Cameron–Liebler k-classes in $\mathrm{PG}(2k+1, q)$

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Abstract

We look at a generalization of Cameron–Liebler line classes to sets of k-spaces, focusing on results in $\mathrm{PG}(2k+1, q)$. Here we obtain a connection to k-spreads which parallels the situation for line classes in $\mathrm{PG}(3, q)$. After looking at some characterizations of these sets and some of the difficulties that arise in contrast to the known results for line classes, we give some connections to various other geometric objects including k-spreads and Erdős–Ko–Rado sets, and prove results concerning the existence of these objects.

1 Introduction

The study of Cameron–Liebler line classes was originally motivated by an attempt by Cameron and Liebler to classify subgroups of $\mathrm{P\Gamma L}(n+1, q)$ having the same number of orbits on points as on lines of $\mathrm{PG}(n, q)$ [6]. Such a group naturally induces a tactical decomposition on the point-line design of $\mathrm{PG}(n, q)$ having the same number of point and line classes. A line class of such a decomposition has many equivalent special properties, and sets of lines sharing these properties are called Cameron–Liebler line classes.

Let A be the matrix with rows indexed by the points of $\mathrm{PG}(n, q)$ and columns indexed by the lines, with entry 1 if the corresponding point and line are incident, and 0 otherwise (viewed as a matrix over $\mathbb{Q}$). In other words, A is the incidence matrix of the point-line geometry $\mathrm{PG}(n, q)$. Then a Cameron–Liebler line class is defined as follows.

Definition 1.1. A set of lines $\mathcal{L}$ in $\mathrm{PG}(n, q)$ is a **Cameron–Liebler line class** if the characteristic function $\chi_{\mathcal{L}}$ belongs to $\mathrm{row}(A)$.

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In an abuse of notation, we will sometimes consider $\boldsymbol{\chi}$ to be a characteristic function of a set of subspaces (with fixed dimension) of $\mathrm{PG}(n,q)$, and sometimes consider it to be a characteristic vector of a set of subspaces (with respect to some ordering of these subspaces).

Definition 1.2. For a point $\boldsymbol{p}$ or a hyperplane H, we will write $[\boldsymbol{p}]_k$ or $[H]_k$ for the set of all k-spaces containing $\boldsymbol{p}$ or the set of all k-spaces contained in H, respectively. For an incident point-hyperplane pair $(\boldsymbol{p},H)$, we will write $[\boldsymbol{p},H]_k$ for $[\boldsymbol{p}]_k \cap [H]_k$.

Trivial examples of Cameron–Liebler line classes are provided by the sets $[\boldsymbol{p}]_1$ for any point $\boldsymbol{p}$, $[H]_1$ for any hyperplane H, $[\boldsymbol{p}]_1 \cup [H]_1$ for a non-incident point-hyperplane pair $(\boldsymbol{p},H)$, and the complements of these sets. While there are no known non-trivial examples in $\mathrm{PG}(n,q)$ when $n>3$, there are examples known in $\mathrm{PG}(3,q)$ for all odd q (see [5, 7, 14]), as well as for $q=4$ [17]. Motivated by these results, we generalize these objects to sets of k-spaces in $\mathrm{PG}(n,q)$. We will focus on the case where $n=2k+1$ so that the k-spaces are half-dimensional, as is the situation with line sets in $\mathrm{PG}(3,q)$.

2 Preliminaries

2.1 Spreads and tactical decompositions

As we will frequently be interested in the size of various collections of subspaces, we introduce the following notation.

Definition 2.1. The symbol $\begin{bmatrix} n \\ k \end{bmatrix}_q$ denotes the number of $(k-1)$-dimensional subspaces of an $(n-1)$-dimensional projective space (dimensions are projective). This value can be computed as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{\prod_{i=1}^{k}(q^{n-k+i}-1)}{\prod_{i=1}^{k}(q^i-1)}.$$

It is well-known that these values satisfy the recursion

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q.$$

Definition 2.2. A **k-spread of** $\mathrm{PG}(n,q)$ is a collection of k-spaces which are mutually disjoint, and partition the point set of $\mathrm{PG}(n,q)$.

Remark 2.3. A k-spread of $\mathrm{PG}(n,q)$ exists if and only if $k+1$ divides $n+1$, and necessarily contains $\frac{q^{n+1}-1}{q^{k+1}-1}$ k-spaces.

Definition 2.4. A *switching set* is a partial k-spread $\mathcal{K}$ for which there exists a *conjugate* partial k-spread $\mathcal{K}'$ such that $\mathcal{K}\cap\mathcal{K}'=\emptyset$, and $\bigcup\mathcal{K}=\bigcup\mathcal{K}'$, in other words, $\mathcal{K}$ and $\mathcal{K}'$ have no common members and cover the same set of points. We say that $\mathcal{K}$ and $\mathcal{K}'$ are a *pair of conjugate switching sets.*

In $\mathrm{PG}(3,q)$, a regulus and its opposite regulus is an example of a pair of conjugate switching sets; the two sets of lines in a double-six [20] gives another example in $\mathrm{PG}(3,4)$. In higher dimensional spaces, switching sets arise in the context of derivable nets, and conjugate pairs can be constructed through a method known as homology replacement [26].

Definition 2.5. Given an incidence structure $\mathcal{D}$, a **tactical decomposition of** $\mathcal{D}$ is a partition of the points of $\mathcal{D}$ into point classes and the blocks of $\mathcal{D}$ into block classes such that the number of points in a point class which are incident with a given block depends only on the class in which the block lies, and the number of blocks in a block class which are incident with a given point depends only on the class in which the point lies.

The following is shown in the typical proof of Fisher's Inequality, see e.g. [4].

Lemma 2.6. *Let $\mathcal{D}$ be a 2-design with incidence matrix A. Then A has full row rank.* ∎

The following result is a well-known consequence of Block's lemma (see [12, p. 21]).

Lemma 2.7 (Block's Lemma)**.** *Suppose $\mathcal{D}$ is an incidence structure whose incidence matrix has full row rank. Then for any tactical decomposition with s point classes and t block classes, the matrix A (block-decomp matrix) has rank s; in particular, $s \leq t$.*

∎

2.2 The q-Kneser graph and Erdős–Ko–Rado sets of k-spaces

Definition 2.8. The q-Kneser graph $qK_{n+1:k+1}$ is the graph whose vertices are the k-spaces of $\mathrm{PG}(n,q)$, with two vertices being adjacent if the corresponding k-spaces are disjoint.

