where $y', z', \dots, w' \notin N_G(\langle x \rangle)$ and none of $y', z', \dots, w'$ has order 2.*

Proof. For clarity and brevity, we present the proof for $k = 3$. It will be obvious how to generalise to $k \geq 3$.

Step 1: *We connect (x, y, z) to (x, y', z') such that $y', z' \notin N_G(\langle x \rangle)$.*

We observe that y and z cannot both be in the normaliser $N_G(\langle x \rangle)$, since G does not normalise $\langle x \rangle$. We note that if $y \in N_G(\langle x \rangle)$ and $z \notin N_G(\langle x \rangle)$, then $yz \notin N_G(\langle x \rangle)$. Thus, we can connect $(x, y, z) \rightarrow (x, yz, z)$, where $yz, z \notin N_G(\langle x \rangle)$. This proves Step 1.

Step 2: *We can connect (x, y', z') to (x, y'', z'') such that $y'', z'' \notin N_G(\langle x \rangle)$ and none of y'', z'' has order 2.*

If y' is an involution, we claim that xy' is not an involution. Indeed, if it were, then $xy'xy' = 1$, which implies $x^{-1} = y'xy' = x^{y'}$. Hence, $x^{y'} \in \langle x \rangle$. That implies y normalises $\langle x \rangle$, which is a contradiction. Thus, if y' has order 2, we may connect (x, y', z') to (x, xy', z') , where xy' does not have order 2. Since $y' \notin N_G(\langle x \rangle)$, it also follows that $xy' \notin N_G(\langle x \rangle)$. We can do the same step with z' instead of y' and arrive at the same assertion. $\square$

Now we show that we can connect every generating k -tuple, $k \geq 5$, to a k -tuple containing at least one element that does not fix a unique point (elements which fix a unique point are either unipotent or have order 6; hence they are characterised by having order 3, 6 or 9). However, again for brevity and clarity, we assume $k = 5$, the proofs for the general case being obvious modifications (but we state all results for general k).

Consider the following “intermediate” condition on a k -tuple $(x, y, z, \dots, w)$.

- (*) Either at least one of the elements $x, y, z, \dots, w$ has order distinct from 2, 3, 6 and 9, or none of the elements are involutions and each subgroup generated by x and $k - 2$ elements of $\{y, z, \dots, w\}$ is a subfield subgroup over a subfield of size at least 27, and the subgroups $\langle x, x^y \rangle, \langle x, x^z \rangle, \dots, \langle x, x^w \rangle$ are not structural subgroups.

Lemma 3.2. *Let $(x, y, z, \dots, w) \in V_k(G)$, $k \geq 5$. If the subgroup $\langle y, z, \dots, w \rangle$ is small, then $(x, y, z, \dots, w)$ is connected to a redundant k -tuple.*

Proof. We set $k = 5$. By Lemma 2.10, $\langle y, z, v, w \rangle$ admits a generating set of 3 elements. Now, by a theorem of Dunwoody [8] $\langle y, z, v, w \rangle$ is connected to a redundant 4-tuple. Hence, (x, y, z, v, w) is connected to a redundant 5-tuple. $\square$

Lemma 3.3. *Let $(x, y, z, \dots, w)$ be a generating k -tuple, $k \geq 5$, such that $\langle y, z, \dots, w \rangle$ is either small or isomorphic to either ${}^2G_2(3)'$ or ${}^2G_2(3)$. Then (x, y, z, v, w) is connected to a redundant k -tuple.*

Proof. We present the proof for $k = 5$ and consider the 5-tuple (x, y, z, v, w) . For $H := \langle y, z, v, w \rangle$ small, this follows from Lemma 3.2. Observe that ${}^2G_2(3) \cong P\Gamma L_2(8) \cong PSL_2(8) \rtimes 3$. Hence, $H \in \{PSL_2(8), PSL_2(8) \rtimes 3\}$. If $H \cong PSL_2(8)$, then by [9], (y, z, v, w) is connected to a redundant 4-tuple, and we are done. So we may assume $H \cong PSL_2(8) \rtimes 3$ and we set $H = H_0 \rtimes \langle \xi \rangle$, with $|\xi| = 3$. We may assume $w \notin H_0$. Then for some $i \in \{0, 1, 2\}$, $w^i y \in H_0$. This yields Nielsen moves to connect (y, z, v, w) to (y', z', v', w) with $y', z', v' \in H_0$. By the classification of subgroups of $PSL_2(8)$, we either have $\langle y', z', v' \rangle \cong PSL_2(8)$, in which case (y', z', v') is connected to a redundant triple by [9] again, or $\langle y', z', v' \rangle$ is solvable and generated by at most two elements, implying by [8] that we again can connect (y', z', v') to a redundant triple, or $\langle y', z', v' \rangle$ is structural, that is, it fixes a point α in the natural permutation representation of $PSL_2(8) \rtimes 3$ on the projective line $PG(1, 8)$. We may assume we are in the latter case. Since y' fixes at most two points of $PG(1, 8)$, either y'^w or ${}^w y'$ does not fix α , and so replacing (y', z', v') with (y'^w, z', v') or $({}^w y', z', v')$, respectively brings us back to one of the two previous possibilities. $\square$

Proposition 3.4. *Every k -tuple of generating elements of G containing only elements of orders 2, 3, 6, 9 is connected in $\tilde{X}_k(G)$ to a k -tuple satisfying Condition (*), or to a redundant k -tuple, $k \geq 5$.*

Proof. We again set $k = 5$. Let the quintuple be given by (x, y, z, v, w) . We first claim that we may assume that not all of x, y, z, v, w are involutions. Indeed, suppose all of them are involutions. If they mutually commuted, they would generate an elementary abelian subgroup of order 32, a contradiction.

Hence we may assume that x and y are non-commuting involutions. Then the quintuple (xy, y, z, v, w) contains an element xy of order at least 3. If $|xy| \notin \{3, 6, 9\}$, then Condition (*) is satisfied. Otherwise the claim follows. So, without loss of generality, we may assume that x is not an involution.

By Lemma 3.1, we may assume that all of y, z, v and w are not involutions. Hence, by Lemma 2.4, each of x, y, z, v, w fixes a unique point (otherwise Condition (*) is satisfied and the proposition follows), which we denote by $\alpha, \beta, \gamma, \delta, \xi$, respectively. Evidently, we do not have $\alpha = \beta = \gamma = \delta = \xi$. If $\alpha = \beta$, then we may assume without loss of generality that $\gamma \neq \alpha$ and we replace (x, y, z, v, w) with (x, y^z, z, v, w) . Repeating this if necessary, this produces a quintuple with $\alpha \notin \{\beta, \gamma, \delta, \xi\}$. This already implies that we may assume that neither $\langle x, y \rangle, \langle x, z \rangle, \langle x, v \rangle$ nor $\langle x, w \rangle$ are contained in a point stabiliser. We divide into cases.

Case 1. *Suppose at least one of x, y, z, v, w has order 9.* We may assume that x has order 9. Since the orders of the small maximal subgroups are not divisible by 9, Lemma 2.2 implies that elements of order 9 only appear in subfield subgroups and point stabilisers. Hence $\langle x, y \rangle$ is a subfield subgroup. Since x^y only fixes the point $y^{-1} \cdot \alpha \neq \alpha$, the group $\langle x, x^y \rangle$ is a subfield subgroup. Similarly all of $\langle x, z \rangle, \langle x, x^z \rangle, \langle x, v \rangle, \langle x, x^v \rangle, \langle x, w \rangle$ and $\langle x, x^w \rangle$ are subfield subgroups and the assertion follows from the fact that overgroups of subfield subgroups (which are not small by definition) are subfield subgroups, as is apparent from the list of maximal subgroups.

Case 2. *None of x, y, z, v, w have order 9, but at least one of them is central.* We may suppose that x is central. By Lemma 2.8, the subgroups $\langle x, y \rangle, \langle x, x^y \rangle, \langle x, z \rangle, \langle x, x^z \rangle, \langle x, v \rangle$, and $\langle x, x^v \rangle$ are subfield subgroups. The assertion again follows.

