Double k-sets in symplectic generalized quadrangles

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Abstract

In classical projective geometry, a double six of lines consists of 12 lines ℓ_1, ℓ_2, …, ℓ_6, m_1, m_2, …, m_6 such that the ℓ_i are pairwise skew, the m_i are pairwise skew, and ℓ_i meets m_j if and only if $i \neq j$. In the 1960's Hirschfeld studied this configuration in finite projective spaces $\mathrm{PG}(3,q)$ showing they exist for almost all values of q, with a couple of exceptions when q is too small. We will be considering double-k sets in the symplectic geometry $\mathrm{W}(q)$, which is constructed from $\mathrm{PG}(3,q)$ using an alternating bilinear form. This geometry is an example of a generalized quadrangle, which means it has the nice property that if we take any line ℓ and any point P not on ℓ, then there is exactly one line through P meeting ℓ. We will discuss all of this in detail, including all of the basic definitions needed to understand the problem, and give a result classifying which values of k and q allow us to construct a double k-set of lines in $\mathrm{W}(q)$.

1 Background and Review

In classical projective geometry a *double six* of lines consists of a set of 12 lines ℓ_1, ℓ_2, $\ldots$, ℓ_6; m_1, m_2, $\ldots$, m_6, such that the ℓ_i are pairwise skew, the m_i are pairwise skew, and ℓ_i meets m_j if and only if $i \neq j$. In the 1960's J.W.P. Hirschfeld studied the existence of double sixes in finite projective spaces (see the articles [1], [2], [3], [4]). He concluded that a double six exists over all fields except for the finite fields of order 2, 3, and 5.

In the present work we shift the basic setting from projective space to the symplectic geometry $\mathrm{W}(q)$ living in $\mathrm{PG}(3,q)$. $\mathrm{W}(q)$ denotes the point-line incidence geometry derived from a symplectic polarity of $\mathrm{PG}(3,q)$, the 3-dimensional projective space over the finite field F_q with $q = p^e$ elements, p a prime. It is well known that this incidence geometry is a generalized quadrangle (GQ) of order q with all points regular for each prime power q, with all lines regular when $p = 2$, and all lines antiregular when p is odd. Conversely, if $\mathscr{S}$ is a GQ with all points regular, then it must be isomorphic to some $\mathrm{W}(q)$. For these results and general background information on GQ see [6]. What is important for the present study is that all lines are regular if and only if each triad of lines (an unordered set of three distinct pairwise disjoint lines) has either $q+1$ transversals or exactly 1 transversal, and all lines are antiregular if and only if each triad of lines has either 0 or exactly 2 transversals.

In 1987 the first author of the present study exhibited a double five in $\mathrm{W}(3)$ along with several combinatorial facts concerning it (see [5]). Here we show that $\mathrm{W}(3)$ has no double six, many double twos and double threes, and then we study the more interesting double fours and double fives.

2 Double k-sets with k = 2, 3, or 6

Let $\mathscr{S} = \mathrm{W}(q)$ for some prime power q, so that either all lines are regular or all lines are antiregular. Then either all triads of lines have 1 or $1+q$ transversals, or all triads of lines have 0 or 2 transversals. Note that $\mathscr{S}$ has too many double twos to mention.

Note: If $\mathscr{S}$ contains a double k, say $\mathscr{D}_k = (\ell_1, \ldots, \ell_k; m_1, \ldots, m_k)$, then $\mathscr{D}'_{k-1} = (\ell_1, \ldots, \ell_{k-1}; m_1, \ldots, m_{k-1})$ is a double $k-1$. Eventually we are going to prove that $\mathrm{W}(q)$ has a double five if and only if $q \equiv 1$ or $3 \mod 6$. So certainly in these cases $\mathrm{W}(q)$ has double threes, double fours and double fives.