When $k+1 \leq (n+1)/2$, the q-Kneser graph represents the distance $k+1$ relation in the Grassmann scheme, and as such has been well-studied. For our purposes, we will be interested in the case where equality holds in this equation, so $n = 2k+1$.

The graph $qK_{2k+2:k+1}$ is distance regular, having diameter $k+1$ and degree $d = q^{(k+1)^2}$. Thus it has $k+2$ distinct eigenvalues. It is clear that the eigenspace E_d corresponding to d is one-dimensional, and is spanned by the all-ones vector $\boldsymbol{j}$. The eigenvalues and the dimensions of the eigenspaces of the q-Kneser graphs were originally described by Delsarte [11], but we will use the following simplified formula found in [22].

Lemma 2.9. *The graph $qK_{2k+2:k+1}$ has smallest eigenvalue $\tau = -q^{(k+1)k}$; the corresponding eigenspace E_τ has dimension $\begin{bmatrix}2k+2\\1\end{bmatrix}_q - 1$.* ∎

Definition 2.10. Let $\mathcal{F}$ be a set of k-spaces in $\mathrm{PG}(n,q)$ which are pairwise non-disjoint. If $\mathcal{F}$ is maximal with this property we say $\mathcal{F}$ is an **Erdős–Ko–Rado set of k-spaces**, in short an $\mathbf{EKR}(k)$ set, in $\mathrm{PG}(n,q)$.

We have that $\mathbf{EKR}(k)$ sets in $\mathrm{PG}(n,q)$ correspond to maximal cocliques in $qK_{n+1:k+1}$. As we will be interested in the structure of the largest $\mathbf{EKR}(k)$ sets, we will frequently apply the following result from [15] (proven in [25]).

Theorem 2.11. *If $\mathcal{F}$ is an* $\mathbf{EKR}(k)$ *set in* $\mathrm{PG}(n,q)$ *with $n \geq 2k+1$, then $|\mathcal{F}| \leq \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$; if equality holds, then either*

1. *$\mathcal{F} = [\boldsymbol{p}]_k$ for some point $\boldsymbol{p} \in \mathrm{PG}(n,q)$; or*

2. *$\mathcal{F} = [H]_k$ for some hyperplane H of $\mathrm{PG}(n,q)$, and $n = 2k+1$.*

Sets of the form $[\boldsymbol{p}]_k$ will be called EKR families*, and those of the form $[H]_k$ will be called* dual-EKR families. ∎

The size of the second largest $\mathbf{EKR}(k)$ sets is known in certain situations; this is known as the **Hilton–Milner bound**.

Theorem 2.12 ([2])**.** *Let $\mathcal{F}$ be an* $\mathbf{EKR}(k)$ *set in $\mathrm{PG}(n,q)$, with $n \geq 2k+2$, such that*

$$|\mathcal{F}| > \begin{bmatrix} n \\ k \end{bmatrix}_q - q^{(k+1)k} \begin{bmatrix} n-k-1 \\ k \end{bmatrix}_q + q^{k+1}.$$

Then $\mathcal{F}$ is an EKR family. ∎

Theorem 2.13 ([3])**.** *Let $k \in \{1,2\}$ and let $\mathcal{F}$ be an* $\mathbf{EKR}(k)$ *set in $\mathrm{PG}(2k+1,q)$ with*

$$|\mathcal{F}| > \begin{bmatrix} 2k+1 \\ k \end{bmatrix}_q - q^{(k+1)k} + q^{k+1}.$$

Then $\mathcal{F}$ is an EKR family or a dual-EKR family. ∎

However, when $n = 2k+1$ and $k \geq 3$, this is still an open problem. We do have a weak Hilton–Milner type bound due to Blokhuis, Brouwer, and Szőnyi [3].

Theorem 2.14. *Let $\mathcal{F}$ be an* $\mathbf{EKR}(k)$ *set in $\mathrm{PG}(2k+1,q)$ with*

$$|\mathcal{F}| > \left(1 + \frac{1}{q}\right) \begin{bmatrix} k+1 \\ 1 \end{bmatrix}_q^k \begin{bmatrix} k \\ 1 \end{bmatrix}_q.$$

Then $\mathcal{F}$ is an EKR family or a dual-EKR family. ∎

This theorem is most useful to us when k is small compared to q, in which case we have the following.

Corollary 2.15. *Let $k+1 < q \log q - q$ and let $\mathcal{F}$ be an* $\mathbf{EKR}(k)$ *set in $\mathrm{PG}(2k+1,q)$ with*

$$|\mathcal{F}| > \frac{1}{2} q^{(k+1)k}.$$

Then $\mathcal{F}$ is an EKR family or a dual-EKR family. ∎

3 Cameron–Liebler k-classes

Let Π_k^n be the collection of k-dimensional subspaces of $\mathrm{PG}(n,q)$ for $0 \leq k \leq n$. We will write simply Π_k when n is understood from the context. Put A_k to be the incidence matrix of points and k-spaces of $\mathrm{PG}(n,q)$, so the rows are indexed by points and the columns by elements of Π_k. The matrix A_k is well known to be the matrix of a 2-design for $1 \le k \le n-1$, and so by Lemma 2.6 has full row rank.

Definition 3.1. A set of k-spaces $\mathcal{L}$ in $\mathrm{PG}(n,q)$ is a **Cameron–Liebler k-class** if the characteristic function $\boldsymbol{\chi}_{\mathcal{L}}$ belongs to $\mathrm{row}(A_k)$.

The following result was shown in [6] for Cameron-Liebler line classes, and provided a good deal of motivation for their further study. Our more general proof is nearly identical to theirs.