Case 3. *None of x, y, z, v, w have order 9 or are central, and one of x, y, z, v, w has order 6.* We may assume $|x| = 6$. Note that x^3 is an involution and hence, by the definition of the Ree unit, stabilises a unique block through α . It follows that x has no orbits of length 3 on the set of blocks through α and x stabilises exactly one block through α . Since, by Lemma 2.2, x^2 only stabilises blocks through α , it now follows that x stabilises a unique block B , which contains α . Hence, if $\langle x, y \rangle$ stabilises a block, it must be B and then $\beta \in B$, by Lemma 2.2 again and the fact that we just proved that elements of order 6 only stabilise one block (and similarly for the other subgroups $\langle x, z \rangle, \langle x, v \rangle$ and $\langle x, w \rangle$). So, suppose $\beta \in B$ and y stabilises $B = B(\alpha, \beta)$. Then we may assume without loss of generality that z does not stabilise B ; consequently one of z or z^{-1} maps β outside B . Suppose without loss of generality that $z \cdot \beta \notin B$. Then we replace (x, y, z, v, w) with $(x, {}^zy, z, v, w)$. Hence, since zy only fixes the point $z \cdot \beta \notin B$, and x only stabilises the block $B \neq B(\alpha, z \cdot \beta)$, we may assume that not both x and y stabilise B . Similarly not both x and z stabilise $B(\alpha, \gamma)$, not both x and v stabilise $B(\alpha, \delta)$, and not both x and w stabilise $B(\alpha, \xi)$. Then $\langle x, y \rangle, \langle x, z \rangle, \langle x, v \rangle$ and $\langle x, w \rangle$ are not structural. Hence, $\langle x, y, z, v \rangle$ is not structural. If it is a small subgroup, then we are done by Lemma 3.2; hence, we may assume it is a subfield subgroup. Similarly, the subgroups $\langle x, y, z, w \rangle, \langle x, y, v, w \rangle$ and $\langle x, z, v, w \rangle$ are subfield subgroups.

Suppose x and x^y stabilise the same block B . Then B contains α and $y^{-1} \cdot \alpha$. Then, since y does not stabilise B , the block $B(\alpha, y \cdot \alpha)$ is distinct from B and hence not stabilised by x . It follows that $\langle x, x^y \rangle$ is not structural. Replacing y with y^{-1} , which is allowed by the Nielsen moves, we obtain that $\langle x, x^y \rangle$ is not structural. On the other hand, if x and x^y do not stabilise the same block, then $\langle x, x^y \rangle$ was not structural in the first place. Similarly, either $\langle x, x^z \rangle$ or $\langle x, x^v \rangle$, either $\langle x, x^u \rangle$ or $\langle x, x^w \rangle$, and either $\langle x, x^w \rangle$ or $\langle x, x^u \rangle$ is not structural. Then an analogous argument as with $\langle x, x^y \rangle$ yields (*).

Case 4. *All of x, y, z, v, w have order 3 and are not central.* Here, each of x, y, z, v, w stabilises q blocks through their respective fixed point. We claim that we may assume that x and y do not share a common fixed block. Indeed, assume all of x, y, z, v, w pairwise share common fixed blocks. Evidently, this cannot be the same block for every pair; hence, we may assume that γ is not contained in $B(\alpha, \beta)$. We apply Lemma 2.7 on the pair of elements x and z , whose fixed points are α and γ , respectively. Since both x and z stabilise the block $B(\alpha, \gamma)$, the lemma implies that each block fixed by x and distinct from $B(\alpha, \gamma)$ intersects a unique block fixed by z and distinct from $B(\alpha, \gamma)$. In particular, β is the unique point on $B(\beta, \gamma) \setminus \{\gamma\}$ whose join to α is fixed by x . Consequently, y^z does not fix any block fixed by x , and since we may replace y with y^z , the claim follows.

Hence, $\langle x, y \rangle$ is not structural and, by Lemma 2.6, and replacing y by its inverse if necessary, the subgroup $\langle x, x^y \rangle$ is not structural. We now claim that we may assume that $\langle x, x^z \rangle$ is not structural. Indeed, this follows from Lemma 2.6 if x does not fix the block $B(\alpha, \gamma)$. So assume that x stabilises $B(\alpha, \gamma)$. Since $\langle x, y \rangle$ is not structural, the subgroup $H := \langle x, y, v, w \rangle$ is not structural and so, by Lemma 3.3, we may assume H isomorphic to ${}^2G_2(q_0)$ for some 3-power $q_0 \geq 27$. By field restriction it follows that H acts doubly transitively on a subunitary Ω_0 of Ω , which contains α as α is the unique fixed point of x . Let $u \in H$ be an element of order 9 fixing α . Then, by Lemma 2.3, u does not preserve the set of lines stabilised by x . Then, replacing z by $z' = z^u$, as is allowed with Nielsen moves, Lemma 2.3 implies that x does not fix $B(\alpha, u \cdot \gamma)$. So, again by Lemma 2.6, and replacing z' by its inverse if necessary, $\langle x, x^{z'} \rangle$ is not structural. The claim follows. Similarly we may assume that $\langle x, x^u \rangle$ and $\langle x, x^w \rangle$ are not structural. Hence none of $\langle x, y, z, v \rangle$, $\langle x, y, z, w \rangle$, $\langle x, y, v, w \rangle$ and $\langle x, z, v, w \rangle$ is structural. By the classification of maximal subgroups of G , any subgroup generated by elements of orders 2, 3, 6, 9 which is not structural is either contained in a small subgroup or contained in a subfield subgroup. In the former case, the corresponding quintuple is redundant by Lemma 3.2, while in the latter case Condition (*) is satisfied. Hence the assertion follows and the proof is complete. $\square$

Proposition 3.5. *Every k -tuple $(x, y, z, \dots, w)$ of generating elements of G containing only elements of orders 2, 3, 6, 9 is connected in $\tilde{X}_k(G)$ to either a redundant k -tuple, or to a k -tuple containing at least one element of order distinct from 2, 3, 6 or 9, $k \geq 5$.*

Proof. We show the proposition for $k = 5$ and the quintuple (x, y, z, v, w) .

By Proposition 3.4, we may assume that $H := \langle x, y, z, v \rangle$ is a subfield subgroup ${}^2G_2(q_0)$, with $q_0 \geq 27$.

We now claim that the coset wH contains at least one element of order distinct from 2, 3, 6 or 9. We proceed by distinguishing the various possible orders of w . We perform explicit computations using the matrices given in [29, Section 9.2.4], see Section A below. Those 7×7 matrices over $\mathbb{F}_q$ describe the action of the full Sylow 3-subgroups P_∞ and P_O , which fix the two points ∞ and O , respectively. Recall that each Sylow 3-subgroup is described by triples $(a, a', a'')_\infty$ and $(a, a', a'')_O$, respectively, with $a, a', a'' \in \mathbb{F}_q$. Also, $Z(P_\infty) = \{(0, 0, a'')_\infty \mid a'' \in \mathbb{F}_q\}$, that $[P_\infty, P_\infty] = \{(0, a', a'')_\infty \mid a', a'' \in \mathbb{F}_q\}$ (and similarly for P_O), and the stabiliser of the block through ∞ and O in P_∞ and P_O is $\{(0, a', 0)_\infty \mid a' \in \mathbb{F}_q\}$ and $\{(0, a', 0)_O \mid a' \in \mathbb{F}_q\}$, respectively. The 2-point stabiliser $G_{\infty, O}$ acts by conjugation on P_∞ as $(a, a', a'')_\infty \mapsto (at^\theta, a't^{\theta+1}, a''t^{\theta+2})_\infty$. Recalling the definition of trace of an element of G from Section A, an element of order 3 or 9 has trace 1 and an element of order 2 or 6 has trace -1 by Observation A.1.

We may assume that w fixes ∞ . Let U be the group of central elements of an arbitrary Sylow 3-subgroup of H not fixing ∞ . Then, by the 2-transitivity of (G, Ω) , we may assume that U fixes O . A generic member u of U is $(0, 0, d)_O$, $d \in \mathbb{F}_q$, but note that not all elements of $\mathbb{F}_q$ provide elements of U ; only q_0 do. However, we may assume that for those q_0 values, the traces of wu and wu^2 are in $\{1, -1\}$. We consider all possible orders for w and prove that under this assumption, each lead to a contradiction.

