A new family of tight sets in $\mathcal{Q}^+(5,q)$

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September 19, 2014

Abstract

In this paper, we describe a new infinite family of $\frac{q^2-1}{2}$-tight sets in the hyperbolic quadrics $\mathcal{Q}^+(5,q)$, for $q \equiv 5$ or $9 \bmod 12$. Under the Klein correspondence, these correspond to Cameron–Liebler line classes of $\mathrm{PG}(3,q)$ having parameter $\frac{q^2-1}{2}$. This is the second known infinite family of nontrivial Cameron–Liebler line classes, the first family having been described by Bruen and Drudge with parameter $\frac{q^2+1}{2}$ in $\mathrm{PG}(3,q)$ for all odd q.

The study of Cameron–Liebler line classes is closely related to the study of symmetric tactical decompositions of $\mathrm{PG}(3,q)$ (those having the same number of point classes as line classes). We show that our new examples occur as line classes in such a tactical decomposition when $q \equiv 9 \bmod 12$ (so $q = 3^{2e}$ for some positive integer e), providing an infinite family of counterexamples to a conjecture made by Cameron and Liebler in 1982; the nature of these decompositions allows us to also prove the existence of a set of type $\left(\frac{1}{2}(3^{2e}-3^e), \frac{1}{2}(3^{2e}+3^e)\right)$ in the affine plane $\mathrm{AG}(2,3^{2e})$ for all positive integers e. This proves a conjecture made by Rodgers in his PhD thesis.

1 Introduction

Let $q = p^h$, p prime, $h \geq 1$, and let $\mathrm{PG}(d,q)$ denote the d-dimensional projective space over the finite field $\mathbb{F}_q$. A *spread* of $\mathrm{PG}(3,q)$ is a set $\mathcal{S}$ of lines of $\mathrm{PG}(3,q)$

*J. De Beule is a postdoctoral fellow of the Research Foundation Flanders – Belgium (FWO). J. Demeyer has been supported as a postdoctoral fellow of the Research Foundation Flanders – Belgium (FWO). The research of M. Rodgers has been supported partially by the FWO project "Moufang verzamelingen" G.0140.09

such that every point of $\mathrm{PG}(3,q)$ is contained in exactly one line of $\mathcal{S}$, i.e. the set of lines of $\mathcal{S}$ partitions the set of points of $\mathrm{PG}(3,q)$. Spreads of $\mathrm{PG}(3,q)$ are well studied, and many families of examples exist.

Cameron and Liebler in [6] originally studied irreducible collineation groups of $\mathrm{PG}(d,q)$ that have equally many point orbits as line orbits. This problem ended up being closely related to the study of symmetric tactical decompositions (tactical decompositions having the same number of point classes as line classes) of $\mathrm{PG}(d,q)$ (viewed as a point-line design), and this work was further generalized to the study of line sets later termed Cameron–Liebler line classes (the following definition is specific for $\mathrm{PG}(3,q)$, though these objects are also defined in higher dimensional projective spaces).

Definition 1.1. Let $\boldsymbol{A}$ be the point-line adjacency matrix of $\mathrm{PG}(d,q)$. A set of lines $\mathcal{L}$ is called a *Cameron–Liebler line class* if the characteristic vector $\boldsymbol{c}_{\mathcal{L}}$ of $\mathcal{L}$ lies in $\mathrm{row}(A)$.

The connection between these three concepts is as follows; a collineation group of $\mathrm{PG}(d,q)$ having equally many orbits on points and lines induces a symmetric tactical decomposition on $\mathrm{PG}(d,q)$, and any line class of such a tactical decomposition is a Cameron–Liebler line class.

Cameron–Liebler line classes of $\mathrm{PG}(3,q)$ have received considerable attention, and this is the situation we will focus on in this work. One important characterization here is that $\mathcal{L}$ is a Cameron–Liebler line class of $\mathrm{PG}(3,q)$ if and only if any spread of $\mathrm{PG}(3,q)$ contains exactly x lines of $\mathcal{L}$ for some integer x called the *parameter* of $\mathcal{L}$. A spread of $\mathrm{PG}(3,q)$ contains q^2+1 lines, so $0 \leq x \leq q^2+1$. Since any spread partitions the set of points of $\mathrm{PG}(3,q)$, it is clear immediately that the set of lines through a given point is a Cameron–Liebler line class with parameter 1. The same holds for its dual: the set of lines in a given plane is a Cameron–Liebler line class with parameter 1. Note that the union of two disjoint Cameron–Liebler line classes with respective parameters x and y is a Cameron–Liebler line class with parameter $x+y$, and that the complement of a Cameron–Liebler line class with parameter x is a Cameron–Liebler line class with parameter q^2+1-x. Hence the union of the set of lines through a point and the set of lines in a plane not containing that point is a Cameron–Liebler line class with parameter 2. The examples seen so far (including their complements) are called trivial examples.

In their work, Cameron and Liebler put forward the following progressively weaker conjectures:

Conjecture 1. *The only Cameron–Liebler line classes in* $\mathrm{PG}(d,q)$ *are the trivial examples.*

Conjecture 2. *A symmetric tactical decomposition of* $\mathrm{PG}(d,q)$ *consists of either*

(i) *a single point and line class;*

(ii) *two point classes* $\{\boldsymbol{p}\}$, $\mathrm{PG}(d,q)\setminus\{\boldsymbol{p}\}$ *and two line classes* $\mathrm{star}(\boldsymbol{p})$, $\mathrm{star}(\boldsymbol{p})^C$ *for some point* $\boldsymbol{p}$*; or (dually)*

(iii) *two point classes H, $\mathrm{PG}(d,q)\setminus H$ and two line classes $\mathrm{line}(H)$, $\mathrm{line}(H)^C$ for some hyperplane H.*

Conjecture 3. *An irreducible collineation group of $\mathrm{PG}(d,q)$ having equally many orbits on points and lines either*

(i) *is line-transitive;*

(ii) *stabilizes a hyperplane π and acts line-transitively on it; or (dually)*

(iii) *fixes a point $\boldsymbol{p}$ and acts line-transitively on the quotient space.*

Conjecture 1 was disproved for the first time in [9] with the construction of a Cameron–Liebler line class with parameter 5 in $\mathrm{PG}(3,3)$. Later in [5], the authors construct an infinite family of Cameron–Liebler line classes. More particularly, they show the existence of a Cameron–Liebler line class with parameter $\frac{q^2+1}{2}$ for odd q. Essentially, their construction is done in a geometric way; the Cameron–Liebler line-class is obtained as the union of all secant lines to a fixed elliptic quadric in $\mathrm{PG}(3,q)$, and a well defined and cleverly chosen subset of the tangent lines to the same elliptic quadric. In [11], another example was constructed in $\mathrm{PG}(3,4)$ having parameter 7. None of these examples arise as line classes in a symmetric tactical decomposition, however.

Meanwhile, the years after [6] have also seen the appearance of many non-existence results ([20], [9], [11], [12], [8], [7], [16], [4]). Currently, the most general result (which includes all the previous ones) is found in [17]. Its main result is the following: if $\mathcal{L}$ is a Cameron–Liebler line class with parameter x then $x \leq 2$ or $x > q\sqrt[4]{\frac{q}{2}} - \frac{2}{3}q$. This result improves all previous non-existence results except for a few cases dealing with small values of q.

In this paper, we prove the existence of Cameron–Liebler line classes with parameter $\frac{q^2-1}{2}$ for $q \equiv 5, 9 \bmod 12$, which is, since [5], the first construction of an infinite family. These examples were first found by Morgan Rodgers, [23]. His examples were obtained under the assumption of symmetry conditions for a hypothetical line class, then proceeding through eigenvalue methods and finally obtaining the examples through a computer search. It was observed that the examples constructed in $\mathrm{PG}(3,9)$ and $\mathrm{PG}(3,81)$ did in fact arise as line classes of a symmetric tactical decomposition having four classes on points and lines, providing the first counterexamples to Conjecture 2. Here, we show that this is the case for all of our examples with $q \equiv 9 \bmod 12$.

Remark 1.2. It is worth noting that Conjecture 3 was proven recently in [2].

2 Tight sets, eigenvectors, and tactical decompositions

Consider a 6-dimensional vector space $V(6,q)$ over the finite field $\mathbb{F}_q$. Let $f : V(6,q) \to \mathbb{F}_q$ be a bilinear form of Witt index 3. Hence the maximal totally isotropic subspaces with relation to f have dimension 3. The totally isotropic subspaces with relation to f induces in $\mathrm{PG}(5,q)$ a set of points, lines and planes contained in a quadratic surface. Up to coordinate transformation, there is only one bilinear form of Witt index 3, and so up to coordinate transformation, there is only one such quadratic surface. We call this geometry the hyperbolic quadric in $\mathrm{PG}(5,q)$, and denote it as $\mathcal{Q}^+(5,q)$. The projective spaces of maximal dimension contained in $\mathcal{Q}^+(5,q)$ are also called *generators*. Hence, the generators of $\mathcal{Q}^+(5,q)$ are planes. The generators of $\mathcal{Q}^+(5,q)$ come in two systems. Two distinct generators from the same system meet in a point, two generators from different systems meet either in a line or are skew. Hence two skew generators of $\mathcal{Q}^+(5,q)$ necessarily belong to a different system, and no three generators mutually skew can be found on $\mathcal{Q}^+(5,q)$. We refer to e.g. [14] to find all these well known properties of quadrics in finite projective spaces. Two points $\boldsymbol{p},\boldsymbol{q}$ of $\mathcal{Q}^+(5,q)$ are collinear if $f(u,v)=0$ with u,v being vector representatives of $\boldsymbol{p},\boldsymbol{q}$. We denote $\boldsymbol{p}^\perp$ the set of points of $\mathcal{Q}^+(5,q)$ collinear with the point $\boldsymbol{p}$ (this includes $\boldsymbol{p}$ itself).

Definition 2.1. A set $\mathcal{T}$ of points of $\mathcal{Q}^+(5,q)$ is an *x-tight set* if for every point $\boldsymbol{p}\in\mathcal{Q}^+(5,q)$ we have

$$|\boldsymbol{p}^\perp\cap\mathcal{T}| = x(q+1)+q^2\boldsymbol{c}_\mathcal{T}(\boldsymbol{p}).$$

The *Klein correspondence* is a bijection from the set of lines of $\mathrm{PG}(3,q)$ to the set of points of $\mathcal{Q}^+(5,q)$. The image of a line is the point with coordinates the Plücker coordinates of the line, and concurrent lines are mapped to collinear points. Generators correspond with the image of either the set of lines through a point, or the set of lines in a plane. The image of a Cameron–Liebler line class with parameter x is an x-tight set of $\mathcal{Q}^+(5,q)$. Hence, the trivial examples are (the union of two skew) generators and their complements. Note that tight sets were introduced in [19].

In constructing i-tight sets of $\mathcal{Q}^+(5,q)$, it is helpful to use the following result, due to Bamberg, Kelly, Law and Penttila [1].

Theorem 2.2. *Let $\mathcal{T}$ be a set of points in $\mathcal{Q}^+(5,q)$ with characteristic vector $\boldsymbol{c}$ and let $\boldsymbol{A}$ be the collinearity matrix of $\mathcal{Q}^+(5,q)$. Then $\mathcal{T}$ is an x-tight set if and only if*

$$(\boldsymbol{c}-\frac{x}{q^2+1}\boldsymbol{j})\boldsymbol{A} = (q^2-1)(\boldsymbol{c}-\frac{x}{q^2+1}\boldsymbol{j}).$$

Proof. By definition, $\mathcal{T}$ is an x-tight set if and only if

$$\boldsymbol{c}\boldsymbol{A} = (q^2-1)\boldsymbol{c}+x(q+1)\boldsymbol{j}.$$

Since $\boldsymbol{j}\boldsymbol{A} = q(q+1)^2\boldsymbol{j}$, the above formula follows immediately. $\square$

Our main theorem concerns tight sets which are disjoint from $(\pi_1\cup\pi_2)$, where π_1 and π_2 are two generators which are disjoint. This allows us to modify this theorem slightly. We will let $\boldsymbol{A}'$ be the matrix obtained from $\boldsymbol{A}$ by throwing away the rows and columns corresponding to points in $(\pi_1\cup\pi_2)$.