Proposition 3.2. *Any class of k-spaces in a symmetric tactical decomposition of* $\mathrm{PG}(n,q)$ *is a Cameron–Liebler k-class.*

Proof. Let $\boldsymbol{v}_1,\dots,\boldsymbol{v}_s$ be the characteristic vectors of the point classes, and let $\boldsymbol{w}_1,\dots,\boldsymbol{w}_s$ be the characteristic vectors of the block classes; we will write c_{ij} for the number of blocks in the jth block class on a point in the ith point class. Then

$$\begin{bmatrix}\boldsymbol{v}_1\\ \vdots\\ \boldsymbol{v}_s\end{bmatrix} A_k = C \begin{bmatrix}\boldsymbol{w}_1\\ \vdots\\ \boldsymbol{w}_s\end{bmatrix},$$

where $C = \begin{bmatrix} c_{ij}\end{bmatrix}$ is nonsingular (see Lemma 2.7). This gives us that

$$C^{-1}\begin{bmatrix}\boldsymbol{v}_1\\ \vdots\\ \boldsymbol{v}_s\end{bmatrix} A_k = \begin{bmatrix}\boldsymbol{w}_1\\ \vdots\\ \boldsymbol{w}_s\end{bmatrix},$$

and so each $\boldsymbol{w}_j$ is a linear combination of $\{\boldsymbol{v}_1A_k,\dots,\boldsymbol{v}_sA_k\} \subseteq \mathrm{row}(A_k)$. ∎

For the remainder, we will focus on Cameron–Liebler k-classes in $\mathrm{PG}(2k+1,q)$.

Proposition 3.3. *Let $\mathcal{L}$ be a Cameron–Liebler k-class in* $\mathrm{PG}(2k+1,q)$. *Then there is some integer x such that $|\mathcal{L}\cap\mathcal{S}| = x$ for every k-spread $\mathcal{S}$.*

Proof. Take an arbitrary k-spread $\mathcal{S}$ and put $\boldsymbol{v} = \boldsymbol{\chi}_{\mathcal{S}} - \frac{1}{\left[{2k+1 \atop k}\right]_q}\boldsymbol{j}$. Then $\boldsymbol{v} \in \ker(A_k^T)$. Since $\mathcal{L}$ is a Cameron–Liebler k-class,

$$\boldsymbol{\chi}_{\mathcal{L}} \in \mathrm{row}(A_k) = (\ker(A_k^T))^{\perp},\ \text{so}\ (\boldsymbol{\chi}_{\mathcal{L}},\ \boldsymbol{v}) = |\mathcal{L}\cap\mathcal{S}| - \frac{|\mathcal{L}|}{\left[{2k+1 \atop k}\right]_q} = 0.$$

Therefore $|\mathcal{L}\cap\mathcal{S}| = \frac{|\mathcal{L}|}{\left[{2k+1 \atop k}\right]_q} = x$ for every k-spread $\mathcal{S}$. ∎

We call this integer x the *parameter* of the Cameron–Liebler k-class; it is clear from Remark 2.3 that $0 \le x \le q^{k+1}+1$.

Theorem 3.4. *Let $\mathcal{L}$ be a Cameron–Liebler k-class in* $\mathrm{PG}(2k+1,q)$ *with parameter x and characteristic function $\boldsymbol{\chi}$. Then*

(i) $|\mathcal{L}| = x\begin{bmatrix}2k+1\\k\end{bmatrix}_q$.

(ii) *For any k-space π, the number of elements of $\mathcal{L}$ disjoint from π is $(x-\boldsymbol{\chi}(\pi))q^{(k+1)k}$.*

(iii) *For any two disjoint k-spaces π_1 and π_2, the number of elements of $\mathcal{L}$ disjoint from both π_1 and π_2 is $(x-\boldsymbol{\chi}(\pi_1)-\boldsymbol{\chi}(\pi_2))\left(q^{(k+1)k/2}\prod_{i=1}^{k}(q^i-1)\right)$.*

Proof. We have that $\mathcal{L}$ shares precisely x elements with every k-spread, so (i) follows directly from the proof of Proposition 3.3. Since the group $\mathrm{PGL}(2k+2,q)$ acts transitively on the triples of pairwise disjoint k-spaces of $\mathrm{PG}(2k+1,q)$, for $i \le 3$, the number n_i of k-spreads containing i fixed pairwise disjoint k-spaces depends only on i and not on the choice of the k-spaces. By counting, we see that

$$n_1/n_2 = q^{(k+1)k}, \text{ and}$$
$$n_2/n_3 = \left(q^{(k+1)k/2}\prod_{i=1}^{k}(q^i-1)\right).$$

Now for any k-space π, if we count the pairs $(\pi',\mathcal{S})$ where π' is a k-space in $\mathcal{L}$ disjoint from π and $\mathcal{S}$ is a k-spread containing π and π', then we see that the number of k-spaces in $\mathcal{L}$ disjoint from π is equal to $(n_1/n_2)(x-\boldsymbol{\chi}(\pi))$; (ii) follows. Finally, we can obtain (iii) by fixing disjoint k-spaces π_1 and π_2, and counting the pairs $(\pi',\mathcal{S})$, where π' is a k-space in $\mathcal{L}$ disjoint from both π_1 and π_2, and $\mathcal{S}$ is a k-spread containing π_1, π_2, and π'. Some further details of the counting techniques of this theorem can be found in [10]. ∎

Let K be the adjacency matrix of $qK_{2k+2:k+1}$ and $A = A_k$, the point-k-space incidence matrix of $\mathrm{PG}(2k+1,q)$. Let $\mathrm{E}_d = \langle \boldsymbol{j}\rangle$ and E_τ be the eigenspaces of K for eigenvalues $d = q^{(k+1)^2}$ and $\tau = -q^{(k+1)k}$, respectively. Then we have the following.

Lemma 3.5. *Let $\mathcal{L}$ be a Cameron–Liebler k-class of* $\mathrm{PG}(2k+1,q)$ *with parameter x, having characteristic function $\boldsymbol{\chi}$. Then*

$$(\boldsymbol{\chi} - \frac{x}{q^{k+1}+1}\boldsymbol{j}) \in \mathrm{E}_\tau.$$

Proof. For each column $\boldsymbol{w}$ of K, $\boldsymbol{w}^T$ corresponds to the characteristic function of the set of k-spaces which are disjoint from some fixed k-space π. So by part (ii) of Theorem 3.4, we have

$$\boldsymbol{\chi}K = (x\boldsymbol{j} - \boldsymbol{\chi})q^{(k+1)k}.$$

We also have that $\boldsymbol{j}$ corresponds to the characteristic function of the complete set of k-spaces, which is a Cameron–Liebler k-class with parameter $q^{k+1}+1$, so

$$\boldsymbol{j}K = (q^{k+1})q^{(k+1)k}\boldsymbol{j}.$$

The result follows from direct computation. ■

Corollary 3.6. *The rows of A form a basis for* $\mathrm{E}_d \oplus \mathrm{E}_\tau$.