- *Assume that w has order 9.* By the action of the 2-point stabiliser $G_{\infty, O}$, we may assume that $w = (1, b, c)_\infty$, for some $b, c \in \mathbb{F}_q$. One can now explicitly calculate the traces of the matrices corresponding to the products $wu^{\pm 1}$ and obtain

$$1 + d^2(c^2 - c^\theta + b^\theta + b^{\theta+1} - b^2 - b - 1) \mp [d + d^\theta(b^2 - b - c - c^\theta)],$$

where the minus sign corresponds to wu and the plus sign to $wu^2 = wu^{-1}$. Since these traces belong to $\{1, -1\}$, the difference of the traces of wu and wu^{-1} belongs to $\{0, 1, -1\}$. Hence, we obtain, setting $b^2 - b - c - c^\theta = f$ (possibly $f = 0$), that $d + fd^\theta = \varepsilon$, with $\varepsilon \in \{0, 1, -1\}$. This implies $d^\theta + fd^3 = \varepsilon^\theta$, yielding

$$(f\varepsilon^\theta - \varepsilon) + d - f^{\theta+1}d^3 = 0.$$

The latter can only be true for at most three nonzero values of d , whereas we have q_0 values of d , a contradiction.

- *Assume that w has order 3 and does not fix every block joining its fixed point ∞ to a point of the subunital stabilised by H .* Similarly as in the previous case, we may assume that $w = (0, b, 1)_\infty$, and both wu and wu^2 have trace either 1 or -1 (with u as above). This implies

$$\begin{cases} 1 + d^2 + b^{\theta+1}d^2 + d^\theta \in \{1, -1\}, \\ 1 + d^2 + b^{\theta+1}d^2 - d^\theta \in \{1, -1\}, \end{cases}$$

which obviously implies $d \in \{0, 1, -1\}$. Since this must be true for all q_0 values for d , this is again a contradiction.

- Assume that w has order 3 and fixes every block joining its fixed point ∞ to a point of the subunital stabilised by H . Here we can take $w = (0, b, 0)_\infty$ and $u = (0, 0, d)_O$. Calculating the trace of wu , we obtain $1 + b^{\theta+1}d^2$, which is only equal to 1 if one of b or d is 0, a contradiction. If the trace of wu is -1 then $b^{\theta+1}d^2 = 1$. Since there are at most two solutions for d , we again reached a contradiction.
- Assume finally that w has order 6. Any order 6 element fixing ∞ can be obtained by multiplying $(0, b, 0)_\infty$ with the involution η that fixes the block $B(\infty, O)$ pointwise, and which is given by the diagonal matrix $\text{Diag}(1, 1, -1, -1, 1, 1, -1)$. The set of elements of the form $(a, 0, c)_\infty$ acts transitively on the set of blocks through ∞ , and conjugating with an appropriate element of the 2-point stabiliser $G_{\infty, O}$, we may assume $a = 1$. Then one calculates that the trace of

$$(1, 0, c)_\infty^{-1}(0, b, 0)_\infty \eta(1, 0, c)_\infty(0, 0, \pm d)_O$$

is equal to

$$1 \mp bd + d^2 - b^2d^2 + b^{\theta+1}d^2 + cd^2 + c^2d^2 - c^\theta d^2 \pm b^\theta d^\theta \pm bcd^\theta$$

where the plus and minus signs should be read consistently. Hence the difference of these traces is equal to

$$bd - b^\theta d^\theta - bcd^\theta$$

Since both traces are either 1 or -1 we obtain that $bd - b^\theta d^\theta - bcd^\theta = k$ where $k \in \{0, 1, -1\}$. Hence also $b^\theta d^\theta - b^3 d^3 - b^\theta c^\theta d^3 = k$. Since $b \neq 0$ we can compute d^θ from the second equation and substitute in the first to obtain either a linear or a cubic equation in d which can be satisfied for at most $3 < q_0$ values of d , again a contradiction.

Hence, the claim that the coset wH contains at least one element of order distinct from 2, 3, 6 or 9, is proved. Let $h \in H$ be such that the order of wh is not 2, 3, 6 or 9. Then we can do Nielsen moves to transform (x, y, z, v, w) into (x, y, z, v, wh) and the proposition follows. $\square$

4. ELIMINATING THE STRUCTURAL SUBGROUPS

In this section, we show that we can always connect a generating k -tuple $(x, y, z, \dots, w)$, where $k \geq 5$ and $|x| \notin \{2, 3, 6, 9\}$, to a generating k -tuple where $\langle x, x^y \rangle, \langle x, x^z \rangle, \dots, \langle x, x^w \rangle$ are not structural subgroups. See Section 2.5 for the list of structural subgroups of the Ree groups. We proceed based on the fixed point sets of x .

4.1. x fixes two points. Proposition 4.1, below, allows us to assume when given a generating k -tuple $(x, y, z, \dots, w)$, $k \geq 3$, where x fixes exactly two points (hence $|x| \neq 2$), that we may connect $(x, y, z, \dots, w)$ to a generating k -tuple $(x, y', z', \dots, w')$ where $\langle x, y' \rangle, \langle x, z' \rangle, \dots, \langle x, w' \rangle$ are not structural subgroups. Moreover, we may assume that $|y'|, |z'|, \dots, |w'| \neq 2$. For this proposition, when given a point α , we denote $G_\alpha = \text{Stab}_G(\alpha)$, and when given two points α, β , we write $G_{\alpha, \beta} = G_\alpha \cap G_\beta$. We denote the stabiliser of $B := B(\alpha, \beta)$ as $G_{B(\alpha, \beta)} = G_B$.

Proposition 4.1. *Let $(x, y, z, \dots, w) \in V_k(G)$, $k \geq 3$, where x fixes two points and $|x| \neq 2$. Suppose that $(x, y, z, \dots, w)$ is not connected to a redundant k -tuple. Then $(x, y, z, \dots, w)$ is connected to $(x, y', z', \dots, w')$, where $\langle x, y' \rangle, \langle x, z' \rangle, \dots, \langle x, w' \rangle$ are not structural subgroups. Moreover, none of $y', z', \dots, w'$ is an involution.*

Proof. By Lemma 3.1, we may assume that $y, z, \dots, w \notin N_G(\langle x \rangle)$ and $|y|, |z|, \dots, |w| \neq 2$. Suppose that x fixes exactly two points α and β and that $H = \langle x, y \rangle$ is a structural subgroup. Set $B := B(\alpha, \beta)$. That implies that H is a subgroup of either G_α , G_β , or G_B by Lemma 2.5. We start by assuming H is a subgroup of G_α . Note $G_\alpha = N_G(P)$, $P \in \text{Syl}_3(G)$. In particular, we have $G_\alpha = P \rtimes G_{\alpha, \beta}$, where $G_{\alpha, \beta}$ is cyclic of order $q - 1$.

Case 1: *Suppose y fixes only the point α .*

Let σ be the involution fixing the block $B(\alpha, \beta)$ pointwise and set $Q = P \rtimes \langle \sigma \rangle$. Then Q is the set of elements of order 2, 3, 6 or 9 in G_α . But, by Lemma 2.4, Q can also be described as the set of elements of G_α not fixing exactly two points. This implies $y \in Q$.

Hence, if xy were in Q , then $xyy^{-1} \in Q$, which would imply that $x \in Q$. This contradicts the fact that x fixes α and β . We conclude that xy is not in Q . In particular, it must have exactly two fixed points, one of which is α . Replacing y with xy , we are reduced to Case 2.

Case 2: *Suppose y fixes two points $\{\alpha, \gamma\}$.*

If $\gamma = \beta$, then H is a cyclic subgroup and $H \leq G_B$ and we can connect $(x, y, z, \dots, w)$ to $(x, x^m y, z, \dots, w)$, where $\langle x^m y \rangle$ generates H (cf. Lemma 2.16). Hence, $(x, y, z, \dots, w)$ is connected to a redundant triple, which implies we may assume that $\gamma \neq \beta$.

We now claim that we may assume that z does not fix α and does not stabilise B . Indeed, if all of $z, \dots, w$ either fix α or stabilise B , then, since $(x, y, z, \dots, w)$ is a generating k -tuple, we may assume on the one hand that z does not fix α , but stabilises B . On the other hand, there is an element v in $\{y, z, \dots, w\}$ that does not stabilise B , but fixes α (by assumption, otherwise the claim follows). If vz is not an involution, then we replace z with vz and the claim follows. Suppose now that vz is an involution. Since vz does not stabilise the block B , we deduce that $vzxvz$ does not stabilise B , hence $vzxvz \neq x^{-1}$, implying that xvz is not an involution. Clearly xvz does not fix α and does not stabilise B . We can now replace z with xvz and the claim follows. Suppose that $z \cdot \alpha = \beta$ and $z^{-1} \cdot \alpha = \beta$. Then z is an involution (since the square of an element of order 6 only fixes one point by Lemma 2.4), a contradiction.