Theorem 2.3. *Let $\mathcal{T}$ be a set of points of $\mathcal{Q}^+(5,q)$ disjoint from $(\pi_1 \cup \pi_2)$, and let $\boldsymbol{c}'$ be the vector obtained from the characteristic vector of $\mathcal{T}$ by removing entries corresponding to points of $(\pi_1 \cup \pi_2)$. Then $\mathcal{T}$ is an x-tight set of $\mathcal{Q}^+(5,q)$ if and only if*

$$(\boldsymbol{c}' - \frac{x}{q^2-1}\boldsymbol{j}')\boldsymbol{A}' = (q^2-1)(\boldsymbol{c}' - \frac{x}{q^2-1}\boldsymbol{j}').$$

Proof. For a vector $\boldsymbol{v}$, denote by $\boldsymbol{v}'$ the vector obtained from $\boldsymbol{v}$ by deleting entries corresponding to points in $\pi_1 \cup \pi_2$. Then $(\boldsymbol{c}\boldsymbol{A})' = \boldsymbol{c}'\boldsymbol{A}'$ since no point of $\mathcal{T}$ lies in $\pi_1 \cup \pi_2$, and $(\boldsymbol{j}\boldsymbol{A})' = \boldsymbol{j}'\boldsymbol{A}' + 2(q+1)\boldsymbol{j}'$. The result then follows from Theorem 2.2. □

Of course $\mathcal{Q}^+(5,q)$ is very large, having $(q^2+1)(q^2+q+1)$ points. The eigenspace of the collinearity matrix associated with the eigenvector (q^2-1) has dimension $q(q^2+q+1)$, and finding vectors in this space corresponding to tight sets amounts to solving a difficult integer programming problem. Thus our work is made easier by considering various decompositions of the incidence structure, as well as decompositions of the incidence matrix.

Definition 2.4. Given an incidence structure $\mathcal{S}$, a **tactical decomposition of $\mathcal{S}$** is a partition of the points of $\mathcal{S}$ into point classes and the blocks of $\mathcal{S}$ into block classes such that the number of points in a point class which are incident with a given block depends only on the class in which the block lies, and the number of blocks in a block class which are incident with a given point depends only on the class in which the point lies.

Definition 2.5. Let $\boldsymbol{A} = [a_{ij}]$ be a matrix, along with a partition of the row indices into subsets $R_1, \ldots, R_t$, and a partition of the column indices into subsets $C_1, \ldots, C_{t'}$. We will call this a **tactical decomposition of $\boldsymbol{A}$** if, for every (i,j), $1 \le i \le t$, $1 \le j \le t'$, the submatrix $[a_{\boldsymbol{p},\ell}]_{\boldsymbol{p}\in R_i,\ \ell \in C_j}$ has constant column sums c_{ij} and row sums r_{ij}.

The most common examples of tactical decompositions of an incidence structure are obtained by taking as point and block classes the orbits of some collineation group (although not all examples arise this way). A tactical decomposition of an incidence structure corresponds to a tactical decomposition of its incidence matrix.

The following result comes from the theory of the interlacing of eigenvalues, which was introduced by Higman and Sims and further developed by Haemers; see [13] for a detailed survey.

Theorem 2.6. *Suppose $\boldsymbol{A}$ can be partitioned as*

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_{11} & \cdots & \boldsymbol{A}_{1k} \\ \vdots & \ddots & \vdots \\ \boldsymbol{A}_{k1} & \cdots & \boldsymbol{A}_{kk} \end{bmatrix}$$

with each $\boldsymbol{A}_{ii}$ square, $1 \le i \le k$, and each $\boldsymbol{A}_{ij}$ having constant column sum c_{ij}. Then any eigenvalue of the matrix $\boldsymbol{B} = [c_{ij}]$ is also an eigenvalue of $\boldsymbol{A}$.

Proof. We use an eigenvector of $\boldsymbol{B}$ to construct and eigenvector of $\boldsymbol{A}$ by expanding the vector. $\square$

3 Characters and Gauss sums

For some basics on characters of a finite field, see [15]; for an in depth look at Gauss sums, see [3]. We will write i for $\sqrt{-1} \in \mathbb{C}$.

Definition 3.1. Let $\mathbb{F}$ be a finite field. A **character** χ of $\mathbb{F}$ is a group homomorphism from $\mathbb{F}^*$ to $\mathbb{C}^*$. We call χ **trivial** if $\chi(x) = 1$ for all $x \in \mathbb{F}^*$.

The characters of a finite field $\mathbb{F}$ form a group $\hat{\mathbb{F}}$ under multiplication, with $\hat{\mathbb{F}} \simeq \mathbb{F}^*$. The **order** of a character χ is the smallest integer d for which $\chi(x)^d = 1$ for all $x \in \mathbb{F}^*$. For any character χ of $\mathbb{F}$, we will write $\overline{\chi}$ for the conjugate character. We will always extend characters to the entire field $\mathbb{F}$ by defining $\chi(0) = 0$, except for the trivial character χ_1 for which we define $\chi_1(0) = 1$.

Definition 3.2. Consider a finite field $\mathbb{F}_{p^r}$ and let χ be a character of $\mathbb{F}_{p^r}$. Let $\mathbb{T}_r : \mathbb{F}_{p^r} \to \mathbb{F}_p$ be the absolute trace function and let $\zeta = \exp(2\pi\mathrm{i}/p)$. The **Gauss sum** of χ is defined to be

$$G_r(\chi) = \sum_{x \in \mathbb{F}_{p^r}} \chi(x) \zeta^{\mathbb{T}_r(x)}.$$

We will need the following result, found in [3, Theorem 1.1.4 (a)]:

Theorem 3.3. *Let χ be a nontrivial character of $\mathbb{F}_{p^r}$. Then*

$$G_r(\chi) G_r(\overline{\chi}) = \chi(-1) p^r.$$

Definition 3.4. Let p be odd, and fix a primitive element α of $\mathbb{F}_{p^r}$. We define the **quadratic character** χ_2 of $\mathbb{F}_{p^r}$ by $\chi_2(0) = 0$, $\chi_2(\alpha^n) = (-1)^n$. We have that χ_2 is the unique character of $\mathbb{F}_{p^r}$ having order 2.

We apply the following result from [3] to evaluate Gauss sums of the quadratic character:

Theorem 3.5. *Let p be an odd prime, χ_2 be the quadratic character on $\mathbb{F}_{p^r}$. Then*

$$G_r(\chi_2) = \begin{cases} (-1)^{r-1}\sqrt{q} & \text{if } p \equiv 1 \bmod 4 \\ (-1)^{r-1}\mathrm{i}^r\sqrt{q} & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

Theorem 3.6 (**Davenport-Hasse product relation**). *Let ψ be a character of $\mathbb{F}_{p^r}$ of order $d > 1$. If χ is a nontrivial character of $\mathbb{F}_{p^r}$ such that $\chi\psi^i$ is nontrivial for all $1 \leq i < d$ then we have*

$$\frac{\chi^d(d) G_r(\chi)}{G_r(\chi^d)} \prod_{i=1}^{d-1} \frac{G_r(\chi\psi^i)}{G_r(\psi^i)} = 1.$$

Lemma 3.7. *Let $q = p^h$, and χ be a character of $\mathbb{F}_{q^3}$ such that the restriction of χ to $\mathbb{F}_q$ is nontrivial. Let T denote the relative trace function $\mathbb{F}_{q^3} \to \mathbb{F}_q$. Then*

$$\sum_{x \in \mathbb{F}_{q^3}} \overline{\chi}(\mathrm{T}(x))\chi(x) = (q-1)\frac{G_{3h}(\chi)}{G_h(\chi)}. \tag{3.1}$$

Proof. We begin by multiplying (3.1) by $G_h(\chi)$. Note that we can omit the terms where $\mathrm{T}(x) = 0$, so

$$\sum_{\lambda \in \mathbb{F}_q} \chi(\lambda)\zeta^{\mathbb{T}_h(\lambda)} \sum_{x \in \mathbb{F}_{q^3}} \overline{\chi}(\mathrm{T}(x))\chi(x) = \sum_{\lambda \in \mathbb{F}_q^*} \sum_{x \in \mathbb{F}_{q^3}, \mathrm{T}(x) \neq 0} \overline{\chi}(\mathrm{T}(x))\chi(\lambda x)\zeta^{\mathbb{T}_h(x)}.$$

We replace λ with $\mathrm{T}(x)\mu$:

$$\begin{aligned} &= \sum_{\mu \in \mathbb{F}_q^*} \sum_{x \in \mathbb{F}_{q^3}, \mathrm{T}(x) \neq 0} \overline{\chi}(\mathrm{T}(x))\chi(\mathrm{T}(x)\mu x)\zeta^{\mathbb{T}_h(\mathrm{T}(x)\mu)} \\ &= \sum_{\mu \in \mathbb{F}_q^*} \sum_{x \in \mathbb{F}_{q^3}, \mathrm{T}(x) \neq 0} \chi(\mu x)\zeta^{\mathbb{T}_h(\mathrm{T}(\mu x))}. \end{aligned}$$

Now we replace x by $\mu^{-1}y$ so that μ disappears from the formula:

$$= (q-1) \sum_{y \in \mathbb{F}_{q^3}, \mathrm{T}(y) \neq 0} \chi(y)\zeta^{\mathbb{T}_{3h}(y)}.$$

Except for the condition that $\mathrm{T}(y) \neq 0$, this is equal to $(q-1)G_{3h}(\chi)$. However,

$$\sum_{y \in \mathbb{F}_{q^3}, \mathrm{T}(y) = 0} \chi(y)\zeta^{\mathbb{T}_{3h}(y)} = \sum_{y \in \mathbb{F}_{q^3}, \mathrm{T}(y) = 0} \chi(y).$$

Let ω be a primitive element of $\mathbb{F}_q$, we have $\chi(\omega) \neq 1$ by assumption. By replacing y with ωy, we get

$$\sum_{y \in \mathbb{F}_{q^3}, \mathrm{T}(y) = 0} \chi(y) = \sum_{y \in \mathbb{F}_{q^3}, \mathrm{T}(y) = 0} \chi(\omega y) = \chi(\omega) \sum_{y \in \mathbb{F}_{q^3}, \mathrm{T}(y) = 0} \chi(y),$$

which implies that the sum of the left hand side must be 0 and that the condition $\mathrm{T}(y) \neq 0$ does not make a difference. $\square$

Definition 3.8. Let $f : \mathbb{F}^* \to \mathbb{C}$, and χ be any character of $\mathbb{F}$. Then the **Fourier transform** of f is defined to be

$$\hat{f}(\chi) = \sum_{x \in \mathbb{F}^*} f(x)\overline{\chi}(x).$$

The Fourier transform is a bijective map from $\mathbb{C}^{\mathbb{F}^*} \to \mathbb{C}^{\hat{\mathbb{F}}}$; that is, if $f : \mathbb{F}^* \to \mathbb{C}$, then $\hat{f} : \hat{\mathbb{F}} \to \mathbb{C}$.

4 Algebraic results

Let q be a prime power with $q \not\equiv 1 \bmod 3$ and $q \equiv 1 \bmod 4$. We will consider the fields $\mathbb{F} = \mathbb{F}_q$ with $\mathbb{F}^* = \langle \omega \rangle$, and $\mathbb{E} = \mathbb{F}_{q^3}$ with $\mathbb{E}^* = \langle \alpha \rangle$, and assume that $\omega = \alpha^{q^2+q+1}$.

4.1 Preliminaries

Define the following functions from $\mathbb{E}$ to $\mathbb{F}$:

$$\begin{aligned} \mathrm{T}(x) =& x^{q^2} + x^q + x, \\ \mathrm{N}(x) =& x^{q^2+q+1}. \end{aligned}$$

Through some simple computations, the following two properties can be verified:

Lemma 4.1. *Let $x, y \in \mathbb{E}$. Then*

$$(y + x)(y + x^q)(y + x^{q^2}) = y^3 + y^2 \, \mathrm{T}(x) + y \, \mathrm{T}(x^{q+1}) + \mathrm{N}(x).$$

In the special case that $y \in \mathbb{F}$, the left hand side is equal to $\mathrm{N}(y + x)$.