Proof. The matrix A has $\left[{2k+2 \atop 1}\right]_q$ rows and, by Lemma 2.6, full row rank, so the rows of A are linearly independent; we also know by Lemma 2.9 that $\dim(\mathrm{E}_\tau) = \left[{2k+2 \atop 1}\right]_q - 1$. Therefore

$$\dim(\mathrm{row}(A)) = \dim(\mathrm{E}_d \oplus \mathrm{E}_\tau).$$

Now each row of A is of the form $\boldsymbol{\chi}_{[\boldsymbol{p}]}$ for some point $\boldsymbol{p}$, so is the characteristic function of a Cameron–Liebler k-class with parameter 1. By Lemma 3.5, such a vector can be written as

$$\boldsymbol{v} + \frac{1}{q^{k+1}+1}\boldsymbol{j}$$

for some eigenvector $\boldsymbol{v}$ of K for τ, thus each row of A belongs to $\mathrm{E}_d \oplus \mathrm{E}_\tau$. Since the dimensions match, we have $\mathrm{row}(A) = \mathrm{E}_\tau \oplus \mathrm{E}_d$ and the result follows. ■

Theorem 3.7. *Let $\mathcal{L}$ be a set of k-spaces in* $\mathrm{PG}(2k+1, q)$ *with characteristic function $\boldsymbol{\chi}$. Then the following properties are equivalent:*

(i) *$\boldsymbol{\chi}$ belongs to* $\mathrm{row}(A)$*;*

(ii) *$\boldsymbol{\chi}$ belongs to* $(\ker(A^T))^\perp$*;*

(iii) *there is some integer x such that $|\mathcal{L} \cap \mathcal{S}| = x$ for every k-spread $\mathcal{S}$;*

(iv) *there is some integer x such that $|\mathcal{L} \cap \mathcal{S}| = x$ for every* regular *k-spread $\mathcal{S}$;*

(v) *for every pair of conjugate switching sets $\mathcal{K}$ and $\mathcal{K}'$, $|\mathcal{L} \cap \mathcal{K}| = |\mathcal{L} \cap \mathcal{K}'|$;*

(vi) *there is some integer x such that, for every k-space π, the number of elements of $\mathcal{L}$ disjoint from π is $(x - \boldsymbol{\chi}(\pi))q^{(k+1)k}$;*

(vii) *there is some integer x such that $(\boldsymbol{\chi} - \frac{x}{q^{k+1}+1}\boldsymbol{j})$ is an eigenvector of K for eigenvalue $\tau = -q^{(k+1)k}$.*

Proof. It is clear that (i) and (ii) are equivalent, since $\mathrm{row}(A) = (\ker(A^T))^\perp$. By Proposition 3.3, (ii) implies (iii), and (iii) clearly implies (iv). By repeating the arguments from Theorem 3.4 using regular spreads, we see that (iv) implies (vi) which, as we saw in the proof of Lemma 3.5, implies (vii). Now (vii) implies that $\boldsymbol{\chi}$ belongs to the sum of the eigenspaces E_d and E_τ of K; by Corollary 3.6, the rows of A form a basis for $\mathrm{E}_d \oplus \mathrm{E}_\tau$, so (vii) implies (i). Finally, since a pair of conjugate switching sets $\mathcal{K}$ and $\mathcal{K}'$ cover the same set of points,

$$\boldsymbol{v} = (\boldsymbol{\chi}_{\mathcal{K}} - \boldsymbol{\chi}_{\mathcal{K}'}) \in \ker(A^T),$$

so (ii) implies (v); since two disjoint k-spreads form a pair of conjugate switching sets, (v) implies (iii). ■

Corollary 3.8. *Let $\mathcal{L}$ be a Cameron–Liebler k-class in* $\mathrm{PG}(2k+1,q)$ *with parameter x, and let $(\boldsymbol{p},H)$ be an incident point-hyperplane pair. Then*

$$|[\boldsymbol{p}]_k \cap \mathcal{L}| + |[H]_k \cap \mathcal{L}| - (q^k+1)|[\boldsymbol{p},H]_k \cap \mathcal{L}| = x\begin{bmatrix} 2k \\ k-1 \end{bmatrix}_q.$$

Proof. We can verify by direct computation that

$$\boldsymbol{v} = \boldsymbol{\chi}_{[\boldsymbol{p}]_k} + \boldsymbol{\chi}_{[H]_k} - (q^k+1)\boldsymbol{\chi}_{[\boldsymbol{p},H]_k} - \frac{(q^k-1)}{(q^{2k+1}-1)}\boldsymbol{j} \in \ker(A^T).$$

Therefore if $\mathcal{L}$ is a Cameron–Liebler k-class, $(\boldsymbol{\chi}_{\mathcal{L}}, \boldsymbol{v}) = 0$. The result follows immediately from the size of $\mathcal{L}$. ■

One interesting note is that, for $k=1$, the above corollary gives a sufficient as well as a necessary condition for a set of lines to be a Cameron–Liebler line class. This relates to the fact that $\ker(A_1^T)$ is an irreducible $\mathrm{PGL}(4,q)$ module (see [27]). However, this is not the case for $k \geq 2$.

4 Initial Results

In this section, we give some initial results on Cameron–Liebler k-classes in $\mathrm{PG}(2k+1,q)$ following from Theorem 3.4, and characterize the examples having parameter $x=1$.

Proposition 4.1. *Let $\mathcal{L}$ be a Cameron–Liebler k-class in* $\mathrm{PG}(2k+1,q)$ *with parameter x. Then the complement $\mathcal{L}^C$ of $\mathcal{L}$ is a Cameron–Liebler k-class with parameter $q^{k+1}+1-x$.* ■

Proposition 4.2. *Given two disjoint Cameron–Liebler k-classes $\mathcal{L}_1$ and $\mathcal{L}_2$ in* $\mathrm{PG}(2k+1,q)$*, with respective parameters x_1 and x_2, the set $\mathcal{L}_1 \cup \mathcal{L}_2$ is a Cameron–Liebler k-class with parameter x_1+x_2.* ■

Corollary 4.3. *We have the following examples of Cameron–Liebler k-classes, which we will call "trivial".*