Lemma 2.1. *$\mathscr{S}$ never contains a double six.*

Proof. Suppose that $\mathscr{S}$ contains a double six $\ell_1, \ell_2, \ldots, \ell_6;\ m_1, m_2, \ldots, m_6$. It follows that (ℓ_1, ℓ_2, ℓ_3) is a triad with three centers m_4, m_5, m_6. This forces us to be in the situation where all lines are regular (so q is even), i.e. any line meeting two of ℓ_1, ℓ_2, ℓ_3 must meet the third one. Hence the lines m_1, m_2, m_3 of the putative double six cannot exist. □

Let $\mathscr{T} = (\ell_1, \ell_2, \ell_3)$ be a triad of lines. We start by considering the existence of double threes.

Case 1: $\mathscr{T}$ has no transversal. Hence q is odd and all lines are antiregular. Let $P_{1,3}$ denote an arbitrary point of ℓ_1, and let m_3 be the line through $P_{1,3}$ meeting ℓ_2 at a point $P_{2,3}$. Clearly m_3 cannot meet ℓ_3 since $\mathscr{T}$ has no transversal. Next let $P_{3,2}$ denote the point on ℓ_3 collinear with $P_{2,3}$, and let m_2 be the line through $P_{3,2}$ meeting ℓ_1, say in the point $P_{1,2}$. Then let $P_{2,1}$ be any point of ℓ_2 different from $P_{2,3}$. So $P_{2,1}$ will be collinear with a point $P_{3,1}$ of ℓ_3 different from $P_{3,2}$, and the line m_1 through $P_{3,1}$ and $P_{2,1}$ will not meet ℓ_1. Hence $(\ell_1, \ell_2, \ell_3; m_1, m_2, m_3)$ is a double three.

Case 2: $\mathscr{T}$ has exactly two transversals, so again s is odd and all lines are antiregular. If we now let m_4 and m_5 be the two transversals of $\mathscr{T}$, using just the $q-1$ points on each of the lines ℓ_1, ℓ_2 and ℓ_3 (and not on m_4 or m_5), we can repeat the construction given in Case 1 to produce a double three.

Case 3: $\mathscr{T}$ has a unique transversal. So all lines are regular. In this case $q = 2^e$. So let $\mathscr{T} = (\ell_1, \ell_2, \ell_3)$ be a triad with a unique transversal m_4. Say m_4 meets ℓ_i at $P_{i,4}$, $i = 1, 2, 3$. Let $P_{1,3}$ be a point of ℓ_1 different from $P_{1,4}$; $P_{2,3}$ the point of ℓ_2 collinear with $P_{1,3}$; and m_3 the line through $P_{1,3}$ and $P_{2,3}$. Clearly m_3 does not meet ℓ_3. Let $P_{3,2}$ be the point of ℓ_3 collinear with $P_{2,3}$; $P_{1,2}$ the point of ℓ_1 collinear with $P_{3,2}$; m_2 the line through $P_{1,2}$ and $P_{3,2}$. Clearly m_2 cannot meet ℓ_2. Let $P_{2,1}$ be the point of ℓ_2 collinear with $P_{1,2}$; $P_{3,1}$ the point of ℓ_3 collinear with $P_{2,1}$; m_1 the line through $P_{2,1}$ and $P_{3,1}$. Clearly m_1 does not meet ℓ_1. So $\mathscr{D}_3 = (\ell_1, \ell_2, \ell_3;\, m_1, m_2, m_3)$ is a double three. We obtained this double three in a particular way. However, now suppose that $\mathscr{D}_3 = (\ell_1, \ell_2, \ell_3;\, m_1, m_2, m_3)$ is any double three and that all lines are regular. Then $\mathscr{T} = (\ell_1, \ell_2, \ell_3)$ and $\mathscr{T}' = (m_1, m_2, m_3)$ must have unique transversals m_4 and ℓ_4, respectively. Suppose that ℓ_4 were to meet m_4. Then (m_1, m_2, m_4) would have two transversals, viz., ℓ_3 and ℓ_4, which is impossible since all triads must have either 1 or $q+1$ transversals. Hence $\mathscr{D}_4 = (\ell_1, \ell_2, \ell_3, \ell_4;\, m_1, m_2, m_3, m_4)$ is a double four.