Lemma 4.2. *For any $x \in \mathbb{E}$, we have*

$$\mathrm{T}(x^2) = \mathrm{T}(x)^2 - 2\, \mathrm{T}(x^{q+1}).$$

Lemma 4.3. *Let $x \in \mathbb{E}$ such that $\mathrm{T}(x) = \mathrm{T}(x^{q+1}) = 0$.*

(i) *If $q \equiv 2 \bmod 3$, then $x = 0$.*

(ii) *If $q = 3^h$, then $x \in \mathbb{F}$.*

Proof. Applying Lemma 4.1 with $y = -x$, we find $\mathrm{N}(x) = x^3$. Therefore $x^3 \in \mathbb{F}$. Since $q \not\equiv 1 \bmod 3$, this implies that $x \in \mathbb{F}$. It follows that $0 = \mathrm{T}(x) = 3x$. $\square$

Corollary 4.4. *Let $x \in \mathbb{E}$ such that $\mathrm{T}(x) = \mathrm{T}(x^2) = 0$.*

(i) *If $q \equiv 5 \bmod 6$, then $x = 0$.*

(ii) *If $q = 3^h$, then $x \in \mathbb{F}$.*

Proof. This follows from Lemma 4.2 and Lemma 4.3. $\square$

4.2 Cyclic model of the projective plane

Let $\mu = \alpha^{q-1}$, so $|\mu| = q^2 + q + 1$. It is then clear that $\mathrm{N}(\mu) = \mu^{q^2+q+1} = 1$.

Lemma 4.5. *The elements of $\langle \mu \rangle \subset \mathbb{E}^*$ are pairwise linearly independent over $\mathbb{F}$.*

Proof. Assume otherwise, so that we have $\mu^i = \lambda\mu^j$, with $i \neq j \pmod{q^2+q+1}$, for some $\lambda \in \mathbb{F}$. Then

$$1 = \mathrm{N}(\mu^i) = \mathrm{N}(\lambda)\,\mathrm{N}(\mu^j) = \lambda^3.$$

Since $q \not\equiv 1 \bmod 3$, this implies that $\lambda = 1$ and so $\mu^i = \mu^j$, a contradiction. $\square$

We can consider the elements of $\langle\mu\rangle$ to represent the points of a projective plane π, with underlying vector space $\mathbb{E}$ over $\mathbb{F}$. The lines of π correspond to solutions to the equation $\mathrm{T}(\lambda x) = 0$ for a fixed $\lambda \in \mathbb{E}^*$, which is an $\mathbb{F}$-linear map from $\mathbb{E} \to \mathbb{F}$. The map $x \mapsto \mu x$ gives a collineation of π which acts sharply transitively on the points, as well as on the lines of π.

Lemma 4.6. *The map $x \mapsto x^2$ induces a permutation on the points of π; the preimage of a line under this map is a conic.*

Proof. To see that this map is a permutation on the points of π, assume that μ^i and μ^j get mapped to the same point, so $\mu^{2i} = \mu^{2j}$. This means that $2i \equiv 2j \bmod q^2+q+1$, which is odd. This can only happen if $i \equiv j \bmod q^2+q+1$, in which case $\mu^i = \mu^j$.

Consider a line ℓ of π described by the equation $\mathrm{T}(\lambda x) = 0$ for some $\lambda \in \mathbb{E}^*$. The map $y \mapsto \mathrm{T}(\lambda y^2)$ defines a quadratic form; there are precisely $q+1$ isotropic points (the preimages of the points in ℓ under the map $x \mapsto x^2$) and so the isotropic points form a nondegenerate conic. $\square$

Lemma 4.7. *The bilinear form on $\mathbb{E}$ given by $(x, y) \mapsto \mathrm{T}(xy)$ is nondegenerate, and has square discriminant.*

Proof. There are q^2 elements of $\mathbb{E}$ with trace 0, so there exists an element of $\mathbb{E}^* \setminus \mathbb{F}^*$ having trace 0; after multiplying by a nonsquare in $\mathbb{F}^*$ if necessary, we can assume that this element is a square. So we have $v \in \mathbb{E}^* \setminus \mathbb{F}^*$ with $\mathrm{T}(v^2) = 0$. By Corollary 4.4, we know that $\mathrm{T}(v) \neq 0$. Let $a = \lambda v$, where λ is chosen so that $\mathrm{T}(a) = 2$. We have that $\mathrm{T}(a^2) = 0$, and it follows from Lemma 4.2 that $\mathrm{T}(a^{q+1}) = 2$. Let $b = a^q$ and $c = 2 - a - a^q$. Then

$$\mathrm{T}(c^2) = 4\,\mathrm{T}(1) - 4\,\mathrm{T}(a) + 2\,\mathrm{T}(a^{q+1}) = 12 - 8 - 8 + 4 = 0.$$

We can also compute

$$\mathrm{T}(ac) = \mathrm{T}(bc) = 2\,\mathrm{T}(a) - \mathrm{T}(a^{q+1}) = 2.$$

The Gram matrix for our bilinear form is given by

$$\begin{bmatrix} \mathrm{T}(a^2) & \mathrm{T}(ab) & \mathrm{T}(ac) \\ \mathrm{T}(ab) & \mathrm{T}(b^2) & \mathrm{T}(bc) \\ \mathrm{T}(ac) & \mathrm{T}(bc) & \mathrm{T}(c^2) \end{bmatrix} = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix},$$

which has determinant 16. $\square$

Corollary 4.8. *Let $a, b, c \in \mathbb{E}^*$ be distinct, with $\mathrm{N}(a) = \mathrm{N}(b) = \mathrm{N}(c) = 1$ and $\mathrm{T}(a^2) = \mathrm{T}(b^2) = \mathrm{T}(c^2) = 0$. Then $2\,\mathrm{T}(ab)\,\mathrm{T}(ac)\,\mathrm{T}(bc)$ is always a nonzero square.*

Proof. We have that a^2, b^2 and c^2 are collinear, so by Lemma 4.6, a, b, and c are noncollinear; this implies that $\{a, b, c\}$ forms a basis for $\mathbb{E}$ over $\mathbb{F}$. Therefore we have

$$\begin{vmatrix} \mathrm{T}(a^2) & \mathrm{T}(ab) & \mathrm{T}(ac) \\ \mathrm{T}(ab) & \mathrm{T}(b^2) & \mathrm{T}(bc) \\ \mathrm{T}(ac) & \mathrm{T}(bc) & \mathrm{T}(c^2) \end{vmatrix} = 2\,\mathrm{T}(ab)\,\mathrm{T}(ac)\,\mathrm{T}(bc)$$

is a square, and nonzero since the form is nondegenerate. □

4.3 Results on $\kappa(x)$

Let χ_4 be the **quartic character** of $\mathbb{E}$; this is defined by $\chi_4(0) = 0$, $\chi_4(\alpha^n) = \mathrm{i}^n$ (and is dependent on our choice of a primitive element α). Notice that $\chi_4(x)^2 = \chi_2(x)$, where χ_2 is the quadratic character of $\mathbb{E}$. We will assume that $\omega = \alpha^{-(q^2+q+1)}$, so that $\chi_4(\omega) = \mathrm{i}^{-(q^2+q+1)} = \mathrm{i}$.

Definition 4.9. For $z \in \{1, \mathrm{i}, -1, -\mathrm{i}\}$, we define functions $\kappa_z : \mathbb{E}^* \to \mathbb{Z}$ by

$$\kappa_z(x) = |\{i : 0 \le i < q^2 + q + 1 \mid \chi_4(\mathrm{T}(\mu^i x))\overline{\chi_4}(\mathrm{T}(\mu^{-i} x)) = z\}|.$$

Lemma 4.10. *For any $\lambda \in \mathbb{F}^*$, $z \in \{1, \mathrm{i}, -1, -\mathrm{i}\}$, we have $\kappa_z(\lambda x) = \kappa_z(x)$.*

Proof. Since $\chi_4(\lambda)\overline{\chi}_4(\lambda) = 1$ we have that, for $0 \le i < q^2 + q + 1$,

$$\begin{aligned} \chi_4(\mathrm{T}(\mu^i \lambda x))\overline{\chi_4}(\mathrm{T}(\mu^{-i} \lambda x)) &= \chi_4(\lambda)\chi_4(\mathrm{T}(\mu^i x))\overline{\chi}_4(\lambda)\overline{\chi_4}(\mathrm{T}(\mu^{-i} x)) \\ &= \chi_4(\mathrm{T}(\mu^i x))\overline{\chi_4}(\mathrm{T}(\mu^{-i} x)). \end{aligned}$$

□

Lemma 4.11. *For every $x \in \mathbb{E}^*$, $\kappa_i(x) = \kappa_{-i}(x)$.*

Proof. Putting $j' = q^2 + q + 1 - j$, we have

$$\chi_4(\mathrm{T}(\mu^j x))\overline{\chi_4}(\mathrm{T}(\mu^{-j} x)) = \chi_4(\mathrm{T}(\mu^{-j'} x))\overline{\chi_4}(\mathrm{T}(\mu^{j'} x)),$$

so the values of j for which

$$\chi_4(\mathrm{T}(\mu^j x))\overline{\chi_4}(\mathrm{T}(\mu^{-j} x)) = i$$

are in one-to-one correspondence with the values of j' for which

$$\chi_4(\mathrm{T}(\mu^{j'} x))\overline{\chi_4}(\mathrm{T}(\mu^{-j'} x)) = -i.$$

□

Theorem 4.12. *For every $x \in \mathbb{E}^*$, we have*

$$\kappa_1(x) - \kappa_{-1}(x) = q \cdot \chi_2(x)\chi_2(\mathrm{T}(x)).$$

Proof. Define

$$\begin{aligned}A(x) &= (q-1)\chi_2(x)(\kappa_0(x)-\kappa_2(x)),\\ B(x) &= q(q-1)\chi_2(\mathrm{T}(x)).\end{aligned}$$

We must show that $A(x) = B(x)$ for all $x \in \mathbb{E}^*$.

We rewrite the expression for $A(x)$ using χ_4:

$$\begin{aligned}A(x) &= (q-1)\chi_2(x)(\kappa_0(x)+\mathrm{i}\kappa_1(x)-\kappa_2(x)-\mathrm{i}\kappa_3(x))\\ &= (q-1)\sum_{\mathrm{N}(a)=1}\chi_4(\mathrm{T}(ax))\overline{\chi_4}(\mathrm{T}(a^{-1}x))\chi_2(x).\end{aligned}$$

We write $q-1$ as $\sum_{\lambda\in\mathbb{F}^*} 1$ and use the fact that elements of norm 1 are always squares:

$$\begin{aligned}A(x) &= \sum_{\lambda\in\mathbb{F}^*}\sum_{\mathrm{N}(a)=1}\chi_4(\mathrm{T}(ax))\overline{\chi_4}(\mathrm{T}(a^{-1}x))\chi_2(x)\\ &= \sum_{\lambda\in\mathbb{F}^*}\sum_{\mathrm{N}(a)=1}\chi_4(\lambda\,\mathrm{T}(ax))\overline{\chi_4}(\lambda^{-1}\,\mathrm{T}(a^{-1}x))\chi_4(\lambda^{-1})\overline{\chi_4}(\lambda)\chi_2(a^{-1}x)\\ &= \sum_{\lambda\in\mathbb{F}^*}\sum_{\mathrm{N}(a)=1}\chi_4(\mathrm{T}(\lambda ax))\overline{\chi_4}(\mathrm{T}(\lambda^{-1}a^{-1}x))\chi_2(\lambda^{-1}a^{-1}x).\end{aligned}$$

If λ runs over all elements of $\mathbb{F}^*$ and a over all elements of $\mathbb{E}^*$ of norm 1, then λa runs over all elements of $\mathbb{E}^*$:

$$A(x) = \sum_a \chi_4(\mathrm{T}(ax))\overline{\chi_4}(\mathrm{T}(a^{-1}x))\chi_2(a^{-1}x).$$

We now substitute $a = b^{-1}x$:

$$A(x) = \sum_b \chi_4(\mathrm{T}(b^{-1}x^2))\overline{\chi_4}(\mathrm{T}(b))\chi_2(b).$$

Let χ be any character of $\mathbb{E}$. Then the Fourier transforms of A and B are defined to be

$$\begin{aligned}\hat{A}(\chi) &= \sum_x A(x)\overline{\chi}(x)\\ &= \sum_x\sum_b \chi_4(\mathrm{T}(b^{-1}x^2))\overline{\chi_4}(\mathrm{T}(b))\chi_2(b)\overline{\chi}(x),\\ \hat{B}(\chi) &= \sum_x B(x)\overline{\chi}(x)\\ &= q(q-1)\sum_x \chi_2(\mathrm{T}(x))\overline{\chi}(x).\end{aligned}$$

Since the Fourier transform is a 1-to-1 operation, we are reduced to showing that $\hat{A}(\chi) = \hat{B}(\chi)$ for all characters χ of $\mathbb{E}$.