Then we define $z' = z$ if $z \cdot \alpha \neq \beta$ and $z' = z^{-1}$ otherwise. Then $z' \cdot \alpha \neq \alpha$ and $z' \cdot \alpha \neq \beta$. Given that x is not an involution, the elements $g_i := z'^i x^i$ for $0 \leq i \leq 2$ are distinct. We see that

$${}^{g_i}y \cdot (g_i \cdot \alpha) = g_i y g_i^{-1} g_i \cdot \alpha = g_i y \cdot \alpha = g_i \cdot \alpha,$$

and similarly, ${}^{g_i}y \cdot (g_i \cdot \gamma) = g_i \cdot \gamma$. Hence, ${}^{g_i}y \in G_{g_i \cdot \alpha, g_i \cdot \gamma}$. Suppose that $g_i \cdot \gamma = g_j \cdot \gamma$, where $0 \leq i < j \leq 2$. We then have $x^{j-i} \cdot \gamma = \gamma$. Hence, we must have $j = i$. Therefore, there exists an integer $\ell \in \{0, 1, 2\}$ such that $g_\ell \cdot \gamma \notin \{\alpha, \beta\}$. Now we also have $g_\ell \cdot \alpha = z' \cdot \alpha \notin \{\alpha, \beta\}$. Hence, ${}^{g_\ell}y$ has

two fixed points, neither of which is α or β . If at least one of the two fixed points of ${}^{g_\ell}y$ is not contained in B , then $\langle x, {}^{g_\ell}y \rangle$ is not a structural subgroup by Lemma 2.5. Therefore, we may assume that ${}^{g_\ell}y$ fixes two points of B and so $\langle x, {}^{g_\ell}y \rangle$ stabilises B .

We claim that $\langle x, ({}^{g_\ell}y)^{z'} \rangle$ neither stabilises a block, nor fixes a point. By our claim above, $z' \cdot B \neq B$. But

$${}^{g_\ell}y \cdot (z' \cdot \alpha) = z' x^k y x^{-k} z'^{-1} \cdot (z' \cdot \alpha) = z' \cdot \alpha;$$

hence, $z' \cdot \alpha$ is one of the fixed points of ${}^{g_\ell}y$, which we assume stabilises B . We conclude that $\{z' \cdot \alpha\} = (z' \cdot B) \cap B$. Recall that $z' \cdot \alpha \notin \{\alpha, \beta\}$. Also, $z' \cdot \beta \in z' \cdot B \setminus B$, so none of α, β is fixed by $({}^{g_\ell}y)^{z'}$. Then the claim follows (because the only fixed block of $({}^{g_\ell}y)^{z'}$ is $z'^{-1} \cdot B \neq B$).

Case 3: Suppose that y does not fix either α or β , but stabilises B .

As in Case 1, we may assume that z does not stabilise B , hence neither does yz . So we replace y with yz , if yz is not an involution, and with xyz , if yz is an involution (and then, as in Case 1, xyz is not an involution). Then either we have that $\langle x, yz \rangle$ or $\langle x, xyz \rangle$, respectively, is not a structural subgroup, or yz or xyz , respectively, fixes either α or β and we end up in Cases 1 and/or 2.

Cases 1, 2 and 3 imply that we can connect to a k -tuple $(x, y', z', \dots, w')$ with $\langle x, y' \rangle$ not structural.

If $\langle x, z' \rangle$ is a structural subgroup, then we can apply the same arguments as for $\langle x, y \rangle$. Note that Case 2 simplifies a lot since we can use y' to play the role of z , and we do not need to replace anything since y does not fix either of α, β and does not stabilise B . Likewise we can apply the same arguments to $\langle x, u \rangle$, for all elements u of $\{z', \dots, w'\}$.

This completes the proof of the proposition. $\square$

Proposition 4.2. *Let $(x, y, z, \dots, w) \in V_k(G)$ where $|x| \neq 2$, $k \geq 3$, and where x fixes exactly two points. Then we may connect $(x, y, z, \dots, w)$ to a generating k -tuple $(x, y', z', \dots, w')$ where $\langle x, x^{y'} \rangle, \langle x, x^{z'} \rangle, \dots, \langle x, x^{w'} \rangle$ are not structural subgroups.*

Proof. Without loss of generality we can assume $k = 3$.

We assume that x fixes the points α and β and stabilises the block $B = B(\alpha, \beta)$. Proposition 4.1 implies that we may assume that $\langle x, y \rangle$ and $\langle x, z \rangle$ are both not structural subgroups where y and z are not involutions. In particular, $y, z \notin N_G(\langle x \rangle)$.

We will proceed in two stages first proving we can connect (x, y, z) to a generating 3-tuple (x, y', z) where $\langle x, x^{y'} \rangle$ is not a structural subgroup. If $\langle x, x^y \rangle$ is not a structural subgroup, then there is nothing to prove. Otherwise, we note that $x^y \in G_{y^{-1}\alpha, y^{-1}\beta}$ and that $\langle x, x^y \rangle$ is a subgroup of G_α, G_β , or G_B . If $x^y \cdot B = B$, then x stabilises $y \cdot B \neq B$, a contradiction to Lemma 2.5. Hence, we may assume that $\langle x, x^y \rangle \leq G_\alpha$. Since $\langle x, y \rangle$ is not a structural group, we have $y \cdot \alpha \neq \alpha$. Hence, we must have $y \cdot \alpha = \beta$. Given that $\langle x, z \rangle$ is not a structural subgroup, we must have $z \cdot \beta \neq \beta$. If $z \cdot \alpha = \beta$ and $z^{-1} \cdot \alpha = \beta$, then z stabilises B and $\langle x, z \rangle$ is structural, a contradiction.

Thus, there exists a $z' \in \{z, z^{-1}\}$ such that $z' \cdot \beta \notin \{\alpha, \beta\}$. We also note that $z' \cdot B \neq B$.

Let $g_i = z'x^iy$ for $0 \leq i \leq 3$. Then $g_i \cdot \alpha = z' \cdot \beta \notin \{\alpha, \beta\}$. We claim that $g_i \cdot \beta \neq g_j \cdot \beta$ for $0 \leq i < j \leq 3$. Suppose otherwise. We then have $x^{j-i}y \cdot \beta = y\beta$. It follows that $y \cdot \beta \in \{\alpha, \beta\}$ since x^{j-i} only fixes α and β . Given that $y \cdot \beta \neq \beta$, we have $y \cdot \beta = \alpha$. Since we assumed $y \cdot \alpha = \beta$, this implies that $y \cdot B = B$, a contradiction.

Therefore, $g_0 \cdot \beta, g_1 \cdot \beta, g_2 \cdot \beta, g_3 \cdot \beta$ are all distinct. Hence, $g_\ell \cdot \beta \notin \{\alpha, \beta\}$ for at least two values of $\ell \in \{0, 1, 2, 3\}$. Similarly, for at most one value of $\ell \in \{0, 1, 2, 3\}$ we have $g_\ell \cdot B = B$. Hence, we see that there exists at least one $\ell \in \{0, 1, 2, 3\}$ such that $g_\ell \cdot \beta \notin \{\alpha, \beta\}$ and $g_\ell \cdot B \neq B$. Since also $g_\ell \cdot \alpha \notin \{\alpha, \beta\}$, we conclude that $\langle x, x^{g_\ell} \rangle$ is not structural.

We then connect

$$(x, y, z) \rightarrow (x, x^\ell y, z) \rightarrow (x, z'x^\ell y^{-1}, z) = (x, g_\ell, z).$$

Setting $y' = g_\ell$, we may apply the same arguments to (x, z, y') and z to obtain a generating 3-tuple (x, y', z') where $\langle x, x^{z'} \rangle$ is not a structural subgroup. $\square$

4.2. x fixes no points. From now on, we again consider vectors in $V_5(G)$, that is, of length 5. This allows us to again use Lemma 3.2 and Lemma 3.3 in the course of our proofs.

