# Geometric authentication codes

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September 21, 2007 / Contact forum

# What is authentication?

- Alice and Bob share a secret private Key K.
- Alice sends to Bob: Source state S and M=e(S,K).
- Bob receives S and M and checks if M=e(S,K).
- Goal for an opponent: Produce a pair (S,e(S,K)).

# Message authentication codes

A message authentication code (MAC) is a 4-tuple $(\mathcal{S}, \mathcal{A}, \mathcal{K}, \mathcal{E})$ with

1. $\mathcal{S}$ a finite set of source states.
2. $\mathcal{M}$ a finite set of messages.
3. $\mathcal{K}$ a finite set of keys.
4. For each $K \in \mathcal{K}$, we have an authentication rule $e_K \in \mathcal{E}$ with $e_K : \mathcal{S} \to \mathcal{M}$.

# Security of a MAC-Perfect MAC

- Let $p_i$ denote the probability of an attacker to construct a pair $(s, e_K(s))$ without knowledge of the key $K$, if he only knows $i$ different pairs $(s_j, e_K(s_j))$.
- If a MAC has attack probabilities $p_i = 1/n_i$ $(0 \leq i \leq l)$ then $|\mathcal{K}| \geq n_0 \cdot \dots \cdot n_l$. If equality holds, the MAC is called perfect.
- For perfect MAC's: $|\mathcal{S}| \leq \frac{n_{l-1}n_l-1}{n_l-1} + l - 1$.

# Important issues of MAC's

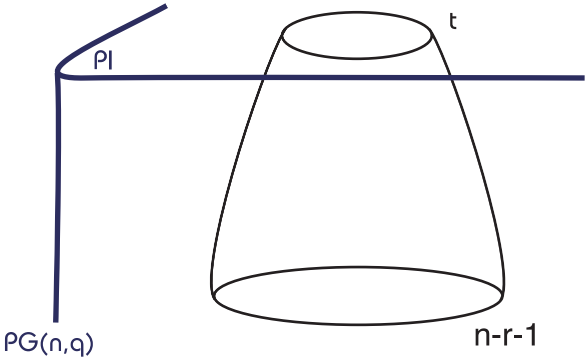
- We assume a uniform distribution for the encoding rules.
- Cartesian: $\mathcal{M}(s_1) \cap \mathcal{M}(s_2) = \emptyset$.
- Perfect authentication schemes are in 1-1 correspondence with certain designs.
- Impersonation and substitution attack.
- Replay attack.

# Gilbert-MacWilliams-Sloane

Fix an $r$-space $\Pi$ in $PG(n, q)$.

- Source states: $t$-spaces in $\Pi$.
- Encoding rules: $(n - r - 1)$-spaces skew from $\Pi$.
- Messages: $(n - r + t)$-subspaces intersecting $\Pi$ in a $t$-space.

# Gilbert-MacWilliams-Sloane II



# A problem with MAC's

- Stockbroker and customer.
- Disputes about orders.
- How to decide in case of such a dispute?

# What are arbitration schemes?

- Alice and Bob don't trust each other.
- A trusted arbiter is needed.
- Bob gives a decoding rule to the arbiter.
- The arbiter gives an encoding rule to Alice.

# Arbitration codes

A message authentication code with arbitration $A^2$*-code* consists of

- $\mathcal{S}$: a set of source states.
- $\mathcal{M}$: a set of encoded messages.
- $\mathcal{E}_{\mathcal{T}}$, a set of encoding rules : 1-1 mappings from $\mathcal{S}$ to $\mathcal{M}$.
- $\mathcal{E}_{\mathcal{R}}$, a set of decoding rules: mappings from $\mathcal{M}$ to $\mathcal{S}$ or reject.

# Security of a MAC with arbitration

- Probabilities for the opponent $P_{O_i}$.
- If dispute between Alice and Bob, then arbiter takes a decision.
- Probability for the sender $P_T$.
- Probabilities for the receiver $P_{R_i}$.