Let $\omega = \alpha^{q^2+q+1}$ be a generator of $\mathbb{F}^*$. If we replace x by ωx in the formula for $\hat{A}(\chi)$ and we use $A(\omega x) = \chi_2(\omega)A(x)$, we get

$$\begin{aligned}\hat{A}(\chi) &= \sum_x A(\omega x)\overline{\chi}(\omega x) \\ &= \sum_x \chi_2(\omega)\overline{\chi}(\omega)A(x)\overline{\chi}(x) = \chi_2(\omega)\overline{\chi}(\omega)\hat{A}(\chi).\end{aligned}$$

If $\chi_2(\omega)\overline{\chi}(\omega) \neq 1$, this implies that $\hat{A}(\chi) = 0$. Analogously, also $\hat{B}(\chi) = 0$ in this case. We can therefore restrict ourselves to the case where $\chi_2(\omega)\overline{\chi}(\omega) = 1$. Since $\chi_2(\omega) = -1$, this means $\overline{\chi}(\omega) = \overline{\chi}(\alpha)^{q^2+q+1} = -1$. It follows that $\overline{\chi}^{q^2+q+1} = \chi_2$.

Since the restriction of $\overline{\chi}$ to $\mathbb{F}$ equals χ_2, we can evaluate $\hat{B}(\chi)$ using Lemma 3.7:

$$\hat{B}(\chi) = q(q-1)^2 \frac{G_{3h}(\overline{\chi})}{G_h(\chi_2)} \tag{4.1}$$

The order of $\overline{\chi}$ is a divisor of $2(q^2+q+1)$. Since the character group is cyclic of order q^3-1, which is a multiple of $4(q^2+q+1)$, it follows that $\overline{\chi} = \sigma^2$ for some character σ. We can choose σ such that $\sigma^{q^2+q+1} = \chi_4$. Since $3(q^2+q+1) \equiv 1 \bmod 4$, the restriction of σ to $\mathbb{F}$ is $\chi_4^3 = \overline{\chi}_4$. (**remark:** This makes $\chi = \overline{\sigma}^2$.)

For $\hat{A}(\chi)$, we get

$$\begin{aligned}\hat{A}(\chi) = \hat{A}(\overline{\sigma}^2) &= \sum_x \sum_b \chi_4(\mathrm{T}(b^{-1}x^2))\overline{\chi_4}(\mathrm{T}(b))\chi_2(b)\sigma(x^2) \\ &= 2 \sum_{s \text{ a square}} \sum_b \chi_4(\mathrm{T}(b^{-1}s))\overline{\chi_4}(\mathrm{T}(b))\chi_2(b)\sigma(s).\end{aligned}$$

If we replace s by ωs in the above sum, the terms remain all invariant. This operation exchanges squares and non-squares. Therefore, $\hat{A}(\sigma^2)$ is equal to

$$\hat{A}(\overline{\sigma}^2) = \sum_s \sum_b \chi_4(\mathrm{T}(b^{-1}s))\overline{\chi_4}(\mathrm{T}(b))\chi_2(b)\sigma(s).$$

After replacing s by by, we get

$$\begin{aligned}\hat{A}(\overline{\sigma}^2) &= \sum_y \sum_b \chi_4(\mathrm{T}(y))\overline{\chi}_4(\mathrm{T}(b))\chi_2(b)\sigma(b)\sigma(y) \\ &= \left(\sum_y \chi_4(\mathrm{T}(y))\sigma(y)\right)\left(\sum_b \overline{\chi}_4(\mathrm{T}(b))\chi_2(b)\sigma(b)\right).\end{aligned}$$

Let $\tau = \sigma \cdot \chi_2$, then the restriction of τ to $\mathbb{F}$ is $\overline{\chi}_4\chi_2 = \chi_4$.

$$\hat{A}(\overline{\sigma}^2) = \left(\sum_y \overline{\sigma}(\mathrm{T}(y))\sigma(y)\right)\left(\sum_b \overline{\tau}(\mathrm{T}(b))\tau(b)\right).$$

We evaluate this using Lemma 3.7:

$$\hat{A}(\overline{\sigma}^2) = (q-1)^2 \frac{G_{3h}(\sigma)G_{3h}(\tau)}{G_h(\sigma)G_h(\tau)} = (q-1)^2 \frac{G_{3h}(\sigma)G_{3h}(\tau)}{G_h(\overline{\chi_4})G_h(\chi_4)}.$$

Applying Theorem 3.3 gives

$$\hat{A}(\overline{\sigma}^2) = q^{-1}(q-1)^2\chi_4(-1)G_{3h}(\sigma)G_{3h}(\tau).$$

Finally, we apply the Davenport–Hasse product formula (Theorem 3.6) which states that

$$\frac{\sigma^2(2)G_{3h}(\sigma)G_{3h}(\sigma\chi_2)}{G_{3h}(\sigma^2)G_{3h}(\chi_2)} = 1 \tag{4.2}$$

Replacing $\overline{\sigma}^2$ back by χ, we get

$$\begin{aligned}\hat{A}(\chi) &= q^{-1}(q-1)^2\chi_4(-1)\overline{\chi}(2)G_{3h}(\chi)G_{3h}(\chi_2)\\ &= q^{-1}(q-1)^2\chi_4(-1)\chi_2(2)G_{3h}(\chi_2)G_{3h}(\chi).\end{aligned}$$

Since $(1+\mathrm{i})^4 = -4$ and $\mathbb{F}$ contains a square root of -1, it follows that $\chi_4(-1)\chi_2(2) = \chi_4(-4) = 1$:

$$\hat{A}(\chi) = q^{-1}(q-1)^2G_{3h}(\chi_2)G_{3h}(\chi).$$

Dividing this by (4.1) gives:

$$\frac{\hat{A}(\chi)}{\hat{B}(\chi)} = \frac{1}{q^2}G_{3h}(\chi_2)G_h(\chi_2).$$

The explicit formulas for quadratic Gauss sums given in Theorem 3.5 show that this always equals 1 if $q \equiv 1 \bmod 4$, hence $\hat{A}(\chi) = \hat{B}(\chi)$. □

5 Our setup

5.1 A model for $\mathcal{Q}^+(5,q)$

As in Section 4, we have $q = p^h$ with $q \equiv 1 \bmod 4$ and $q \not\equiv 1 \bmod 3$ (so $q \equiv 5$ or $9 \bmod 12$). We again define $\mathbb{F} = \mathbb{F}_q$ and $\mathbb{E} = \mathbb{F}_{q^3}$ with primitive elements ω and α, respectively (where ω is chosen to be α^{q^2+q+1}), and the relative trace and norm functions from $\mathbb{E} \to \mathbb{F}$ by

$$\begin{aligned}\mathrm{T}(x) &= x^{q^2} + x^q + x,\\ \mathrm{N}(x) &= x^{q^2+q+1}.\end{aligned}$$

Let $\mathrm{PG}(5,q)$ have the underlying $\mathbb{F}$-vector space $V = \mathbb{E}^2$, and consider the quadratic form Q on V given by

$$\mathrm{Q}((u,v)) = \mathrm{T}(uv).$$

The polar form B of Q is then given by

$$\mathrm{B}((u_1,v_1),(u_2,v_2)) = \mathrm{T}(u_1v_2) + \mathrm{T}(v_1u_2).$$

Lemma 5.1. *The form* Q *defined above is nondegenerate, and the associated quadric is a* $\mathcal{Q}^+(5, q)$*.*

Proof. If there exists $(u, v) \in V$ with $\mathrm{B}((u, v), (x, y)) = 0$ for all $(x, y) \in V$, then $\mathrm{T}(uy) + \mathrm{T}(vx) = 0$ for all $(x, y) \in V$. Setting $x = 0$ forces us to have $\mathrm{T}(uy) = 0$ for all $y \in \mathbb{E}^*$, thus $u = 0$. Likewise setting $y = 0$ can be seen to force $v = 0$, and so $(u, v) = (0, 0)$. Thus Q is nondegenerate. We can see that $\{(x, 0) \,:\, x \in \mathbb{E}\}$ is a totally singular subspace of V with dimension 3, so (V, Q) has maximal Witt dimension. Therefore we have that Q is hyperbolic, with the associated quadric being a $\mathcal{Q}^+(5, q)$. □

5.2 A group

As in Section 4.2, we put $\mu = \alpha^{q-1}$, so $|\mu| = q^2 + q + 1$. Recall that $\mathrm{N}(\mu) = 1$, and that the elements of $\langle \mu \rangle$ are pairwise $\mathbb{F}$-linearly independent over $\mathbb{E}$.

Lemma 5.2. *The map* $c : (u, v) \mapsto (\mu u, \mu^{-1} v)$ *is a projective isometry of* $\mathcal{Q}^+(5, q)$*, and* $\langle c \rangle$ *acts semi-regularly on the points of the space. There are* $q^2 + 1$ *orbits of* $\langle c \rangle$ *on* $\mathcal{Q}^+(5, q)$*; these include the generators* π_1 *and* π_2 *given by*

$$\pi_1 = \{(x, 0) : \; x \in \mathbb{E}^*\} \text{ and}$$
$$\pi_2 = \{(0, y) : \; y \in \mathbb{E}^*\}.$$

A collection of representatives for the remaining $(q^2 - 1)$ *orbits is given by*

$$\{(1, \lambda a^2) : \; a \in \mathcal{S}, \lambda \in \mathbb{F}^*\},$$

where

$$\mathcal{S} = \{a : \; a \in \mathbb{E}^* \mid \mathrm{N}(a) = 1, \mathrm{T}(a^2) = 0\}.$$

Proof. It is easy to see that c is an isometry. Now, if c^i fixes some point $(u, v) \in \mathcal{Q}^+(5, q)$, then $(\mu^i u, \mu^{-i} v) = (\lambda u, \lambda v)$ for some $\lambda \in \mathbb{F}^*$. But by Lemma 4.5, we cannot have $\mu^i \in \mathbb{F}^*$ unless $i = 0$. So $\langle c \rangle$ acts semi-regularly on the points of $\mathcal{Q}^+(5, q)$ and so must have $q^2 + 1$ orbits since each orbit has size $q^2 + q + 1$ and there are $(q^2 + 1)(q^2 + q + 1)$ points in total in the space.

Now π_1 and π_2 are each clearly point orbits of $\langle c \rangle$. Any point $\boldsymbol{p} \in \mathcal{Q}^+(5, q) \setminus (\pi_1 \cup \pi_2)$ has the form (u, v), with $u, v \in \mathbb{E}^*$ satisfying $\mathrm{T}(uv) = 0$, and we can multiply by a scalar in $\mathbb{F}$ to assume that $\mathrm{N}(u) = 1$; this implies that $u = \mu^i$ for some $0 \le i < q^2 + q + 1$. We can also write $v = \lambda \mu^j$ for some $\lambda \in \mathbb{F}^*$ and $0 \le j < q^2 + q + 1$. Then (u, v) maps under c^{-i} to $(1, \lambda \mu^{i+j})$, with $\mathrm{T}(\mu^{i+j}) = 0$. Since $\langle \mu \rangle$ is a cyclic group with odd order, there is an $a \in \langle \mu \rangle$ with $a^2 = \mu^{i+j}$, therefore (u, v) is in the same orbit under $\langle c \rangle$ as $(1, \lambda a^2)$, with $\mathrm{N}(a) = 1$ and $\mathrm{T}(a^2) = 0$.