(i) $\emptyset$*, having parameter* 0*;*

(ii) $[\boldsymbol{p}]_k$ *for a point $\boldsymbol{p}$, having parameter* 1*;*

(iii) $[H]_k$ *for a hyperplane H, having parameter* 1*;*

(iv) $[\boldsymbol{p}]_k \cup [H]_k$ *for a non-incident point-hyperplane pair $(\boldsymbol{p},H)$, having parameter* 2*;*

(v) *the complements of any of the above sets, having parameters $q^{k+1}+1$, q^{k+1}, q^{k+1}, and $q^{k+1}-1$, respectively.* ■

Proposition 4.4. *Let $\mathcal{L}$ be a Cameron–Liebler k-class in* $\mathrm{PG}(2k+1,q)$ *having parameter* 1*. Then $\mathcal{L}$ is either the set $[\boldsymbol{p}]_k$ for some point $\boldsymbol{p}$, or the set $[H]_k$ for some hyperplane H.*

Proof. Suppose $\mathcal{L}$ is a Cameron–Liebler k-class with parameter 1. By Theorem 3.4, $\mathcal{L}$ cannot contain two disjoint k-spaces. Therefore, $\mathcal{L}$ consists of $\begin{bmatrix} 2k+1 \\ k+1 \end{bmatrix}_q$ k-spaces that pairwise intersect nontrivially, and the result then follows immediately from Theorem 2.11. ■

For the Cameron–Liebler line classes, for which $k = 1$, it is known that an example with parameter 1 or 2 must be trivial. There has also been considerable work showing non-existence results for nontrivial examples with small parameter ([27], [13], [17], [18], [9], [8], [23], [1]). Currently, the best known results have been given in [23] and [24], and are shown by exploiting the connection between the lines of $\mathrm{PG}(3, q)$ and the points in the hyperbolic quadric $\mathcal{Q}^+(5, q)$ given by the Klein correspondence. Combining these results gives the following.

Theorem 4.5. *A Cameron–Liebler line class in* $\mathrm{PG}(3, q)$ *with parameter* $x \leq \max\{q, q\sqrt[3]{\frac{q}{2}} - \frac{2}{3}q\}$ *is trivial.*

There is also a recent powerful result given in [16] restricting the parameters.

Theorem 4.6. *Suppose* $\mathcal{L}$ *is a Cameron–Liebler line class in* $\mathrm{PG}(3, q)$ *with parameter* x*. Then for every plane and every point of* $\mathrm{PG}(3, q)$*, we have*

$$\binom{x}{2} + n(n - x) \equiv 0 \pmod{q + 1},$$

where n *is the number of lines in* $\mathcal{L}$ *in the plane or through the point, respectively.*

This modular equation rules out roughly half of the possible values for x.

5 Some considerations for $k = 2$

Recall that if $\mathcal{L}$ is a Cameron–Liebler 2-class in $\mathrm{PG}(5, q)$ with parameter x, then

$$|\mathcal{L}| = x\begin{bmatrix} 5 \\ 2 \end{bmatrix}_q = xq^6 + xq^5 + 2xq^4 + 2xq^3 + 2xq^2 + xq + x.$$

Lemma 5.1. *Let* $\mathcal{L}$ *be a Cameron–Liebler* 2*-class in* $\mathrm{PG}(5, q)$ *with parameter* x*, and let* $\pi_1, \ldots, \pi_t$ *be a collection of* t *pairwise skew planes in* $\mathcal{L}$*. Put* S_i *to be the set of planes in* $\mathcal{L}$ *which intersect* π_i*, and* S_{ij} *to be the set of planes in* $\mathcal{L}$ *intersecting both* π_i *and* π_j*. Then*

$$|S_i| = q^6 + xq^5 + 2xq^4 + 2xq^3 + 2xq^2 + xq + x \text{ and}$$
$$|S_{ij}| = 2q^5 + (x + 2)q^4 + (3x - 2)q^3 + 2xq^2 + xq + x.$$

Proof. This follows directly from Theorem 3.4. ■

Notice that if $\mathcal{L}$ is a Cameron–Liebler 2-class in $\mathrm{PG}(5, q)$ with parameter x, then we can always find x pairwise skew planes; we simply take the intersection of $\mathcal{L}$ with any 2-spread of the space. The next lemma shows that when x is small enough, this is the best we can do.

Lemma 5.2. *Let $\mathcal{L}$ be a Cameron–Liebler 2-class in $\mathrm{PG}(5,q)$ with parameter x. If $x \leq q^{2/3}$, no $x+1$ distinct planes of $\mathcal{L}$ are pairwise skew.*

Proof. Suppose that $\mathcal{L}$ contains $x+1$ distinct skew planes $\pi_1, \dots, \pi_{x+1}$, and define S_i and S_{ij} as above. Then $\bigcup_{i=1}^{x+1} S_i \subseteq \mathcal{L}$, giving the lower bound $|\mathcal{L}| \geq (x+1)|S_i| - \frac{(x+1)x}{2}|S_{ij}|$. This implies that

$$q^6 \leq x\left(q^5 + \frac{(x^2-x+2)}{2}q^4 + \frac{(3x^2-3x-2)}{2}q^3 + (x^2-x)q^2 + \frac{(x^2-x)}{2}q + \frac{(x^2-x)}{2}\right),$$

which gives a contradiction if $x \leq q^{2/3}$. ∎

Lemma 5.3. *Let $\mathcal{L}$ be a Cameron–Liebler 2-class in $\mathrm{PG}(5,q)$ with parameter $x \leq q^{2/3}$, containing x pairwise skew planes $\pi_1, \dots, \pi_x$. Put L_i to be the set of planes in $\mathcal{L}$ which intersect π_i and are skew to π_j for $j \neq i$. Then each set L_i is contained in a unique EKR family or dual-EKR family of planes.*

Proof. By Lemma 5.2, $\mathcal{L}$ does not contain a set of $x+1$ pairwise skew planes; so the planes in each set L_i mutually intersect nontrivially. Therefore, by Theorem 2.13, when

$$|L_i| > \begin{bmatrix}5\\2\end{bmatrix}_q - q^6 + q^3 = q^5 + 2q^4 + 3q^3 + 2q^2 + q + 1,$$

this set L_i is uniquely contained in either an EKR family or dual-EKR family. Since for each set L_i,

$$|L_i| \geq |S_i| - \sum_{j\neq i}|S_{ij}| = |S_i| - (x-1)|S_{ij}|,$$

we can apply the result from Lemma 5.1 to see that this bound is met when

$$q^6 > (x-1)q^5 + (x^2-x)q^4 + (3x^2-7x+5)q^3 + (2x^2-4x+2)q^2 + (x^2-2x+1)q + (x^2-2x+1).$$

This is satisfied when $x \leq q^{2/3}$. ∎

Theorem 5.4. *There are no Cameron–Liebler 2-classes in $\mathrm{PG}(5,q)$ with parameter x for $3 \leq x \leq \sqrt{q}$.*

Proof. Let $\mathcal{L}$ be a Cameron–Liebler 2-class with parameter $1 < x \leq \sqrt{q}$ containing pairwise skew planes $\pi_1, \dots, \pi_x$, and define S_i, S_{ij}, and L_i as before. Then, by Lemma 5.3, each set L_i is contained in a unique EKR family or dual-EKR family $\mathcal{F}_i$.