Proposition 4.3. *Let $(x, y, z, v, w) \in V_5(G)$ be minimal and not connected to a redundant vector, where x does not fix any point (in particular $|x| \neq 2$). Then either all subgroups $\langle x, y, z, v \rangle$, $\langle x, y, z, w \rangle$, $\langle x, y, v, w \rangle$, $\langle x, z, v, w \rangle$, $\langle x, x^y \rangle$, $\langle x, x^z \rangle$, $\langle x, x^v \rangle$ and $\langle x, x^w \rangle$ are non-structural, or we can connect (x, y, z, v, w) to a 5-tuple containing an element of G fixing exactly two points of $U_R(q)$.*

Proof. By Lemma 3.1, we may assume that $y, z, v, w \notin N_G(\langle x \rangle)$ and that $|y|, |z|, |v|, |w| \neq 2$. Since x does not fix any point, its order divides either $\frac{1}{2}(q+1)$, $q + \sqrt{3q} + 1$, or $q - \sqrt{3q} + 1$ by Lemma 2.4. In the latter two cases, x is not contained in any structural subgroup, so that all of $\langle x, x^y \rangle$, $\langle x, x^z \rangle$, $\langle x, x^v \rangle$, $\langle x, x^w \rangle$, $\langle x, y, z, v \rangle$, $\langle x, y, z, w \rangle$, $\langle x, y, v, w \rangle$ and $\langle x, z, v, w \rangle$ are not structural by the minimality assumption.

Hence, we may assume that $|x|$ divides $\frac{1}{2}(q+1)$. We claim that, if $\langle y, z, v, w \rangle$ is not a subfield subgroup, then we can connect (x, y, z, v, w) to either a 5-tuple containing an element fixing exactly two points of $U_R(q)$, or a 5-tuple (x, y', z, v, w) such that $\langle y', z, v, w \rangle$ is a subfield subgroup. Indeed, suppose first that $\langle y, z, v, w \rangle$ fixes some point α . Then we can connect (x, y, z, v, w) to (x, xyx^{-1}, z, u, v) , and the subgroup $H := \langle xyx^{-1}, z, v, w \rangle$ does not stabilise α or any other point of Ω (since each element of $\{y, z, v, w\}$ fixes exactly one point; indeed, if at least one element of $\{y, z, v, w\}$ fixed two points, then the proposition would trivially follow).

Suppose now that $\langle y, z, v, w \rangle$ stabilises a block C , but fixes no point. Then we obtain that $\langle y, z, v, w \rangle \leq \text{PSL}_2(q) \times 2$. Now, by [30], see also [7], the subgroups of $\text{PSL}_2(q)$ that do not fix a point are subgroups of dihedral groups, or a subgroup of $\text{Alt}(4)$, or are isomorphic to $\text{PSL}_2(q_0)$, for some 3-power q_0 with

$q_0 - 1$ a divisor of $q - 1$. Since in the two former cases, $\langle y, z, v, w \rangle$ is at most 3-generated, we may, by [8], assume that $\langle y, z, v, w \rangle \in \{\mathrm{PSL}_2(q_0), \mathrm{PSL}_2(q_0) \times 2\}$. If $\langle y, z, v, w \rangle \cong \mathrm{PSL}_2(q_0)$, then, by [10], we can connect (y, z, v, w) to a redundant quadruple. We now show that $\langle y, z, v, w \rangle \cong \mathrm{PSL}_2(q_0) \times 2$ leads to our assertion.

Let η be the unique involution fixing C pointwise. We can write (y, z, v, w) as

$$(\eta^{\epsilon_y} y_0, \eta^{\epsilon_z} z_0, \eta^{\epsilon_v} v_0, \eta^{\epsilon_w} w_0),$$

where $\epsilon_y, \dots, \epsilon_w \in \{0, 1\}$ and $y_0, \dots, w_0 \in \mathrm{PSL}_2(q_0)$. It follows from the main result of [10] that we can connect (y_0, z_0, v_0, w_0) to (y'_0, z'_0, v'_0, w'_0) with Nielsen moves, where y'_0 fixes exactly two points of C . The same Nielsen moves take (y, z, v, w) to (y^*, z^*, v^*, w^*) , where $y'_0(y^*)^{-1}, \dots, w'_0(w^*)^{-1} \in \{1, \eta\}$. But then clearly, y^* fixes exactly two points and the proposition follows.

So we have shown that we may assume that $\langle y, z, v, w \rangle$ is not a structural subgroup. Since by Lemma 3.2 it is not a small subgroup either the claim follows, that is, we may assume that $\langle y, z, v, w \rangle$ is a subfield subgroup.

So, let $\langle y, z, v, w \rangle \cong {}^2\mathrm{G}_2(q_1) := H$, for some appropriate power q_1 of 3, and denote by $U_R(q_1)$ the corresponding Ree subunitary of $U_R(q)$. Lemma 3.3 immediately yields $q_1 \geq 27$. By Lemma 2.9, x stabilises three mutually disjoint blocks B, B', B'' . Let b be the size of the stabiliser of a block of $U_R(q_1)$ in H . We claim that the size of the stabiliser in H of a block of $U_R(q)$ outside $U_R(q_1)$ is not larger than b . Indeed, it is easy to check that the only maximal subgroup of H having a size larger than b is, up to conjugation, the normaliser $N(P)$ of a Sylow 3-subgroup P . However, by Lemma 2.2, an arbitrary block of $U_R(q)$ is stabilised by a subgroup of P of maximal size q_1 . It follows that no subgroup of $N(P)$ larger than $q_1(q_1 - 1)$ stabilises a block of $U_R(q)$ and the claim is proved.

Next we bound the number of elements h of H with the property that y maps $h \cdot C$ to $h \cdot C'$, with $C, C' \in \{B, B', B''\}$ (we do not exclude $C = C'$). We may assume that there is at least one such element $h \in H$. Then $y^h \cdot C = C'$. By the previous paragraph, there are at most b choices for y^h . However, for each such choice, there are exactly $|C_H(y)|$ choices for h . Since y is not an involution, Proposition 2.11 implies that, if y is not a central element of a Sylow 3-subgroup, then $|C_H(y)| \leq 2q_1^2$, whereas if y is such a central element, then $|C_H(y)| = q_1^3$. However, in the latter case, there is no such element h if $C = C'$, as follows from Lemma 2.2. Now, there are nine ordered pairs (C, C') with $C, C' \in \{B, B', B''\}$, and there are six ordered pairs (C, C') with $C, C' \in \{B, B', B''\}$ and $C \neq C'$. Hence, if y is not central of order 3, then there are at most $9 \cdot 2q_1^2 \cdot b$ elements $h \in H$ such that $\{h \cdot B, h \cdot B', h \cdot B''\}$ is not disjoint from $\{yh \cdot B, yh \cdot B', yh \cdot B''\}$, whereas if y is central of order 3, then there are at most $6q_1^3 \cdot b$ such elements. Since $6q_1^3 \geq 18q_1^2$, we conclude that there are at most $6q_1^3 \cdot b$ such elements $h \in H$.

We now apply the argument of the previous paragraph to each of $y^{-1}, z^{-1}, v^{-1}, w^{-1}$ and we obtain that there are at most

$$24bq_1^3 = 48|\mathrm{PSL}_2(q_1)|q_1^3 = 24q_1^4(q_1^2 - 1) < q_1^3(q_1^3 + 1)(q_1 - 1) = |{}^2\mathrm{G}_2(q_1)|$$

elements $h \in H$ such that at least one of $y^{-1}, z^{-1}, v^{-1}, w^{-1}$ maps at least one of $h \cdot B, h \cdot B', h \cdot B''$ to a member of $\{h \cdot B, h \cdot B', h \cdot B''\}$.

Thus there exists $g \in H$ such that every element of $\{y^{-1}, z^{-1}, v^{-1}, w^{-1}\}$ maps the set

$$\Sigma := \{g^{-1} \cdot B, g^{-1} \cdot B', g^{-1} \cdot B''\}$$

to a disjoint set of blocks. It follows that

$$\langle x^g, (x^g)^y \rangle, \quad \langle x^g, (x^g)^z \rangle, \quad \langle x^g, (x^g)^v \rangle, \quad \langle x^g, (x^g)^w \rangle$$

are all non-structural.