To see that, for $a, a' \in \mathcal{S}$, $(1, \lambda a^2)$ and $(1, \lambda' a'^2)$ are not in the same orbit under $\langle c \rangle$ unless $\lambda = \lambda'$ and $a = a'$, we recall that $\mu^i \in \mathbb{F}^*$ implies that $i = 0$. Therefore we would have to have $\lambda a^2 = \lambda' a'^2$. But since $\mathrm{N}(a) = \mathrm{N}(a') = 1$, we must have $\mathrm{N}(\lambda) = \mathrm{N}(\lambda')$ or equivalently, $\lambda^3 = \lambda'^3$. Since $q \not\equiv 1 \bmod 3$, this implies that $\lambda = \lambda'$ and $a^2 = a'^2$, thus $a = \pm a'$. But since $\mathrm{N}(a) = \mathrm{N}(a') = 1$, we must have $a = a'$. □

Lemma 5.3. *The map $z : (u, v) \mapsto (u, \omega^4 v)$ is a projective similarity of $\mathcal{Q}^+(5, q)$ which centralizes c. Define the group $G = \langle c, z\rangle$. Then G has $2 + 4(q + 1)$ point orbits on $\mathcal{Q}^+(5, q)$, including the planes π_1 and π_2. The remaining $4(q + 1)$ orbits each have size $\frac{(q-1)}{4}(q^2 + q + 1)$, and are represented by the points*

$$\{(1, \omega^s a^2) : \ 0 \le s < 4, \ a \in \mathcal{S}\},$$

where $\mathcal{S}$ is as defined in Lemma 5.2.

Proof. It is clear that z is a projective similarity of order $(q - 1)/4$, and that z commutes with c, therefore z permutes the orbits of $\langle c\rangle$. Furthermore, z stabilizes the planes π_1 and π_2 pointwise.

Now consider two orbits $\mathcal{O}_1$, $\mathcal{O}_2$ of $\langle c\rangle$ on $\mathcal{Q}^+(5, q) \setminus (\pi_1 \cup \pi_2)$ represented by points $(1, \lambda a^2)$ and $(1, \lambda' a'^2)$, respectively, with $a, a' \in \mathcal{S}$ and $\lambda, \lambda' \in \mathbb{F}^*$. If $\mathcal{O}_1 \mapsto \mathcal{O}_2$ under z^k for some $0 \le k < (q - 1)/4$, then $(1, \omega^{4k}\lambda a^2) \in \mathcal{O}_2$. This can only happen if $\omega^{4k}\lambda a^2 = \lambda' a'^2$. Since $\mathrm{N}(a^2) = \mathrm{N}(a'^2) = 1$, we must have $\mathrm{N}(\omega^{4k}\lambda) = \mathrm{N}(\lambda')$; but $x \mapsto \mathrm{N}(x)$ is a permutation of $\mathbb{F}$ since $q \not\equiv 1 \bmod 3$, so $\lambda' = \omega^{4k}\lambda$ for some $0 \le k < (q - 1)/4$ and $a' = a$. Therefore the orbit under G containing the point $(1, \lambda a^2)$, with $a \in \mathcal{S}$, can be described by writing $\lambda = \omega^{4k+s}$; then the point orbit in question has unique representative $(1, \omega^s a^2)$. The number of orbits follows immediately. □

5.3 A tactical decomposition

As we are interested in finding eigenvectors of the collineation matrix $\boldsymbol{A}$ of $\mathcal{Q}^+(5, q)$, we will define a useful tactical decomposition of this matrix. If we think of $\boldsymbol{A}$ as the incidence matrix of the structure whose "points" and "blocks" are both the set of points of $\mathcal{Q}^+(5, q)$ with incidence being given by collinearity in the quadric, then it is clear that any collineation group of $\mathcal{Q}^+(5, q)$ will induce a tactical decomposition of $\boldsymbol{A}$.

Let

$$\{a_1, \ \ldots, a_{q+1}\} = \{a : \ a \in \mathbb{E}^* \mid \mathrm{N}(a) = 1, \ \mathrm{T}(a^2) = 0\}. \tag{5.1}$$

Remark 5.4. It can be seen from Lemma 5.2 that the size of the set on the right hand side of 5.1 is in fact $q + 1$.

We wish to consider the tactical decomposition of $\boldsymbol{A}$ induced by the group G defined in Lemma 5.3. As seen in Lemma 5.3, the orbits of G on $\mathcal{Q}^+(5, q) \setminus (\pi_1 \cup \pi_2)$ are represented uniquely by the points $(1, \ w^s a_i^2)$ for $0 \le s < 4$, $1 \le i \le q+1$. Order these points

$$\begin{array}{lll} (1, a_1^2), & \ldots, & (1, a_{q+1}^2), \\ (1, \omega a_1^2), & \ldots, & (1, \omega a_{q+1}^2), \\ (1, \omega^2 a_1^2), & \ldots, & (1, \omega^2 a_{q+1}^2), \\ (1, \omega^3 a_1^2), & \ldots, & (1, \omega^3 a_{q+1}^2). \end{array} \tag{5.2}$$

Let $\boldsymbol{B}$ be the column sum matrix associated with the tactical decomposition of $\mathcal{Q}^+(5, q)$ induced by the group G, after throwing away the rows and columns corresponding to points in $(\pi_1 \cup \pi_2)$ (as in Theorem 2.3).

Definition 5.5. For $a, b \in \mathbb{E}^*$ with $\mathrm{T}(a) = \mathrm{T}(b) = 0$, and $0 \le s \le 3$, define

$$\kappa_s(a,b) := \left|\{(k,i) : 0 \le k < (q-1)/4,\ 0 \le i < q^2+q+1 \mid \mathrm{T}(\mu^i a) = -\omega^{4k+s}\,\mathrm{T}(\mu^{-i} b)\}\right|$$

Lemma 5.6. *The matrix $\boldsymbol{B}$ can be described as follows: for $0 \le s \le 3$, let $\boldsymbol{B}_s = \left[\kappa_s(a_i^2, a_j^2)\right]_{1 \le i,j \le q+1}$, where the a_i are as described in Equation 5.1. Then*

$$\boldsymbol{B} = \begin{bmatrix} (\boldsymbol{B}_0 - \boldsymbol{I}) & \boldsymbol{B}_1 & \boldsymbol{B}_2 & \boldsymbol{B}_3 \\ \boldsymbol{B}_3 & (\boldsymbol{B}_0 - \boldsymbol{I}) & \boldsymbol{B}_1 & \boldsymbol{B}_2 \\ \boldsymbol{B}_2 & \boldsymbol{B}_3 & (\boldsymbol{B}_0 - \boldsymbol{I}) & \boldsymbol{B}_1 \\ \boldsymbol{B}_1 & \boldsymbol{B}_2 & \boldsymbol{B}_3 & (\boldsymbol{B}_0 - \boldsymbol{I}) \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}_0' & \boldsymbol{B}_1 & \boldsymbol{B}_2 & \boldsymbol{B}_3 \\ \boldsymbol{B}_3 & \boldsymbol{B}_0' & \boldsymbol{B}_1 & \boldsymbol{B}_2 \\ \boldsymbol{B}_2 & \boldsymbol{B}_3 & \boldsymbol{B}_0' & \boldsymbol{B}_1 \\ \boldsymbol{B}_1 & \boldsymbol{B}_2 & \boldsymbol{B}_3 & \boldsymbol{B}_0' \end{bmatrix}. \tag{5.3}$$

Proof. This can be seen by observing the relationship between the functions κ_z and the polar form B of the quadratic form Q associated with $\mathcal{Q}^+(5,q)$. ☐

5.4 Eigenvectors of $\boldsymbol{B}$

We have that $\boldsymbol{B}$ is real symmetric and block-circulant. Eigenvectors of block-circulant matrices have been studied in [24], and can be found using the following result:

Theorem 5.7. *Let ζ be a fourth root of unity in $\mathbb{C}$, and let*

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_0 & \boldsymbol{A}_1 & \boldsymbol{A}_2 & \boldsymbol{A}_3 \\ \boldsymbol{A}_3 & \boldsymbol{A}_0 & \boldsymbol{A}_1 & \boldsymbol{A}_2 \\ \boldsymbol{A}_2 & \boldsymbol{A}_3 & \boldsymbol{A}_0 & \boldsymbol{A}_1 \\ \boldsymbol{A}_1 & \boldsymbol{A}_2 & \boldsymbol{A}_3 & \boldsymbol{A}_0 \end{bmatrix}.$$

Then if $\boldsymbol{v}$ is an eigenvector of $\boldsymbol{H} = \sum_{s=0}^{3} \zeta^s \boldsymbol{A}_s$, then

$$\boldsymbol{w} = \begin{bmatrix} \zeta^0 \boldsymbol{v} \\ \zeta^1 \boldsymbol{v} \\ \zeta^2 \boldsymbol{v} \\ \zeta^3 \boldsymbol{v} \end{bmatrix}$$

is an eigenvector of $\boldsymbol{A}$.

Putting $\zeta = \mathrm{i}$, we will find eigenvectors of $\boldsymbol{B}$ by considering the matrix

$$\boldsymbol{H} = \boldsymbol{B}_0' + \sum_{s=1}^{3} \mathrm{i}^s \boldsymbol{B}_s = \left(\sum_{s=0}^{3} \mathrm{i}^s \boldsymbol{B}_s\right) - \boldsymbol{I}. \tag{5.4}$$

The entries $\kappa_s(a_i^2, a_j^2)$ of $\boldsymbol{B}_s$ for $0 \le s < 3$ can be analyzed by the following relationship to the functions $\kappa_z(x)$ for $z \in \{1, \mathrm{i}, -1, -\mathrm{i}\}$ given in Definition 4.9.

Theorem 5.8. *Let $a, b \in \mathbb{E}^*$ with $\mathrm{N}(a) = \mathrm{N}(b) = 1$ and $\mathrm{T}(a^2) = \mathrm{T}(b^2) = 0$. Then*

$$\kappa_s(a^2, b^2) = \kappa_{\mathrm{i}^s \chi_2(2)}(ab) + \left(\frac{q-1}{4}\right) \epsilon_{ab}$$

where

$$\epsilon_{ab} = \left|\{j : \ \mathrm{T}(\mu^j ab) = \mathrm{T}(\mu^{-j} ab) = 0\}\right|.$$

Proof. Take a and b as above. Then $\kappa_s(a^2, b^2)$ counts the number of pairs (k, i), $0 \le k < (q-1)/4$, $0 \le i < q^2 + q + 1$, such that

$$\mathrm{T}(\mu^i a^2) + \omega^{4k+s}\, \mathrm{T}(\mu^{-i} b^2) = 0.$$

Since $\mathrm{N}(ab^{-1}) = 1$, there exists n with $0 \le n < q^2 + q + 1$ such that $\mu^n = ab^{-1}$. Then $\mu^{-(n-i)} a^2 = \mu^i ab$ and $\mu^{(n-i)} b^2 = \mu^{-i} ab$, so we can replace i with $(i - n)$ so that we are instead counting pairs (k, i) satisfying

$$\mathrm{T}(\mu^i ab) = -\omega^{4k+s}\, \mathrm{T}(\mu^{-i} ab). \tag{5.5}$$

Now, consider a pair (k, i) satisfying this equation. If $\mathrm{T}(\mu^{-i} ab) = 0$, then we must also have $\mathrm{T}(\mu^i ab) = 0$, and the choice of k is immaterial. Thus there are $\left(\frac{q-1}{4}\right) \epsilon_{ab}$ such pairs. If $\mathrm{T}(\mu^{-i} ab) \neq 0$ then, applying χ_4 to both sides of Equation 5.5, we have

$$\chi_4(\mathrm{T}(\mu^i ab)) = \chi_4(-\omega^{4k+s}) \chi_4(\mathrm{T}(\mu^{-i} ab)),$$

and since $\chi_4(\mathrm{T}(\mu^{-i} ab)) \neq 0$,

$$\chi_4(\mathrm{T}(\mu^i ab)) \overline{\chi}_4(\mathrm{T}(\mu^{-i} ab)) = \chi_4(-\omega^{4k+s}) = \mathrm{i}^s \chi_4(-1).$$

Now since $q \equiv 1 \bmod 4$, $\chi_4(-1) = \pm 1$, and $\chi_4(-1) = 1$ if and only if $q \equiv 1 \bmod 8$.