Now suppose there is a plane $\pi \in \mathcal{L} \setminus (\bigcup_i L_i)$. Since no $x+1$ planes of $\mathcal{L}$ are pairwise skew, π must intersect at least two of the planes in $\{\pi_1, \dots, \pi_x\}$, and so $\pi \in S_{ij}$ for some i, j. Furthermore, if $\pi \notin \mathcal{F}_i$, then by Theorem 2.13, π intersects at most $\begin{bmatrix}5\\2\end{bmatrix}_q - q^6 + q^3$

elements of L_i. By Theorem 3.4, π intersects $x\left[{5 \atop 2}\right]_q - (x-1)q^6$ planes of $\mathcal{L}$ in total, so if $\pi \in \mathcal{L} \setminus (\bigcup_{i=1}^{x} \mathcal{F}_i)$,

$$\left(x\begin{bmatrix}5\\2\end{bmatrix}_q - (x-1)q^6\right) - x\left(\begin{bmatrix}5\\2\end{bmatrix}_q - q^6 + q^3\right) = q^6 - xq^3$$

of the planes in $\mathcal{L}$ intersecting π are not contained in any set L_i, and so belong to some S_{ij}. This forces us to have

$$q^6 - xq^3 \leq |\bigcup_{i<j} S_{ij}| \leq \frac{x(x-1)}{2}|S_{ij}|.$$

Putting in the formula for $|S_{ij}|$, we see that

$$q^6 - xq^3 \leq \frac{x(x-1)}{2}\left(2q^5 + (x+2)q^4 + (3x-2)q^3 + 2xq^2 + xq + x\right),$$

or

$$\begin{aligned} q^6 - xq^3 \leq x^2q^5 &+ \frac{x^2}{2}\left((x+2)q^4 + (3x-2)q^3 + 2xq^2 + xq + x\right) \\ &- \frac{1}{2}\left(2xq^5 + x(x+2)q^4 + x(3x-2)q^3 + 2x^2q^2 + x^2q + x^2\right). \end{aligned}$$

Since $x \leq \sqrt{q}$, this implies that

$$\begin{aligned} q^6 - xq^3 \leq q^6 &+ \frac{q}{2}\left((x+2)q^4 + (3x-2)q^3 + 2xq^2 + xq + x\right) \\ &- \frac{1}{2}\left(2xq^5 + x(x+2)q^4 + x(3x-2)q^3 + 2x^2q^2 + x^2q + x^2\right). \end{aligned}$$

and so

$$(x-2)q^5 + (x^2 - x + 2)q^4 + 3x(x-2)q^3 + x(2x-1)q^2 + x(x-1)q + x^2 \leq 0,$$

a contradiction. Therefore we must have $\mathcal{L} \subseteq \bigcup_i \mathcal{F}_i$. Since

$$|\mathcal{L}| = x\begin{bmatrix}2k+1\\k\end{bmatrix}_q,$$

and $|\mathcal{F}_i| \leq \left[{2k+1 \atop k}\right]_q$ for all i, we see that $\mathcal{L}$ is the union of x pairwise disjoint sets $\mathcal{F}_1, \ldots, \mathcal{F}_x$, each an EKR family or dual-EKR family. This is impossible for $x \geq 3$. ∎

This same proof can be used to characterize the Cameron-Liebler 2-classes in $\mathrm{PG}(5, q)$ having parameter $x = 2$, as long as q is large enough.

Corollary 5.5. *If $q \geq 4$, any Cameron–Liebler 2-class in $\mathrm{PG}(5, q)$ having parameter 2 is trivial.* ∎

This leaves the cases where $q = 2$ or $q = 3$; we can use a computer search to rule out nontrivial examples of Cameron–Liebler 2-classes with parameter $x = 2$ in $\mathrm{PG}(5, 2)$, as follows:

Assume that $\mathcal{L}$ is a Cameron–Liebler 2-class in $\mathrm{PG}(5,q)$ containing disjoint planes π_1, π_2. There are q^6 planes in $\mathcal{L}$ disjoint from π_1, and a further q^6 planes in $\mathcal{L}$ disjoint from π_2. Since no three planes of $\mathcal{L}$ are disjoint, this leaves $2\left[\begin{smallmatrix}5\\2\end{smallmatrix}\right]_q - 2q^6$ planes in $\mathcal{L}$ which intersect both π_1 and π_2 nontrivially. For any such plane π', there are three possibilities.

(i) π' meets each of π_1 and π_2 in a single point;

(ii) π' meets π_1 in a single point and π_2 in a line;

(iii) π' meets π_1 in a line and π_2 in a single point.

A simple count reveals that there are a total of $2(q^2+q+1)^2 < 2\left[\begin{smallmatrix}5\\2\end{smallmatrix}\right]_q - 2q^6$ planes of $\mathrm{PG}(5,q)$ falling into the latter two categories, so there is at least one plane π' of $\mathcal{L}$ meeting each of π_1 and π_2 in a single point. We can therefore assume that, up to isomorphism, $\mathcal{L}$ contains the planes

$$\begin{aligned}
\pi_1 &= \langle (1,0,0,0,0,0), (0,1,0,0,0,0), (0,0,1,0,0,0) \rangle \\
\pi_2 &= \langle (0,0,0,1,0,0), (0,0,0,0,1,0), (0,0,0,0,0,1) \rangle \\
\pi' &= \langle (0,0,1,0,0,0), (0,0,0,1,0,0), (0,1,0,0,1,0) \rangle.
\end{aligned}$$