We collect these observations into a lemma.

Lemma 2.2. *Let $\mathscr{S} = \mathrm{W}(q)$ for some prime power q. If $\mathscr{T} = (\ell_1, \ell_2, \ell_3)$ is any triad that does not have $1+q$ transversals, then $\mathscr{T}$ is contained in a double three. If q is even, then each double three is contained in a unique double four. In particular,* $\mathrm{W}(q)$ *contains double fours when $q = 2^e$.*

In the course of determining just when $\mathrm{W}(q)$ contains double fives we show that for all prime powers q it is true that $\mathrm{W}(q)$ contains double fours.

3 Double Fives

Our first result here implies that if $\mathrm{W}(q)$ has a double five, then q must be odd.

Lemma 3.1. *Let $\mathscr{S} = \mathrm{W}(q)$ for some prime power q. Suppose that $\mathscr{D}_5 = (\ell_1, \ldots, \ell_5;\, m_1, \ldots, m_5)$ is a double five. Since $\mathscr{T} = (\ell_1, \ell_2, \ell_3)$ cannot have $1+q$ transversals but does have two (viz., m_4 and m_5), it must be the case that all lines are antiregular, which implies that q is odd (see [6]).*

Theorem 3.2. *The symplectic geometry* $\mathrm{W}(q)$ *has a double five if and only if* $q \equiv 1$ *or* $3 \bmod 6$.

Proof. Our proof requires the extensive use of coordinates. As each two symplectic forms are equivalent up to a change of basis, we will represent ours without losing generality by

$$\bar{x} \circ \bar{y} = x_0 y_1 - x_1 y_0 + x_2 y_3 - x_3 y_4.$$

A line ℓ is called *isotropic* provided $\bar{x} \circ \bar{y} = 0$ for all $\bar{x}$ and $\bar{y}$ on ℓ, and the points of $\mathrm{PG}(3,q)$ along with the set of all isotropic lines form the symplectic geometry $\mathrm{W}(q)$. The idea is to coordinatize the lines of a putative double five as generally as possible and determine those restrictions on q which are necessary and sufficient for the existence of a double five of isotropic lines.

The configuration of lines we wish to consider is this: five pairwise skew isotropic lines, denoted ℓ_1, ℓ_2, …, ℓ_5, with each four of the ℓ_i having exactly one transversal m_k not meeting ℓ_k. We will refer to the point $\ell_i \cap m_j$, where $i \neq j$, as $P_{i,j}$.

We will use some important properties of the symplectic group to put convenient coordinates on as many points as possible, without losing any generality. Firstly, the projective symplectic group $\mathrm{PSp}(4,q)$ is transitive on all of the points of $\mathrm{W}(q)$; the stabilizer $\mathrm{PSp}(4,q)_{\{P\}}$ of a fixed point P is transitive on points not collinear with P; and the stabilizer $\mathrm{PSp}(4,q)_{\{P\},\{Q\}}$ of two noncollinear points P and Q is triply transitive on the points collinear with both P and Q. These results allow us to label

$$\begin{aligned}
P_{1,5} &= \ell_1 \cap m_5 = (1:0:0:0).\\
P_{2,4} &= \ell_2 \cap m_4 = (0:1:0:0),\\
P_{1,4} &= \ell_1 \cap m_4 = (0:0:0:1), \text{ and}\\
P_{2,5} &= \ell_2 \cap m_5 = (0:0:1:0).
\end{aligned}$$

We have now coordinatized the following lines:

$$\begin{aligned}
\ell_1 &= \langle (1:0:0:0),\ (0:0:0:1) \rangle,\\
\ell_2 &= \langle (0:1:0:0),\ (0:0:1:0) \rangle,\\
m_4 &= \langle (0:1:0:0),\ (0:0:0:1) \rangle, \text{ and}\\
m_5 &= \langle (1:0:0:0),\ (0:0:1:0) \rangle.
\end{aligned}$$

It can be easily verified that ℓ_1 and ℓ_2 are skew, as are m_4 and m_5.