Notice that, with $q = p^h$ for some prime p, if h is even then $q \equiv 1 \bmod 8$ and every element of $\mathbb{F}_p$ is a square in $\mathbb{F}_q$ so $\chi_2(2) = 1$. On the other hand if h is odd, $\mathbb{F}_q$ is an odd-degree field extension of $\mathbb{F}_p$, so 2 is a square in $\mathbb{F}_q$ if and only if it is a square in $\mathbb{F}_p$. Since $h - 1$ is even, $q = p \cdot p^{h-1} \equiv p \bmod 8$, thus $\chi_2(2) = (-1)^{\frac{p^2-1}{8}} = 1$ if and only if $q \equiv 1 \bmod 8$. In either case, $\chi_4(-1) = \chi_2(2)$. $\square$

Corollary 5.9. *Let $a, b \in \mathbb{E}^*$ with $\mathrm{N}(a) = \mathrm{N}(b) = 1$ and $\mathrm{T}(a^2) = \mathrm{T}(b^2) = 0$. Then we have the following:*

(i) $\kappa_1(a^2, b^2) = \kappa_3(a^2, b^2)$;

(ii) $\kappa_0(a^2, b^2) - \kappa_2(a^2, b^2) = q \cdot \chi_2(2\,\mathrm{T}(ab))$;

(iii) *for each $c \in \mathbb{E}^* \setminus \{a, b\}$ with $\mathrm{N}(c) = 1$, $\mathrm{T}(c^2) = 0$, we have*

$$\kappa_0(a^2, c^2) - \kappa_2(a^2, c^2) = \kappa_0(b^2, c^2) - \kappa_2(b^2, c^2) \iff \chi_2(2\,\mathrm{T}(ab)) \neq -1.$$

Proof. We have (i) following directly from Lemma 4.11.

To see that (ii) holds, notice that since a and b have norm 1 they are both squares, $\chi_2(ab) = 1$; then by Theorem 4.12 we have

$$\begin{aligned}\kappa_0(a^2, b^2) - \kappa_2(a^2, b^2) &= \kappa_{\chi_2(2)}(ab) - \kappa_{-\chi_2(2)}(ab)\\ &= \chi_2(2)\left(\kappa_1(ab) - \kappa_{-1}(ab)\right)\\ &= q \cdot \chi_2(2\,\mathrm{T}(ab)).\end{aligned}$$

Finally to obtain (iii), consider $c \in \mathbb{E}^* \setminus \{a, b\}$ with $\mathrm{N}(c) = 1$ and $\mathrm{T}(c^2) = 0$. We can assume that $a \neq b$ since if so, $\kappa_0(a^2, c^2) - \kappa_2(a^2, c^2) = \kappa_0(b^2, c^2) - \kappa_2(b^2, c^2)$ and $\chi_2(2\,\mathrm{T}(ab)) = 0$. Then by Lemma 4.6, $\{a, b, c\}$ forms a basis for $\mathbb{E}$ over $\mathbb{F}$, and so applying Corollary 4.8, $\chi_2(2\,\mathrm{T}(ab)\,\mathrm{T}(ac)\,\mathrm{T}(bc)) = 1$. By (ii),

$$\kappa_0(a^2, c^2) - \kappa_2(a^2, c^2) = \kappa_0(b^2, c^2) - \kappa_2(b^2, c^2)$$

if and only if $\chi_2(2\,\mathrm{T}(ac)) = \chi_2(2\,\mathrm{T}(bc))$ and, since both are nonzero, this implies that $\chi_2(\mathrm{T}(ac))\chi_2(\mathrm{T}(bc)) = 1$. Thus we also have $\chi_2(2\,\mathrm{T}(ab)) = 1$. □

This result shows that $\boldsymbol{B}_1 = \boldsymbol{B}_3$ so we have that Equation (5.4) becomes

$$\boldsymbol{H} = \boldsymbol{B}_0 - \boldsymbol{B}_2 - \boldsymbol{I} = \left[\kappa_0(a_i^2, a_j^2) - \kappa_2(a_i^2, a_j^2)\right] - \boldsymbol{I}. \tag{5.6}$$

Applying Corollary 5.9 (ii), we have

$$\boldsymbol{H} = \left[q \cdot \chi_2(2\,\mathrm{T}(a_i a_j))\right] - \boldsymbol{I}. \tag{5.7}$$

Lemma 5.10. *Let $\boldsymbol{H}$ be as in Equation* (5.7)*, with the rows and columns indexed by the a_i from Equation* (5.1) *in the natural way. Put*

$$\begin{aligned}X_1 &= \{a_i :\ 1 \le i \le q+1 \mid \chi_2(2\,\mathrm{T}(a_1 a_i)) \neq -1\} \text{ and}\\ X_2 &= \{a_i :\ 1 \le i \le q+1 \mid \chi_2(2\,\mathrm{T}(a_1 a_i)) = -1\},\end{aligned}$$

and let $\boldsymbol{K}$ be the adjacency matrix of the graph $K_{X_1} \oplus K_{X_2}$ and $\boldsymbol{K}'$ be the adjacency matrix of the complementary graph. Then

$$\boldsymbol{H} = q\boldsymbol{K} - q\boldsymbol{K}' - I. \tag{5.8}$$

Proof. It is clear that the diagonal entries of $\boldsymbol{H}$ are -1, so we only need to consider the entries $\boldsymbol{H}_{ij}$ for $i \neq j$; also, since $\boldsymbol{H}$ is symmetric, we can assume that $i < j$. All entries of $\boldsymbol{H}$ that are not on the diagonal are equal to $\pm q$.

Assume that $\boldsymbol{H}_{ij} = -q$; then $\chi_2(2\,\mathrm{T}(a_i a_j)) = -1$. If $i = 1$, then $a_i \in X_1$ and $\chi_2(2\,\mathrm{T}(a_1 a_j)) = -1$, so $a_j \in X_2$. On the other hand, if $i \neq 1$ then $a_1 \in \mathbb{E}^* \setminus \{a_i, a_j\}$ and, applying Corollary 5.9 (iii), we must have $\chi_2(2\,\mathrm{T}(a_1 a_i)) \neq \chi_2(2\,\mathrm{T}(a_1 a_j))$, with both nonzero by Corollary 4.8. Therefore we can assume (WLOG) that $a_i \in X_1$ and $a_j \in X_2$. □

5.5 Proof of the main theorem

Theorem 5.11. *Let $q = p^h$ be a prime power with $q \equiv 5$ or $9 \bmod 12$. Then the hyperbolic quadric $\mathcal{Q}^+(5,q)$ has a decomposition $\pi_1 \cup \pi_2 \cup \mathcal{T}_1 \cup \mathcal{T}_2$ into disjoint tight sets, with π_1 and π_2 being generators, and $\mathcal{T}_1 \simeq \mathcal{T}_2$ being $\frac{1}{2}(q^2-1)$-tight sets.*

Proof. Define the sets X_1 and X_2 and matrices $\boldsymbol{K}$ and $\boldsymbol{K}'$ and $\boldsymbol{H}$ as in Lemma 5.10; then eigenvectors of the matrix $\boldsymbol{H} = q \cdot \boldsymbol{K} - q \cdot \boldsymbol{K}' - I$ can be used to obtain eigenvectors of the matrix $\boldsymbol{B}$ defined in Lemma 5.6, which is a column sum matrix associated to a tactical decomposition of the collinearity matrix $\boldsymbol{A}$ of $\mathcal{Q}^+(5,q)$ after throwing away rows and columns corresponding to points in disjoint generators π_1 and π_2.

Now we form the vector

$$\boldsymbol{v} = \boldsymbol{c}_{X_1} - \frac{1}{2}\boldsymbol{j} = \frac{1}{2}\left(\boldsymbol{c}_{X_1} - \boldsymbol{c}_{X_2}\right),$$

where $\boldsymbol{c}_{X_1}$ and $\boldsymbol{c}_{X_2}$ are the characteristic vectors of the sets X_1 and X_2. We have that

$$\begin{aligned}
\boldsymbol{v}\boldsymbol{H} &= \frac{1}{2}\left(\boldsymbol{c}_{X_1} - \boldsymbol{c}_{X_2}\right)\left(q\boldsymbol{K} - q\boldsymbol{K}' - I\right) \\
&= \frac{1}{2}[(q\left(|X_1|-1\right)\boldsymbol{c}_{X_1} - q|X_1|\boldsymbol{c}_{X_2} - \boldsymbol{c}_{X_1}) - (q\left(|X_2|-1\right)\boldsymbol{c}_{X_2} - q|X_2|\boldsymbol{c}_{X_1} - \boldsymbol{c}_{X_2})] \\
&= \frac{1}{2}\left(q\left(|X_1|+|X_2|-1\right)-1\right)\boldsymbol{c}_{X_1} - \left(\left(|X_1|+|X_2|-1\right)-1\right)\boldsymbol{c}_{X_2} \\
&= \frac{1}{2}(q^2-1)(\boldsymbol{c}_{X_1} - \boldsymbol{c}_{X_2}).
\end{aligned}$$

So $\boldsymbol{v}$ is an eigenvector of $\boldsymbol{H}$ for eigenvalue (q^2-1) and by Theorem 5.7, putting

$$\boldsymbol{w}_1 = \begin{bmatrix} \boldsymbol{v} \\ 0 \\ -\boldsymbol{v} \\ 0 \end{bmatrix}, \qquad \boldsymbol{w}_2 = \begin{bmatrix} 0 \\ \boldsymbol{v} \\ 0 \\ -\boldsymbol{v} \end{bmatrix},$$

we have that $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are eigenvectors of $\boldsymbol{B}$ for this same eigenvalue. Applying Theorem 2.3, vectors of the form

$$\boldsymbol{w} = \pm\boldsymbol{w}_1 \pm \boldsymbol{w}_2$$

correspond to $\frac{1}{2}(q^2-1)$-tight sets of $\mathcal{Q}^+(5,q)$ which are disjoint from $(\pi_1 \cup \pi_2)$.

Put $\mathcal{T}_1$ to be the tight set corresponding to $\boldsymbol{w}_1 + \boldsymbol{w}_2$ and $\mathcal{T}_2$ to be the tight set corresponding to $-\boldsymbol{w}_1 - \boldsymbol{w}_2$; then it is clear that $\mathcal{T}_1$ and $\mathcal{T}_2$ are disjoint. Comparing the vectors $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ to the ordering of the entries of $\boldsymbol{B}$ shown in Equation (5.2), we see that

$$\begin{aligned}
\mathcal{T}_1 &= \left(\cup_{a_i \in X_1}\left((1,a_i)^G \cup (1,\omega a_i)^G\right)\right) \cup \left(\cup_{a_j \in X_2}\left((1,\omega^2 a_j)^G \cup (1,\omega^3 a_j)^G\right)\right) \text{ and} \\
\mathcal{T}_2 &= \left(\cup_{a_i \in X_2}\left((1,a_i)^G \cup (1,\omega a_i)^G\right)\right) \cup \left(\cup_{a_j \in X_1}\left((1,\omega^2 a_j)^G \cup (1,\omega^3 a_j)^G\right)\right),
\end{aligned}$$

where G is the group defined in Lemma 5.3. It is clear that the projective similarity $(x, y) \mapsto (x, \omega^2 y)$ sends the set $\mathcal{T}_1$ to $\mathcal{T}_2$. $\square$

Note that there are a couple of decisions made in the proof of this theorem. The first is the ordering of the a_i; the choice of a_1 affects the definitions of the sets X_1 and X_2. However, ordering these elements differently can only possibly interchange the role of X_1 and X_2 (and so interchange the tight sets $\mathcal{T}_1$ and $\mathcal{T}_2$). Also, in defining the two tight sets, we could just as easily put $\mathcal{T}_1'$ to be the tight set corresponding to $\boldsymbol{w}_1 - \boldsymbol{w}_2$ and $\mathcal{T}_2'$ to be the tight set corresponding to $-\boldsymbol{w}_1 + \boldsymbol{w}_2$; this would make

$$\mathcal{T}_1' = \left(\cup_{a_i \in X_1} \left((1, a_i)^G \cup (1, \omega^3 a_i)^G\right)\right) \cup \left(\cup_{a_j \in X_2} \left((1, \omega^2 a_j)^j \cup (1, \omega a_j)^G\right)\right) \text{ and}$$
$$\mathcal{T}_2' = \left(\cup_{a_i \in X_2} \left((1, a_i)^G \cup (1, \omega^3 a_i)^G\right)\right) \cup \left(\cup_{a_j \in X_1} \left((1, \omega^2 a_j)^j \cup (1, \omega a_j)^G\right)\right).$$

It is clear in this case that, under the projective similarity $(x, y) \mapsto (x, \omega y)$, $\mathcal{T}_1' \mapsto \mathcal{T}_1$ and $\mathcal{T}_2' \mapsto \mathcal{T}_2$.

