The smallest minimal blocking sets of $\mathrm{Q}(2n, q)$ for small odd q.

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Definitions

An ovoid $\mathcal{O}$ of a polar space is a set of points such that every maximal totally isotropic subspace meets $\mathcal{O}$ in exactly one point. If $\mathcal{O}$ is an ovoid of $\mathrm{Q}(2n,q)$ then $|\mathcal{O}| = q^n + 1$

The polar spaces $\mathrm{Q}^-(2n,q)$ $(n \geqslant 2)$, $\mathrm{W}(2n+1,q) \cong \mathrm{Q}(2n,q)$ $(n \geqslant 2$, q even), $\mathrm{W}(2n+1,q)$ $(n \geqslant 1$, q odd) and $\mathrm{U}(2n,q)$ $(n \geqslant 2)$ have no ovoids (J. Thas).

A blocking set $\mathcal{K}$ of a polar space is a set of points such that every maximal totally isotropic subspace meets $\mathcal{K}$ in at least one point. If $\mathcal{K}$ is a blocking set of $\mathrm{Q}(2n,q)$ then $|\mathcal{K}| = q^n + 1 + r$.

$\mathcal{K}$ is minimal iff $\mathcal{K} \setminus \{p\}$ is not a blocking set for all $p \in \mathcal{K}$,

Known Results

- The characterization of the smallest minimal blocking sets of $\mathrm{Q}^{-}(2n, q)$ and $\mathrm{W}(2n+1, q)$ (q even) (Metsch)

- The characterization of the smallest minimal blocking sets of $\mathrm{Q}(6, q)$, q even, $q \geqslant 32$ (De Beule and Storme)

- Lower bound for the smallest minimal blocking set of $\mathrm{W}(2n+1, q)$ (q odd). (Metsch and also Govaerts and Storme)

Ovoids of $\mathrm{Q}(2n, q)$, q odd

Theorem 1. *(Gunawardena and Moorhouse) $\mathrm{Q}(2n, q)$, q odd, $n \geqslant 4$ has no ovoids.*

Theorem 2. *(O' Keefe and Thas) If every ovoid of $\mathrm{Q}(4, q)$, q odd, is an elliptic quadric, then $\mathrm{Q}(6, q)$ has no ovoids.*

Corollary 1. *$\mathrm{Q}(6, 5)$ and $\mathrm{Q}(6, 7)$ have no ovoids*

$\mathrm{Q}(6, 3)$ has ovoids.

Starting Point

Theorem 3. *(Eisfeld, Storme, Szőnyi and Sziklai) A blocking set of* $\mathrm{Q}(4,q)$*,* q *even,* $q \geqslant 32$*, of size* q^2+1+r*, with* $0 < r \leqslant \sqrt{q}$*, contains an ovoid.*

replacement for q odd is needed!

Theorem 4. *If* $\mathcal{B}$ *is a minimal blocking set of* $\mathrm{Q}(4,3)$ *different from an ovoid, then* $|\mathcal{B}| > 11$*.*

Theorem 5. *(computerresult) If* $\mathcal{B}$ *is a minimal blocking set of* $\mathrm{Q}(4,q)$*,* $q = 5, 7$*, different from an ovoid of* $\mathrm{Q}(4,q)$*, then* $|\mathcal{B}| > q^2+2$*.*

Looking in tangent cones

Supose $\mathcal{K}$ is a minimal blocking set of $\mathrm{Q}(2n+2,q)$, of size $q^{n+1}+1+r$, $r<q^{n-1}$.

Lemma 1. *If* $p \in \mathcal{K}$*, then* $|T_p(\mathrm{Q}(2n+2,q)) \cap \mathcal{K}| \leqslant r$*.*

Lemma 2. *If* $p \notin \mathcal{K}$*, then* p *projects* $\mathcal{K}$ *onto* $\mathcal{K}_p$*, a minimal blocking set of* $\mathrm{Q}(2n,q) \subset T_p(\mathrm{Q}(2n+2,q))$

The lowest dimension: $\mathrm{Q}(6,q)$ and $\mathrm{Q}(8,q)$

Theorem 6. *Let $\mathcal{K}$ be a minimal blocking set (different from an ovoid) of $\mathrm{Q}(6,q)$, $q = 3,5,7$, f size $|\mathcal{K}| \leqslant q^3 + q$, Then there is a point $p \in \mathrm{Q}(6,q) \setminus \mathcal{K}$ with the following property: $T_p(\mathrm{Q}(6,q)) \cap \mathrm{Q}(6,q) = p\mathrm{Q}(4,q)$ and $\mathcal{K}$ consists of all the points of the lines L on p meeting $\mathrm{Q}(4,q)$ in an ovoid $\mathcal{O}$, minus the point p itself, and $|\mathcal{K}| = q^3 + q$.*

Theorem 7. *Let $\mathcal{K}$ be a minimal blocking of $\mathrm{Q}(8,3)$, of size $|\mathcal{K}| \leqslant q^4 + q$, Then there is a point $p \in \mathrm{Q}(8,3) \setminus \mathcal{K}$ with the following property: $T_p(\mathrm{Q}(8,3)) \cap \mathrm{Q}(8,3) = p\mathrm{Q}(6,3)$ and $\mathcal{K}$ consists of all the points of the lines L on p meeting $\mathrm{Q}(6,3)$ in an ovoid $\mathcal{O}$, minus the point p itself, and $|\mathcal{K}| = q^4 + q$.*

$\mathrm{Q}(2n+2,q)$

Theorem 8. *The smallest minimal blocking set of* $\mathrm{Q}(2n+2,q)$ $q=5,7$, $n \geqslant 3$, q *odd, is a cone* $\pi_{n-3}\mathcal{O} \setminus \pi_{n-3}$, $\mathcal{O}$ *an elliptic quadric,* $\mathcal{O} \subset \mathrm{Q}(4,q)$, *with* $T_{\pi_{n-3}}(\mathrm{Q}(2n+2,q)) = \pi_{n-3}\mathrm{Q}(4,q)$

Theorem 9. *The smallest minimal blocking set of* $\mathrm{Q}(2n+2,3)$, $n \geqslant 4$, q *odd, is a cone* $\pi_{n-4}\mathcal{O} \setminus \pi_{n-4}$, $\mathcal{O}$ *an ovoid of* $\mathrm{Q}(4,q)$, *with* $T_{\pi_{n-4}}(\mathrm{Q}(2n+2,q)) = \pi_{n-4}\mathrm{Q}(4,q)$