From here, we will label points

$$\begin{aligned}
P_{3,5} &= (1:0:\alpha:0) \text{ and}\\
P_{1,3} &= (1:0:0:\beta),
\end{aligned}$$

which can be seen to force

$$\begin{aligned}
P_{3,4} &= (0:1:0:-\alpha^{-1}) \text{ and}\\
P_{2,3} &= (0:1:\beta^{-1}:0), \text{ so}\\
\ell_3 &= \langle (1:0:\alpha:0),\ (0:1:0:-\alpha^{-1}) \rangle \text{ and}\\
m_3 &= \langle (1:0:0:\beta),\ (0:1:\beta^{-1}:0) \rangle.
\end{aligned}$$

It can be verified that ℓ_3 and m_3 will be skew so long as q is odd.

Next, we will label the point

$$P_{1,2} = (1:0:0:\gamma),$$

where γ is nonzero and $\gamma \neq \beta$, which forces us to have

$$\begin{aligned}
P_{3,2} &= (1:\alpha\gamma:\alpha:-\gamma) \text{ and}\\
m_2 &= \langle (1:0:0:\gamma),\ (1:\alpha\gamma:\alpha:-\gamma) \rangle.
\end{aligned}$$

At this point we should remark that in order for ℓ_2 and m_2 to not be concurrent, we must have $2 \neq 0$, showing again that q must be odd.

We will next coordinatize ℓ_4. We may arbitrarily label

$$P_{4,5} = (1:0:\delta:0),$$

requiring only that $\delta \neq 0$ and $\delta \neq \alpha$. This can be seen to force

$$\begin{aligned}P_{4,3} &= (1 : -\beta\delta : -\delta : \beta) \text{ and} \\ P_{4,2} &= (2 - \alpha\delta^{-1} : \alpha\gamma : \alpha : -\alpha\gamma\delta^{-1})\end{aligned}$$

to be the unique points on m_3 and m_2 collinear with $P_{4,5}$. However, since we must have these two points collinear with each other as well, we must have $\delta = \alpha(1 - \beta^{-1}\gamma)$.

We would like to compile the coordinates we have determined thus far, as these will completely determine the coordinates for the rest of the points in our configuration.

On ℓ_1:

$$\begin{aligned}P_{1,5} &= (1 : 0 : 0 : 0) \\ P_{1,4} &= (0 : 0 : 0 : 1) \\ P_{1,3} &= (1 : 0 : 0 : \beta) \\ P_{1,2} &= (1 : 0 : 0 : \gamma)\end{aligned}$$

On ℓ_2:

$$\begin{aligned}P_{2,5} &= (0 : 0 : 1 : 0) \\ P_{2,4} &= (0 : 1 : 0 : 0) \\ P_{2,3} &= (0 : 1 : \beta^{-1} : 0)\end{aligned}$$

On ℓ_3:

$$\begin{aligned}P_{3,5} &= (1 : 0 : \alpha : 0) \\ P_{3,4} &= (0 : 1 : 0 : -\alpha^{-1}) \\ P_{3,2} &= (1 : \alpha\gamma : \alpha : -\gamma)\end{aligned}$$

On ℓ_4:

$$\begin{aligned}P_{4,5} &= (1 : 0 : \alpha(1 - \beta^{-1}\gamma) : 0) \\ P_{4,3} &= (1 : -\alpha\beta(1 - \beta^{-1}\gamma) : -\alpha(1 - \beta^{-1}\gamma) : \beta) \\ P_{4,2} &= (1 - 2\beta^{-1}\gamma : \alpha\gamma(1 - \beta^{-1}\gamma) : \alpha(1 - \beta^{-1}\gamma) : -\gamma).\end{aligned}$$

None of the coordinates for points on ℓ_5 and m_1 have yet been determined, and the only requirement we have imposed on our field up to this point is that $2 \not\equiv 0$.