Indeed, since x^g fixes no point and its order divides $(q+1)/2$, the only structural maximal subgroups containing x^g are block stabilisers. Hence, if $\langle x^g, (x^g)^a \rangle$, $a \in \{y, z, v, w\}$, were structural, it would be contained in a block stabiliser. Consequently, the corresponding block would be fixed by both x^g and $(x^g)^a$. But Corollary 2.13 asserts that x^g only stabilises the blocks of Σ , whereas $(x^g)^a$ only stabilises the blocks of $a^{-1} \cdot \Sigma$, which is disjoint from Σ by the choice of g .

The proof is complete. $\square$

5. CONNECTING TO A REDUNDANT VECTOR

In this section, we connect a generating k -tuple $(x, y, z, \dots, w)$, $k \geq 5$, where $\langle x, x^y \rangle, \langle x, x^z \rangle, \dots, \langle x, x^w \rangle$ are non-structural subgroups to a redundant k -tuple. For the sake of simplicity, we do this for $k = 5$, the proof of the general case $k \geq 5$ being a direct and obvious modification of the case $k = 5$. Proposition 3.5 implies that we may assume that there exists an element of (x, y, z, v, w) which fixes either no points or exactly two points. Thus, up to Nielsen moves, we may assume that x is that element. By Proposition 4.2 and Proposition 4.3, we may assume that $L_1 = \langle x, x^y \rangle$, $L_2 = \langle x, x^z \rangle$, $L_3 = \langle x, x^v \rangle$, and $L_4 = \langle x, x^w \rangle$ are not structural subgroups. Let $K_1 = \langle x, z, v, w \rangle$, $K_2 = \langle x, y, v, w \rangle$, $K_3 = \langle x, y, z, w \rangle$, and $K_4 = \langle x, y, z, v \rangle$. We then see that K_i are not structural subgroups since they contain non-structural subgroups for $1 \leq i \leq 4$.

Lemma 3.3 implies that we may assume that K_i are subfield subgroups over fields of size at least 27 for $1 \leq i \leq 4$. In particular, we have $K_i \cong {}^2\text{G}_2(q_i)$ where $q_1 \geq q_2 \geq q_3 \geq q_4 \geq 27$.

Proposition 5.1. *Using the above notation and assumptions, if $q_1 > q_4$, then one of the following holds:*

- (a) (x, y, z, v, w) is connected to a redundant 5 tuple.
- (b) (x, y, z, v, w) is connected to a generating 5-tuple (x, y', z, v, w) such that $\langle x, y' \rangle$ is a subfield subgroup, where

$$|\langle x, y', z, v \rangle| + |\langle x, y', z, w \rangle| + |\langle x, y', v, w \rangle| + |\langle x, z, v, w \rangle| > \sum_{i=1}^4 |K_i|.$$

Proof. It suffices to prove that we can connect (x, y, z, v, w) to (x, y', z, v, w) such that $\langle x, x^y \rangle = \langle x, x^{y'} \rangle$ and

$$\begin{cases} |C_{\langle x, z, v, w \rangle}(x)| \leq |C_{\langle x, y', z, w \rangle}(x)|, \\ |C_{\langle x, z, v, w \rangle}(x)| \leq |C_{\langle x, y', z, v \rangle}(x)|, \text{ and} \\ |C_{\langle x, z, v, w \rangle}(x)| \leq |C_{\langle x, y', w, v \rangle}(x)|. \end{cases}$$

By ?? $C_{\langle x, z, v, w \rangle}(x)$ is one of the following:

$$C_{\langle x, z, v, w \rangle}(x) = \begin{cases} C_{q_1-1} & |x| \mid \frac{q-1}{2} \\ C_2^2 \times C_{\frac{q+1}{4}} & |x| \mid \frac{q+1}{4} \\ C_{q_1 \pm \sqrt{3q_1} + 1} & x \in M_i, i = \pm 1 \\ P' \in \text{Syl}_3(2\text{G}_2(q_1)) & x \in Z(P), P \in \text{Syl}_3(G) \\ [P', P'] \times \langle \eta \rangle, |\eta| = 2 & x \in [P, P] \setminus Z(P), P \in \text{Syl}_3(G) \\ Z(P') \cdot \langle x \rangle & x \in P \setminus [P, P], P \in \text{Syl}_3(G) \\ \langle x \rangle \times \text{PSL}_2(q_1) & |x| = 2 \end{cases}$$

Therefore, we have a few cases. The first case takes the first and the third possibility above together and corresponds to $C_{\langle x, z, v, w \rangle}(x)$ being cyclic. The second case is when $C_{\langle x, z, v, w \rangle}(x) \cong C_2^2 \times C_{\frac{q+1}{4}}$. The last four possibilities above occur when x is either an involution or has order 3, 6 or 9. However, Proposition 3.5 implies that we may assume that x does not have order 2, 3, 6, or 9.

Case 1: $C_{\langle x, z, v, w \rangle}(x)$ is cyclic.

In this case, we have $C_{\langle x, z, v, w \rangle}(x) \in \{C_{q_1-1}, M_{-1}, M_1\}$. Since x and x^y are conjugate and $\langle x, x^y \rangle$ is not a structural group, by Lemma 2.14 and our assumption on K_i being subfield subgroups there exists an element $d \in \langle x, x^y \rangle$ such that $x^y = x^d$. Let $u = yd^{-1}$, which gives $u \in C_{\langle x, y \rangle}(x)$. Writing $C_{\langle x, z, v, w \rangle}(x) = \langle c \rangle$, we claim that $\langle u, c \rangle$ is cyclic, and we proceed based on the isomorphism type of $C_{\langle x, z, v, w \rangle}$. Since x does not have order 2, 3, 6 or 9, we have $C_G(x) \in \{C_{q-1}, M_{-1}, M_1, C_2^2 \times C_{\frac{q+1}{4}}\}$.

If $C_{\langle x, z, v, w \rangle}(x) = C_{q_1-1}$, we then have $|x| \mid \frac{q_1-1}{2}$, which implies $2|x| \mid q_1 - 1$. Since $q_1 = 3^{2m+1}$ and $q = 3^{2n+1}$, where $2m+1 \mid 2n+1$, we have $q_1 - 1 \mid q - 1$. Therefore, $2|x| \mid q - 1$; hence, $|x| \mid \frac{q-1}{2}$. Hence, $C_G(x) \cong C_{q-1}$, which is cyclic. Since $\langle u, c \rangle \leq C_G(x)$, it follows that $\langle c, u \rangle$ is cyclic.

Now suppose that $C_{\langle x, z, v, w \rangle}(x)$ is cyclic of order $q_1 \pm \sqrt{3q_1} + 1$. In this case the order of x divides $q \pm \sqrt{3q} + 1$ and not $q + 1$, so $C_G(x)$ is again cyclic, and so is $\langle c, u \rangle$.

Lemma 2.16 then implies there exists a natural number m such that $\langle c, u \rangle = \langle c^m u \rangle$. Since $c^m \in \langle x, z, v, w \rangle$, we may connect (x, y, z, v, w) to $(x, c^m y, z, v, w)$. Since

$$d \in \langle x, x^y \rangle = \langle x, x^{c^m y} \rangle \leq \langle x, c^m y, z, w \rangle \cap \langle x, c^m y, z, v \rangle \cap \langle x, c^m y, w, v \rangle,$$

we see that $\langle x, c^m y, z, w \rangle$, $\langle x, c^m y, z, v \rangle$ and $\langle x, c^m y, w, v \rangle$ are subgroups that contain d which are not structural. Thus,

$$c^m u = c^m y y^{-1} u = c^m y d^{-1} \in \langle x, y c^m, z, w \rangle \cap \langle x, y c^m, z, v \rangle \cap \langle x, y c^m, w, v \rangle.$$

Given that $C_{\langle x, z, v, w \rangle}(x) = \langle c \rangle \in \langle c^m u \rangle$, we have

$$C_{\langle x, z, v, w \rangle}(x) \leq C_{\langle x, c^m y, z, w \rangle}(x) \cap C_{\langle x, c^m y, z, v \rangle}(x) \cap C_{\langle x, c^m y, w, v \rangle}(x).$$