By defining $\boldsymbol{p}_1 = (0,0,1,0,0,0)$, $\boldsymbol{p}_2 = (0,0,0,1,0,0)$, $H_1 = \{(x_0,x_1,x_2,x_3,x_4,0)\}$, and $H_2 = \{(0,x_1,x_2,x_3,x_4,x_5)\}$, $\mathcal{L}$ is trivial if and only if one of $[\boldsymbol{p}_1]_2$, $[\boldsymbol{p}_2]_2$, $[H_1]_2$, or $[H_2]_2$ is contained in $\mathcal{L}$. Using the computational software Gurobi [19], we can search for examples of Cameron–Liebler 2-classes with parameter 2 by looking for 0-1 vectors $\boldsymbol{\chi}$ satisfying

$$\begin{aligned}
\boldsymbol{\chi}(K + q^6 \cdot I - q^6 / \begin{bmatrix}5\\2\end{bmatrix}_q \cdot J) = 0 \text{ and}& \\
\boldsymbol{\chi}\boldsymbol{j}^T = x\begin{bmatrix}5\\2\end{bmatrix}_q.&
\end{aligned}$$

This formula is just a rewriting of Theorem 3.7 (vii). By requiring that the entries of $\boldsymbol{\chi}$ corresponding to π_1, π_2, and π' are equal to 1, and that the inner product of $\boldsymbol{\chi}$ with the characteristic vectors for $[\boldsymbol{p}_1]_2$, $[\boldsymbol{p}_2]_2$, $[H_1]_2$, and $[H_2]_2$ is always less than $\left[\begin{smallmatrix}5\\2\end{smallmatrix}\right]_q$, any examples we find will be nontrivial.

This computation runs very quickly with no nontrivial examples found when $q = 2$. For $q = 3$ this is computationally a very difficult problem.

6 Cameron–Liebler k-classes for $k > 2$

We now look at Cameron–Liebler k-classes in $\mathrm{PG}(2k+1,q)$ when $k > 2$.

Lemma 6.1. *Let $\mathcal{L}$ be a Cameron–Liebler k-class in $\mathrm{PG}(2k+1,q)$ with parameter x, and let $\pi_1, \ldots, \pi_t$ be a collection of t pairwise skew k-spaces in $\mathcal{L}$. Let S_i be the set of k-spaces* *in $\mathcal{L}$ which intersect π_i, and let S_{ij} be the set of k-spaces in $\mathcal{L}$ intersecting both π_i and π_j. Then*

$$|S_i| = x\begin{bmatrix}2k+1\\k\end{bmatrix}_q - (x-1)q^{(k+1)k} \text{ and}$$
$$|S_{ij}| = x\begin{bmatrix}2k+1\\k\end{bmatrix}_q - 2(x-1)q^{(k+1)k} + (x-2)q^{(k+1)k/2}\prod_{i=1}^{k}(q^i-1).$$

Proof. This follows directly from Theorem 3.4. ■

Despite $\begin{bmatrix}2k+1\\k\end{bmatrix}_q$ being a polynomial of degree $q^{(k+1)k}$, it is difficult to prove results for general k. We will work with an upper bound instead of calculating these polynomials explicitly. To accomplish this, we use the following inequalities found in [21].

Lemma 6.2. *For $n_1 < \cdots < n_k$, we have*

$$(q^{n_1}-1)\prod_{j=2}^{k}(q^{n_j}\pm 1) \le q^{\sum_{j=1}^{k} n_j} \text{ and}$$
$$(q^{n_1}+1)\prod_{j=2}^{k}(q^{n_j}\pm 1) \ge q^{\sum_{j=1}^{k} n_j}.$$

■

Applying this result, we obtain the following inequalities.

Corollary 6.3.

$$\frac{(q^{n_1}-1)}{(q^{n_1}+1)}q^{\sum_{i=1}^{k} n_i} \le (q^{n_1}-1)\prod_{i=2}^{k}(q^{n_i}\pm 1) \le q^{\sum_{i=1}^{k} n_i};$$
$$q^{\sum_{i=1}^{k} n_i} \le (q^{n_1}+1)\prod_{i=2}^{k}(q^{n_i}\pm 1) \le \frac{(q^{n_1}+1)}{(q^{n_1}-1)}q^{\sum_{i=1}^{k} n_i}.$$

Proof. We write

$$(q^{n_1}-1)\prod_{j=2}^{k}(q^{n_j}\pm 1) = \frac{(q^{n_1}-1)}{(q^{n_1}+1)}(q^{n_1}+1)\prod_{j=2}^{k}(q^{n_j}\pm 1),$$

and apply the inequality above to obtain the first result. A similar method yields the second result as well. ■

From this, we can prove the following bound.

Corollary 6.4.

$$\begin{bmatrix}2k+1\\k\end{bmatrix}_q \le \frac{(q+1)}{(q-1)}q^{(k+1)k} = q^{(k+1)k} + \frac{2}{(q-1)}q^{(k+1)k}.$$

Proof. We have that

$$\begin{bmatrix} 2k+1 \\ k \end{bmatrix}_q = \frac{\prod_{i=1}^{k}(q^{k+1+i}-1)}{\prod_{i=1}^{k}(q^i-1)}.$$

An upper bound on the numerator is given by

$$q^{\sum_{i=1}^{k}(k+1+i)} = q^{\frac{3}{2}(k+1)k},$$

while a lower bound on the denominator is

$$\frac{(q-1)}{(q+1)}q^{\sum_{i=1}^{k} i} = \frac{(q-1)}{(q+1)}q^{(k+1)k/2}.$$

The result follows immediately. ■

We can apply this upper bound in our consideration of Cameron–Liebler k-classes with small parameter x.