The next line to tackle will be m_1. The line m_1 intersects (of the lines we have constructed thus far) ℓ_2, ℓ_3, and ℓ_4. We will first label

$$P_{2,1} = (0 : 1 : h : 0),$$

where h is nonzero, and $h \neq \beta^{-1}$. This gives

$$P_{3,1} = (1 : -\alpha h^{-1} : \alpha : h^{-1})$$

as the unique point on ℓ_3 collinear with $P_{2,1}$, and

$$P_{4,1} = (\beta h : -\alpha\beta(1-\beta^{-1}\gamma) : \alpha(\beta h - 2)(1-\beta^{-1}\gamma) : \beta)$$

as the unique point on ℓ_4 collinear with $P_{2,1}$. For these two points to also be collinear, we must have

$$2\alpha\gamma h + 2\alpha - 2\alpha\beta^{-1}\gamma = 0.$$

Since $2 \not\equiv 0$ we can deduce that $\gamma h = \beta^{-1}\gamma - 1$, and so $h = \beta^{-1} - \gamma^{-1}$.

At this point we have the freedom to label these three points on m_1 in terms of α, β and γ, and we will do so. We have:

$$\begin{aligned}
P_{2,1} &= (0 : 1 : \beta^{-1} - \gamma^{-1} : 0) \\
P_{3,1} &= (\beta^{-1} - \gamma^{-1} : -\alpha : \alpha(\beta^{-1} - \gamma^{-1}) : 1) \\
P_{4,1} &= (\beta(\beta^{-1} - \gamma^{-1}) : -\alpha\beta(1-\beta^{-1}\gamma) : \alpha(\beta(\beta^{-1} - \gamma^{-1}) - 2)(1-\beta^{-1}\gamma) : \beta) \\
&\equiv (\beta^{-1} - \gamma^{-1} : -\alpha(1-\beta^{-1}\gamma) : \alpha(\beta^{-2}\gamma - \gamma^{-1}) : 1).
\end{aligned}$$

At this point we know the coordinates of all points on ℓ_1, ℓ_2, ℓ_3 and ℓ_4 in terms of the arbitrary nonzero field elements α, β and γ in F_q, q odd.

Note: Before proceeding we notice that $\mathscr{D}_4 = (\ell_1, \ell_2, \ell_3, \ell_4; m_1, m_2, m_3, m_4)$ is a double four for all odd prime powers q. Hence by Lemma 2.2 we see that for each prime power q there is a double four in $\mathrm{W}(q)$.

Now ℓ_5 is the last line we need, and is a transversal of m_i for $i \neq 5$. Let us take take a point $P_{5,4} = (0 : 1 : 0 : \delta)$ on m_4. We see that

$$\begin{aligned}
P_{5,3} &= (1 : -\beta\delta^{-1} : -\delta^{-1} : \beta) \equiv (\delta : -\beta : -1 : \beta\delta) \\
P_{5,2} &= (-\alpha\delta : \alpha\gamma : \alpha : -\gamma(2+\alpha\delta)) \\
P_{5,1} &= (\beta^{-1} - \gamma^{-1} : -2\alpha - \delta^{-1} : -\delta^{-1}(\beta^{-1} - \gamma^{-1}) : 1)
\end{aligned}$$

are the unique points collinear with $P_{5,4}$ on m_3, m_2, and m_1, respectively. Requiring $P_{5,2}$ and $P_{5,3}$ to be collinear means we must have

$$2\alpha\beta\delta - 2\alpha\gamma\delta - 2\gamma = 0,$$

therefore $\delta = \frac{\alpha^{-1}\gamma}{\beta - \gamma}$. But we also need $P_{5,1}$ and $P_{5,3}$ to be collinear, so we require

$$2\beta\gamma^{-1} + 2\alpha\delta = 0$$

as well. Thus we must have $\delta = -\alpha^{-1}\beta\gamma^{-1}$.