Thus, we have $|C_{\langle x, z, v, w \rangle}(x)| \leq |C_{\langle x, c^m y, z, v \rangle}(x)|$. Similarly,

$$|C_{\langle x, z, v, w \rangle}(x)| \leq |C_{\langle x, c^m y, w, v \rangle}(x)| \quad \text{and} \quad |C_{\langle x, z, v, w \rangle}(x)| \leq |C_{\langle x, c^m y, z, w \rangle}(x)|.$$

Lemma 2.15 implies

$$|\langle x, z, v, w \rangle| \leq |\langle x, c^m y, z, v \rangle|, \quad |\langle x, z, v, w \rangle| \leq |\langle x, c^m y, w, v \rangle|,$$

and

$$|\langle x, z, v, w \rangle| \leq |\langle x, c^m y, z, w \rangle|.$$

Case 2: $C_{\langle x, z, v, w \rangle}(x) = C_2^2 \times C_{\frac{q_1+1}{4}}$

This case happens when $|x| \mid \frac{q_1+1}{4}$. Since $\langle x, x^y \rangle$ is not a structural subgroup, by Lemma 2.14 and our assumption that the K_i are subfield subgroups there exists an element $d \in \langle x, x^y \rangle$ such that $x^y = x^d$. Let $u = yd^{-1}$, which gives $u \in C_{\langle x, y \rangle}(x)$. Let $c \in \langle x, z, v, w \rangle$ be an order $\frac{1}{4}(q_1 + 1)$ element such that $C_{\langle x, z, v, w \rangle}(x) = C_2^2 \times \langle c \rangle$, where C_2^2 is generated by some two involution η_1, η_2 . Since $|x|$ divides $\frac{1}{2}(q_1 + 1)$, it also divides $\frac{1}{2}(q + 1)$, and hence, there exists an order $\frac{1}{4}(q + 1)$ element $b \in G$ such that $C_G(x) = \langle \eta_1, \eta_2 \rangle \times \langle b \rangle$. Then we have $c = b^k$, for some $k \in \mathbb{N}$, and $u = \eta^\epsilon b^j$, for some $j \in \mathbb{N}$, $\eta \in \langle \eta_1, \eta_2 \rangle$ and $\epsilon \in \{0, 1\}$. It follows that $\langle c, u \rangle \leq \langle \eta \rangle \times \langle b \rangle$, which is cyclic.

Lemma 2.16 then implies that there exists a natural number m such that $\langle c, u \rangle = \langle c^m u \rangle$, and the rest of the proof of Case 1 can be copied. $\square$

In case $q_1 = q_4$ we obtain the following proposition.

Proposition 5.2. *With respect to the above notation, if $q_1 = q_4$, then (x, y, z, v, w) is connected to a redundant 5-tuple*

Proof. The groups K_1, K_2, K_3 , and K_4 are all isomorphic to ${}^2G_2(q_1)$. Let $H = \langle x, v, w \rangle \leq K_1 \cap K_2$. Since $L_3 \leq H$ and L_3 is not structural, H is either small or a subfield subgroup. Assume first that H is a subfield subgroup, and hence self-normalising.

Since K_1 and K_2 are subfield subgroups of the same order, there exists an element $g \in G$ so that $K_1^g = K_2$. Thus, since $H \leq K_1$, we have $H^g \leq K_2$. However, H is also a subgroup of K_2 . Therefore, H and H^g are conjugate in K_2 . Hence, as K_2 is a subfield subgroup, there exists an element $h \in K_2$ such that $H^g = H^h$. So, $gh^{-1} \in N_G(H) = H \leq K_2$. Consequently, $g \in K_2$. We now have $K_1 = K_2^{g^{-1}} = K_2$. Since $K_1 = \langle x, z, v, w \rangle$ and $K_2 = \langle x, y, v, w \rangle$, we have $G = \langle K_1, K_2 \rangle$. The assertion follows.

Hence we may assume that H is small. Then H is solvable. If it is generated by two elements, then Dunwoody [8] implies that (x, v, w) is connected to a redundant triple, and we are done in this case. Hence we may assume that H is generated by at least three elements, and so, by Lemma 2.10, $H \leq N_{K_1}(M_0)$, with M_0 a cyclic Hall subgroup of K_1 . Similarly, we may assume that $H' := K_1 \cap K_3$ is a small subgroup generated by at least three elements, and so $H' \leq N_{K_1}(M'_0)$, with M'_0 a cyclic Hall subgroup of K_1 . Now $x \in H$ implies that x does not fix any point (remember x was chosen

such that it fixes either exactly two points—which is not the case here as H does not contain such elements—or no points). Hence its order divides $\frac{q_1+1}{2}$. Corollary 2.12 now yields $M_0 = M'_0$, which implies, using [8] again, that we may connect (x, z, v, w) to a redundant quadruple since, by Lemma 2.10, every subgroup of $N(M_0)$ is 3-generated.

The proposition is proved. $\square$

We now come to the proof of our Main Theorem.

Proof. Let $(x, y, z, v, w) \in V_5(G)$ be a generating 5-tuple. Using Proposition 3.5 implies that we may assume that x either fixes exactly two points or fixes no points. Moreover, using Proposition 4.2 and Proposition 4.3 we may also assume, with previous notation, that L_i and K_i are not structural subgroups, $i = 1, 2, 3, 4$. We note that if any of the subgroups K_1, K_2, K_3, K_4 are small, we then have that (x, y, z, v, w) is connected to a redundant 5-tuple, by Lemma 3.3. Therefore, we may assume that K_1, K_2, K_3 , and K_4 are subfield subgroups. We may apply Proposition 5.1, possibly interchanging the roles of y, z, v and w , finitely many times until we either can connect (x, y, z, v, w) to a redundant 5-tuple, or $|K_1| = |K_2| = |K_3| = |K_4|$. Proposition 5.2 implies that (x, y, z, v, w) is connected to a redundant 5-tuple, as desired. $\square$

APPENDIX A. MATRICES FOR THE POINT STABILISERS

In this appendix we review the explicit construction of ${}^2G_2(q)$ given in [29, Section 9.2.4]. It is used in the present paper to prove various claims about the action of ${}^2G_2(q)$ on $U_R(q)$, in particular actions on blocks of this unit. We start with defining an explicit model of $U_R(q)$. Recall $q = 3^{2e+1}$ and $\theta : \mathbb{F}_q \rightarrow \mathbb{F}_q : x \mapsto x^{3^{e+1}}$. We then define $U_R(q)$ to be the set of points of the projective 6-space $\text{PG}(6, q)$ with coordinates given by $(1, 0, 0, 0, 0, 0, 0)$ and

$$(f_1(a, a', a''), -a', -a, -a'', 1, f_2(a, a', a''), f_3(a, a', a'')) =: p(a, a', a''),$$

$a, a', a'' \in \mathbb{F}_q$ and

$$\begin{cases} f_1(a, a', a'') = -a^{2\theta+4} - aa''^\theta + a^{\theta+1}a'^\theta + a''^2 + a'^{\theta+1} - a'a^{\theta+3} - a^2a'^2, \\ f_2(a, a', a'') = -a^{\theta+3} + a'^\theta - aa'' + a^2a', \\ f_3(a, a', a'') = -a^{2\theta+3} - a''^\theta + a^\theta a'^\theta + a'a'' + aa'^2. \end{cases}$$

We set $\infty = (1, 0, 0, 0, 0, 0, 0)$ and $O = (0, 0, 0, 0, 1, 0, 0)$. Then the Sylow 3-subgroup P_∞ of ${}^2G_2(q)$ fixing ∞ is given by the set of linear transformations $(a, a', a'')_\infty$ (acting on the left) with corresponding matrices

$$\begin{pmatrix} 1 & p & q & a'' & f_1(a, a', a'') & a' - a^{\theta+1} & a \\ 0 & 1 & a^\theta & 0 & -a' & 0 & 0 \\ 0 & 0 & 1 & 0 & -a & 0 & 0 \\ 0 & -a & a' - a^{\theta+1} & 1 & -a'' & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & a^2 & r & a & f_2(a, a', a'') & 1 & 0 \\ 0 & -a'' - aa' & s & -a' & f_3(a, a', a'') & -a^\theta & 1 \end{pmatrix},$$

where

$$\begin{cases} p = a^{\theta+3} - a'^{\theta} - aa'' - a^2a', \\ q = -a^{2\theta+3} + a''^{\theta} + a^{\theta}a'^{\theta} + a'a'' - aa'^2 - a^{\theta+2}a' - a^{\theta+1}a'', \\ r = a^{\theta+2} + a'' - aa', \\ s = -a^{\theta}a'' + a'^2 - a^{\theta+1}a'. \end{cases}$$