# Combinatorial bounds

**Theorem**

*We have the following lower bounds for the number of encoding and decoding rules.*

$$|\mathcal{E}_R| \geq (P_{O_0} P_{O_1} \cdots P_{O_{t-1}} P_T)^{-1},$$

$$|\mathcal{E}_T| \geq (P_{O_0} P_{O_1} \cdots P_{O_{t-1}} P_{R_0} P_{R_1} \cdots P_{R_{t-1}})^{-1}.$$

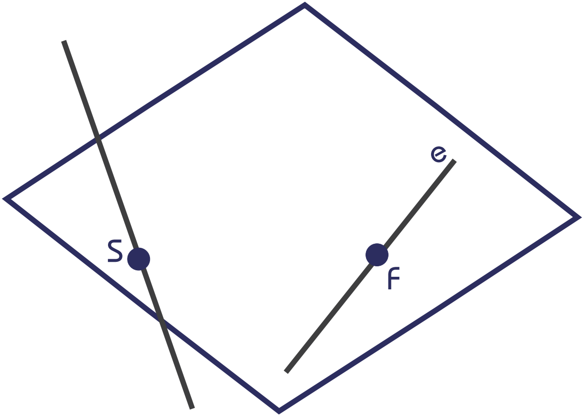
If equality holds in both inequalities above, then we call the arbitration scheme *$t$-fold perfect*.

# A first scheme with arbitration I (T. Johansson)

Fix a line $L_0$ in $PG(3, q)$.

- Source states: Points on $L_0$.
- Receiver's decoding rule: Point $F$ not on $L_0$.
- Transmitter's encoding rule: A line $e$ not intersecting $L_0$.
- Messages: planes spanned by a source state $S$ and an encoding rule $e$.
- $e$ valid under $F$ if $F$ on $e$.

# A first scheme with arbitration II



# Generalized dual arcs

- A generalised dual arc $\mathcal{D}$ of order $l$ with dimensions $d_1 > d_2 > \cdots > d_{l+1}$ of $PG(n, q)$ is a set of subspaces of dimension $d_1$ such that:
  1. each $j$ of these subspaces intersect in a subspace of dimension $d_j$, $1 \leq j \leq l + 1$,
  2. each $l + 2$ of these subspaces have no common intersection.

## Definition

A generalised dual arc of order $l$ with parameters $(n = d_0, \ldots, d_{l+1})$ is *regular* if, in addition, it satisfies the property that if $\pi$ is the intersection of $j$ elements of $\mathcal{D}$, $j \leq l$, then $\pi$ is spanned by the subspaces of dimension $d_{j+1}$ which are the intersections of $\pi$ with the remaining elements of $\mathcal{D}$.

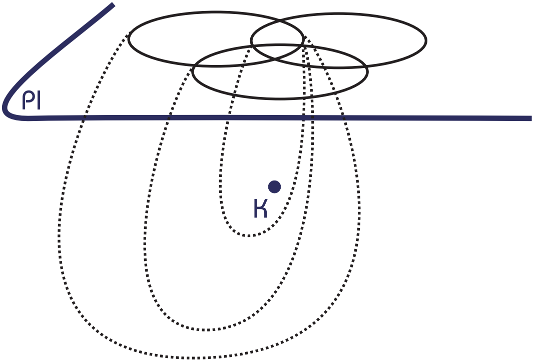
# Use of a GDA to construct a MAC

## Theorem

*(A. Klein, J.S., L. Storme) Let* $\Pi$ *be a hyperplane of* $PG(n+1,q)$ *and let* $\mathcal{D}$ *be a generalised dual arc of order* $l$ *in* $\Pi$ *with parameters* $(n, d_1, \ldots, d_{l+1})$.
*The elements of* $\mathcal{D}$ *are the source states and the points of* $PG(n+1,q)$ *not in* $\Pi$ *are the keys. The message that belongs to a source state and a key is the generated* $(d_1+1)$*-dimensional subspace.*
*This defines a perfect MAC with attack probabilities*