Lemma 6.5. *Let $\mathcal{L}$ be a Cameron–Liebler k-class of* $\mathrm{PG}(2k+1,q)$*. For $x \leq q^{1/3}$, no $x+1$ distinct k-spaces of $\mathcal{L}$ are pairwise skew.*

Proof. Assume $\mathcal{L}$ is a Cameron–Liebler k-class of $\mathrm{PG}(2k+1,q)$ with parameter $x \leq q^{1/3}$. Let $\pi_1, \ldots, \pi_{x+1}$ be $x+1$ pairwise skew k-spaces in $\mathcal{L}$, and define S_i and S_{ij} as in Lemma 6.1. We have that

$$|\mathcal{L}| \;=\; x\begin{bmatrix} 2k+1 \\ k \end{bmatrix}_q \;\geq\; (x+1)|S_i| - \frac{(x+1)x}{2}|S_{ij}|.$$

Putting in the known sizes of these sets and rearranging, we see that

$$\frac{x^2(x-1)}{2}\begin{bmatrix} 2k+1 \\ k \end{bmatrix}_q \;\geq\; (x^2-1)(x-1)q^{(k+1)k} - \frac{(x+1)x(x-2)}{2}q^{(k+1)k/2}\prod_{i=1}^{k}(q^i-1).$$

Applying our upper bounds on $\left[\begin{smallmatrix} 2k+1 \\ k \end{smallmatrix}\right]_q$ and $\prod_{i=1}^{k}(q^i-1)$, we see that this implies

$$x^2(x-1)\left(q^{(k+1)k} + \frac{2}{(q-1)}q^{(k+1)k}\right) \;\geq\; (x^3-x^2+2)q^{(k+1)k},$$

or equivalently $x^2(x-1) \geq (q-1)$, which is a contradiction since $x \leq q^{1/3}$. ■

Lemma 6.6. *Let $\mathcal{L}$ be a Cameron–Liebler k-class in* $\mathrm{PG}(2k+1,q)$ *with parameter $x \leq q^{1/3}$, containing x pairwise skew k-spaces $\pi_1, \ldots, \pi_x$. For each $1 \leq i \leq x$, put L_i to be the set of k-spaces in $\mathcal{L}$ intersecting π_i nontrivially and being skew to π_j for all $j \neq i$. If $k < q\log q - q$, then each set L_i is contained in a unique EKR family or dual-EKR family of k-spaces.*

Proof. For each i, $|L_i| \geq |S_i| - (x-1)|S_{ij}|$, so

$$|L_i| \geq (2x-3)(x-1)q^{(k+1)k} - (x-1)(x-2)q^{(k+1)k/2}\prod_{i=1}^{k}(q^i-1) - x(x-2)\begin{bmatrix} 2k+1 \\ k \end{bmatrix}_q.$$

Applying our upper bounds on $\begin{bmatrix} 2k+1 \\ k \end{bmatrix}_q$ and $\prod_{i=1}^{k}(q^i-1)$, we see that

$$|L_i| \ \geq q^{(k+1)k} - \frac{2x(x-2)}{q-1}q^{(k+1)k}.$$

The k-spaces of each set L_i pairwise intersect nontrivially. Therefore by Corollary 2.15, when this lower bound exceeds $\frac{1}{2}q^{(k+1)k}$, L_i must be uniquely contained in an EKR family or dual-EKR family. This lower bound is large enough when $x < 1 + \frac{\sqrt{q+3}}{2}$, so certainly when $x \leq q^{1/3}$. ■

Theorem 6.7. *When $k < q\log q - q$, there are no Cameron–Liebler k-classes having parameter $3 \leq x \leq (q/2)^{1/3}$ in $\mathrm{PG}(2k+1, q)$.*

Proof. Let $\mathcal{L}$ be a Cameron–Liebler k-class in $\mathrm{PG}(2k+1, q)$ with parameter $1 \leq x \leq (q/2)^{1/3}$. Take x pairwise skew k-spaces $\pi_1, \ldots, \pi_x$ in $\mathcal{L}$ and for $1 \leq i \leq x$, define L_i as in Lemma 6.6. Note that the sets L_i are pairwise disjoint. For each set L_i, let $\mathcal{F}_i$ be the unique EKR family or dual-EKR family of k-spaces containing L_i.

Now assume that we have some k-space π in $\mathcal{L}$ with $\pi \notin \bigcup_i \mathcal{F}_i$. There exists a k-spread containing π; this k-spread shares x k-spaces with $\mathcal{L}$, giving rise to a collection L_π of pairwise intersecting k-spaces of $\mathcal{L}$, containing π, uniquely contained in an EKR family or dual-EKR family $\mathcal{F}_\pi$. Then for each $1 \leq i \leq x$, $L_\pi \cap L_i \subset \mathcal{F}_\pi \cap \mathcal{F}_i$, and

$$|\mathcal{F}_\pi \cap \mathcal{F}_i| \leq \begin{bmatrix} 2k \\ k \end{bmatrix}_q = \frac{q^{k+1}-1}{q^{2k+1}-1}\begin{bmatrix} 2k+1 \\ k \end{bmatrix}_q.$$

We must have

$$|L_\pi| + \sum_i |L_i| - \sum_i |L_\pi \cap L_i| \leq |L_\pi \cup (\bigcup_i L_i)| \leq |\mathcal{L}|.$$

Applying our lower bound on the values $|L_i|$ (which also applies to $|L_\pi|$) and our upper bound on the values $|L_\pi \cap L_i|$, we see that

$$(x+1)\left((x-1)^2q^{(k+1)k} - x(x-2)\begin{bmatrix} 2k+1 \\ k \end{bmatrix}_q\right) - x\frac{(q^{k+1}-1)}{(q^{2k+1}-1)}\begin{bmatrix} 2k+1 \\ k \end{bmatrix}_q \leq x\begin{bmatrix} 2k+1 \\ k \end{bmatrix}_q,$$

which we can rewrite as

$$(x^3-x^2-x+1)q^{(k+1)k} \leq \left(x^3-x^2-x+x\frac{(q^{k+1}-1)}{(q^{2k+1}-1)}\right)\begin{bmatrix} 2k+1 \\ k \end{bmatrix}_q.$$

We now apply our upper bound on $\begin{bmatrix} 2k+1 \\ k \end{bmatrix}_q$ and rearrange to obtain

$$(q-(2x^3-2x^2-2x+1))q^{(k+1)k} \leq x\frac{(q^{k+1}-1)(q+1)}{(q^{2k+1}-1)}q^{(k+1)k} \leq xq^{(k+1)k}.$$

This implies that $q \leq (2x^2-1)(x-1) < 2x^3$, a contradiction since $x \leq (q/2)^{1/3}$. ■

Corollary 6.8. *If $q > 7$ and $k < q \log q - q$, a Cameron–Liebler k-class in* $\mathrm{PG}(2k+1, q)$ *with $x = 2$ is trivial.*

Proof. This result follows from noticing that, in the last line of the previous proof, we actually obtain a contradiction whenever $q > (2x^2 - 1)(x - 1)$; for $x = 2$, this occurs when $q > 7$. ∎

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