The Sylow 3-subgroup P_O of ${}^2\mathbf{G}_2(q)$ fixing O is given by the set of linear transformations $(a, a', a'')_O$ (again acting on the left) with corresponding matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ f_2(a, a', a'') & 1 & 0 & -a & 0 & a^2 & r \\ f_3(a, a', a'') & -a^{\theta} & 1 & a' & 0 & -a'' - aa' & s \\ a'' & 0 & 0 & 1 & 0 & a & a^{\theta+1} - a' \\ f_1(a, a', a'') & a' - a^{\theta+1} & a & -a'' & 1 & p & q \\ -a' & 0 & 0 & 0 & 0 & 1 & a^{\theta} \\ -a & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Its action on $U_R(q)$ can best be seen using an alternative description of $U_R(q)$ (see also [29, Section 9.2.4]), namely $U_R(q)$ consists of the point $(0, 0, 0, 0, 1, 0, 0)$ and the points

$$q(a, a', a'') := (1, f_2(a, a', a''), f_3(a, a', a''), a'', f_1(a, a', a''), -a', -a),$$

$a, a', a'' \in \mathbb{F}_q$. These points are the images of $(1, 0, 0, 0, 0, 0, 0)$ under the maps $(a, a', a'')_O$, as is readily checked.

One also checks that the matrix $\text{diag}(1, 1, -1, -1, 1, 1, -1)$ (using standard notation) defines a projective involution σ preserving $U_R(q)$ and fixing the points ∞ and $p(0, a', 0)$, $a' \in \mathbb{F}_q$. These points hence constitute by definition the block $B(\infty, O)$.

Now we note that the matrices of the unipotent elements displayed above all have determinant equal to 1. Since ${}^2\mathbf{G}_2(q) \cong \langle P_{\infty}, P_O \rangle$, we obtain ${}^2\mathbf{G}_2(q) \leq \mathbf{SL}_7(q)$. Also, since 7 does not divide $q - 1$, the matrix in $\mathbf{SL}_7(q)$ of every element of ${}^2\mathbf{G}_2(q)$ is uniquely determined by the action of that element on the projective space $\mathbf{PG}(6, q)$. In particular, the matrix in $\mathbf{SL}_7(q)$ of σ defined above is $\text{diag}(-1, -1, 1, 1, -1, -1, 1)$ and so has trace -1 . Multiplying σ with an appropriate unipotent element to obtain an element of order 6, leads to the following observation.

Observation A.1. *If $x \in {}^2\mathbf{G}_2(q)$ has order 3 or 9, then (the corresponding matrix of) x (in $\mathbf{SL}_7(q)$) has trace 1. If $x \in {}^2\mathbf{G}_2(q)$ has order 2 or 6, then (the corresponding matrix of) x (in $\mathbf{SL}_7(q)$) has trace -1 .*

One verifies easily the following multiplication law: $(a, a', a'')_{\infty}(b, b', b'')_{\infty} =$

$$(a + b, a' + b' + a^{\theta}b, a'' + b'' - ab' + a'b - a^{\theta+1}b)_{\infty}.$$

Note that this differs slightly from the expression in [29, Section 9.2.4] as we consider here action on the left, whereas the action is on the right in [29, Section 9.2.4]. By letting the above product act on $p(0, 0, 0)$, we deduce

$$(a, a', a'')_{\infty} \cdot p(b, b', b'') = p(a + b, a' + b' + a^{\theta}b, a'' + b'' - ab' + a'b - a^{\theta+1}b).$$

The inverse is given by

$$(a, a', a'')_{\infty}^{-1} = (-a, -a' + a^{\theta+1}, -a'')_{\infty}.$$

It is easy to check now that

$$\begin{cases} Z(P_{\infty}) = \{(0, 0, a'')_{\infty} \mid a'' \in \mathbb{F}_q\}, \\ [P_{\infty}, P_{\infty}] = \{(0, a', a'')_{\infty} \mid a', a'' \in \mathbb{F}_q\} \\ \{x^3 \mid x \in P_{\infty}\} = Z(P_{\infty}). \end{cases}$$

The two-point stabiliser $G_{\infty, O}$ is given the following action on $U_R(q)$.

$$h(t) \cdot p(b, b', b'') = p(bt^{\theta}, b't^{\theta+1}, b''t^{\theta+2}), t, b, b', b'' \in \mathbb{F}_q.$$

We find that $h(-1)$ is the involution '*sigma*' considered above, hence fixing the point set $\{\infty\} \cup \{p(0, b', 0) \mid b' \in \mathbb{F}_q\} = B(\infty, O)$. The other blocks through ∞ can be obtained by the image of $B(\infty, O)$ under the elements $(a, 0, a'')_{\infty}$, $a, a'' \in \mathbb{F}_q$ (since $(0, a', 0)_{\infty}$ stabilises $B(\infty, O)$). A generic block like that looks like

$$\{\infty\} \cup \{p(a, b', a'' - ab') \mid b' \in \mathbb{F}_q\} =: B_{a, a''}.$$

The following is immediate.

Observation A.2. *Every nontrivial member of $G_{\infty, O}$ fixes exactly two points (∞ and O), except $h(-1)$, which fixes $B(\infty, O)$ pointwise, but no point off that block. Moreover, $G_{\infty, O}$ acts freely on $U_R(q) \setminus B(\infty, O)$.*

The second assertion of the previous observation follows from the fact that, if $p(bt^{\theta}, b't^{\theta+1}, b''t^{\theta+2}) = p(bu^{\theta}, b'u^{\theta+1}, b''u^{\theta+2})$, for arbitrary $b, b', b'', t, u \in \mathbb{F}_q$, then either $b = b'' = 0$, or $t = u$ (noting that $t \mapsto t^{\theta+2}$ is injective).

Using the explicit form of the multiplication, inverse and action of the two-point stabiliser, one calculates that the elements $(1, 0, 0)_{\infty}$, $(1, 1, 0)_{\infty}$ and $(1, -1, 0)_{\infty}$ are representatives of the three different conjugacy classes of elements of order 9 of P_{∞} , whereas $(0, 0, 1)_{\infty}$, $(0, 1, 0)_{\infty}$ and $(0, -1, 0)_{\infty}$ are representatives of the three different conjugacy classes of elements of order 3 of P_{∞} . Also,

$$p(b, b', b'') \mapsto p(-b, b' + 1, -b'' - b) \text{ and } p(b, b', b'') \mapsto p(-b, b' - 1, -b'' + b)$$

are representatives of the two conjugacy classes of elements of order 6.

Finally we make the following observation.

Observation A.3. *The point $p(0, a', a'')$, with $a'' \neq 0$, coincides with the point $q(0, b', b'')$, $b'' \neq 0$, if, and only if, $b' = b''^{\theta-1} = a''^{1-\theta} = a'^{-1}$.*

Proof. We write the points $p(0, a', a'')$ and $q(0, b', b'')$ in projective coordinates and obtain

$$\begin{cases} p(0, a', a'') = (a''^2 + a'^{\theta+1}, -a', 0, -a'', 1, a'^{\theta}, -a''^{\theta} + a'a'') =: p, \\ q(0, b', b'') = (1, b'^{\theta}, -b''^{\theta} + b'b'', b''^2 + b'^{\theta+1}, -b', 0) =: q. \end{cases}$$

If $p = q$, then clearly $a'a'' = a''^{\theta}$ and $b'b'' = b''^{\theta}$. Hence, $a' = a''^{\theta-1}$ and $b' = b''^{\theta-1}$. Since projective coordinates are determined up to a non-zero

scalar factor, we can rewrite the projective coordinates of p and q as follows:

$$\begin{cases} p = (-a''^2, -a''^{\theta-1}, 0, -a'', 1, a''^{3-\theta}, 0), \\ q = (-b''^{-2}, -b''^{1-\theta}, 0, -b''^{-1}, 1, b''^{\theta-3}, 0), \end{cases}$$

from which the assertion readily follows, noting that the converse is easy. $\square$

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