$$p_i = q^{d_{i+1}-d_i}.$$

# Use of a GDA to construct a MAC



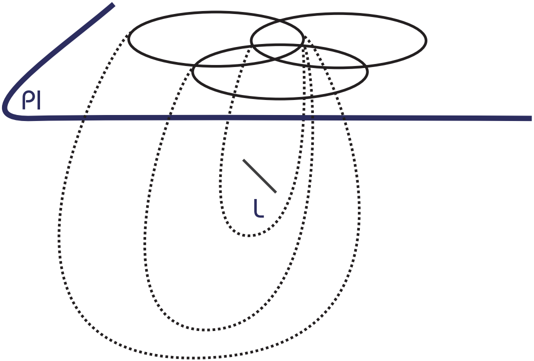
# Use of a GDA to construct a MAC with arbitration

## Theorem

*(A. Klein, J.S., L. Storme) Let* $\Pi$ *be a codimension 2 space of* $PG(n+2,q)$ *and let* $\mathcal{D}$ *be a generalised dual arc of order* $l$ *in* $\Pi$ *with parameters* $(n, d_1, \ldots, d_{l+1})$.
*The elements of* $\mathcal{D}$ *are the source states and the lines of* $PG(n+2,q)$ *skew to* $\Pi$ *are the keys. The message that belongs to a source state and a key is the generated* $(d_1+2)$*-dimensional subspace.*
*This defines a perfect MAC with attack probabilities*

$$p_{O_i} = q^{d_{i+1}-d_i},\ p_T = \frac{1}{q+1},\ p_{R_i} = q^{d_{i+1}-d_i}.$$

# Use of a GDA to construct a MAC with arbitration



# Examples of a GDA I

- The mapping $\zeta : PG(2,q) \to PG(5,q)$ with

$$\zeta([x_0, x_1, x_2]) = [x_0^2, x_1^2, x_2^2, x_0x_1, x_0x_2, x_1x_2]$$

  defines the quadratic Veronesean $V_2^4$.
- This defines a configuration of $q^2 + q + 1$ planes in $PG(5,q)$ such that
  - They generate $PG(5,q)$.
  - Each two intersect in a point.
  - Each three are skew.

# Examples of a GDA II

- Consider the map $\zeta : PG(2,q) \to PG(9,q)$ with

$$\zeta([x_0,x_1,x_2]) = [x_0^3, x_1^3, x_2^3, x_0^2x_1, x_0^2x_2, \cdots, x_2^2x_1, x_0x_1x_2]$$

- This defines a configuration of $q^2+q+1$ 5-dimensional spaces in $PG(9,q)$ such that
    - Each two intersect in a plane
    - Each three in a point
    - Each four are skew.

# Construction of a GDA I

- $PG(V)$ resp. $PG(W)$ a $d$-dimensional resp. $\left(\binom{d+l+1}{l+1} - 1\right)$-dimensional space.
- We define $\zeta : PG(V) \to PG(W)$ by

$$\zeta : [\sum_{i=0}^{d} x_i e_i] \mapsto [\sum_{0 \le i_0 \le \cdots \le i_l \le d]} x_{i_0} \cdot \ldots \cdot x_{i_l} e_{i_0,\ldots,i_l}].$$

- For each $x \in V$, we denote by $x^{\perp}$ the subspace of $V$ perpendicular to $x$ with respect to $b$. So

$$x^{\perp} = \{y \in V \mid b(x,y) = 0\}.$$

# Construction of a GDA II

- For each point $P = [x]$ of $PG(V)$, we define a subspace $D(P)$ of $PG(W)$ by

$$D(P) = \{[z] \in W \mid B(z, \zeta(y)) = 0 \text{ for all } y \in x^{\perp}\}. \qquad (1)$$

**Theorem**

*The set $\mathcal{D} = \{D(P) \mid P \in PG(V)\}$ is a regular generalised dual arc with dimensions $d_i = \binom{d+l+1-i}{l+1-i} - 1$.*

