APPROACHES TO BUILDINGS

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Overview

Buildings were introduced by Jacques Tits in order to provide a unified geometric framework for understanding semisimple complex Lie groups and, later, semisimple algebraic groups over an arbitrary field. The definition evolved gradually during the 1950s and 1960s and reached a mature form in about 1965. At that time Tits thought of a building as a simplicial complex with a family of subcomplexes called *apartments*, subject to a few axioms. Each apartment is made up of *chambers*, which are the top-dimensional simplices. This viewpoint is the *simplicial approach* to buildings.

In the 1980s Tits introduced a new way of thinking about buildings, in which one forgets about all the simplices except the chambers, and one forgets about the apartments. One remembers only certain combinatorial information about how the chambers fit together. This can be encoded in a *Weyl-group-valued distance function*, subject to a few axioms. For lack of a better name, I will refer to this as the *combinatorial approach* to buildings.

A third way of thinking about buildings is gotten by taking geometric realizations of the structures described in the previous two paragraphs. It turns out that this can always be done so as to obtain a metric space with nice geometric properties. The possibility of doing this has been known for a long time in special cases where the apartments are spheres or Euclidean spaces or hyperbolic spaces. (One speaks then of spherical, Euclidean, or hyperbolic buildings.) But Mike Davis discovered that it can be done in general. In this *metric approach* to buildings, apartments again play a prominent role but are viewed as metric spaces rather than simplicial complexes.

The three approaches to buildings are distinguished by how one thinks of a chamber. In the simplicial approach, chambers are maximal simplices. In the combinatorial approach, chambers are just elements of an abstract set, or vertices of a graph. And in the metric approach, chambers are metric spaces.

My goal in these lectures is to provide an introduction to buildings from all three of these points of view. The various approaches complement each other and are all useful.

Prerequisites

The main prerequisite for the lectures is familiarity with Coxeter groups, especially the finite ones. Recall that a *Coxeter group* is a group W that is generated by a set S of elements of order 2, subject to relations that specify the orders of the pairwise products st $(s, t \in S)$. A finite Coxeter group is the same thing as a finite reflection group together with a choice of fundamental chamber C. The distinguished generating set S consists of the reflections with respect to the walls of C.

Associated to every Coxeter group W is a simplicial complex Σ, called a *Coxeter complex*. I will begin the lectures by briefly recalling the construction of Σ. People who have not seen it before should be able follow the discussion by relying on the finite case for motivation. In this case W is a (finite) linear reflection group acting on a Euclidean vector space, and Σ is the unit sphere, decomposed into simplices by the reflecting hyperplanes.

Coxeter complexes are important to us because, in the simplicial approach to buildings, the apartments are Coxeter complexes.

Outline of the lectures

0. Coxeter complexes (sketch)
1. Buildings as simplicial complexes
    - the axioms
    - examples
    - colorings
    - minimal galleries and reduced words
    - the complete system of apartments
    - spherical buildings and the opposition relation
    - projections
    - Weyl distance
2. Buildings as W-metric spaces
    - the axioms
    - examples
    - residues and projections
    - extension of isometries
    - construction of apartments
    - equivalence between the simplicial and W-metric approaches
3. Euclidean (or affine) buildings
    - Euclidean Coxeter groups
    - Euclidean buildings as CAT(0) metric spaces
    - examples
    - the Bruhat–Tits fixed-point theorem
    - construction of apartments
    - the spherical building at infinity
4. Metric realizations of buildings
    - general theory
    - spherical, Euclidean, and hyperbolic realizations
    - the dual Coxeter complex
    - the Davis realization