Let $\phi : \mathbb{F}_{q^3} \to \mathbb{F}_{q^3}$: $x \mapsto x^q$. Hence ϕ has order 3. The map $e : (u, v) \mapsto (\phi(u), \phi(v))$ is a semi-similarity of the formed space (V, f). It induces a collineation of $\mathcal{Q}^+(5, q)$. It is straightforward to check that ϕ is a permutation of $\mathcal{S}$. But since $T(\phi(x)) = T(x)$, and ϕ maps the squares of $\mathbb{F}_{q^3}$ to squares, ϕ is also a permutation of the sets X_1 and X_2. Also, it can be seen that the map $o : (u, v) \mapsto (v, \omega u)$ is a collineation of $\mathcal{Q}^+(5, q)$ that stabilizes $\mathcal{T}_1$ and $\mathcal{T}_2$. The following corollary is now obvious.

Corollary 5.12. *The tight sets constructed in Theorem 5.11 are stabilized by the group $\langle c, z, e, o\rangle$. This group has order $3\frac{(q-1)}{2}(q^2 + q + 1)$.*

Note that the map o interchanges the two sets of generators of $\mathcal{Q}^+(5, q)$, and thus does not induce a collineation of $\mathrm{PG}(3, q)$ under the Klein correspondence.

6 The Klein correspondence and Cameron–Liebler line classes

The Klein correspondence maps points of $\mathcal{Q}^+(5, q)$ to lines of $\mathrm{PG}(3, q)$, with collinear points of $\mathcal{Q}^+(5, q)$ corresponding to intersecting lines of $\mathrm{PG}(3, q)$. Each generator of $\mathcal{Q}^+(5, q)$ then corresponds to a set of q^2+q+1 pairwise intersecting lines of $\mathrm{PG}(3, q)$, and so it's image is either of the form star($\boldsymbol{p}$) for some point $\boldsymbol{p}$ or line(π) for some plane π depending on the system the generator came from. We will assume that the generators in the same system as π_1 correspond to the point stars of $\mathrm{PG}(3, q)$, and that the generators in the same system as π_2 correspond to the planes. It is then clear that any collineation of $\mathcal{Q}^+(5, q)$ induces either a collineation or a correlation of $\mathrm{PG}(3, q)$ depending on whether it stabilizes or interchanges the two systems of generators.

Under the Klein correspondence, a x-tight set corresponds to a set of lines in $\mathrm{PG}(3, q)$ called a Cameron–Liebler line class with parameter x; while there are many equivalent definitions, we will find the following characterizations first described in [20] most useful.

Theorem 6.1. *A set of lines $\mathcal{L}$ is a Cameron–Liebler line class with parameter x if and only if the following equivalent conditions are met:*

(i) *for every incident point-plane pair $(\boldsymbol{r}, \tau)$ we have*

$$|\operatorname{star}(\boldsymbol{r}) \cap \mathcal{L}| + |\operatorname{line}(\tau) \cap \mathcal{L}| = x + (q+1)|\operatorname{pencil}(\boldsymbol{r}, \tau) \cap \mathcal{L}|;$$

(ii) *for every line $\ell \in \mathrm{PG}(3, q)$, the number of lines $m \in \mathcal{L}$ distinct from ℓ and intersecting ℓ nontrivially is given by*

$$x(q+1) + (q^2-1)\boldsymbol{c}_{\mathcal{L}}(\ell).$$

The result of Theorem 5.11 gives a decomposition of the points of $\mathcal{Q}^+(5, q)$ into π_1, π_2, $\mathcal{T}_1$, $\mathcal{T}_2$, where $\mathcal{T}_1$ and $\mathcal{T}_2$ are isomorphic $\frac{(q-1)}{2}$-tight sets. We have that $\mathcal{T}_1$ and $\mathcal{T}_2$ are stabilized by an abelian group $G = \langle c, z\rangle$, where c and z are the maps defined (as in Lemma 5.3) by

$$c : (x, y) \mapsto (\mu x, \mu^{-1} y) \qquad\qquad z : (x, y) \mapsto (x, \omega^4 y).$$

Under the Klein correspondence, π_1 corresponds to the set $\operatorname{star}(\boldsymbol{p})$ of lines through a common point $\boldsymbol{p}$ in $\mathrm{PG}(3, q)$, π_2 to the set $\operatorname{line}(\pi)$ of lines in a common plane $\pi \not\ni \boldsymbol{p}$, and $\mathcal{T}_1$ and $\mathcal{T}_2$ to isomorphic Cameron–Liebler line classes $\mathcal{L}_1$ and $\mathcal{L}_2$ each having parameter $\frac{(q^2-1)}{2}$. The group $G = \langle c, z\rangle$ induces a collineation group of $\mathrm{PG}(3, q)$ (which for convenience we will also call G) stabilizing these four sets of lines.

Theorem 6.2. *The orbits of G on the points of $\mathrm{PG}(3, q)$ are as follows:*

(i) *$\boldsymbol{p}$ is fixed;*

(ii) *the points on π all fall into a single orbit of size $q^2 + q + 1$;*

(iii) *the remaining points fall into four orbits, each having size $\frac{(q-1)}{4}(q^2+q+1)$.*

Proof. We will prove this by considering the action of G on the system of generators of $\mathcal{Q}^+(5, q)$ corresponding to points in $\mathrm{PG}(3, q)$. Any generator in this system different from π_1 must meet π_1 in a single point, and must either be disjoint from π_2 or else meet π_2 in a line. Since S is transitive on the points of π_1, we consider an arbitrary point $\boldsymbol{s} = (\mu^s, 0) \in \pi_1$. There are $q+1$ generators on $\boldsymbol{s}$ in each system; the system we are interested in contains π_1, a unique generator meeting π_2 in a line, and $q-1$ others. The stabilizer in G of $\boldsymbol{s}$ is simply $\langle z\rangle$, which fixes π_1 and stabilizes the unique plane on $\boldsymbol{s}$ which meets π_2 in a line. Consider one of the other planes $\pi' = \langle \boldsymbol{s}, (x_1, y_1), (x_2, y_2)\rangle$. We must have

$$\mathrm{T}(\mu^s y_1) = \mathrm{T}(\mu^s y_2) = \mathrm{T}(x_1 y_1) = \mathrm{T}(x_2 y_2) = \mathrm{T}(x_1 y_2) + \mathrm{T}(x_2 y_1) = 0.$$

Since π' is disjoint from $\pi_2 = \{(0, a) : a \in \mathbb{E}^*\}$, we must have that $\{\mu^s, x_1, x_2\}$ is linearly independent over $\mathbb{F}$. Therefore,

$$x_1 \notin \{c : \mathrm{T}(c y_2) = 0\} = \langle \mu^s, x_2\rangle.$$

Now z^k maps $(x_2 y_2)$ to $(x_2, \omega^{4k} y_2)$, and since $\mathrm{T}(x_1 y_2) \neq 0$,

$$\mathrm{T}(\omega^{4k} x_1 y_2) + \mathrm{T}(x_2 y_1) = (\omega^{4k} - 1)\,\mathrm{T}(x_1 y_2) \neq 0$$

unless $\omega^{4k} = 1$, in which case z^k is the identity map. Thus $\langle z \rangle$ acts semiregularly on the planes through $\boldsymbol{s}$ distinct from π_1 and disjoint from π_2. The result follows immediately from $|\langle z \rangle|$, the transitivity of G on π_1, and the Klein mapping from $\mathcal{Q}^+(5, q)$ to $\mathrm{PG}(3, q)$. □

We want to consider the number of lines of $\mathcal{L}_1$ through the various points of $\mathrm{PG}(3, q)$; some information can be deduced immediately.

Lemma 6.3.

(i) $|\operatorname{line}(\pi) \cap \mathcal{L}_1| = 0$;

(ii) $|\operatorname{star}(\boldsymbol{p}) \cap \mathcal{L}_1 = 0$;

(iii) *for any plane* $\tau \ni \boldsymbol{p}$, $|\operatorname{line}(\tau) \cap \mathcal{L}_1| = \frac{(q^2-1)}{2}$;

(iv) *for any point* $\boldsymbol{r} \in \pi$, $|\operatorname{star}(\boldsymbol{r}) \cap \mathcal{L}_1| = \frac{(q^2-1)}{2}$.

Proof. (i) and (ii) follow directly from our construction of $\mathcal{T}_1$. (iii) and (iv) follow from applying Theorem 6.1 to $(\boldsymbol{p}, \tau)$ and $(\boldsymbol{r}, \pi)$, respectively. □

We need to find the number of lines of $\mathcal{L}_1$ through the points $\boldsymbol{r} \neq \boldsymbol{p}$ with $\boldsymbol{r} \notin \pi$. If we apply Theorem 6.1 to $(\boldsymbol{r}, \tau)$ for any plane τ on $\ell = \langle \boldsymbol{p}, \boldsymbol{r} \rangle$, we see that

$$|\operatorname{star}(\boldsymbol{r}) \cap \mathcal{L}_1| = (q+1)|\operatorname{pencil}(\boldsymbol{r}, \tau) \cap \mathcal{L}_1|.$$

This tells us that $|\operatorname{star}(\boldsymbol{r}) \cap \mathcal{L}_1|$ is divisible by $(q+1)$. Since the points $\boldsymbol{r}$ fall into four orbits under G, we will let

$$(q+1)a_1 \leq (q+1)a_2 \leq (q+1)a_3 \leq (q+1)a_4$$

be the values of $|\operatorname{star}(\boldsymbol{r}) \cap \mathcal{L}|$ for $\boldsymbol{r}$ in each of these four orbits. We can also see from this formula that $|\operatorname{pencil}(\boldsymbol{r}, \tau) \cap \mathcal{L}_1|$ depends only on $\boldsymbol{r}$ and not on the choice of the plane τ containing ℓ. We can gain some information on the values of $a_1, \dots, a_4$ by utilizing the following concept introduced by Gavrilyuk and Mogilnykh in [10].

Definition 6.4. Let $\mathcal{L}$ be a Cameron–Liebler line class with parameter x in $\mathrm{PG}(3, q)$, and let ℓ be a line of $\mathrm{PG}(3, q)$. Number the points on ℓ as $\boldsymbol{p}_1, \dots \boldsymbol{p}_{q+1}$, and the planes containing ℓ as $\pi_1, \dots \pi_{q+1}$. We define the *pattern* of $\mathcal{L}$ with respect to ℓ to be the $(q+1) \times (q+1)$ matrix $\mathcal{T}(\ell)$ given by

$$\mathcal{T}(\ell) = \begin{bmatrix} t_{1,1} & \dots & t_{1,q+1} \\ \vdots & \dots & \vdots \\ t_{q+1,1} & \dots & t_{q+1,q+1} \end{bmatrix},$$

where $t_{i,j} = |\mathcal{L} \cap \operatorname{pencil}(\boldsymbol{p_i}, \pi_j) \setminus \{\ell\}|$.

Theorem 6.5. *For every line $\ell \in \mathrm{PG}(3,q)$, the pattern $\mathcal{T}(\ell)$ has the following properties:*

(i) $0 \le t_{i,j} \le q$ *for all* i,j*;*

(ii) $\sum_{i,j} t_{i,j} = x(q+1) + \boldsymbol{c}_{\mathcal{L}}(\ell)(q^2-1)$*;*

(iii) $\sum_j t_{k,j} + \sum_i t_{i,l} = x + (q+1)t_{k,l} + (q-1)\boldsymbol{c}_{\mathcal{L}}(\ell)$ *for all* k,l*;*

(iv) $\sum_{i,j} t_{i,j}^2 = (x - \boldsymbol{c}_{\mathcal{L}}(\ell))^2 + q(x - \boldsymbol{c}_{\mathcal{L}}(\ell)) + \boldsymbol{c}_{\mathcal{L}}(\ell)q^2(q+1)$.