# A characterization of Veronesean surfaces

## Theorem

*(J. A. Thas-H. Van Maldeghem) Let $\mathcal{F}$ be a set of $\frac{q^{n+1}-1}{q-1}$ n-dimensional spaces generating $PG(N = \frac{n(n+3)}{2}, q)$, such that*

1. *two distinct elements of $\mathcal{F}$ intersect in a point,*
2. *three distinct elements of $\mathcal{F}$ have an empty intersection.*
3. *Two technical conditions which can be dropped in some cases.*

*Then $\mathcal{F}$ consists of $V_{n-1}$ subspaces to a Veronesean surface $V_n^{2^n}$ if q is odd, if q is even there is also an exception with the nucleus subspace.*

# Extension result on Veronesean surfaces

**Theorem**

*(A. Klein, J.S., L. Storme) A set of* $\frac{q^{n+1}-1}{q-1} - \delta$ *n-dimensional spaces in* $PG(N = \frac{n(n+3)}{2}, q)$ *satisfying the above properties can always be extended if* $\delta \leq \frac{q}{2} - 1$*.*

# Algebraic characterisation of the GDA $(9, 5, 2, 0)$

**Theorem**

*Every regular generalised dual arc $\mathcal{D}$ with parameters $(9, 5, 2, 0)$ in $PG(9, q)$, $q > 3$, $q$ odd, which contains $q^2 + q + 1$ elements, is isomorphic to the one given in the construction.*

**Corollary**

*A regular generalised dual arc in $PG(9, q)$, $q > 3$, $q$ odd, with parameters $(9, 5, 2, 0)$ contains at most $q^2 + q + 1$ elements.*

# More general algebraic characterisation

We work inductively.

- Basic step: Theorem of Thas-Van Maldeghem.
- We get generalised dual arcs with missing elements in the subspaces.
- Find the remaining elements in these subspaces using our result.

# Generalized quadrangles

A GQ of order $(s, t)$ is an incidence structure $S = (P, B, I)$ for which $I$ is a symmetric point-line incidence relation satisfying the following axioms.

(GQ1) Each point is incident with $t + 1$ lines ($t \geq 1$) and two distinct points are incident with at most one line.

(GQ2) Each line is incident with $s + 1$ points ($s \geq 1$) and two distinct lines are incident with at most one point.

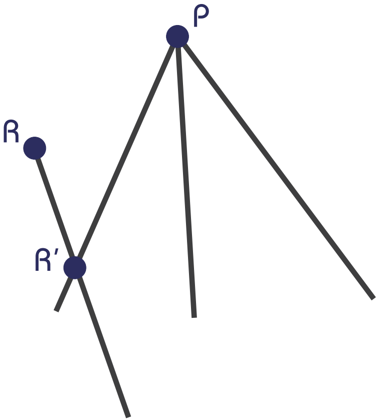
(GQ3) If $p$ is a point and $L$ is a line not incident with $p$, then there is a unique point-line pair $(q, M)$ such that $pIMIqIL$.

# A scheme by Desoete using GQ's

Take a fixed point $p$ in a GQ.

- Source states: Lines through $p$.
- Encoding rules: Points not collinear with $p$.
- Messages: The points of $p^{\perp} \setminus \{p\}$.

# A scheme by Desoete



# An $A$-scheme using ovoids in subGQ's (JS-K.Thas)

Consider a set $\{S_1, \cdots, S_r\}$ of $r > 0$ distinct subGQs of order $(s, \frac{t}{s})$ of the GQ $S$ of order $(s > 1, t > 1)$

- Source states: subGQs $S_j$.
- Keys: Points in $S \setminus \cup_{i=1}^{r} S_i$.
- Messages: Ovoids in the GQs $S_j$ subtended by a point outside their union.
- This yields 1-fold perfect schemes with very good $p_0$.

# Several schemes

All kinds of geometries and combinatorial structures can be used.

- Unitary and symplectic space.
- Latin squares.
- Rational normal curves.
- ...