So in order for us to have a double five we have imposed two different conditions on δ, we need $\delta = \frac{\alpha^{-1}\gamma}{\beta - \gamma}$ as well as $\delta = -\alpha^{-1}\beta\gamma^{-1}$. For these both to hold we must have

$$\frac{\alpha^{-1}\gamma}{\beta - \gamma} = -\alpha^{-1}\beta\gamma^{-1},$$

which is equivalent to

$$\beta^2\gamma^{-2} - \beta\gamma^{-1} + 1 = 0.$$

Conveniently, we may consider this a quadratic polynomial in the variable $\beta\gamma^{-1}$ with coefficients in the prime base field. This allows us to compute the discriminant of this polynomial, which is -3. Thus this polynomial has exactly one distinct root if $q \equiv 0 \mod 3$, and two distinct roots if -3 is a nonzero square in the field of order q. Using the Law of Quadratic Reciprocity it is straightforward to determine that the quadratic equation will have a solution if and only if q is congruent to 0 or 1 mod 3. Since q must be odd, this is equivalent to having q congruent to 1 or 3 mod 6, as claimed in the theorem. □

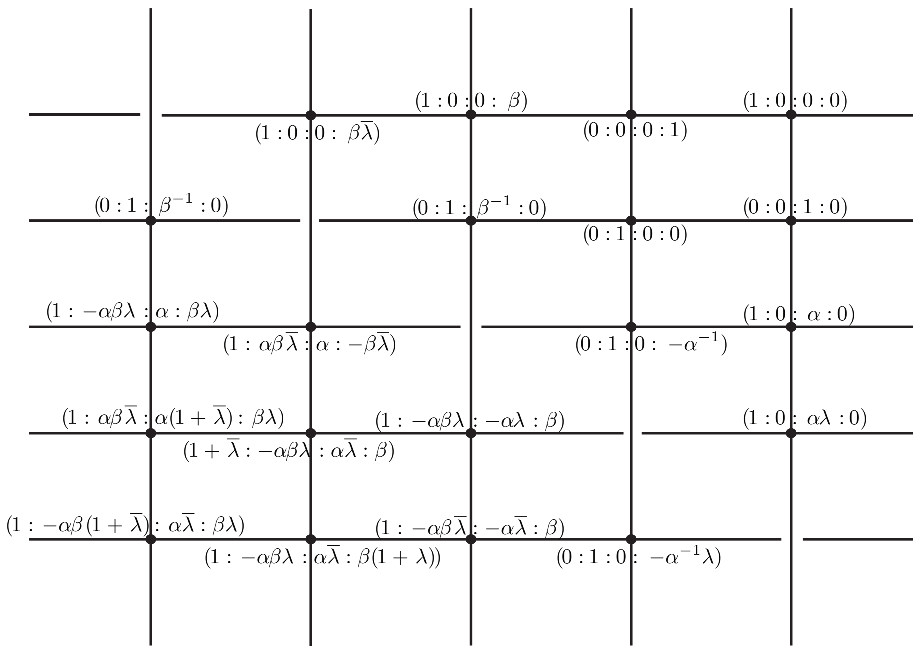

Figure 1: Coordinates for a double five; α and β are arbitrary (nonzero) field elements, and λ, $\overline{\lambda}$ are the two roots of $x^2 - x + 1$.

We should now remark that, in the situations where a double five exists, we can write $\lambda = \beta\gamma^{-1}$ as a chosen root of $x^2 - x + 1$, which allows us to replace $\gamma = \beta\lambda^{-1}$ and $\delta = -\alpha^{-1}\beta\gamma^{-1} = -\alpha^{-1}\lambda$. This allows us to coordinatize the points of our double five as shown in Fig. 1.

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