Lemma 6.6. *Let ℓ be a line on $\boldsymbol{p}$; we will number the points on ℓ as $\boldsymbol{r}_1, \dots, \boldsymbol{r}_{q+1}$ and the planes as $\tau_1, \dots, \tau_{q+1}$. We will take $\boldsymbol{r}_1 = \boldsymbol{p}$, $\boldsymbol{r}_2 = \ell \cap \pi$, and $\tau_1 = \pi$. Then the pattern of $\mathcal{L}_1$ with respect to ℓ is given (up to a permutation of the columns) by*

$$\mathcal{T}(\ell) = \begin{bmatrix} 0 & \frac{q-1}{2} & a_1 & \dots & a_1 & a_2 & \dots & a_2 & a_3 & \dots & a_3 & a_4 & \dots & a_4 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & \frac{q-1}{2} & a_1 & \dots & a_1 & a_2 & \dots & a_2 & a_3 & \dots & a_3 & a_4 & \dots & a_4 \end{bmatrix}^T$$

where each a_i is repeated in $\frac{q-1}{4}$ rows.

Proof. Since $|\operatorname{star}(\boldsymbol{p}) \cap \mathcal{L}_1| = 0$, it is clear that $|\operatorname{pencil}(\boldsymbol{p}, \tau) \cap \mathcal{L}_1| = 0$ for every plane τ on ℓ. Also, $\boldsymbol{r}_2 \in \pi$ so $|\operatorname{star}(\boldsymbol{r}_2) \cap \mathcal{L}_1$; applying Theorem 6.1, we see that $|\operatorname{pencil}(\boldsymbol{r}_2, \tau) \cap \mathcal{L}_1| = \frac{q-1}{2}$ for all τ on ℓ. For $\boldsymbol{r}_i$, $2 < i \le q+1$, the value $|\operatorname{pencil}(\boldsymbol{r}_i, \tau) \cap \mathcal{L}_1| \in \{a_1, a_2, a_3, a_4\}$ depends only on which orbit $\boldsymbol{r}_i$ falls into under G. The proof of Theorem 6.2 makes it clear that the points of $\ell \setminus \{\boldsymbol{r}_1, \boldsymbol{r}_2\}$ are evenly distributed among these four orbits. □

Lemma 6.7. *The values a_1, a_2, a_3, a_4 satisfy the following properties:*

(i) $a_4 = q - a_1$ *and* $a_3 = q - a_2$*;*

(ii) $a_1(q-a_1) + a_2(q-a_2) = \frac{q(q-1)}{2}$*;*

(iii) $\frac{q-\sqrt{2q-1}}{2} \le a_1 \le \frac{(q-\sqrt{q})}{2} \le a_2 \le \frac{(q-1)}{2}$*;*

Proof. (i) comes from noticing that, for any point $\boldsymbol{r} \neq \boldsymbol{p}$ with $\boldsymbol{r} \notin \pi$, every line through $\boldsymbol{r}$ that does not contain $\boldsymbol{r}$ is in either $\mathcal{L}_1$ or $\mathcal{L}_2$; therefore if $|\operatorname{star}(\boldsymbol{r}) \cap \mathcal{L}_1| = a \le \frac{q-1}{2}$, we have $|\operatorname{star}(\boldsymbol{r}) \cap \mathcal{L}_2| = (q-a) \ge \frac{q-1}{2}$. Since $\mathcal{L}_1 \simeq \mathcal{L}_2$, this means there exists a point $\boldsymbol{r}'$ with $|\operatorname{star}(\boldsymbol{r}') \cap \mathcal{L}_1| = (q-a)$. We see that (ii) holds by applying Theorem 6.5 (iv) and combining the result with (i) using $x = \frac{(q^2-1)}{2}$. For (iii), it is clear that $a_2 \le \frac{(q-1)}{2}$ since $a_2 \le a_3 = (q-a_2)$ (and q is odd). This implies that $a_2(q-a_2) \le \frac{(q^2-1)}{4}$ and so $a_1(q-a_1) \ge \frac{q^2-2q+1}{4}$, therefore $a_1 \ge \frac{q+\sqrt{2q-1}}{2}$. Finally, if $a_2 < \frac{(q-\sqrt{q})}{2}$ then $a_2(q-a_2) < \frac{q(q-1)}{4}$;

this forces us to have $a_1(q-a_1) > \frac{q(q-1)}{4}$, which is impossible since $a_1 \le a_2$. Therefore we have $a_1 \le \frac{(q-\sqrt{q})}{2} \le a_2$. $\square$

A closer look at Lemma 6.7 (ii) tells us that

$$a_2^2 \equiv -a_1^2 \bmod q.$$

When $q \equiv 9 \bmod 12$, we must have $q = 3^{2e}$ for some integer e. In this case, the only way for $a_2^2 \equiv -a_1^2 \bmod q$ is for $a_1 \equiv a_2 \equiv 0 \bmod \sqrt{q}$; by Lemma 6.7 (iii), this forces $a_1 = a_2 = \frac{(q-\sqrt{q})}{2}$ (and so $a_3 = a_4 = \frac{(q+\sqrt{q})}{2}$).

Theorem 6.8. *For any positive integer e, there is a symmetric tactical decomposition of* $\mathrm{PG}(3, 3^{2e})$ *having four line classes given by* $\mathrm{star}(\boldsymbol{p})$, $\mathrm{line}(\pi)$, $\mathcal{L}_1$, $\mathcal{L}_2$, *and four point classes given by* $\{\boldsymbol{p}\}$, π, $\mathcal{P}_1 = \{\boldsymbol{r} \in \mathrm{PG}(3,q) \mid |\mathrm{star}(\boldsymbol{r}) \cap \mathcal{L}_1| = (3^{2e}+1)(3^{2e}-3^e)\}$, $\mathcal{P}_2 = \{\boldsymbol{r} \in \mathrm{PG}(3,q) \mid |\mathrm{star}(\boldsymbol{r}) \cap \mathcal{L}_1| = (3^{2e}+1)(3^{2e}+3^e)\}$.

Proof. The comments above make it clear that the given decomposition is point-tactical, with Table 1 giving the number of lines from each line class on a point from each point class. To see that the decomposition is also block-tactical, consider a line ℓ in $\mathrm{PG}(3, 3^{2e})$. If $\boldsymbol{p} \in \ell$, then ℓ contains $\boldsymbol{p}$ and a single point of π; by our remarks in the proof of Theorem 6.2, the remaining $(3^{2e}-1)$ points are distributed evenly between $\mathcal{P}_1$ and $\mathcal{P}_2$. If $\ell \subset \pi$, then all $(3^{2e}+1)$ points of ℓ are in π. If $\ell \in \mathcal{L}_1$, then we have ℓ meeting π in a single point and the remaining 3^{2e} points on ℓ are either in $\mathcal{P}_1$ or $\mathcal{P}_2$. Let $m = |\ell \cap \mathcal{P}_1|$ and $n = |\ell \cap \mathcal{P}_2|$, so $m + n = 3^{2e}$. Since $\ell \notin \mathcal{L}_2$, we have that ℓ meets $\frac{(3^{2e}+1)(3^{4e}-1)}{2}$ lines of $\mathcal{L}_2$; the point in $\ell \cap \pi$ is on $\frac{(3^{4e}-1)}{2}$ lines of $\mathcal{L}_2$, the m points in $\ell \cap \mathcal{P}_1$ are each on $\frac{(3^{2e}+1)(3^{2e}+3^e)}{2}$, and the n points in $\ell \cap \mathcal{P}_2$ are each on $\frac{(3^{2e}+1)(3^{2e}-3^e)}{2}$. This allows us to compute

$$\frac{(3^{4e}-1)}{2} + m\frac{(3^{2e}+1)(3^{2e}+3^e)}{2} + n\frac{(3^{2e}+1)(3^{2e}-3^e)}{2} = \frac{(3^{2e}+1)(3^{4e}-1)}{2}, \text{ or}$$

$$m(3^{2e}+3^e) + (3^{2e}-m)(3^{2e}-3^e) = 3^{4e} - 3^{2e}.$$

From this, we see that $m = \frac{3^{2e}-3^e}{2}$ and $n = \frac{3^{2e}+3^e}{2}$. A similar argument allows us to compute the intersection numbers of a line $\ell \in \mathcal{L}_2$ with the four point classes, showing that our decomposition is block-tactical (with Table 2 giving the number of points from each point class on a line from each line class) as well as point-tactical. $\square$

A set of type (m,n) in a projective or affine space is a set $\mathcal{K}$ of points such that every line of the space contains either m or n points of $\mathcal{K}$; we require that $m < n$, and that both values occur (these sets are also frequently called two-intersection sets or two-character sets). For projective spaces, there are many examples of these types of sets with q both even and odd. However the situation is quite different for affine spaces. When q is even, we obtain a set of type $(0,2)$ in $\mathrm{AG}(2,q)$ from a hyperoval of the corresponding projective plane, and similarly

	star($\boldsymbol{p}$)	line(π)	$\mathcal{L}_1$	$\mathcal{L}_2$
$\{\boldsymbol{p}\}$	$3^{4e}+3^{2e}+1$	0	0	0
π	1	$3^{2e}+1$	$\frac{3^{4e}-1}{2}$	$\frac{3^{4e}-1}{2}$
$\mathcal{P}_1$	1	0	$\frac{(3^{2e}+1)(3^{2e}-3^e)}{2}$	$\frac{(3^{2e}+1)(3^{2e}+3^e)}{2}$
$\mathcal{P}_2$	1	0	$\frac{(3^{2e}+1)(3^{2e}+3^e)}{2}$	$\frac{(3^{2e}+1)(3^{2e}-3^e)}{2}$

Table 1: *Lines per point for the symmetric tactical decomposition induced on* $\mathrm{PG}(3, 3^{2e})$.

	star($\boldsymbol{p}$)	line(π)	$\mathcal{L}_1$	$\mathcal{L}_2$
$\{\boldsymbol{p}\}$	1	0	0	0
π	1	$3^{2e}+1$	1	1
$\mathcal{P}_1$	$\frac{3^{2e}-1}{2}$	0	$\frac{3^{2e}-3^e}{2}$	$\frac{3^{2e}+3^e}{2}$
$\mathcal{P}_2$	$\frac{3^{2e}-1}{2}$	0	$\frac{3^{2e}+3^e}{2}$	$\frac{3^{2e}-3^e}{2}$

Table 2: *Points per line for the symmetric tactical decomposition induced on* $\mathrm{PG}(3, 3^{2e})$.

a set of type $(0, n)$ from a maximal arc of degree n. Examples of sets of type (m, n) in affine planes of odd order, on the other hand, are extremely scarce. The only previously known examples are those described in [21] (in affine planes of order 9), along with an example in $\mathrm{AG}(2, 81)$ described in [22] and [23]. A result from [21] gives us the following:

Lemma 6.9. *If we have a set $\mathcal{K}$ of type (m, n) in an affine plane of order q, then $k = |\mathcal{K}|$ must satisfy*

$$k^2 - k(q(n + m - 1) + n + m) + mnq(q + 1) = 0.$$

Corollary 6.10. *There exists a set of type $(3^{2e} - 3^e, 3^{2e} + 3^e)$ having size $\frac{(3^{4e}-3^{2e})}{2}$ in $\mathrm{AG}(2, 3^{2e})$.*

Proof. If we take a plane $\tau \neq \pi$ that does not contain $\boldsymbol{r}$ then all points of the affine plane $\tau' = \tau \setminus (\tau \cap \pi)$ lie in either $\mathcal{P}_1$ or $\mathcal{P}_2$, and all lines lie in either $\mathcal{L}_1$ or $\mathcal{L}_2$. Putting $\mathcal{K} = \mathcal{P}_1 \cap \tau'$, the values in Table 2 give the intersection numbers of the lines of the plane with the set $\mathcal{K}$. The size of $\mathcal{K}$ follows from Lemma 6.9. □

7 Final remarks

While finishing this manuscript, Koji Momihara, Tao Feng and Qing Xiang informed us that they had proven at almost the same time, and independently, the existence of a Cameron-Liebler line class of $\mathrm{PG}(3, q)$ with parameter $\frac{q^2-1}{2}$ for $q \equiv 5$ or $9 \bmod 12$, see [18]. They became aware of our result through the availability of the abstracts of the conference "Combinatorics 2014", held in june in Gaeta, Italy, where our result was presented, and were so kind to inform us about their result. As their approach is slightly more algebraic, and our approach more geometric, we all decided that publishing both manuscripts independently from each other was justified.

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